A Minimal Pair in the Quotient Structure $M/NCup$

Rongfang Bie[1,⋆] and Guohua Wu[2,⋆⋆]

[1] School of Information Science and Technology
Beijing Normal University
Beijing 100875, People's Republic of China
rfbie@bnu.edu.cn
[2] Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore 639798, Republic of Singapore
guohua@ntu.edu.sg

Abstract. In this paper, we prove the existence of a minimal pair of c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that both of them are cuppable, and no incomplete c.e. degree can cup both of them to $\mathbf{0}'$. As a consequence, $[\mathbf{a}]$ and $[\mathbf{b}]$ form a minimal pair in $M/NCup$, the quotient structure of the cappable degrees modulo noncuppable degrees. We also prove that the dual of Lempp's conjecture is true.

Keywords: Computably enumerable degrees, quotient structure, minimal pairs.

1 Introduction

Friedberg (1956) and Muchnik (1957) proved independently that there are two incomparable c.e. degrees. This answers Post's question positively. Improving this, Sacks [13,14] showed that every nonzero c.e. degree $\mathbf{a}$ is the joint of two incomparable c.e. degrees and that the computably enumerable degrees are dense. After seeing this, in 1965, Shoenfield conjectured that for any finite partial orderings $P \subseteq Q$, with the least element 0 and the greatest element 1, any embedding of P into $\mathcal{R}$ (the set of all c.e. degrees) can be extended to an embedding of Q into $\mathcal{R}$. Shoenfield also listed two consequences of this conjecture:

C1. There are no incomparable c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \cap \mathbf{b}$ (the infimum of $\mathbf{a}, \mathbf{b}$) exists;
C2. For any c.e. degrees $\mathbf{0} < \mathbf{c} < \mathbf{a}$, there is a c.e. degree $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b} \cup \mathbf{c} = \mathbf{a}$.

C1 is refuted by the existence of minimal pairs (Lachlan [7], Yates [20], independently). Therefore, Shoenfield's conjecture cannot be true. Here we say that two nonzero c.e. degrees $\mathbf{a}, \mathbf{b}$ form a *minimal pair* if $\mathbf{a}, \mathbf{b}$ have infimum $\mathbf{0}$.

Say that a degree $\mathbf{a}$ is *cappable* if $\mathbf{a}$ is $\mathbf{0}$ or a half of a minimal pair. A degree is *noncappable* if it is not cappable. The dual notion of the cappable degrees is the cuppable

⋆ Bie is supported by NNSF Grant No. 19931020, No. 10001006 and No. 60273015 of China.
⋆⋆ Correspondence author. Wu is partially supported by a a start-up grant No. M48110008 and a research grant No. RG58/06 from NTU.

S.B. Cooper, B. Löwe, and A. Sorbi (Eds.): CiE 2007, LNCS 4497, pp. 53–62, 2007.

degrees. That is, a c.e. degree $\mathbf{c}$ is cuppable if there is an incomplete c.e. degree $\mathbf{b}$ such that $\mathbf{c} \cup \mathbf{b} = \mathbf{0}'$. C2 implies that all the nonzero c.e. degrees are cuppable, which turns out to be wrong, because Yates and Cooper (see [3]) proved the existence of a nonzero c.e. degree cupping no incomplete c.e. degree to $\mathbf{0}'$. In [1], Ambos-Spies, Jockusch, Shore and Soare proved that a c.e. degree is noncappable if and only if it can be cupped to $\mathbf{0}'$ via a low c.e. degree. An immediate consequence of this is that all the noncuppable degrees are cappable, and hence each c.e. degree is either cappable or cuppable, which was first proved by Harrington.

Note that all the cappable degrees and all the noncuppable degrees form ideals in $\mathcal{R}$, M and $NCup$ respectively. It becomes interesting to study the corresponding quotient structures: $\mathcal{R}/M$, $\mathcal{R}/NCup$. Schwarz provided in [15] several structural properties of $\mathcal{R}/M$. Particularly, Schwarz pointed out that Sacks splitting is true in $\mathcal{R}/M$, but there is no minimal pair in this structure. Sui and Zhang proved in [19] that C2 listed above is true in $\mathcal{R}/M$. Lempp asked in [17] whether the Shoenfield conjecture holds in $\mathcal{R}/M$. This problem was solved by Yi in [22] who claims that Shoenfield conjecture is also not true in $\mathcal{R}/M$.

