Some Notes on Degree Spectra of the Structures

Iskander Kalimullin

Kazan State University, Kazan, 420008, Kremlevskaya str. 18, Russia
Iskander.Kalimullin@ksu.ru

Abstract. In the paper the problem of existence of an algebraic structure with the degree spectra $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$ is studied for arbitrary degree $\mathbf{b}$.

1 Restrictions on the Degree Spectra

A *representation* of a countable algebraic structure $\mathcal{A}$ is any isomorphic copy of $\mathcal{A}$ with the universe, which is a subset of ω (the set of natural numbers with zero). Under *degree spectrum* $\mathbf{Sp}(\mathcal{A})$ of a countable algebraic structure $\mathcal{A}$ we understand the collection of Turing degrees of atomic diagrams of all representations of $\mathcal{A}$.

The following well-known result presents the first restriction on possible degree spectra.

Theorem 1 (Knight [8])**.** *Let $\mathcal{A}$ be a countable structure in a finite language. Then precisely one of the following two statements holds:*

1. *For any two Turing degrees $\mathbf{c} \leq \mathbf{d}$, if $\mathbf{c} \in \mathbf{Sp}(\mathcal{A})$, then also $\mathbf{d} \in \mathbf{Sp}(\mathcal{A})$ (i.e., the degree spectrum is closed upwards).*
2. $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{0}\}$. *(The structures with this property is called trivial).*

Each finite structure is an obvious example of a trivial structure, but there are also infinite trivial structures, such as the infinite complete graph.

In this paper we will consider only the nontrivial countable structures in finite languages. Theorem 1 shows that for nontrivial structures the degree spectrum is simply a collection of all degrees $\mathbf{x}$ such that the structure is $\mathbf{x}$-computable.

One of important and interesting area of studying non-computable structures is to describe which collections of degrees closed upward are realizable as a degree spectra of structures (or, of some special kind of structures such as linear orderings, Boolean algebras, groups, etc.). It is easy to check that the class of such collections is closed under intersection: for any structures $\mathcal{A}$ and $\mathcal{B}$ there a structure $\mathcal{C}$ such that $\mathbf{Sp}(\mathcal{A}) \cap \mathbf{Sp}(\mathcal{B}) = \mathbf{Sp}(\mathcal{C})$.

There are is a lot of various papers devoted to this direction (see e.g. [1], [3], [4], [9], [11], [13] etc.). In particular, by [11] for any degree $\mathbf{a}$ the collection $\{\mathbf{x} : \mathbf{x} \geq \mathbf{a}\}$ is realizable as spectrum of a structure.

There are also more surprising examples: Slaman [12] and, independently, Wehner [14], constructed structures with the degree spectrum $\{\mathbf{x} : \mathbf{x} > \mathbf{0}\}$. An easy relativization shows that for any degree $\mathbf{b}$ the collection $\{\mathbf{x} : \mathbf{x} > \mathbf{b}\}$ is

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also realizable as a spectrum. In the present paper we will try to obtain a more strong relativization: for a degree $\mathbf{b}$ to find a structure with the degree spectrum $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$. We will see that for some degrees $\mathbf{b}$ this is not possible (Corollaries 4 and 5).

This problem is related to the following question posed by Miller [9]:

Question 1. (Miller). Does for any incomparable degrees $\mathbf{a}$ and $\mathbf{b}$ there exist a linear ordering $\mathcal{L}$ such that $\mathbf{a} \in \mathbf{Sp}(\mathcal{L})$ and $\mathbf{b} \notin \mathbf{Sp}(\mathcal{L})$?

As it follows from the next theorem the answer on this question is negative. Hence, there is a degree $\mathbf{b}$ such that, at least, the collection $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$ is not realizable as a spectrum of linear ordering.

