Hierarchies in Fragments of Monadic Strict NP

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Abstract. We expose a strict hierarchy within *monotone monadic strict* NP *without inequalities* (MMSNP), based on the number of second-order monadic quantifiers. We do this by studying a finer strict hierarchy within a class of *forbidden patterns problems* (FPP), based on the number of permitted colours. Through an adaptation of a preservation theorem of Feder and Vardi, we are able to prove that this strict hierarchy also exists in *monadic strict* NP (MSNP). Our hierarchy results apply over a uniform signature involving a single binary relation, that is over digraphs.

1 Introduction

Answering a question of Fagin, Martin Otto proved in [1] that there is a strict hierarchy in *monadic* NP (MNP) – the monadic fragment of existential second-order logic – based on the second-order quantifier rank, i.e. the number of second-order quantifiers. The strictness of the hierarchy is proved with a uniform signature involving two binary relations. It is worth noting that this hierarchy was known to collapse to its first level in the very restricted case of word structures (strings). In fact, MNP with a single second-order quantifier is as powerful as the whole of monadic second-order logic – not just its existential fragment – on word structures, capturing exactly the class of regular languages [2,3,4].

In this paper we search for a similar, second-order quantifier-rank-based, hierarchy within *monadic strict* NP (MSNP), and its monotone inequality-free fragment (MMSNP). We note that the problems involved in Otto's proof are not monotone, and in any case require first-order existential quantification – placing them outside MSNP. We achieve our hierarchy theorems by proving a strict hierarchy within a class of *forbidden patterns problems* (FPP), introduced in [5,6] to provide a combinatorial characterisation for MMSNP, based on the number of permitted colours. Specifically, we are able to prove that the digraph colourability problem $k+1$-COL is expressible in the $k+1$th level of FPP, but is not expressible in the kth level. We can then derive that 2^{k+1}-COL is expressible in the $k+1$th level of MMSNP (respectively, MSNP), but is not expressible in the kth level. We work in FPP to expose a finer hierarchy than that in MMSNP; informally we demonstrate a complexity jump between the problems k-COL and $k+1$-COL, and not just between 2^k-COL and 2^{k+1}-COL. For the reader more familiar with the Ehrenfeucht-Fraïssé method, we provide in the appendix an overview of how the hierarchy result for MMSNP (and consequently MSNP) may be so obtained. Like Otto, we do not require extensional

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relations of increasing arity for our hierarchy results; we work with a uniform signature involving a single binary relation, i.e. on digraphs.

MMSNP was studied by Feder and Vardi [7] because of its close relationship with the non-uniform constraint satisfaction problem CSP(T) – for an input structure A, does A admit a homomorphism to a fixed template T? Not only can every CSP(T) be easily recast as some MMSNP query ψ_T, but it is now known that, for every ψ in MMSNP, there is a T_ψ such that the query evaluation problem for ψ and CSP(T_ψ) are polynomial-time equivalent [7,8]. If one were to consider classes of non-uniform constraint satisfaction problems CSP^k in which the template was restricted to being of size $\leq k$, then it would be virtually immediate that this hierarchy was strict and, indeed, separated by the problems k-Col. Essentially, one could not express the problem $k+1$-Col as a CSP whose template T has less than $k+1$ vertices, since then the $k+1$-clique K_{k+1}, which is plainly $k+1$-colourable, would manifest as a no-instance. The relationship between the size of a CSP template T and the number of colours required to express it as a FPP, is explored in [5]. Certainly any CSP whose template T is of size k can be expressed as a FPP whose forbidden patterns involve k colours. However, this relationship is not known to hold in the converse, and, therefore, proving the corresponding strict hierarchy in FPP does not appear to be trivial.

Following the necessary preliminaries, the paper is organised as follows. In Section 3 we prove the hierarchy result for FPP and in Section 4 we derive the related result for MMSNP. In Section 5 we demonstrate how to adapt a certain preservation theorem of Feder and Vardi to derive the same hierarchy in MSNP. At the end of the paper sits an appendix, in which we show how our results may be obtained through the ubiquitous Ehrenfeucht-Fraïssé games.

