Pseudojump Operators and Π^0_1 Classes

Douglas Cenzer[1,⋆], Geoffrey LaForte[2], and Guohua Wu[3,⋆⋆]

[1] Department of Mathematics, University of Florida
P.O. Box 118105, Gainesville, Florida 32611, USA
cenzer@math.ufl.edu

[2] Department of Computer Science, University of West Florida
Pensacola, Florida 32514, USA
glaforte@coginst.uwf.edu

[3] School of Physical and Mathematical Sciences
Nanyang Technological University, Singapore 639798
guohua@ntu.edu.sg

Abstract. For a pseudojump operator V^X and a Π^0_1 class P, we consider properties of the set $\{V^X : X \in P\}$. We show that there always exists $X \in P$ with $V^X \leq_T \mathbf{0}'$ and that if P is Medvedev complete, then there exists $X \in P$ with $V^X \equiv_T \mathbf{0}'$. We examine the consequences when V^X is Turing incomparable with V^Y for $X \neq Y$ in P and when $W^X_e = W^Y_e$ for all $X, Y \in P$. Finally, we give a characterization of the jump in terms of Π^0_1 classes.

Keywords: Computability, Π^0_1 Classes.

Pseudojump operators have been of great interest in computability theory and were explicitly introduced by Jockusch and Soare in [7]. If ϕ^X_e is the eth partial computable functional with oracle X, then $W^X_e = \{n : \phi^X_e(n) \downarrow\}$ and the eth pseudojump operator J_e maps X to $X \oplus W^X_e$. In particular, the jump operator $J(X) = X' = \{e : \phi^X_e(e) \downarrow\}$ is also a pseudojump operator. We will often denote a pseudojump operator by V and let V^X denote the pseudojump of X. Friedberg [3] constructed a noncomputable c.e. set A such that $A' \equiv_T \mathbf{0}'$. The fundamental theorem for pseudojumps, from [7], states that for any index e, there exists a noncomputable c.e. set A such that $J_e(A) \equiv_T \mathbf{0}'$. This generalizes the result of Friedberg that $A' \equiv_T \mathbf{0}'$ for some noncomputable c.e. set A. On the other hand, if V^X is obtained from the construction of a low^X set, then $(V^A)' = A'$, so that if $V^A \equiv_T \mathbf{0}'$, then $A' = \mathbf{0}''$. In each of these examples, $X <_T V^X$ for all X. We will say that a pseudojump operator V is *strongly nontrivial* if $X <_T V^X$ for all X. In the recent paper [2], it was shown that for any pseudojump operator V with $A <_T V^A$ for all c.e. sets A, there exist Turing incomparable c.e. sets A and B such that $V^A \equiv_T V^B \equiv_T \mathbf{0}'$.

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The study of pseudojump operators is a natural extension of the study of c.e. sets and degrees, which are fundamental in computability theory. Another natural extension is the study of effectively closed sets (Π^0_1 classes), which are sets of reals and play an important role in many areas of computable mathematics. The degrees of members of Π^0_1 classes is of great interest here. For example, every Π^0_1 class $Q \subseteq 2^{\mathbb{N}}$ has a member of c.e. degree, but there exist Π^0_1 classes with no computable member. A survey of results on Π^0_1 classes may be found in [1].

In this paper, we consider the interaction between pseudojump operators and Π^0_1 classses, in particular how pseudojump operators act on Π^0_1 classes. Recent work of Simpson [8] on the Medvedev degrees of Π^0_1 classes has characterized the complete degrees in several ways. The main result is that if V is a pseudojump operator and P is a Medvedev complete Π^0_1 class, then there exists $X \in P$ with $V^X \equiv_T \mathbf{0}'$. (It follows that there exist infinitely many such $X \in P$.)

We also give a new characterization of the jump in terms of Π^0_1 classes and consider for a Π^0_1 class Q, properties of the set $\{V^X : X \in Q\}$. That is, we examine the consequences of having $W_e^X = W_e^Y$ for all $X \in Q$ and of having W_e^X Turing incomparable with W_e^Y for all $X \neq Y$ in Q.

It is easy to find a nonempty Π^0_1 class P and a pseudojump operator V such that $V^X \not\equiv_T \mathbf{0}'$ for any $X \in P$. For example, if P contains only computable elements and V^X is low^X, then $X' \equiv \mathbf{0}'$ for all $X \in P$. Our intuition is that if P is complicated enough, then it should have a member with $V^X \equiv_T \mathbf{0}'$.

