Some Problems with Entropy Measures for the Atanassov Intuitionistic Fuzzy Sets

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Abstract. This paper is a continuation of our previous papers on entropy of the Atanassov intuitionistic fuzzy sets (A-IFSs, for short)[1]. We discuss the necessity of taking into account all three functions (membership, non-membership and hesitation margin) describing A-IFSs while considering the entropy.

Keywords: Intuitionistic fuzzy sets, entropy.

1 Introduction

Fuzziness, a feature of imperfect information, results from the lack of a crisp distinction between the elements belonging and not belonging to a set (i.e. the boundaries of a set under consideration are not sharply defined). A measure of fuzziness often used and cited in the literature is called an entropy (first mentioned by Zadeh [27]).

De Luca and Termini [4] introduced some requirements which capture our intuitive comprehension of a degree of fuzziness. Kaufmann (1975) (cf. [11]) proposed to measure a degree of fuzziness of a fuzzy set A by a metric distance between its membership function and the membership (characteristic) function of its nearest crisp set. Yager [26] viewed a degree of fuzziness in terms of a lack of distinction between the fuzzy set and its complement. Higashi and Klir [3] extended Yager's concept to a general class of fuzzy complements. Yager's approach was also further developed by Hu and Yu [8]. Indeed, it is the lack of distinction between sets and their complements that distinguishes fuzzy sets from crisp sets. The less the fuzzy set differs from its complement, the fuzzier it is. Kosko [10] investigated the fuzzy entropy in relation to a measure of subsethood. Fan at al. [5], [6], [7] generalized Kosko's approach.

Here we discuss measures of fuzziness for intuitionistic fuzzy sets which are a generalization of fuzzy sets. We recall a measure of entropy we introduced (Szmidt and Kacprzyk [16], [22]). We compare our approach with Zeng and Li [28] approach. We discuss the reasons of differences and the counter-intuitive results obtained in the case of Zeng and Li's entropy which boils down to entropy given by Hung [9] (cf. Szmidt and Kacprzyk [22] for further discussion).

[1] There is currently a discussion on the appropriateness of the name *intuitionistic fuzzy set* introduced by Atanassov. However, this is beyond the scope of this paper which is just concerned with an application of the concept.

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2 A Brief Introduction to A-IFSs

One of the possible generalizations of a fuzzy set in X (Zadeh [27]), given by

$$A^{'} = \{< x, \mu_{A'}(x) > | x \in X\} \quad (1)$$

where $\mu_{A'}(x) \in [0, 1]$ is the membership function of the fuzzy set $A^{'}$, is an A-IFS, i.e. Atanassov's intuitionistic fuzzy set, (Atanassov [1], [2]) A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \quad (2)$$

where: $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ such that

$$0 \underline{<} \mu_A(x) + \nu_A(x) \underline{<} 1 \quad (3)$$

and $\mu_A(x)$, $\nu_A(x) \in [0, 1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

Obviously, each fuzzy set may be represented by the following A-IFS

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \quad (4)$$

For each A-IFS in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \quad (5)$$

an *intuitionistic fuzzy index* (or a *hesitation margin*) of $x \in A$, and it expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [2]). It is obvious that $0 \underline{<} \pi_A(x) \underline{<} 1$, for each $x \in X$.

In our further considerations we will use the complement set A^C [2]

$$A^C = \{< x, \nu_A(x), \mu_A(x) > | x \in X\} \quad (6)$$

In our further considerations we will use the normalized Hamming distance between fuzzy sets A, B in $X = \{x_{1,}, \ldots, x_n\}$ Szmidt and Baldwin [13], [14], Szmidt and Kacprzyk [15], [21]:

$$\begin{aligned} l_{IFS}(A, B) = \frac{1}{2n} \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \\ + \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|) \end{aligned} \quad (7)$$

For (7) we have: $0 \underline{<} l_{IFS}(A, B) \underline{<} 1$. Clearly the normalized Hamming distance (7) satisfies the conditions of the metric. In Szmidt and Kacprzyk [15], Szmidt and Baldwin [13], [14], and especially in Szmidt and Kacprzyk [21] it is shown why when calculating distances between IFSs we should take into account all three functions describing A-IFSs.

Applications of A-IFSs to group decision making, negotiations, etc. are presented in (Szmidt and Kacprzyk [17,19,20]).

3 Entropy

The entropy measures the whole missing information which may be necessary to have no doubts when classifying an element, i.e. to say that an element fully belongs or fully does not belong to a set considered.