2 Preliminaries

In this paper, the only structures we consider are finite, non-empty digraphs. A *digraph* G consists of a finite *vertex set* $V(G)$ together with an *edge set* $E(G) \subseteq V(G) \times V(G)$. For a positive integer k, let $[k]$ be the set $\{0, \ldots, k-1\}$. A *k-coloured digraph* is a pair (G, c^k) where G is a digraph and c^k is a function from $V(G)$ to the colour set $[k]$ (note that we do not require that a colouring be 'proper', i.e. we do not force adjacent vertices to take different colours). If the range of c^k is the singleton $\{i\}$, then we refer to (G, c^k) as *i-monochrome*.

A homomorphism between the digraphs G and H is a function $h : V(G) \rightarrow V(H)$ such that, for all $x, y \in V(G)$, $(x, y) \in E(G)$ implies $(h(x), h(y)) \in E(H)$. A homomorphism between the k-coloured digraphs (G, c^k_G) and (H, c^k_H) is a digraph homomorphism $h : G \rightarrow H$ that also respects the colouring of G, i.e., for all $x \in V(G)$, $c^k_H(h(x)) = c^k_G(x)$. Existence (respectively, non-existence) of a homomorphism between entities P and Q is denoted $P \longrightarrow Q$ (respectively, $P \not\longrightarrow Q$).

Let K_k be the antireflexive k-clique, that is the digraph with vertex set $[k]$ and edge set $\{(i, j) : i \neq j\}$. Define the problem k-Col to be the set of digraphs G which admit a homomorphism to K_k. We describe digraphs G s.t. $G \in$ 2-Col as *bipartite*. An edge of the form (x, x) in a digraph G is described as a *self-loop*; a digraph with no self-loops is said to be *antireflexive*. It is a simple observation that a digraph with a self-loop cannot map homomorphically into an antireflexive digraph.

Let $\mathscr{R}^k$ be some finite set of k-coloured digraphs. Define the *forbidden patterns problem* $FPP(\mathscr{R}^k)$ to be the set of digraphs G for which there exists a k-colouring c_G^k such that, for all $(H, c_H^k) \in \mathscr{R}^k$, $(H, c_H^k) \not\longrightarrow (G, c_G^k)$. Intuitively, $FPP(\mathscr{R}^k)$ is the class of digraphs for which there exists a k-colouring that forbids homomorphism from all of the k-coloured digraphs of $\mathscr{R}^k$, whence we refer to $\mathscr{R}^k$ as the set of *forbidden patterns*. Define FPP^k to be the class of problems $FPP(\mathscr{R}^k)$, where $\mathscr{R}^k$ ranges over all finite sets of k-coloured digraphs, and let FPP be $\cup_{i\in\omega}\mathrm{FPP}^i$.

The logic *k-monadic* NP (MNP^k) will be considered that fragment of monadic existential second-order logic that allows at most k second-order quantifiers. The logic *k-monadic strict* NP (MSNP^k) is that fragment of MNP^k that involves prenex sentences whose first-order quantification is purely universal. We may therefore consider MSNP^k to be the the class of sentences φ of the form

$$\exists \mathbf{M} \forall \mathbf{v}\ \Phi(\mathbf{M}, \mathbf{v}),$$

where $\mathbf{M}$ is an k-tuple of monadic relation symbols and Φ is quantifier-free. In these logics, we refer to k as the *second-order quantifier rank*. The logic *k-monotone* MSNP *without inequalities* (MMSNP^k) is defined similarly, but with the additional restriction that Φ be of the form

$$\bigwedge_i \neg(\alpha_i(\mathbf{v}) \wedge \beta_i(\mathbf{M}, \mathbf{v})),$$

where: α_i is a conjunction of positive atoms, involving neither equality nor relations from $\mathbf{M}$; and β_i is a conjunction of positive or negative atoms, involving only relations from $\mathbf{M}$. Define MNP (respectively, MSNP, MMSNP) to be $\cup_{i\in\omega}\mathrm{MNP}^i$ (respectively, $\cup_{i\in\omega}\mathrm{MSNP}^i$, $\cup_{i\in\omega}\mathrm{MMSNP}^i$). The following hierarchy theorem for MNP is due to Otto.

Theorem 1 ([1]). *For all k,* $\mathrm{MNP}^k \subseteq \mathrm{MNP}^{k+1}$ *but* $\mathrm{MNP}^k \neq \mathrm{MNP}^{k+1}$.

Furthermore, the following is straightforward.