For Π^0_1 classes with no computable members, we still might not have a c. e. member or even a member of c.e. degree with $V^X \equiv_T \mathbf{0}'$. We can find examples of such *special* Π^0_1 classes with no members X of c. e. degree such that $V^X \equiv_T \mathbf{0}'$. Jockusch [4] constructed a Π^0_1 class P with no c. e. members at all. Jockusch and Soare [5] constructed a Π^0_1 class Q such that for any c. e. degree $\mathbf{b}$ and any $X \in P$, if $X \leq_T \mathbf{b}$, then $\mathbf{b} = 0'$. Thus if X has c. e. degree and $X \in Q$, then $X \equiv_T \mathbf{0}'$, so that if $V^X \leq_T \mathbf{0}'$, then $V^X \equiv_T X$, so that V fails to be strongly non-trivial. Recall that the Low Basis Theorem of Jockusch and Soare [6] shows that any nonempty Π^0_1 class $P \subseteq 2^{\mathbb{N}}$ must contain a member of low degree. The previous result implies that this member need not have c.e. degree.

Since $V^X \leq_T X'$ for any set X and any pseudojump operator V, the following is an immediate corollary of the low basis theorem. We sketch a proof in preparation for the main theorem. Let K denote the Halting Problem $\{e : \phi_e(e)\downarrow\}$.

Proposition 1. *For any pseudojump operator V and any nonempty Π^0_1 class P, there exists $X \in P$ with $V^X \leq_T K$.*

Proof. This is an easy modification of the Low Basis Theorem [6]. Let $P = [T]$ and fix e such that $V^X = W_e^X = \{m : \phi_e^X(m)\downarrow\}$. For each a, define the computable tree

$$U_a = \{\sigma \in \{0,1\}^* : \phi_e^\sigma(a)\uparrow\}.$$

Then $[U_a] = \{X : \phi_e^X(a)\uparrow\}$. Now define a sequence of Π^0_1 trees $\{S_n : n < \omega\}$ as follows. Let $S_0 = T$ and for each n, define

$$S_{n+1} = \begin{cases} S_n \cap U_n, & \text{if } S_n \cap U_n \text{ is infinite,} \\ S_n, & \text{otherwise.} \end{cases}$$

Now let $S = \cap_n S_n$ and $Q = [S] = \cap_n [S_n]$. By assumption, P is nonempty so that T is infinite and it follows from the construction, by induction, that each S_n is infinite. Thus Q is nonempty.

The construction is computable in K and therefore $\{n : S_n \cap U_n \text{ is infinite}\}$ is computable in K. Now for $X \in [S_{n+1}]$, it is clear that if $S_n \cap U_n$ is infinite, then $n \notin V^X$. On the other hand, if $S_n \cap U_n$ is finite, then $[S_n] \cap [U_n] = \emptyset$, so that for $X \in [S_n]$, $n \in V^X$. This gives a definition of V^X using K. Note that for any $X, Y \in Q$, we have $V^X = V^Y$. □

We now turn to the main result. Let $\mathcal{Q}$ be the computable Boolean algebra of clopen sets in $\{0,1\}^{\mathbb{N}}$. A clopen set is simply a finite union of intervals. A Π^0_1 class P is said to be *productive* if there is a computable *splitting* function $g : \mathbb{N} \to \mathcal{B}$ such that, for any e, if $P_e \cap P$ is nonempty, then both $P_e \cap P \cap g(e)$ and $P_e \cap P - g(e)$ are nonempty. Simpson showed that a Π^0_1 class is productive if and only if it is Medvedev complete. The Medvedev complete classes are the most *difficult* in the sense that if Q is Medvedev complete and P is any Π^0_1 class, then there exists a computable map Φ mapping Q into P.