Both $\mathcal{R}/M$ and $\mathcal{R}/NCup$ have the least and the greatest elements. In $\mathcal{R}/NCup$, the least element is the set of all noncuppable degrees, and the greatest element contains only one element, $\mathbf{0}'$. It is also easy to see that in $\mathcal{R}/NCup$, every nonzero element is cuppable to the greatest element. In [10], Li, Wu and Yang proved that there is a minimal pair in $\mathcal{R}/NCup$, and hence Shoenfield conjecture does not hold in $\mathcal{R}/NCup$ neither. Recently, in [11], Li, Wu and Yang prove that the diamond lattice can be embedded into $\mathcal{R}/NCup$ preserving 0 and 1. The construction involves several new features.

Theorem 1. *There are two c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$ *(hence* $[\mathbf{a}] \cup [\mathbf{b}] = [\mathbf{0}']$*), and* $[\mathbf{a}] \cap [\mathbf{b}] = [\mathbf{0}]$*, where* $[\mathbf{a}], [\mathbf{b}], [\mathbf{0}], [\mathbf{0}']$ *are the equivalence classes of* $\mathbf{a}, \mathbf{b}, \mathbf{0}, \mathbf{0}'$ *in the quotient structure* $\mathcal{R}/NCup$*.*

There are several fundamental questions left open, like whether Sacks splitting is true in $\mathcal{R}/NCup$ (it is true in $\mathcal{R}/M$, as proved in [15]) or whether C2 is true in $\mathcal{R}/NCup$.

Since $NCup$ is also an ideal of M, and M is a upper semi-lattice, we ask what the quotient structure $M/NCup$ looks like. It has a least element, but it seems that there is no greatest element in this structure. If $\mathbf{a}$ is a c.e. degree, from now on, we use $[\mathbf{a}]$ to denote the equivalence class in $M/NCup$ containing $\mathbf{a}$. Thus, $[\mathbf{0}]$ is the set of all noncuppable degrees. In this paper, we first prove that there is a minimal pair in $M/NCup$.

Theorem 2. *There are two cuppable c.e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a}$ *and* $\mathbf{b}$ *form a minimal pair in the c.e. degrees and* $[\mathbf{a}]$, $[\mathbf{b}]$ *form a minimal pair in* $M/NCup$*.*

Thus, Shoenfield Conjecture is not true in $M/NCup$. The following is the crucial step to prove Theorem 2.

Theorem 3. *There are two cuppable degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a}$ *and* $\mathbf{b}$ *form a minimal pair and that no incomplete c.e. degree can cup both* $\mathbf{a}$ *and* $\mathbf{b}$ *to* $\mathbf{0}'$*.*

The strategy to ensure that no incomplete c.e. degree can cup both $\mathbf{a}$ and $\mathbf{b}$ to $\mathbf{0}'$ is different from the one provided in [10]. We will outline the proof of Theorem 3 in Section 2.

We prove now that $\mathbf{a}$ and $\mathbf{b}$ degrees provided in Theorem 3 are exactly the ones we want in Theorem 2. Since $\mathbf{a}$ and $\mathbf{b}$ are both cuppable and cappable, $[\mathbf{a}], [\mathbf{b}]$ are nonzero elements in $M/NCup$. To prove that $[\mathbf{a}] \cap [\mathbf{b}] = [\mathbf{0}]$, suppose for a contradiction that there is a c.e. degree $\mathbf{c}$ such that $[\mathbf{0}] < [\mathbf{c}] \leq [\mathbf{a}], [\mathbf{b}]$. Then $\mathbf{c}$ is cuppable, and we assume that $\mathbf{c} \cup \mathbf{w} = \mathbf{0}'$ with $\mathbf{w} < \mathbf{0}'$. Since $[\mathbf{c}] \leq [\mathbf{a}], [\mathbf{b}]$, there are noncuppable degrees $\mathbf{m}_1$, $\mathbf{m}_2$ such that $\mathbf{c} \leq \mathbf{a} \cup \mathbf{m}_1$, $\mathbf{c} \leq \mathbf{b} \cup \mathbf{m}_2$, and hence, $\mathbf{a} \cup \mathbf{m}_1 \cup \mathbf{m}_2 \cup \mathbf{w} = \mathbf{0}'$, $\mathbf{b} \cup \mathbf{m}_1 \cup \mathbf{m}_2 \cup \mathbf{w} = \mathbf{0}'$. Let $\mathbf{v} = \mathbf{m}_1 \cup \mathbf{m}_2 \cup \mathbf{w}$. Then $\mathbf{v}$ cups both $\mathbf{a}$ and $\mathbf{b}$ to $\mathbf{0}'$. According to Theorem 3, $\mathbf{v} = \mathbf{m}_1 \cup \mathbf{m}_2 \cup \mathbf{w}$ is complete. Since both $\mathbf{m}_1, \mathbf{m}_2$ are noncuppable, $\mathbf{w} = \mathbf{0}'$. A contradiction. This completes the proof of Theorem 2.