Theorem 2 ([7])**.** *For each degree* $\mathbf{a} > \mathbf{0}$ *there is a degree* $\mathbf{b}$ *incomparable with* $\mathbf{a}$ *such that* $\mathbf{b}' \leq \mathbf{a}'$ *and for any linear ordering* $\mathcal{L}$

$$\mathbf{a} \in \mathbf{Sp}(\mathcal{L}) \Longrightarrow \mathbf{b} \in \mathbf{Sp}(\mathcal{L}).$$

Corollary 1. *There is a low degree* $\mathbf{b}$ *(i.e.,* $\mathbf{b}' = \mathbf{0}'$*), such that* $\mathbf{Sp}(\mathcal{L}) \neq \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$ *for any linear ordering* $\mathcal{L}$*.*

Note that, in comparison with the linear orderings, in the general case (Theorem 7) for any low degree $\mathbf{b}$ we have some structure $\mathcal{A}_{\mathbf{b}}$ such that $\mathbf{Sp}(\mathcal{A}_{\mathbf{b}}) = \{\mathbf{x} \not\leq \mathbf{b}\}$.

The proof of Theorem 2 is just a more uniform version of the Richter's result, which is about only one ordering:

Theorem 3 (Richter [11])**.** *For every degree* $\mathbf{a} > \mathbf{0}$ *and any linear ordering* $\mathcal{L}$ *such that* $\mathbf{a} \in \mathbf{Sp}(\mathcal{L})$ *there is a degree* $\mathbf{b}$ *such that* $\mathbf{a} \not\leq \mathbf{b}$ *and* $\mathbf{b} \in \mathbf{Sp}(\mathcal{L})$*. (And hence, the collection* $\{\mathbf{x} : \mathbf{x} \geq \mathbf{a}\}$ *is realizable as a degree spectrum of a linear ordering if and only if* $\mathbf{a} = \mathbf{0}$*).*

Returning to algebraic structures in general, one can recall the following folklore result (see e.g. [13]).

Theorem 4. (Folklore). *Let* $\mathbf{A}$ *be a nonempty countable collection of degrees without least element and* $\mathcal{A}$ *be a structure such that* $\mathbf{A} \subseteq \mathbf{Sp}(\mathcal{A})$*. Then there is a degree* $\mathbf{b}$ *such that* $\mathbf{a} \not\leq \mathbf{b}$ *for all* $\mathbf{a} \in \mathbf{A}$*, and* $\mathbf{b} \in \mathbf{Sp}(\mathcal{A})$*.*

Corollary 2. (Folklore). *If* $\mathbf{A}$ *is a nonempty countable collection of degrees without least element, then the collection* $\cup_{\mathbf{a} \in \mathbf{A}} \{\mathbf{x} : \mathbf{x} \geq \mathbf{a}\}$ *is not realizable as a spectrum of an algebraic structure.*

Corollary 3. (Folklore). *If* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *are incomparable, then the collection* $\{\mathbf{x} : \mathbf{x} \geq \mathbf{a}_0\} \cup \{\mathbf{x} : \mathbf{x} \geq \mathbf{a}_1\}$ *is not realizable as a spectrum of an algebraic structure.*

For Theorem 4 the same uniformization is also possible:

Theorem 5 ([7]). *Let* $\mathbf{A}$ *be a nonempty countable collection of degrees without least element. Then there is a degree* $\mathbf{b}$ *such that* $\mathbf{a} \not\leq \mathbf{b}$ *for all* $\mathbf{a} \in \mathbf{A}$*, and for any algebraic structure* $\mathcal{A}$

$$\mathbf{A} \subseteq \mathbf{Sp}\,(\mathcal{A}) \Longrightarrow \mathbf{b} \in \mathbf{Sp}\,(\mathcal{A}).$$

Applying the last theorem with $\mathbf{A} = \{\mathbf{a}_0, \mathbf{a}_1\}$ for a pair of incomparable degrees $\mathbf{a}_0$ and $\mathbf{a}_1$, we get following

Corollary 4. *There is a degree* $\mathbf{b}$*, such that* $\mathbf{Sp}\,(\mathcal{A}) \neq \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$ *for any algebraic structure* $\mathcal{A}$*.*

The construction of the degree $\mathbf{b}$ in the proof of the Theorem 5 essentially uses a list of all structures (up to isomorphism) which are computable relative to any element of $\mathbf{A}$. By this reason, it is very difficult to give an upper bound for the degree $\mathbf{b}$ from Corollary 4.