Proposition 1. *For all k, we have the inclusions* $\mathrm{FPP}^k \subseteq \mathrm{FPP}^{k+1}$*,* $\mathrm{MSNP}^k \subseteq \mathrm{MSNP}^{k+1}$ *and* $\mathrm{MMSNP}^k \subseteq \mathrm{MMSNP}^{k+1}$.

Proof. For the first part, let $FPP(\mathscr{R}^k)$ be a problem of FPP^k. Construct $\mathscr{R}^{k+1}$ from $\mathscr{R}^k$ by the addition of a k-monochrome copy of K_1. Since the extra colour is now forbidden, it is plain to see that $FPP(\mathscr{R}^{k+1}) = FPP(\mathscr{R}^k)$.

For the second part, let $\exists M_0 \ldots \exists M_{k-1} \forall \mathbf{v}\ \Phi$ be a sentence of MSNP^k, and v be one of the variables of $\mathbf{v}$. Then $\exists M_0 \ldots \exists M_{k-1} \exists M_k \forall \mathbf{v}\ \Phi \wedge \neg M_k(v)$ is an equivalent sentence of MSNP^{k+1}.

The third part may be proved in the same manner. □

The contribution of this paper will be to prove that these inclusions are strict.

The following result ties together FPP and MMSNP.

Theorem 2 ([5]). *The class of problems expressible in* MMSNP^k *coincides exactly with the class of problems that are finite unions of problems in* FPP$^{2^k}$.

Example 1. Let us consider the problem 2-Col. This is a forbidden patterns problem $FPP(\mathscr{R}^2)$, where $\mathscr{R}^2$ consists of two coloured digraphs (P_1, c_0^2) and (P_1, c_1^2), which are 0- and 1-monochrome, respectively, where P_1 is the digraph with vertex set $\{0, 1\}$ and a single edge $(0, 1)$. In the following depiction, we may view the white vertices as coloured 0, and the black vertices as coloured 1.

$$\mathscr{R}^2 \ := \ \{ \ \circ\!\longrightarrow\!\circ \ , \ \bullet\!\longrightarrow\!\bullet \ \}$$

2-Col may also be expressed by the sentence φ of MMSNP^1:

$$\exists M \forall u \forall v \ \neg(E(u,v) \wedge M(u) \wedge M(v)) \ \wedge \ \neg(E(u,v) \wedge \neg M(u) \wedge \neg M(v)).$$

Define the *chromatic number* of a digraph G to be the minimal k such that $G \in k$-Col. We define the *symmetric closure* of a digraph G, denoted $Sym(G)$, over the same vertex set as G, but with edge set $\{(x, y), (y, x) : (x, y) \in E(G)\}$. For $k \geq 3$, let C_k be the undirected k-cycle, that is the digraph with vertex set $[k]$ and edge set $\{(i, j), (j, i) : j = i + 1 \bmod k\}$. Define the *odd girth* of an antireflexive, non-bipartite digraph G to be the minimal odd k s.t. C_k is (isomorphic to) an induced subdigraph of $Sym(G)$ (note that this is always defined). We define the odd girth of an antireflexive, non-bipartite coloured digraph likewise. It may be easily verified that, if two digraphs G and H have odd girth γ_G and γ_H, respectively, with $\gamma_G \leq \gamma_H$, then $G \not\longrightarrow H$.

We require the following lemma, originally proved by Erdös through the probabilistic method [9], but for which the citation provides a constructive proof.

Lemma 1 (See [10]). *For all i, one may construct a digraph B_i whose chromatic number and odd girth both strictly exceed i.*

3 A Strict Hierarchy in FPP

In this section we aim to prove that there is a strict hierarchy in FPP given by the number of colours allowed in the set $\mathscr{R}$. We will establish this through the following theorem.

Theorem 3. *For each $k \geq 1$, $k+1$-Col $\in$ FPP^{k+1} but $k+1$-Col $\notin$ FPP^k.*

Proof. ($k+1$-Col $\in$ FPP^{k+1}.) This follows similarly to Example 1. $k+1$-Col is expressed by $FPP(\mathscr{R}^{k+1})$, where $\mathscr{R}^{k+1}$ consists of $k+1$ coloured digraphs (P_1, c_0^2), ..., (P_1, c_k^2), in which, for $0 \leq i \leq k$, (P_1, c_i^2) is i-monochrome.