The existence of noncappable degrees was first proved by Yates in [21], and this existence enables us to prove that any nonzero c.e. degree $\mathbf{c}$ bounds a cappable degree. To see this, we consider two cases. If $\mathbf{c}$ itself is cappable, then we are done since cappable degrees are downwards closed. Otherwise, let $\mathbf{a}$ be any cappable degree. Since $\mathbf{c}$ is assumed to be noncappable, there is a nonzero c.e. degree $\mathbf{b}$ below both $\mathbf{a}$ and $\mathbf{c}$, and hence $\mathbf{b}$ is cappable. An almost the same argument proves that any nonzero noncappable degree bounds a minimal pair. Hence, Lachlan's nonbounding degrees are all cappable.

Li and Wang (see Li [9]) proved that it is impossible to take the nonuniformity in the previous argument away. The nature behind this nonuniformity is that the direct permitting method and the minimal pair construction are not consistent. In 1996, Lempp raised the following conjecture:

Conjecture 1. (Lempp, see Slaman [17]) For any c.e. degrees $\mathbf{a}, \mathbf{b}$ with $\mathbf{a} \not\leq \mathbf{b}$, there is a cappable degree $\mathbf{c} \leq \mathbf{a}$ such that $\mathbf{c} \not\leq \mathbf{b}$.

Li refuted Lempp's conjecture in [9] by constructing c.e. degrees $\mathbf{a}, \mathbf{b}$ with $\mathbf{a} \not\leq \mathbf{b}$ such that any cappable degree below $\mathbf{a}$ is also below $\mathbf{b}$.

Unlike the cappable degrees, not every nonzero c.e. degree bounds noncuppable degrees and such degrees are called *plus-cupping* degrees[1]. However, it is true that above any incomplete c.e. degree, there is an incomplete cuppable degree. We can prove this as follows: let $\mathbf{c}$ be a given incomplete c.e. degree. The case when $\mathbf{c}$ itself is cuppable is trivial. If $\mathbf{c}$ is noncuppable, then let $\mathbf{a}$ be any incomplete cuppable degree, then $\mathbf{a} \cup \mathbf{c}$ is also incomplete and cuppable. We will provide a uniform construction in Theorem 4.

In [5], Downey and Lempp considered the dual notion of the plus-cupping degrees, the plus-capping degrees and proved that no plus-capping degrees exist. In this paper, we prove that the dual of Lempp conjecture is true.

Theorem 4. *For any incomplete c.e. degrees* $\mathbf{a}, \mathbf{b}$ *with* $\mathbf{a} \not\geq \mathbf{b}$*, there is an incomplete cuppable degree* $\mathbf{c} > \mathbf{a}$ *such that* $\mathbf{c} \not\geq \mathbf{b}$.

The proof of Theorem 4 employs Sacks coding strategy, Sacks preservation strategy, and splitting $\mathbf{0}'$ into $\mathbf{c}$ and $\mathbf{e}$. While we will make $\mathbf{c} > \mathbf{a}$, we will not require that $\mathbf{e}$ is above $\mathbf{a}$ (Harrington's nonsplitting theorem says that we cannot do this), which leaves

[1] Harrington's original notion of plus-cupping degrees is even stronger: $\mathbf{a}$ is plus-cupping, in the sense of Harrington, if for any c.e. degrees $\mathbf{b}$, $\mathbf{c}$, if $\mathbf{0} < \mathbf{b} \leq \mathbf{a} \leq \mathbf{c}$, there is a c.e. degree $\mathbf{e}$ below $\mathbf{c}$ cupping $\mathbf{b}$ to $\mathbf{c}$. The notion given in this paper was given by Fejer and Soare in [6].

enough space for us to combine the splitting strategy with the Sacks coding strategy. We will outline the proof of Theorem 4 in Section 3.