The following result is more weak than Theorem 5, but it has more constructive proof. This allows to bound the degree $\mathbf{b}$ by the double-jump.

Theorem 6 ([7]). *For any degree* $\mathbf{a}_0 > \mathbf{0}$ *there are degrees* $\mathbf{a}_1 \leq \mathbf{a}_0''$ *and* $\mathbf{b} \leq \mathbf{a}_0''$ *such that* $\mathbf{a}_0 \not\leq \mathbf{b}$*,* $\mathbf{a}_1 \not\leq \mathbf{b}$*, and for any algebraic structure* $\mathcal{A}$

$$\{\mathbf{a}_0, \mathbf{a}_1\} \subseteq \mathbf{Sp}\,(\mathcal{A}) \Longrightarrow \mathbf{b} \in \mathbf{Sp}\,(\mathcal{A}).$$

Corollary 5. *There is a degree* $\mathbf{b} \leq \mathbf{0}''$*, such that* $\mathbf{Sp}\,(\mathcal{A}) \neq \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$ *for any algebraic structure* $\mathcal{A}$*.*

An essential idea of the proof of Theorem 6 is to use the following not difficult lemma:

Lemma. *For any set* A *there is a noncomputable set* $A_1 \leq_T A''$*, and there is a partially* A''*-computable function* θ *such that for any* $e \in \omega$

$$W_e^A \text{ is c.e. in } A_1 \iff \theta(e)\downarrow \iff W_e^A = W_{\theta(e)}.$$

Here W_e^A *is the standard numbering of all* A*-c.e. sets. In particular, the condition above is an effective version of* $\deg(A) \cap \deg(A_1) = \mathbf{0}$.

In fact, the degree $\mathbf{a}_1$ in Theorem 6 is the degree of the set A_1 from the lemma applied with $A \in \mathbf{a}_0$. Such set A_1 allows to bound existential types of structures $\mathcal{A}$ such that $\{\mathbf{a}_0, \mathbf{a}_1\} \subseteq \mathbf{Sp}\,(\mathcal{A})$: they must be c.e. and θ gives their c.e. indices.

2 The Structures with the Degree Spectra $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$

In spite of Corollaries 4 and 5 there are a lot of nonzero degrees $\mathbf{b}$ such that the collection $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$ is a degree spectrum of a structure. Moreover, we can build such structures for any low degree $\mathbf{b}$.

Theorem 7 ([5]). *For any low degree* $\mathbf{b}$ *there is a structure* $\mathcal{A}$ *such that* $\mathbf{Sp}\,(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$.

The proof of this theorem is based on the same ideas as the proof of Wehner's result [14] on the structure with the degree spectrum $\mathbf{Sp}\,(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} > \mathbf{0}\}$.

Namely, we first fix some effective coding $\mathcal{S} \mapsto \Gamma(\mathcal{S})$ of countable families $\mathcal{S}$ of subsets of ω into a irreflexive symmetric graphs $\Gamma(\mathcal{S})$ (see [3], [10]), such that for any degree $\mathbf{x}$

$$\mathbf{x} \in \mathbf{Sp}\,(\Gamma(\mathcal{S})) \iff \mathcal{S} \text{ is uniformly c.e. in } \mathbf{x}.$$