($k+1$-Col $\notin$ FPP^k.) Suppose that $k+1$-Col $\in$ FPP^k, and is expressed by $FPP(\mathscr{R}^k)$ where $\mathscr{R}^k$ is a finite set of k-coloured digraphs. We are therefore claiming that, for all digraphs G:

$$(*) \quad G \in k+1\text{-Col} \ \text{ iff } \ \text{exists } c_G^k \text{ s.t. } \forall\ (H, c_H^k) \in \mathscr{R}^k \quad (H, c_H^k) \not\longrightarrow (G, c_G^k).$$

First, we aim to prove that, for every i, $\mathscr{R}^k$ must contain some i-monochrome bipartite digraph. Suppose, for some i, it does not. Let the maximum odd girth of the coloured digraphs of $\mathscr{R}^k$ be γ; if all members of $\mathscr{R}^k$ possess a self-loop or are bipartite, set $\gamma := 3$. Set μ to be $1 + \max\{k, \gamma\}$. By Lemma 1, we can construct a graph B_μ whose chromatic number and odd girth both strictly exceed μ. We now deduce from $(*)$ the absurdity $B_\mu \in k+1$-Col, since the i-monochrome colouring of B_μ forbids homomorphism from all of the coloured digraphs of $\mathscr{R}^k$ (recall that any bipartite members of $\mathscr{R}^k$ are not i-monochrome).

Now we aim to prove that $K_{k+1} \notin FPP(\mathscr{R}^k)$. Consider any k-colouring c^k of K_{k+1}; there must be distinct vertices x and y such that $c^k(x) = c^k(y)$, let their colour be i. But we know that $\mathscr{R}^k$ contains an i-monochrome bipartite digraph, which plainly maps homomorphically into (K_{k+1}, c^k) (in fact into its i-monochrome subdigraph K_2 induced by $\{x, y\}$). By definition, we deduce that $K_{k+1} \notin FPP(\mathscr{R}^k)$.

Finally, we reach a contradiction since K_{k+1} is plainly in $k+1$-Col. □

4 A Strict Hierarchy in MMSNP

We now show how to adapt the previous proof to generate the following[1].

Theorem 4. *For $k \geq 0$, 2^{k+1}-Col $\in$ MMSNP^{k+1} but 2^{k+1}-Col $\notin$ MMSNP^k.*

Proof. (2^{k+1}-Col $\in$ MMSNP^{k+1}.) This follows similarly to Example 1. 2^{k+1}-Col may be expressed by the following sentence of MMSNP^{k+1}:

$$\exists M_0 \ldots \exists M_k \forall u \forall v \bigwedge_{i \in [2^{k+1}]} \neg(E(u,v) \wedge \Psi_i(M_0, \ldots, M_k, u, v)),$$

where $\Psi_i(M_0, \ldots, M_k, u, v)$ is

$$(\neg)^{i_0} M_0(u) \wedge (\neg)^{i_0} M_0(v) \wedge \; (\neg)^{i_1} M_1(u) \wedge (\neg)^{i_1} M_1(v) \wedge \ldots \wedge \; (\neg)^{i_k} M_k(u) \wedge (\neg)^{i_k} M_k(v)$$

where i_j is the $j+1$th digit in the binary expansion of i.

(2^{k+1}-Col $\notin$ MMSNP^k.) Suppose that 2^{k+1}-Col $\in$ MMSNP^k. By Theorem 2, this implies that 2^{k+1}-Col is the union, for some s, of the forbidden pattern problems $FPP(\mathscr{R}_0^{2^k})$, ..., $FPP(\mathscr{R}_{s-1}^{2^k})$. In a similar vein to before, we can deduce that, for each j ($0 \leq j < s$), $\mathscr{R}_j^{2^k}$ contains, for each i ($0 \leq i < 2^k$), an i-monochrome bipartite digraph. The proof concludes as before. □

Remark 1. Our proof can actually go further, yielding not just 2^{k+1}-Col $\notin$ MMSNP^k, but also $2^k + 1$-Col $\notin$ MMSNP^k.

5 A Strict Hierarchy in MSNP

We say that a class of finite digraphs $\mathcal{C}$ is *closed under inverse homomorphism* iff whenever we have $G \longrightarrow H$ and $H \in \mathcal{C}$ we also have $G \in \mathcal{C}$. Similarly, a class of

[1] By abuse of notation, we write that a class of digraphs belongs to a logic precisely when that class is expressible in the logic, e.g. 2^{k+1}-Col $\in$ MMSNP^{k+1}.

finite digraphs is *antireflexive* iff each digraph within it is antireflexive. It follows straight from our definition that, for each k, the class k-Col is both antireflexive and closed under inverse homomorphism. The following is from [11], and is an example of a preservation theorem.

