# Reachability Problems: An Update

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**Abstract.** There has been a great deal of progress in the fifteen years that have elapsed since Wigderson published his survey on the complexity of the graph connectivity problem [Wig92]. Most significantly, Reingold solved the longstanding question of the complexity of the $s$-$t$ connectivity problem in undirected graphs, showing that this is complete for logspace (L) [Rei05].

This survey talk will focus on some of the remaining open questions dealing with graph reachability problems. Particular attention will be paid to these topics:

- Reachability in planar directed graphs (and more generally, in graphs of low genus) [ADR05, BTV07].
- Reachability in different classes of grid graphs [ABC$^+$06].
- Reachability in mangroves [AL98].

The problem of finding a path from one vertex to another in a graph is the first problem that was identified as being complete for a natural subclass of P; it was shown to be complete for nondeterministic logspace (NL) by Jones [Jon75]. Restricted versions of this problem were subsequently shown to be complete for other natural complexity classes such as $NC^1$ and L. More than three decades have passed since the publication of Jones' work, and for most of that time, the outstanding open problem about graph reachability centered on the complexity of the reachability problem in *undirected* graphs. This problem was finally resolved by Reingold [Rei05], who showed that it is complete for L.

There are several other natural graph reachability problems whose complexity remains uncharacterized. The purpose of this lecture is to present some open questions about graph reachability problems, and to survey some recent progress toward understanding these problems.

We make use of reachability problems in order to understand familiar subclasses of NL, such as L (deterministic logspace), $AC^0$ (the class of problems solvable by constant-depth polynomial-size circuits of unbounded fan-in AND and OR gates), $TC^0$ (the class of problems solvable by constant-depth threshold circuits of polynomial size), and $NC^1$ (the class of problems solvable by Boolean formulae of polynomial size). Two other complexity classes turn out to play important roles in our study of reachability problems: UL and RUL. UL (unambiguous logspace) is the class of problems solvable by NL machines with the property that, on every input, they have at most one accepting computation path [AJ93]. Although this seems to be a severe limitation, there is evidence that NL $=$ UL [RA00, ARZ99]. RUL ("Reach" unambiguous logspace) was introduced in [BJLR91] by imposing a more restrictive condition

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on UL machines; no configuration can be reached by two distinct computation paths on any input (even on rejecting computation paths). Thus on a RUL machine, the subgraph of reachable configurations always forms a directed tree rooted at the start configuration.

$$\mathrm{AC}^0 \subseteq \mathrm{TC}^0 \subseteq \mathrm{NC}^1 \subseteq \mathrm{L} \subseteq \mathrm{RUL} \subseteq \mathrm{UL} \subseteq \mathrm{NL}.$$

All of these classes are known to be closed under complement, except UL. All of these classes except UL also contain sets that are complete under $\mathrm{AC}^0$ reductions; this is trivial except for the case of RUL [Lan97]. UL does contain a set that is complete under *nonuniform* $\mathrm{AC}^0$ reductions [RA00].

Planarity is one of the most important and most frequently studied graph-theoretic restrictions, but only very recently has there started to be any evidence that the planar case might be easier than the unrestricted reachability problem. Reachability in planar digraphs is now known to be solvable in UL [BTV07]. Planar reachability is logspace-equivalent to the restricted problem of reachability in *grid graphs*, as well as to the more general problem of determining reachability for graphs embedded on a torus (i.e., genus 1 graphs) [ADR05]. Interestingly, nothing is known about graphs of genus 2; it is possible that computing reachability for genus 2 graphs is hard for NL.