For example, we can define $\Gamma(\mathcal{S})$ as the graph with the vertices A, $\mathsf{B}_{i,j,X}$ (where $i, j \in \omega$, $X \in \mathcal{S}$), $\mathsf{C}_{i,j,X}$ (where $i \in \omega$, $j \in X \in \mathcal{S}$) and the edges $\{\mathsf{A}, \mathsf{B}_{i,0,X}\}$ (where $i \in \omega$, $X \in \mathcal{S}$), $\{\mathsf{B}_{i,j,X}, \mathsf{B}_{i,j+1,X}\}$ (where $i, j \in \omega$, $X \in \mathcal{S}$), $\{\mathsf{B}_{i,j,X}, \mathsf{C}_{i,j,X}\}$ (where $i \in \omega$, $j \in X \in \mathcal{S}$).

Then for a low degree $\mathbf{b}$ it is sufficient to find a countable family $\mathcal{S}$ such that for all degrees $\mathbf{x}$

$$\mathcal{S} \text{ is uniformly c.e. in } \mathbf{x} \iff \mathbf{x} \not\leq \mathbf{b}. \tag{1}$$

For the case $\mathbf{b} = \mathbf{0}$ Wehner [14], in fact, considered the family

$$\mathcal{F} = \{\{n\} \oplus F : n \in \omega \;\&\; F \subseteq \omega \;\&\; F \text{ is finite} \;\&\; F \neq W_n\},$$

where W_n is the standard numbering of all c.e. sets. By the Recursion Theorem, we immediately get that $\mathcal{F}$ is not uniformly c.e. (otherwise for every n we can effectively enumerate a set not equal to W_n). Note that, in the original proof Wehner used a direct diagonalization instead of using the Recursion Theorem. By this reason his definition is more complicate, but it can be equivalently reduced to the same form as the family $\mathcal{F}$.

Now to build a family $\mathcal{S} = \mathcal{F}_B$ satisfying the equivalence (1) with $\mathbf{b} = \deg(B)$, $\mathbf{b}' = \mathbf{0}'$, it is sufficient to consider the easy analogue of $\mathcal{F}$:

$$\mathcal{F}_B = \{\{n\} \oplus F : n \in \omega \;\&\; F \subseteq \omega \;\&\; F \text{ is finite} \;\&\; F \neq W_n^B\},$$

where W_n^B is the standard numbering of all B-c.e. sets. To prove that for all $\mathbf{x}$

$$\mathcal{F}_B \text{ is uniformly c.e. in } \mathbf{x} \iff \mathbf{x} \not\leq \deg(B).$$

it is necessary to use the fact that, if $B' \equiv_T \emptyset'$, then the predicate $K_0^B(m,n) \iff m \in W_n^B$ is a Δ_2^0-predicate.

By this reason, Theorem 7 can not be extended to non-low degrees $\mathbf{b}$ by the same way. For example, for $\mathbf{b} = \mathbf{0}'$ the predicate $K_0^{\emptyset'}$ is Σ_2^0-complete, although the theorem can be extended to such $\mathbf{b}$.

Theorem 8 ([6]). *For any c.e. degree* $\mathbf{b}$ *there is a structure* $\mathcal{A}$ *such that* $\mathbf{Sp}\,(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$.

For a c.e. set B the following family $\mathcal{S} = \mathcal{E}_B$ satisfies the equivalence (1) with $\mathbf{b} = \deg(B)$:

$$\mathcal{E}_B = \{\{n\} \oplus F : n \in \omega \;\&\; F \in \mathcal{P} \;\&\; F \neq W_n^B\},$$

where $\mathcal{P}$ is the family of all c.e. set, which are images of injective primitive recursive functions. Here such $\mathcal{P}$ is used because it is an example of sufficiently rich family which is uniformly c.e. and contains only infinite sets (in contrast with the infinite computable sets and the infinite c.e. sets).