Theorem 5 ([11]). *For every $\psi \in$ MSNP s.t. the class of finite models of ψ is closed under inverse homomorphism, there exists $\psi' \in$ MMSNP s.t. ψ and ψ' agree on all finite models.*

In fact, we will require a variant on their proof, to derive the following theorem (whose proof we defer to the end of this section).

Theorem 6. *For every $\psi \in \text{MSNP}^k$ s.t. the class of finite models of ψ is both antireflexive and closed under inverse homomorphism, there exists $\psi' \in \text{MMSNP}^k$ s.t. ψ and ψ' agree on all finite models.*

We are now in a position to state and prove the main result of this section.

Theorem 7. *For each $k \geq 0$, 2^{k+1}-Col $\in \text{MSNP}^{k+1}$ but 2^{k+1}-Col $\notin \text{MSNP}^k$.*

Proof. Membership follows as in Theorem 4

(2^{k+1}-Col $\notin \text{MSNP}^k$.) Note that 2^{k+1}-Col is an antireflexive class that is closed under inverse homomorphism. By the previous theorem that implies that it may be expressed in MSNP^{k+1} only if it may be expressed in MMSNP^{k+1}, which we know it can not – by Theorem 4. □

Proof (of Theorem 6). We show how to adapt the proof of Theorem 3 of [11]. In [11], they demonstrate how, if ψ_0 is a sentence of MSNP whose finite models form a class closed under inverse homomorphism, to construct a sequence of sentences culminating with ψ_5 in MMSNP s.t. ψ_0 and ψ_5 agree on all finite models. Unfortunately, it is not the case that the second-order quantifier rank is preserved: in their construction of ψ_2 from ψ_1 it may be necessary to introduce new second-order monadic relations. In all other of their translations, the second-order quantifier rank is preserved. Our proof uses the additional constraint of antireflexivity to amend the translation from ψ_1 to ψ_2 into something very simple that does preserve the second-order quantifier rank. This is the only area in which our proof differs from theirs. We now sketch the proof.

Starting with a sentence ψ_0 of MSNP^k whose finite models form an antireflexive class that is closed under inverse homomorphism, we will describe a sequence of sentences culminating in ψ_5 that is in MMSNP^k and agrees with ψ_0 on all finite digraphs. We may assume that ψ_0 is in prenex form with its quantifier-free part in conjunctive normal form, where we interpret each clause as a negated conjunction. That is, ψ_0 is of the form

$$\exists \mathbf{M} \forall \mathbf{v} \bigwedge_i \neg (\bigwedge_j \alpha_{ij}(\mathbf{M}, \mathbf{v}))$$

where each α_{ij} is atomic.

From ψ_0 we generate ψ_1 by enforcing that, if distinct u and v occur in some negated conjunct, then $u \neq v$ also occurs in that conjunct. If this is not already the case, then we split the negated conjunct in two, one involving $u \neq v$ and the other involving $u = v$, whereupon, in the latter case, we may substitute all occurrences of v with u, dispensing with the equality.

From ψ_1 we generate ψ_2 by removing any atomic instances of $\neg E(v, v)$. From ψ_2 we generate ψ_3 by removing any negated conjuncts that contain either an instance $v \neq v$ or both atoms $E(u, v)$ and $\neg E(v, u)$. From ψ_3 we generate ψ_4 by removing all negative atoms. Finally, from ψ_4 we generate ψ_5 by removing all inequalities. It is transparent that ψ_5 is in MMSNP^k. It remains for us to settle the following.

Lemma 2. *Let ψ_0 be a sentence of MSNP^k in the required form. Then, on the class of finite digraphs,*

(i) ψ_0 is equivalent to ψ_1,
(ii) ψ_1 is equivalent to ψ_2 (since ψ_1 describes an antireflexive class),
(iii) ψ_2 is equivalent to ψ_3,
(iv) ψ_3 is equivalent to ψ_4, and
(v) ψ_4 is equivalent to ψ_5 (since ψ_4 describes a class closed under inverse homomorphism).