3.1 Zeng and Li's Entropy Measure

We cite here Zeng and Li's entropy measure [28] for an A-IFSs A (notation used in [28] is changed so that they are consistent with those in this paper):

$$E_{ZL}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i) + \mu_A(x_i) + \pi_A(x_i) - 1| \tag{8}$$

Having in mind that for A-IFSs we have $\mu_.(x_i) + \nu_.(x_i) + \pi_.(x_i) = 1$, Zeng and Li's entropy measure (8) becomes

$$E_{ZL}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i) - \nu_A(x_i)| \tag{9}$$

In other words, Zeng and Li's similarity measure (9) does not take into account the values of $\pi_A(x_i)$. Only the values of the memberships and non-memberships are taken into account.

In Szmidt and Kacprzyk [22] we discussed in more detail the above measure (9). Although all the mathematical "constructions" of this measure are correct, the question arises if we may use any mathematically correct approach to represent the measures which by definition are to render some properties that have a concrete semantic meaning, and are in most cases to be useful. It seems that the mathematical correctness is in this context for sure a necessary but not a sufficient condition. The same conclusions are drawn in Szmidt and Kacprzyk [23] for similarity measures.

Now we will recall briefly another approach (cf. Szmidt and Kacprzyk [22] which is not only mathematically correct but at the same time rendering the sense of entropy not as a pure mathematical construction but as a measure to be useful in practice.

3.2 Szmidt and Kacprzyk's Entropy for A-IFSs

In Szmidt and Kacprzyk [22] we gave a motivation and revised some conditions for entropy measures for A-IFSs. Here we only recall one of the possible entropy measures fulfilling the new conditions (cf. Szmidt and Kacprzyk [22]) and rendering the very meaning of entropy.

Entropy for an A-IFS A with n elements may be given as (Szmidt and Kacprzyk [16]):

$$E(A) = \frac{1}{n}\sum_{i=1}^{n}\frac{d(F_i, F_{i,near})}{d(F_i, F_{i,far})} \tag{10}$$

where $d(F_i, F_{i,near})$ is a *distance* from F_i to its the nearer point $F_{i,near}$ among $M(1,0,0)$ and $N(0,1,0)$, and $d(F_i, F_{i,far})$ is the *distance* from F_i to its the farer point $F_{i,far}$ among $M(1,0,0)$ and $N(0,1,0)$.

A ratio-based measure of entropy (10) satisfies the entropy axioms formulated in Szmidt and Kacprzyk [22]. For the detailed explanations we refer an interested reader to Szmidt and Kacprzyk [15], [16], [18], [22].

4 Results

Now we will verify if the results produced by (9) and (10) are consistent with our intuition. We examine entropy of single elements x_i of an A-IFS, each described via (μ_i, ν_i, π_i), namely:

$$x_1 : (0.7, 0.3, 0) \tag{11}$$
$$x_2 : (0.6, 0.2, 0.2) \tag{12}$$
$$x_3 : (0.5, 0.1, 0.4) \tag{13}$$
$$x_4 : (0.4, 0, 0.6) \tag{14}$$

We assume that x_i represents the $i-th$ house we consider to buy. On the one extreme, for house x_1 the first house 70% of the attributes have desirable values, and 30% of attributes have undesirable values. On the other extreme, for house x_4 we only know that it has 40% of the desirable attributes and we do not know about 60% of the attributes we are interested in. The entropy calculated due to (9) gives the following results:

$$E_{ZL}(x_1) = 1 - |0.7 - 03| = 0.6 \tag{15}$$
$$E_{ZL}(x_2) = 1 - |0.6 - 0.2| = 0.6 \tag{16}$$
$$E_{ZL}(x_3) = 1 - |0.5 - 0.1| = 0.6 \tag{17}$$
$$E_{ZL}(x_4) = 1 - |0.4 - 0| = 0.6 \tag{18}$$

Results (15)–(18) suggest that the entropy of all $x_1, \ldots, x_4$ is the same though this is counter-intuitive! It seems that the entropy of the situation expressed by x_1, i.e., 70% positive attributes, 30% negative attributes is less than the entropy of x_4, i.e., 40% of positive attributes, and 60% unknown. Case (x_1) is "clear" in the sense that we know for sure that 30% negative attributes prevents house x_1 to be our "dream house" while in case of (x_4) we only know for sure that it has 40% of desirable attributes, and 60% is unknown. So we may conclude that it is quite possible that (x_4) may: fulfill in 100% our demands (if all 60% of the unknown attributes happen to be desirable), or may fulfill in 40% our demands and does not fulfill 60% of our demands (if 60% of unknown attributes turn out to be undesirable), or in general – 40%$+\alpha$ can fulfill and 0%$+\beta$ does not fulfill our demands where $\alpha + \beta = 60\%$ and α, $\beta \geq 0$. So we intuitively feel that it is easier to classify house x_1 as fulfilling our demands (30% is missing) than to classify house x_4 to the set of houses fulfilling (worth buying) or not fulfilling (not worth buying) our demands.