Theorem 1. *Let V be a pseudojump operator V and let P be a Medvedev complete Π^0_1 class. Then there exists $X \in P$ with $V^X \equiv_T K$.*

Proof. Let $P = P_c = [T]$ be Medvedev complete and let g be a splitting function for P. We now give a modification of the proof of Proposition 1 above. The idea is that the Halting Problem K will be coded into V^X via a function $f : \mathbb{N} \to \mathcal{Q}$, computable in V^X, such that

$$X \in f(n) \iff n \in K.$$

Fix e such that $V^X = W^X_e$ and let U_a be defined as above. Now define the sequences $\{R_n : n < \omega\}$ and $\{Q_n : n < \omega\}$ of Π^0_1 classes as follows. Let $R_0 = P = P_c$ and let

$$R_n = \begin{cases} Q_n \cap [U_n], & \text{if } Q_n \cap [U_n] \text{ is nonempty,} \\ Q_n, & \text{otherwise.} \end{cases}$$

Let $R_n = P_{r(n)}$. By the construction, R_n is a nonempty subset of P, so that $R_n \cap g(r(n))$ and $R_n - g(r(n))$ are both nonempty subsets of P. Then define

$$Q_{n+1} = \begin{cases} R_n \cap g(r(n)), & \text{if } n \in K, \\ R_n - g(r(n)), & \text{otherwise.} \end{cases}$$

As before, let $Q = \cap_n Q_n$. It follows by induction that each tree each Q_n is nonempty and hence Q is nonempty. Once again, the construction is computable in K and it follows as in the proof of Proposition 1 that, for $X \in Q$, $V^X \leq_T K$ and that, for any $X \in Q$,

$$(*) \qquad V^X = \{n : Q_n \cap [U_n] \text{ is nonempty}\}.$$

On the other hand, suppose that $X \in Q$ and we use V^X as an oracle. Note that $X \leq_T V^X$ so that we can also use X in our computation from V^X. Then we can recursively compute the function $r(n)$ as follows. Informally, we can compute R_n using V^X and then we can compute Q_{n+1} using X.

More formally, we may define functions r and q, computable from V^X, so that $R_n = P_{r(n)}$ and $Q_n = P_{q(n)}$. That is, $Q_0 = P$, so $q(0) = c$. Given $q(n)$, we have

$$P_{r(n)} = \begin{cases} P_{q(n)} \cap [U_n], & \text{if } n \in V^X, \\ P_{q(n)}, & \text{otherwise.} \end{cases}$$

Then we have

$$P_{q(n+1)} = \begin{cases} P_{r(n)} \cap g(r(n)), & \text{if } X \in g(r(n)), \\ P_{r(n)} - g(r(n), & \text{otherwise.} \end{cases}$$

It follows that the functions $q(n)$ and $r(n)$ are computable from V^X. Finally $K \leq_T V^X$ since

$$n \in K \iff X \in g(r(n)).$$

Note that in fact $V^X \equiv_T K$ for all $X \in Q$.

To obtain infinitely many X with $V^X \equiv_T K$, note that for any σ such that $P \cap I(\sigma) \neq \emptyset$, $P \cap I(\sigma)$ is also Medvedev complete. This is because the splitting function for P is easily adapted to a splitting function for $P \cap I(\sigma)$. This means that for every σ such that $P \cap I(\sigma) \neq \emptyset$, there exists $X \in I(\sigma)$ with $V^X \equiv_T K$. Thus there are infinitely many such $X \in$P. □

Although the class Q constructed in the theorem is not a Π^0_1 class, it is a strong Π^0_2 class with the property that $\{V^X : X \in Q\}$ is a singleton and this unique V^X is $\leq_T K$. It seems natural to consider the question of a Π^0_1 class P where V^X is unique for $X \in P$. A classical result is that if $P = \{X\}$ itself is a singleton, then X is computable. By our definition, $V^X = V^Y$ implies that $X = Y$, so we consider just W^X_e.

Proposition 2. *Let P be a Π^0_1 class and suppose that $W^X_e = W^Y_e = W_P$ for all $X, Y \in P$.*

(a) The unique W^X_e for $X \in P$ is a c.e. set.
(b) If $X \leq_T W^X_e$ for all X, then $X \leq_T W_P$, so that P is countable and therefore has a computable member.
(c) Suppose that $X \leq_T W^X_e$ for all X and further that $W^R_e <_T K$ for any recursive R. Then $W_P <_T K$.

Proof. Fix a computable tree T such that $P = [T]$.

(a) Claim: $a \in V^X \iff (\exists n)[(\forall \sigma \in \{0,1\}^n \cap T \;\rightarrow\; a \in V^\sigma)$.

Suppose first that $a \in V^X$ for all $X \in P$. Then by compactness, there exists m such that $a \in V^{X \lceil m}$ for all $X \in P$. Let $S = \{\sigma \in \{0,1\}^m : P \cap I(\sigma) \neq \emptyset\} =$

$\{X\lceil m : X \in P\}$. For $\sigma \in \{0,1\}^m - S$, T contains only finitely many extensions of σ. Thus we can find $n > m$ such that $\tau\lceil m \in S$ for all $\tau \in \{0,1\}^n \cap T$. This n satisfies the formula above.