Our notation and terminology are standard and generally follow Cooper [4] and Soare [18].

2 Proof of Theorem 3

In this section, we give the outline of the proof of Theorem 3. We will construct incomplete c.e. sets A, B, C, D, E, F and p.c. functionals Γ, Δ to satisfy the following requirements:

$$G : K = \Gamma^{A,C};$$
$$H : K = \Delta^{B,D};$$
$$P_e : E \neq \Phi_e^C;$$
$$Q_e : E \neq \Phi_e^D;$$
$$R_e : \Phi_e^A = \Phi_e^B = f \text{ is total} \Rightarrow f \text{ is computable};$$
$$S_e : \Phi_e^{A,W} = \Psi_e^{B,W} = K \oplus F \Rightarrow W \geq_T K;$$

where $e \in \omega$, $\{(\Phi_e, \Psi_e, W_e) : e \in \omega\}$ is an effective enumeration of triples (Φ, Ψ, W), Φ, Ψ p.c. functionals, W a c.e. set. K is a fixed creative set.

It is easy to see that the P, Q-requirements ensure that C and D are incomplete, thus, by the G, H-requirements, both A and B are cuppable. The R-requirements ensure that A and B form a minimal pair, and the S-requirements ensure that no incomplete c.e. set can cup both A and B to K. Therefore, A and B are exactly the sets Theorem 3 requires.

2.1 The G and H-Strategies

The G and H-strategies will be dedicated to the construction of the functionals Γ, Δ respectively, which will reduce K to $A \oplus C$ and $B \oplus D$. As the constructions of Δ and Γ are the same, we only describe the construction of Γ, which will be defined such that for any x, $\Gamma^{A,C}(x)$ is defined and equals to $K(x)$.

Let $\{K_s\}_{s\in\omega}$ be a recursive enumeration of K. Γ will be defined by stages as follows: at stage s,

1. If there are xs such that $\Gamma^{A,C}(x)[s] \downarrow\neq K_s(x)$, then let k be the least such x, enumerate $\gamma(k)$ into C, and let $\Gamma^{A,C}(x)$ be undefined for all $x \geq k$.
2. Otherwise, let k be the least number x such that $\Gamma^{A,C}(x)[s]$ is not defined, then define $\Gamma^{A,C}(k)[s] = K_s(k)$ with $\gamma(k)[s]$ a fresh number.

The G-strategy (and H-strategy) has the highest priority, in the sense that at any time, a number x enters K will require us to put a number $\leq \gamma(x)$ $(\leq \delta(x))$ into C (D respectively) immediately, and no other strategies can stop, or even delay, such actions.

We note that the G-strategy itself never enumerates any element into C. In the construction, from time to time, we need to enumerate certain γ-markers into A to lift the γ-uses, in order to prevent the P-strategies from being injured by the G-strategy.

Returning to the construction of Γ, we will ensure that the corresponding γ-use function to have the following basic properties:

1. For any k, s, if $\Gamma^{A,C}(k)[s]$ is defined, then $\gamma(k)[s] \notin A_s \cup C_s$;
2. For any x, y, if $x < y$, and $\gamma(y)[s]$ is defined, then $\gamma(x)$ is also defined at this stage, and $\gamma(x)[s] < \gamma(y)[s]$;
3. Whenever we define $\gamma(k)$, we define it as a fresh number, the least number bigger than any number being used so far;
4. $\Gamma^{A,C}(x)$ is undefined at stage s iff at this stage, there is an $y \leq x$ such that $\gamma(y)$ is enumerated into A or C.

If Γ is constructed as total, the (1) – (4) above will ensure that $\Gamma^{A,C} = K$ and G is satisfied.

2.2 The P and Q-Strategies

All the P and Q-strategies will ensure that C and D are not complete. Again, since the Q-strategies are the same as the P-strategies, we only describe how the P-strategies are satisfied.