We finish the section by the following remark. Theorems 7 and 8 give examples when a nontrivial union of two degree spectra is again a degree spectrum (by Corollary 3 this is not possible for such unions as $\{\mathbf{x} : \mathbf{x} \geq \mathbf{a}_0\} \cup \{\mathbf{x} : \mathbf{x} \geq \mathbf{a}_1\}$). Indeed, it is sufficient to take three different low (or c.e.) degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{c}$. Then, obviously,

$$\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\} \cup \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\} = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{c}\},$$

and each of these three collections is a degree spectrum. It follows from the next section (Corollary 7 and Theorem 11), that if $\mathbf{a}$ and $\mathbf{b}$ are low then the union $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\} \cup \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}$ is a degree spectrum even though $\mathbf{a} \cap \mathbf{b}$ does not exist.

3 Some Other Degree Spectra Derived from the Families

The proof of Theorem 7 is based on the fact, that the predicate "$m \in W_n^B$" is Δ_2^0, if $B' \equiv \emptyset'$. This established the idea to change the numbering $\varepsilon_B(n) = W_n^B$ by an arbitrary numbering $\nu : \omega \to 2^\omega$ such that the predicate "$m \in \nu(n)$" is Δ_2^0, so called *a computable numbering of Δ_2^0 sets*. Let

$$\mathcal{F}(\nu) = \{\{n\} \oplus F : n \in \omega \;\&\; F \subseteq \omega \;\&\; F \text{ is finite} \;\&\; F \neq \nu(n)\},$$

and hence for the family $\mathcal{F}_B$ from the previous section we have $\mathcal{F}_B = \mathcal{F}(\varepsilon_B)$.

Note that the class of degree spectra of graphs $\Gamma(\mathcal{F}(\nu))$ is closed under intersection. Moreover, it is easy to check, that for any numberings $\nu, \eta : \omega \to 2^\omega$

$$\mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu))) \cap \mathbf{Sp}\,(\Gamma(\mathcal{F}(\eta))) = \mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu + \eta))),$$

where $\nu + \eta$ is the standard sum of numberings: for all $n \in \omega$

$$(\nu + \eta)(2n) = \nu(n); \quad (\nu + \eta)(2n + 1) = \eta(n).$$

Theorem 9 describes the degree spectra of the graphs $\Gamma(\mathcal{F}(\nu))$, where the predicate "$m \in \nu(n)$" is Δ_2^0.

Theorem 9 ([5]). *Let ν be a computable numbering of Δ_2^0 sets. Then for a set $X \subseteq \omega$ the following conditions are equivalent:*

1. $\deg(X) \in \mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu)))$;
2. *there is a computable function $f : \omega^2 \to \omega$ such that for all $m, n \in \omega$ we have $W_{f(n,m)}^X \neq \nu(n)$, $\{k \in \omega : k < m\} \subseteq W_{f(n,m)}^X$, and $W_{f(n,m)}^X$ is finite;*
3. *there is a computable function $f : \omega^2 \to \omega$ such that for all $m, n \in \omega$ we have $W_{f(n,m)}^X \neq \nu(n)$ and $\{k \in \omega : k < m\} \subseteq W_{f(n,m)}^X$.*

As a corollary, we get that for computable numberings ν of Δ^0_2 sets degree spectra of graphs $\Gamma(\mathcal{F}(\nu))$ have the same behavior on non-low degrees.

Corollary 6. *If* $\mathbf{x}' > \mathbf{0}'$ *and* ν *is a computable numbering of* Δ^0_2 *sets, then* $\mathbf{x} \in \mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu)))$.

Indeed, if $\emptyset' <_T X'$ then a computable function f, such that

$$W^X_{f(n,m)} = X' \cup \{k \in \omega : k < m\},$$

satisfies the condition 3 of Theorem 9 (since $X' \notin \Delta^0_2$).