The entropy calculated from (10) gives the following results:

$$E(x_1) = \frac{|1-0.7| + |0-0.3| + |0-0|}{|0-0.7| + |1-0.3| + |0-0|} = 0.43 \tag{19}$$

$$E(x_2) = \frac{|1-0.6| + |0-0.2| + |0-0.2|}{|0-0.6| + |1-0.2| + |0-0.2|} = 0.5 \tag{20}$$

$$E(x_3) = \frac{|1-0.5| + |0-0.1| + |0-0.4|}{|0-0.5| + |1-0.1| + |0-0.4|} = 0.56 \tag{21}$$

$$E(x_4) = \frac{|1-0.4| + |0-0| + |0-0.6|}{|0-0.4| + |1-0| + |0-0.6|} = 0.6 \tag{22}$$

Results (19)–(22) seem to better reflect our intuition - the purchase decision is the easiest in the first case (entropy is the smallest) and the most difficult in the fourth case (the biggest entropy). This may be depicted as in Fig. 1. It is worth stressing that entropy (10) is a special case of a similarity measure (we refere an interested reader to Szmidt and Kacprzyk [18] for more details). Certainly, Fig. 1 a) and b) represent A-IFS entropy only for such $\mu(x)$ and $\nu(x)$ for which $\mu(x) + \nu(x)\underline{<}1$ (in Figures 1 a) and b) we illustarted the shape of (9) and (10) for $\mu(x) \in [0, 1]$ and $\nu(x) \in [0, 1]$ so to better render the shape differences of the two functions – a more general case of the situation discussed in the example above on buying a house).

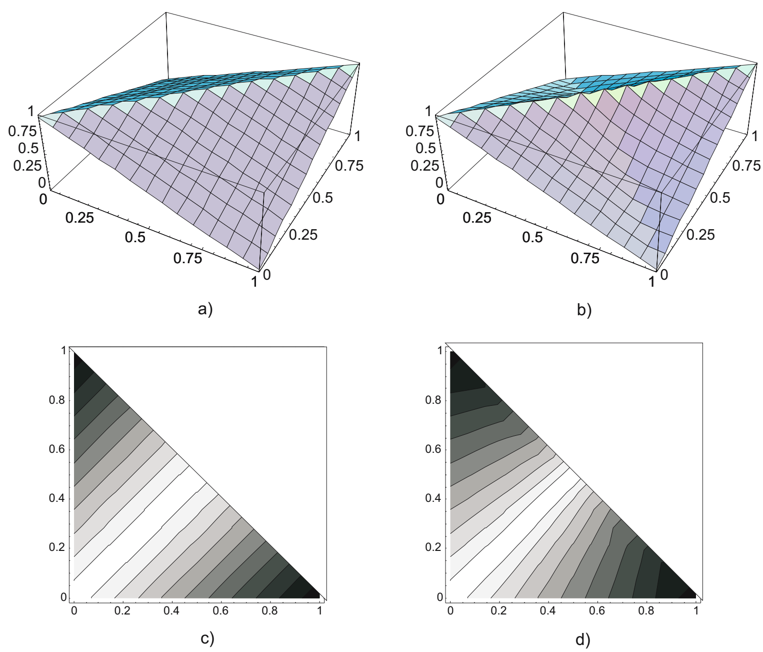

Fig. 1. Entropy calculated from (9): a) and c)– countour plot, entropy calculated from (10): b) and d) – countour plot

It seems that when calculating entropy of A-IFSs one should take into account all three functions (membership, non-membership and hesitation margin) describing an A-IFSs. Only then full information preventing from univocal classification of an element as belonging or not belonging to a set is taken into account (due to the very sense of entropy). This point of view has been also justified in, e.g., image processing via A-IFSs (cf. Vlachos and Sergiadis [25]).

5 Concluding Remarks

We considered the problem of measuring entropy for A-IFSs. It turns out that just the same as it was while considering the possible representations of A-IFSs (Szmidt and Kacprzyk [15], Tasseva at al. [24]), distances between A-IFSs (Szmidt and Kacprzyk [15], [21])), and similarity (Szmidt and Kacprzyk [23]), while considering entropy one should take into account all three functions (membership, non-membership and hesitation margin). Omitting e.g., hesitation margin may lead to counter-intuitive results.

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