A single P-strategy, P_e say, will be devoted to find an x such that $C(x) \neq \Phi_e^E(x)$. It is a variant of the Friedberg-Muchnik strategy, modified to cooperate with the G-strategy (later in the S-strategies, we will see how to modify a P-strategy to work consistently with the S-strategies). Recall that the G-strategy always enumerates the γ-markers into C, but it may happen that a P-strategy wants to preserve a computation $\Phi_e^C(x)$, but the G-strategy wants to put a small number into C or A to code K. If we enumerate such a number into C, this enumeration can change the computation $\Phi_e^C(x)$. With this in mind, when a P-strategy wants to preserve a computation $\Phi_e^C(x)$, we enumerate a small number into A first, to lift up the γ-uses, to make sure that the computation $\Phi_e^C(x)$ will not be changed by the G-strategy. A P-strategy works as follows:

1. Choose k, as a fresh number. Whenever a number $n \leq k$ enters K, go to 2.
2. Appoint a witness, $x > k$ say as a fresh number.
3. Wait for a stage s at which $\Phi_e^C(x)[s] \downarrow= 0$.
4. Enumerate $\gamma(k)[s]$ into A and x into E, and stop.

By $x > k$, the enumeration of $\gamma(k)[s]$ into A lifts all $\gamma(n)$, $(n \geq k)$, to big numbers and so, if after stage s, no $n \leq k$ enters K, then since every $\gamma(n)$ with $n \geq k$ is defined as big numbers after stage s, $\Phi_e^C(x)$ is protected from the enumeration of the G-strategy. Therefore,

$$\Phi_e^C(x) = \Phi_e^C(x)[s] = 0 \neq 1 = E_{s+1}(x) = E(x).$$

P_e is satisfied.

Now consider the case when some $n \leq k$ enters K, at stage $s' > s$ say. Then at this stage, $\gamma(n)[s']$, which may be less than $\varphi_e(x)[s]$, is enumerated into C, according to the G-strategy. This enumeration can change the computation $\Phi_e^C(x)$. If so, we go to 2 by choosing another witness for P_e, since $\Phi_e^C(x)$ may converge later to 1, and we cannot obtain a disagreement between E and Φ_e^C at x. Such a process can happen at most k many times, and after the last time, when $\Phi_e^C(x')$ converges to 0 again, we enumerate $\gamma(k)[s]$ and x' into C, and the computation of $\Phi_e^C(x')$ can never be injured by the G-strategy afterwords, and P_e is satisfied forever.

As usual, we call the parameter k above the "killing point" of this P-strategy. When a number $\leq k$ enters K, then we reset this strategy by invalidating all of the parameters we have defined, except k. As discussed above, since k is fixed, this strategy can be reset at most $k+1$ many times.

2.3 The *R*-Strategies

There is no conflict between the G, H-strategies (coding K into $A \oplus C$ and $B \oplus D$) and the R-strategies since in general, we only put the γ, δ-markers into C and D to rectify Γ and Δ, and we only put numbers into A or B when a P strategy or Q-strategy acts. This is consistent with the minimal pair construction.

2.4 The *S*-Strategies

In the following, we describe how to make the P, Q-strategies work consistently with the G, H and the S-strategies. First, an S-strategy will construct a p.c. function Θ such that if $\Phi_e^{A,W} = \Psi_e^{B,W} = K \oplus F$ is true, then Θ^W will be totally defined and compute K correct.

Let α be an S_e strategy. First we define the length agreement function as follows:

Definition 1. *(1)* $\ell(\alpha, s) = \max\{x :$ *for all* $y < x$, $\Phi_e^{A,W}(y)[s] = \Psi_e^{B,W}(y)[s] = K_s \oplus F_s(y)\}$*; (2)* $m(\alpha, s) = \max\{0, \ell(\alpha, t) : t < s$ *and* t *is an* α*-stage*$\}$.

Say that s is α-expansionary if $s = 0$ or $\ell(\alpha, s) > m(\alpha, s)$ and s is an α-stage. Θ is defined at the α-expansionary stages. That is, for a particular n, if $\Theta^W(n)$ is not defined at stage s, and $\ell(\alpha, s) > 2n$, then we define $\Theta^W(n)[s] = K_s(n)$ with use $\theta_s(n) = s$. After $\Theta^W(n)$ is defined, only W's changes below $\varphi_e(n)[s]$ or $\psi_e(n)[s]$ can redefine $\Theta^W(n)$ with a new use.

The trouble is that we can enumerate numbers into A and B, by P and Q-strategies, respectively, and can lift the uses $\varphi_e(n)$ and $\psi_e(n)$ to bigger numbers (bigger than s). Now W may change below these new uses (but above s), and at the next α-expansionary stage, we can see that n enters K, and both $\Phi_e^{A,W}(2n)$ and $\Psi_e^{B,W}(2n)$ converge and equal to $K(n) = 1$. However, since W has not change below s, $\Theta^W(n)$ is kept defined as 0. Thus Θ^W is wrong at n, and we should avoid such a scenario.