(i) and *(iii)* are trivial (and appear in [11]). Part *(ii)* is transparent. Parts *(iii)* and *(iv)* are non-trivial and appear as Lemmas 8 and 7, respectively, in [11]. □

6 Further Work

The problems in Otto's proof of the strict hierarchy in MNP are colouring problems of a kind. However, they demand highly regular structures that, in some sense, make them less natural than the problems k-Col. It would be interesting to know whether the problems 2^k-Col separate the hierarchy in MNP; that is, whether 2^{k+1}-Col can be proved inexpressible in MNP^k. Our attempts to use Ehrenfeucht-Fraïssé games (even Ajtai-Fagin games) to settle this have not, thus far, succeeded.

References

1. Otto, M.: A note on the number of monadic quantifiers in monadic Σ_1^1. Information Processing Letters 53(6), 337–339 (1995)
2. Büchi, J.R.: Weak second-order arithmetic and finite automata. Zeitschrift für mathematische Logik und Grundladen der Mathematik 6, 66–92 (1960)
3. Elgot, C.: Decision problems of finite-automata design and related arithmetics. Trans. Amer. Math. Soc. 98, 21–51 (1961)
4. Thomas, W.: Classifying regular events in symbolic logic. Journal of Computer and System Sciences 25, 360–376 (1982)
5. Madelaine, F.: Constraint satisfaction problems and related logic. PhD thesis, University of Leicester (2003)
6. Madelaine, F., Stewart, I.A.: Constraint satisfaction, logic and forbidden patterns (SIAM Journal of Computing) 33 pp (To appear)

7. Feder, T., Vardi, M.Y.: The computational structure of monotone monadic SNP and constraint satisfaction: a study through datalog and group theory. SIAM J. Comput. 28 (1999)
8. Kun, G.: Constraints, MMSNP and expander structures (2006)
9. Erdös, P.: Graph theory and probability. Canad. J. Math. 11, 34–38 (1959)
10. Hell, P., Nešetřil, J.: Graphs and Homomorphisms. OUP (2004)
11. Feder, T., Vardi, M.: Homomorphism closed vs existential positive (2003)
12. Fagin, R.: Monadic generalized spectra. Z. Math. Logik Grund. Math. 21, 89–96 (1975)

Appendix

An Ehrenfeucht-Fraïssé Game for MMSNP

Ehrenfeucht-Fraïssé games traditionally provide the preferred method for separating logics. Here we state the relevant Ehrenfeucht-Fraïssé game for MMSNP and give its methodology theorem. We are then able to give an alternative proof of Theorem 4.

For digraphs G and H, the game $\mathcal{G}^m_q(G,H)$ is played between two players, Spoiler and Duplicator, and proceeds as follows.

– Spoiler chooses a 2^m-colouring $c^{2^m}_G$ of G;
– Duplicator responds with a 2^m-colouring $c^{2^m}_H$ of H.
– Spoiler places q pebbles $a_0,\ldots,a_{q-1}$ on H;
– Duplicator responds with q pebbles $b_0,\ldots,b_{q-1}$ on G.

Duplicator wins iff the resultant relation $\{(a_0,b_0),\ldots,(a_{q-1},b_{q-1})\}$ is a partial homomorphism from $(H,c^{2^m}_H)$ to $(G,c^{2^m}_G)$.

Let MMSNP^m_q be that fragment of MMSNP^m in which the first-order part of the sentences has quantifier-rank bounded by q (owing to the restricted syntax of MMSNP, we may equivalently consider q to be a bound on the number of first-order variables). The next theorem ties together the game and the logic; a proof, based on the connection between FPP and MMSNP appears at the end of the section (although it is possible to derive a more conventional proof similar to that given by Fagin for his original game for MNP [12]).

Theorem 8 (Methodology). *For digraphs G and H the following are equivalent.*

– *Duplicator has a winning strategy in the game $\mathcal{G}^m_q(G,H)$.*
– *For all $\varphi \in \text{MMSNP}^m_q$, $G \models \varphi$ implies $H \models \varphi$.*

We are now in a position to give another proof of Theorem 4.