For some computable numberings ν of Δ^0_2 sets the description of $\mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu)))$ can be made more easy. Namely, we say that a numbering ν *is an LR-numbering*, if for some computable functions $L, R : \omega \to \omega$ we have

$$\nu(n) = \nu(L(n)) \oplus_1 \nu(R(n))$$

for each $n \in \omega$, where

$$X \oplus_1 Y = \{\langle 2x, y\rangle : \langle x, y\rangle \in X\} \cup \{\langle 2x+1, y\rangle : \langle x, y\rangle \in Y\}$$

is the bijection between $2^\omega \times 2^\omega$ and 2^ω. In fact, for the next theorem no matter which of $X \oplus_1 Y$ or the standard $X \oplus Y = \{2x : x \in X\} \cup \{2x+1 : x \in Y\}$ is used in the definition of LR-numberings, but we prefer to use $\oplus_1$ instead of $\oplus$ for the sake of Corollary 7.

Theorem 10 ([5])**.** *Let* ν *be a computable LR-numbering of* Δ^0_2 *sets. Then for a degree* $\mathbf{x}$ *the following conditions are equivalent:*

1. $\mathbf{x} \in \mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu)))$;
2. *the family of all* $\mathbf{x}$*-c.e. sets is not a subset of the image of* ν *(i.e. there is an* $\mathbf{x}$*-c.e. set* $Z \notin \{\nu(n) : n \in \omega\}$*).*

Note that the numbering $\nu(n) = W^B_n$ is an LR-numbering for any $B \subseteq \omega$. Thus, Theorem 10 is a generalization of Theorem 7.

Let $X \oplus_2 Y$ be the another bijection between $2^\omega \times 2^\omega$ and 2^ω:

$$X \oplus_2 Y = \{\langle x, 2y\rangle : \langle x, y\rangle \in X\} \cup \{\langle x, 2y+1\rangle : \langle x, y\rangle \in Y\}.$$

For numberings ν and η define the numbering $\nu \times \eta$ as follows: for every $n, m \in \omega$

$$(\nu \times \eta)(\langle n, m\rangle) = \nu(n) \oplus_2 \nu(m).$$

By the obvious identity

$$(A \oplus_1 B) \oplus_2 (C \oplus_1 D) = (A \oplus_2 C) \oplus_1 (B \oplus_2 D)$$

it follows, that if ν and η are LR-numberings then $\nu \times \eta$ is also an LR-numbering. Now the next corollary follows immediately:

Corollary 7. *Let ν and η be computable LR-numberings of Δ^0_2 sets. Then $\nu+\eta$ and $\nu\times\eta$ are also computable LR-numberings of Δ^0_2 sets, and*

$$\mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu))) \cap \mathbf{Sp}\,(\Gamma(\mathcal{F}(\eta))) = \mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu+\eta))),$$

$$\mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu))) \cup \mathbf{Sp}\,(\Gamma(\mathcal{F}(\eta))) = \mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu\times\eta))).$$

Note that for the LR-numberings $\varepsilon_B = W^B_n$, $B' \equiv_T \emptyset'$, Corollary 7 can be strengthen:

Theorem 11 ([5])**.** *If $B' \equiv_T \emptyset'$, then for any computable numberings ν of Δ^0_2 sets*

$$\mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu))) \cup \mathbf{Sp}\,(\Gamma(\mathcal{F}(\varepsilon_B))) = \mathbf{Sp}\,(\Gamma(\mathcal{F}(\nu\times\varepsilon_B))).$$

4 Further Questions

The questions from this paragraph are closely related to the results from the previous three paragraphs. Namely, seeing Theorem 2 it is interesting to find two different degrees which compute the same (up to isomorphism) collection of linear orderings. By a result of Knight (see e.g. [1]) this two degrees must be incomparable.