We apply a strategy of constructing the noncuppable degrees to get around of this problem. That is, before we enumerate a number into A or B, we first enumerate appropriate numbers into F to force W to change on small numbers, so that the trouble situation we described above will never happen. In the construction of noncuppable degrees, we need to satisfy the following requirements:

$$\Phi^{A,W} = K \oplus F \;\Rightarrow\; \exists \Gamma(\Gamma^A = K).$$

To be consistent with this kind of the requirements, when a P-strategy chooses x as an attacker, at stage s_0 say, to make A not computable, it also chooses z, and at the next expansionary stage *(we measure the length of agreement between* $\Phi^{A,W}$ *and* $K \oplus F$

at each stage, and we say a stage is expansionary, if the length of agreement is bigger than previous ones and also bigger than $2z + 1$*)*, we allow the definition of Γ to be extended. Now suppose we want to put a number x into A, what we do first is to put z into F, and wait for the next expansionary stage, which will provides W-changes on numbers small enough to undefine all $\Gamma(m)$ defined after stage s_0. We only put x into A after W has such changes, that is, after we say the next expansionary stage.

In our argument, we will do almost the same thing. Let β be a P-strategy (for Q-strategies, the same, except that we will put $\delta(k)$ into B), and suppose that $\beta \supset \alpha_n \supseteq \alpha_{n-1} \supseteq \cdots \alpha_1 \supseteq \alpha_0$, where $\alpha_n, \alpha_{n-1}, \cdots, \alpha_1, \alpha_0$ are the S-strategy with priority higher than β. Then, β first defines its "killing point" k and then its attackers $x_0, x_1, \cdots, x_k$, and an auxiliary number $z_0, z_1, \cdots, z_k$. Suppose β does this at stage s_0. Now, for each $i \le n$, we say that a stage s is α_i-expansionary if the length of agreement between $\Phi_{\alpha_i}^{A,W_{\alpha_i}}$, $\Psi_{\alpha_i}^{B,W_{\alpha_i}}$ and $K \oplus F$ is greater than $2z_j + 1$ for each j with $0 \le j \le k$.

When β finds that $\Phi_\beta^C(x_k)$ converges to 0 for the first time, at stage s_1 say, then β puts x_k into E, $\gamma(k)$ into A immediately, to lift $\gamma(k)$ to a big number, and puts z_k into F, and for each i, wait for the next α_i-expansionary stage. β itself is satisfied, unless β is reset because of changes of K below k (if so, we wait for such a stage s_1 again, but with x_k and z_k replaced with x_{k-1} and z_{k-1} respectively). Now consider those α_i-strategies. Fix i, and assume that s_2 is the next α_i-expansionary stage. Then between stages s_1 and s_2, no small numbers are enumerated into B, and hence we must have a change of the corresponding W_{α_i} on some small numbers. It means that between stages s_1 and s_2, all of the Θ_{α_i} defined by α_i after stage s_0 are undefined, and therefore, Θ can be redefined correctly. βs action is consistent with these α_i strategies.

This completes the basic ideas of proof of Theorem 3. We can now implement the whole construction by a tree argument.

3 Proof of Theorem 4

Given c.e. sets A, B with $A \not\ge_T B$, we will construct c.e. sets C and E, and a partial computable functional Γ, to satisfy the following requirements:

$$G : K = \Gamma^{C,E};$$
$$P_e : C \neq \Phi_e^E;$$
$$Q_e : C \neq \Phi_e^A;$$
$$R_e : B \neq \Phi_e^{A,C};$$

where $e \in \omega$, $\{\Phi_e : e \in \omega\}$ is an effective enumeration of p.c. functionals Φ, and K is a fixed creative set.

Note that the G and P-strategies ensure that C and hence $A \oplus C$ is cuppable, while the Q-strategies ensure that $A \oplus C$ is strictly above A, and the R-strategies ensure that $A \oplus C$ is incomplete.

We have seen in Section 2 how to make the P, Q-strategies consistent with the G, H-strategies respectively.