Next suppose that n exists as in the formula. Then for every $X \in P$, $a \in V^{X\lceil n}$ and therefore $a \in V^X$.

(b) There can be only countably many $X \leq_T W_P$, so it follows from (a) that P is countable and hence P has a computable member.

(c) Finally, let R be a computable member of P which exists by (b). Then for any $X \in P$, $V^X = V^R <_T K$. □

For the other extreme, suppose that V^X is Turing incomparable with V^Y for all $X \neq Y$ in P. It was also shown in [6] that there exist Π^0_1 classes containing continuum many elements, with each pair Turing incomparable. This will serve as an example with $V^X = X$.

Of course if $V^X = X'$, then any Π^0_1 class Q must contain X with $V^X = K$ and therefore if nontrivial, Q must contain distinct X, Y with $V^X \equiv_T K \equiv_T V^Y$.

Proposition 3. *Let W^X denote either W^X_e or $X \oplus W^X_e$ and suppose that P is an infinite Π^0_1 class such that W^X and W^Y are Turing incomparable for any $X, Y \in P$. Then there is no $X \in P$ such that $K \leq_T W^X$.*

Proof. Suppose by way of contradiction that $K \leq_T W^X$ for some $X \in P$. Since P is infinite, there is some $Y \in P$ with $Y \neq X$. Let n be the least such that $X(n) \neq Y(n)$ and let $Q = P \cap I(Y\lceil n+1)$. By Proposition 1, there exists $Z \in Q$ with $W^Z \leq_T K \leq_T V^X$. □

Finally, we observe that Π^0_1 classes may be used to define the jump and also pseudojumps.

Proposition 4. *For any set X, $\{e : X \in P_e\} \equiv_T X'$.*

Proof. Let $W^X = \{e : X \in P_e\}$. Then $W^X \leq_T X'$ since

$$e \in W^X \iff (\forall n) X\lceil n \notin W_e.$$

For the completeness, use the s-m-n theorem to define a computable function f such that

$$P_{f(e)} = \{X : \phi^X_e(e) \uparrow\}.$$

Then

$$e \in X' \iff f(e) \notin W^X$$

gives a reduction of X' to W^X. □

One can define a pseudojump using Π^0_1 classes as follows. Let $\pi_i(P)$ be the projection of P onto the ith coordinate, where $\pi_i(X) = Y$ means that $X = \langle X_1, X_2, \ldots\rangle$ and $Y = X_i$.

Then let

$$V_e^X = \{i : X \in \pi_i(P_e)\}.$$

It can be seen that $V_e^X \equiv_T X'$ when P_e is a particular Medvedev complete class, such that $\pi_i(P)$ runs over all Π_1^0 classes. It is an interesting question whether every pseudojump can be expressed in this form.

References

1. Cenzer, D., Remmel, J.B.: Π_1^0 classes in mathematics, In: Ershov, Y., Goncharov, S., Nerode, A., Remmel, J. (eds.) Handbook of Recursive Mathematics, Part Two, Elsevier Studies in Logic. vol. 139 pp. 623-821. (1998)
2. Coles, R., Downey, R., Jockusch, C., LaForte, G.: Completing pseudojump operators. Ann. Pure and Appl. Logic 136, 297–333 (2005)
3. Friedberg, R.M.: A criterion for completeness of degrees of unsolvability. J. Symbolic Logic 22, 159–160 (1957)
4. Jockusch, C.G.: Π_1^0 classes and boolean combinations of recursively enumerable sets. J. Symbolic Logic 39, 95–96 (1974)
5. Jockusch, C.G., Soare, R.: Degrees of members of Π_1^0 classes. Pacific J. Math 40, 605–616 (1972)
6. Jockusch, C., Soare, R.: Π_1^0 classes and degrees of theories. Trans. Amer. Math. Soc 173, 35–56 (1972)
7. Jockusch, C., Shore, R.: Pseudojump operators I: the r.e. case. Trans. Amer. Math. Soc 275, 599–609 (1983)
8. Simpson, S.: Mass problems and randomness. Bull. Symbolic Logic 11, 1–27 (2005)
9. Soare, R.: Recursively Enumerable Sets and Degrees. Springer, Heidelberg (1987)