No deterministic algorithm for planar reachability has been found that uses less than $\log^2 n$ space (although a logspace algorithm was presented in [ABC$^+$06] for the special case of planar acyclic digraphs having a single source). One class of digraphs where a better deterministic algorithm has been found is the class of *mangroves*. A graph is a mangrove if, for every vertex $v$, both the subgraph of vertices *reachable from* $v$ and the subgraph of vertices that *reach* $v$ are trees. (Equivalently, for every pair of vertices $(u, v)$, there is at most one path from $u$ to $v$.) A deterministic algorithm for reachability on mangroves was presented in [AL98] that uses space $\log^2 n/\log\log n$, and the same paper builds on this to show RUL $\subseteq$ DSPACE$(\log^2 n/\log\log n)$.

Because grid graph reachability is logspace-equivalent to planar reachability, it suffices to concentrate on grid graphs in trying to find a better algorithm for planar reachability. It is interesting to note, however, that whereas planar reachability is hard for L under $\mathrm{AC}^0$ reductions, this is not known to hold for grid graph reachability. A detailed study of grid graph reachability was undertaken in [ABC$^+$06]. There, it was shown that many restricted versions of grid graph reachability (such as the undirected case, the outdegree one case, and the case where both indegree and outdegree are exactly one) are equivalent under $\mathrm{AC}^0$ reductions, thus giving rise to a natural cluster of problems intermediate between L and $\mathrm{NC}^1$ (i.e., known to be hard for $\mathrm{NC}^1$ and lying in L but not known to be hard for L). A very restricted grid graph reachability problem was also shown to be complete for $\mathrm{TC}^0$ under $\mathrm{AC}^0$-Turing reductions.

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## References


[ABC$^+$06] Allender, E., Mix Barrington, D., Chakraborty, T., Datta, S., Roy, S.: Grid graph reachability problems. In: IEEE Conference on Computational Complexity, pp. 299–313 (2006)

[ADR05] Allender, E., Datta, S., Roy, S.: The directed planar reachability problem. In: Proc. 25th annual Conference on Foundations of Software Technology and Theoretical Computer Science (FST&TCS). LNCS, vol. 1373, pp. 238–249. Springer, Heidelberg (2005)

[AJ93] Àlvarez, C., Jenner, B.: A very hard log-space counting class. Theoretical Computer Science 107, 3–30 (1993)

[AL98] Allender, E., Lange, K.-J.: RUSPACE($\log n$) is contained in DSPACE($\log^2 n / \log \log n$). Theory of Computing Systems 31, 539–550 (1998)

[ARZ99] Allender, E., Reinhardt, K., Zhou, S.: Isolation, matching, and counting: Uniform and nonuniform upper bounds. Journal of Computer and System Sciences 59(2), 164–181 (1999)

[BJLR91] Buntrock, G., Jenner, B., Lange, K.-J., Rossmanith, P.: Unambiguity and fewness for logarithmic space. In: Budach, L. (ed.) FCT 1991. LNCS, vol. 529, pp. 168–179. Springer, Heidelberg (1991)

[BTV07] Bourke, C., Tewari, R., Vinodchandran, N.V.: Directed planar reachability is in unambiguous logspace. In: IEEE Conference on Computational Complexity (to appear 2007)

[Jon75] Jones, N.D.: Space bounded reducibility among combinatorial problems. Journal of Computer and System Sciences 11, 68–85 (1975)

[Lan97] Lange, K.-J.: An unambiguous class possessing a complete set. In: 14th International Symposium on Theoretical Aspects of Computer Science (STACS). LNCS, vol. 1200, pp. 339–350. Springer, Heidelberg (1997)

[RA00] Reinhardt, K., Allender, E.: Making nondeterminism unambiguous. SIAM Journal of Computing 29, 1118–1131 (2000)

[Rei05] Reingold, O.: Undirected st-connectivity in log-space. In: Proceedings 37th Symposium on Foundations of Computer Science, pp. 376–385. IEEE Computer Society Press, Washington, DC, USA (2005)

[Wig92] Wigderson, A.: The complexity of graph connectivity. In: Havel, I.M., Koubek, V. (eds.) Symposium on Mathematical Foundations of Computer Science. LNCS, vol. 629, pp. 112–132. Springer, Heidelberg (1992)