3.1 The *Q*-Strategies

All the Q-strategies ensure that C is not reducible to A, and a single Q-strategy is exactly the Sacks coding strategy. That is, a single Q-strategy will run (infinitely) many cycles, i, each of which will choose a witness x_i, with the purpose of making $C(x_i) \neq \Phi_e^A(x_i)$, and all these cycles will define a p.c. functional Δ to threaten the assumption that A is incomplete. If a cycle i fails to make $C(x_i) \neq \Phi_e^A(x_i)$, then this cycle will code $K(i)$ into A via Δ. A cycle i works as follows:

1. Choose x_i as a fresh number.
2. Wait for a stage s such that $\Phi_e^A(x_i)[s]$ converges to 0.
3. Define $\Delta^A(i)[s] = K_s(i)$ with use $\delta(i) = \varphi_e(i)[s]$. If A changes below $\delta(i)$ before 5, then go back to 2.
4. Start cycle $i+1$ and wait for $K(i)$ to change.
5. Enumerate x_i into C.
6. Wait for A to change below $\delta(i)$.
7. Define $\Delta^A(i) = K(i) = 1$ with use $\delta(i) = -1$ and start cycle $i+1$. In this case, the A-changes will undefine $\Delta^A(j)$ for each $j \geq i$.

If cycle i waits at 2 forever, then $\Phi_e^A(x_i)$ does not converge to 0 and hence $C(x_i) = 0 \neq \Phi_e^A(x_i)$ and Q_e is satisfied. In this case, cycle does not care whether $\Delta^A(i)$ is defined, since this cycle can satisfy the Q_e requirement directly, in stead of relying on Δ^A.

If cycle i waits at 6 forever, then $\Phi_e^A(x_i)$ does converge to 0, $C(x_i) = 1 \neq 0 = \Phi_e^A(x_i)$ and hence Q_e is again satisfied. In this case, cycle i does not care whether $\Delta^A(i)$ computes $K(i)$ correctly neither.

If cycle i waits at 4 forever or 7 happens, then $\Delta^A(i)$ is defined and equals to $K(i)$. In these two cases, cycle i cannot satisfy Q_e, but succeed in defining $\Delta^A(i) = K(i)$.

Without loss of generality, suppose that no cycle waits at 2 or 6 forever. If every cycle eventually waits at 4 or 7 permanently, then for each i, $\Delta^A(i)$ is defined and equals to $K(i)$, and hence $K = \Delta^A$, and A is complete. A contradiction. Therefore, there are cycles going from 4 back to 2 infinitely often, which makes $\Delta^A(i)$ undefined. However, in this case, $\Phi_e^A(x_i)$ diverges, which shows that $\Phi_e^A(x_i) \neq C(x_i)$, and Q_e is satisfied again.

3.2 The *R*-Strategies

All the R-strategies will ensure that B is not reducible to $A \oplus C$. From this we can see that $A \oplus C$ is incomplete, and hence, the Q and the R strategies will ensure that $A \oplus C$ is strictly between A and K.

A single Q-strategy is simply the Sacks preservation strategy, which also runs (infinitely) many cycles, to define a partial function Θ to threaten $B \not\leq_T A$. Fix i. Cycle i works as follows:

1. Wait for a stage s such that $\Phi_e^{A,C}(i)[s] \downarrow = B_s(i)$.
2. Put a restraint on C to prevent small numbers being enumerated into C, to preserve the computation $\Phi_e^{A,C}(i)$. Define $\Theta^A(i)[s] = B_s(i)$ with use $\delta_s(i) = \varphi_{e,s}(i)$. Start cycle $i+1$.
3. Wait for A to change below $\varphi_{e,s}(i)$ or B to change at i.
4. If A changes first, then go back to 1. In this case, $\Theta^A(i)$ is undefined, and all cycles after i are canceled.

5. If B changes at i first, then we get a temporary disagreement between $\Phi_e^{A,C}(i)$ and $B(i)$. Again, wait for A to change below $\varphi_{e,s}(i)$.
   5a. If A never changes, then we will have $\Phi_e^{A,C}(i) = 0 \neq 1 = B(i)$, and R_e is satisfied.
   5b. Otherwise, go back to 1. In this case, the A-changes also undefine $\Theta^A(j)$ for all $j \geq i$.

If cycle i goes back from 4 or 5b to 1 infinitely often, then $\Theta^A(i)$ is not defined. However, in this case, $\Phi_e^{A,C}(i)$ diverges and hence, $\Phi_e^{A,C}(i) \neq B(i)$. If cycle i waits at 1 or 5a forever, then $\Phi_e^{A,C}(i) \neq B_s(i)$ is again true. In any case, R_e is satisfied. If cycle i waits at 3 forever from some stage on, then $\Theta^A(i)$ is defined and equals to $B(i)$.