Theorem 9 (a.k.a. Theorem 4). *For each $m \geq 0$, 2^{m+1}-*Col* $\notin \text{MMSNP}^m$.*

Proof. Suppose that 2^{m+1}-Col were expressible by a sentence $\psi^m_q \in \text{MMSNP}^m_q$, for some q. Set μ to be $1+\max\{2^{m+1},q\}$. Note that the digraph B_μ, constructed as in Lemma 1, is not in 2^{m+1}-Col. We aim to prove that Duplicator has a winning strategy in the game $\mathcal{G}_q^m(K_{2^{m+1}}, B_\mu)$, which, taken with the previous methodology theorem, leads to a contradiction.

Let Spoiler give a 2^m-colouring of $K_{2^{m+1}}$, and let x and y be some distinct vertices that are given some same colour i (such vertices clearly must exist). Duplicator chooses the i-monochrome colouring of B_μ. Now Spoiler places q pebbles on B_μ. Crucially, because of the enormous odd girth of B_μ, the subdigraph induced by these q pebbles must be bipartite. It therefore homomorphically maps onto the subdigraph K_2 of $K_{2^{m+1}}$ induced by the set $\{x, y\}$, and we are done. $\square$

Proof (of Theorem 8). Let FPP_q^m be that subclass of FPP^m in which all the forbidden patterns in $\mathscr{R}^m$ have size bounded by q. We require the following, more sophisticated, version of Theorem 2, also proved in [5].

- The class of problems expressible in MMSNP_q^m coincides exactly with the class of problems that are finite unions of problems in $\mathrm{FPP}_q^{2^m}$.

By this result, it suffices to prove that the following are equivalent.

(i) Duplicator has a winning strategy in the game $\mathcal{G}_q^m(G, H)$.

(ii) For all $\mathscr{R}_0^{2^m}, \ldots, \mathscr{R}_{s-1}^{2^m}$, whose members are of size bounded by q, we have that $G \in \bigcup_{i\in[s]} FPP(\mathscr{R}_i^{2^m})$ implies $H \in \bigcup_{i\in[s]} FPP(\mathscr{R}_i^{2^m})$.

$[(i) \Rightarrow (ii)]$ Consider a winning strategy for Duplicator in the game $\mathcal{G}_q^m(G, H)$, and any sequence of sets of forbidden patterns,each of whose members is bounded in size by q, $\mathscr{R}_0^{2^m}, \ldots, \mathscr{R}_{s-1}^{2^m}$. Further assume that $G \in \bigcup_{i\in[s]} FPP(\mathscr{R}_i^{2^m})$. It follows that there is some $i \in [s]$ s.t. $G \in FPP(\mathscr{R}_i^{2^m})$. We will prove that $H \in FPP(\mathscr{R}_i^{2^m})$ whereupon $H \in \bigcup_{i\in[s]} FPP(\mathscr{R}_i^{2^m})$ is immediate. Take the 2^m-colouring $c_G^{2^m}$ of G that witnesses its membership of $FPP(\mathscr{R}_i^{2^m})$, and consider Duplicator's response c_H^{2m} on H to it in her winning strategy in the game $\mathcal{G}_q^m(G, H)$. We claim that this witnesses the membership of H in $FPP(\mathscr{R}_i^{2^m})$; for, otherwise, if some forbidden pattern – of size bounded by q – of $\mathscr{R}_i^{2^m}$ mapped homomorphically into $(H, c_H^{2^m})$ then it would also map homomorphically into $(G, c_G^{2^m})$, by the winning strategy of Duplicator, which is a contradiction.

$[\neg(i) \Rightarrow \neg(ii)]$. Given a winning strategy for Spoiler in the game $\mathcal{G}_q^m(G, H)$, we will construct a set of forbidden patterns $\mathscr{R}^{2^m}$, each of whose size is bounded by q, such that $G \in FPP(\mathscr{R}^{2^m})$ but $H \notin FPP(\mathscr{R}^{2^m})$. Taking Spoiler's winning strategy, consider the size $\leq q$ induced subdigraph H' of H that he pebbles with $a_0, \ldots, a_{q-1}$. Let $\mathscr{R}^{2^m}$ be the set of all 2^m-colourings of H'. Now, $G \in FPP(\mathscr{R}^{2^m})$ and this is witnessed by Spoiler's initial colouring $c_G^{2^m}$ of G in his winning strategy. But $H \notin FPP(\mathscr{R}^{2^m})$ since any colouring of H admits homomorphism from itself, and consequently from the same colouring restricted to its induced subdigraph H'. $\square$