Question 2. Are there two incomparable degrees $\mathbf{a}$ and $\mathbf{b}$ such that for any linear ordering $\mathcal{L}$

$$\mathbf{a} \in \mathbf{Sp}\,(\mathcal{L}) \iff \mathbf{b} \in \mathbf{Sp}\,(\mathcal{L})?$$

Also, it is not so clear how to find the degree $\mathbf{b}$ in Theorem 5 more effectively. In particular:

Question 3. Let $\mathbf{a}_0$ and $\mathbf{a}_1$ be incomparable arithmetical degrees. Is there an arithmetical degree $\mathbf{b}$ such that $\mathbf{a}_0 \not\leq \mathbf{b}$, $\mathbf{a}_1 \not\leq \mathbf{b}$, and for any algebraic structure $\mathcal{A}$

$$\{\mathbf{a}_0, \mathbf{a}_1\} \subseteq \mathbf{Sp}\,(\mathcal{A}) \Longrightarrow \mathbf{b} \in \mathbf{Sp}\,(\mathcal{A})?$$

The related questions are about possible extensions of Thorems 7 and 8:

Question 4. Does for any degree $\mathbf{b} \leq \mathbf{0}'$ there exist a structure $\mathcal{A}$ such that

$$\mathbf{Sp}\,(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{b}\}?$$

Question 5. Is there a structure $\mathcal{A}$ such that

$$\mathbf{Sp}\,(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{0}''\},$$

or, at least, $\mathbf{Sp}\,(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{0}^{(n)}\}$ for some $n \geq 2$?

Finally, Theorem 10 allows for any uniformly Δ^0_2 family $\mathcal{C}$, which is closed under left and right parts of $\oplus_1$ (i.e., if $X \oplus_1 Y \in \mathcal{C}$ then $X \in \mathcal{C}$ and $Y \in \mathcal{C}$), to create a structure with the degree spectra

$$\mathbf{S}(\mathcal{C}) = \{\mathbf{x} : (\exists Z \notin \mathcal{C})[Z \text{ is } \mathbf{x}\text{-c.e.}]\}.$$

For example, consider the family Δ^{-1}_ω of all ω-c.e. sets. Recall, that a set A is ω-c.e. if there are computable functions f and g such that for all $x \in \omega$

$$A(x) = \lim_s f(x,s) \text{ and card } \{s : f(x,s) \neq f(x,s+1)\} < g(x).$$

Then $\mathbf{S}(\Delta^{-1}_\omega)$ consists from the degrees of sets X such that the Turing jump X' is not ω-c.e. Note that the condition $X' \in \Delta^{-1}_\omega$ is not equivalent to a computability of X (for example, for the sets constructed for the Original Friedberg-Muchnik Theorem).

The family Δ^{-1}_ω is the first infinite level of Ershov Hierarchy [2]. The closest levels are Σ^{-1}_ω and Π^{-1}_ω. Namely, $A \in \Sigma^{-1}_\omega$, if there are a computable function f and a partially computable fucntion g such that for all $x \in \omega$ we have $A(x) = \lim_s f(x,s)$,

$$x \in A \Longrightarrow g(x) \text{ is defined, and}$$

$$g(x) \text{ is defined} \Longrightarrow \text{card } \{s : f(x,s) \neq f(x,s+1)\} < g(x).$$

The level Π^{-1}_ω consists from the complements of sets from Σ^{-1}_ω. The families Σ^{-1}_ω and Π^{-1}_ω are again uniformly Δ^0_2 families $\mathcal{C}$, closed under left and right parts of $\oplus_1$, but it is not clear, are the collections $\mathbf{S}(\Sigma^{-1}_\omega)$ and $\mathbf{S}(\Pi^{-1}_\omega)$ equal to $\mathbf{S}(\Delta^{-1}_\omega)$:

Question 6. Is there a set $X \subseteq \omega$ such that $X' \in \Sigma^{-1}_\omega - \Delta^{-1}_\omega$?

Question 7. Is there a set $X \subseteq \omega$ such that $X' \in \Pi^{-1}_\omega - \Delta^{-1}_\omega$?

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