Because $B \not\leq_T A$, it cannot be true that every cycle would wait at 3 forever. Suppose i is the least such cycle. Then as discussed above, R_e is satisfied since $\Phi_e^{A,C}(i) \neq B(i)$.

This completes the description of the basic strategies to prove Theorem 4. The whole construction can proceed on a priority tree.

References

1. Ambos-Spies, K., Jockusch Jr., C.G., Shore, R.A., Soare, R.I.: An algebraic decomposition of the recursively enumerable degrees and the coincidence of several degree classes with the promptly simple degrees. Trans. Amer. Math. Soc. 281, 109–128 (1984)
2. Cooper, S.B.: Minimal pairs and high recursively enumerable degrees. J. Symbolic Logic 39, 655–660 (1974)
3. Cooper, S.B.: On a theorem of C. E. M. Yates. Handwritten notes (1974)
4. Cooper, S.B.: Computability Theory, Chapman & Hall/CRC Mathematics, Boca Raton, FL, New York, London (2004)
5. Downey, R., Lempp, S.: There is no plus-capping degree. Arch. Math. Logic 33, 109–119 (1994)
6. Fejer, P.A., Soare, R.I.: The plus-cupping theorem for the recursively enumerable degrees. In: Logic Year 1979–80: University of Connecticut, pp. 49–62 (1981)
7. Lachlan, A.H.: Lower bounds for pairs of recursively enumerable degrees. Proc. London Math. Soc. 16, 537–569 (1966)
8. Lachlan, A.H.: Bounding minimal pairs. J. Symb. Logic 44, 626–642 (1979)
9. Li, A.: On a conjecture of Lempp. Arch. Math. Logic 39, 281–309 (2000)
10. Li, A., Wu, G., Yang, Y.: On the quotient structure of computably enumerable degrees modulo the noncuppable ideal. In: Cai, J.-Y., Cooper, S.B., Li, A. (eds.) TAMC 2006. LNCS, vol. 3959, pp. 731–736. Springer, Heidelberg (2006)
11. Li, A., Wu, G., Yang, Y.: Embed the diamond lattice into R/NCup preserving 0 and 1. In preparation
12. Miller, D.: High recursively enumerable degrees and the anticupping property. In: Logic Year 1979–80: University of Connecticut, pp. 230–245 (1981)
13. Sacks, G.E.: On the degrees less than 0̸. Ann. of Math. 77, 211–231 (1963)
14. Sacks, G.E.: The recursively enumerable degrees are dense. Ann. of Math. 80, 300–312 (1964)
15. Schwarz, S.: The quotient semilattice of the recursively enumerable degrees modulo the cappable degrees. Trans. Amer. Math. Soc. 283, 315–328 (1984)
16. Shoenfield, J.R.: Applications of model theory to degrees of unsolvability. In: Addison, J.W., Henkin, L., Tarski, A. (eds.) The Theory of Models, Proceedings of the 1963 International Symposium at Berkeley, Studies in Logic and the Foundations of Mathematics, pp. 359–363. Holland Publishing, Amsterdam (1965)

17. Slaman, T.: Questions in recursion theory. In: Cooper, S.B., Slaman, T.A., Wainer, S.S. (eds.) Computability, enumerability, unsolvability. Directions in recursion theory. London Mathematical Society Lecture Note Series, vol. 224, pp. 333–347. Cambridge University Press, Cambridge (1996)
18. Soare, R.I.: Recursively Enumerable Sets and Degrees. Perspective in Mathematical Logic, Springer-Verlag, Berlin, Heidelberg, New York
19. Sui, Y., Zhang, Z.: The cupping theorem in R/ M. J. Symbolic Logic 64, 643–650 (1999)
20. Yates, C.E.M.: A minimal pair of recursively enumerable degrees. J. Symbolic Logic 31, 159–168 (1966)
21. Yates, C.E.M.: On the degrees of index sets. Trans. Amer. Math. Soc. 121, 309–328 (1966)
22. Yi, X.: Extension of embeddings on the recursively enumerable degrees modulo the capable degrees. In: Computability, enumerability, unsolvability, Directions in recursion theory (eds. Cooper, Slaman, Wainer), London Mathematical Society Lecture Note Series 224, pp. 313–331