Shifting and Lifting of Cellular Automata*

Luigi Acerbi[1], Alberto Dennunzio[1], and Enrico Formenti[2,**]

[1] Università degli Studi di Milano–Bicocca
Dipartimento di Informatica, Sistemistica e Comunicazione,
Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy
dennunzio@disco.unimib.it, luigi.acerbi@gmail.com
[2] Université de Nice-Sophia Antipolis
Laboratoire I3S,
2000 Route des Colles, 06903 Sophia Antipolis, France
enrico.formenti@unice.fr

Abstract. We consider the family of all the Cellular Automata (CA) sharing the same local rule but have different memory. This family contains also all the CA with memory $m \leq 0$ (one-sided CA) which can act both on $A^{\mathbb{Z}}$ and on $A^{\mathbb{N}}$. We study several set theoretical and topological properties for these classes. In particular we investigate if the properties of a given CA are preserved when we consider the CA obtained by changing the memory of the original one (shifting operation). Furthermore we focus our attention to the one-sided CA acting on $A^{\mathbb{Z}}$ starting from the one-sided CA acting on $A^{\mathbb{N}}$ and having the same local rule (lifting operation). As a particular consequence of these investigations, we prove that the long-standing conjecture [Surjectivity $\Rightarrow$ Density of the Periodic Orbits (DPO)] is equivalent to the conjecture [Topological Mixing $\Rightarrow$ DPO].

Keywords: discrete time dynamical systems, cellular automata, topological dynamics, deterministic chaos.

1 Introduction and Motivations

Cellular automata (CA) are a simple formal model for complex systems. Their are used in many scientific fields ranging from biology to chemistry or from physics to computer science.

A CA is made of an infinite set of finite automata distributed over a regular lattice $\mathcal{L}$. All finite automata are identical. Each automaton assumes a *state*, chosen from a finite set A, called the *set of states* or the *alphabet*. A *configuration* is a snapshot of all states of the automata *i.e.* a function from $\mathcal{L}$ to A. In the present paper, $\mathcal{L} = \mathbb{Z}$ or $\mathcal{L} = \mathbb{N}$.

* This work has been supported by the Interlink/MIUR project "Cellular Automata: Topological Properties, Chaos and Associated Formal Languages", by the ANR Blanc "Projet Sycomore" and by the PRIN/MIUR project "Formal Languages and Automata: Mathematical and Applicative Aspects".
** Corresponding author.

S.B. Cooper, B. Löwe, and A. Sorbi (Eds.): CiE 2007, LNCS 4497, pp. 1–10, 2007.

A *local rule* updates the state of an automaton on the basis of its current state and the ones of a fixed set of neighboring automata which are individuated by the neighborhood frame $N = \{-m, -m+1, \ldots, -m+d\}$, where $m \in \mathbb{Z}$, $d \in \mathbb{N}$, and $r = \max\{m, d-m\}$ are the *memory*, the *diameter*, and the *radius* of the CA, respectively. Formally, the local rule is a function $f : A^{d+1} \to A$.

All the automata of the lattice are updated synchronously. In other words, the local rule f induces a *global* rule $F_m : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ describing the evolution of the whole system from time t to $t+1$:

$$\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F_m(x)_i = f(x_{i-m}, \ldots, x_{i-m+d}) \ .$$

The *shift map* σ is one among the simplest examples of CA and it is induced by the rule $f : A \to A$, defined as $\forall a \in A,\ f(a) = a$, with $m = -1$, $d = 0$. Remark that σ can also be induced by the rule $f : A^2 \to A$ defined as $\forall a, b \in A,\ f(a,b) = b$ with $m = 0$, $d = 1$. We prefer to use the former representation since it minimizes the neighborhood size.

For any CA on $A^{\mathbb{Z}}$, the structure $\langle A^{\mathbb{Z}}, F_m \rangle$ is a (discrete time) dynamical system. From now on, for the sake of simplicity, we identify a CA with the dynamical system induced by itself or even with its global rule F_m.

The local rule of a CA can be convenient represented by a look-up table. Anyway, the look-up table does not uniquely define the CA. Indeed, for each value of m we have a different CA. It is therefore natural to wonder what dynamical properties are conserved by the CA obtained by changing the value of m but keeping the same look-up table for the local rule. This paper tries to answer this question.

Remark that the solution to the problem is absolutely not trivial. For instance, a periodic Coven automaton is a CA defined by the following local rule ($A = \{0,1\}$, $m = 0$ and $d \in \mathbb{N}$): $f(a_0, a_1, \ldots, a_d) = a_0 \oplus \prod_{k=1}^{d}(a_k \oplus w_k \oplus 1)$, where $\oplus$ is the usual xor operation and $w = w_1 w_2 \ldots w_d \in \{0,1\}^d$ is a periodic word[1]. Despite of the fact that for aperiodic Coven automata almost everything is known [2], very little is known about the periodic case even for the simplest example *i.e.* when $w = 11$. Call $\mathcal{A}$ this last automaton and F_0 its global rule. Taking the same look-up table as $\mathcal{A}$ but $m = -1$, we obtain the elementary CA rule called $ECA120$. Most of the dynamical properties of $ECA120$ are well-known (see [7], for instance). More formally, one can write that $ECA120 = \sigma \circ F_0$. For this reason we say that $ECA120$ is a *shifted version* of F_0.

In [13], Sablik studies the behavior of the shift operation over look-up tables *w.r.t.* the equicontinuity property and gives precise bounds for conservation and non-conservation. In this paper, we focus on the periodic behavior. We show that the proof of an old-standing conjecture about denseness of periodic orbits (DPO) can be reduced to the study of the class of topologically mixing CA *i.e.* a very small class with very special dynamical behavior. Maybe this would simplify the task of proving the conjecture. The result is obtained as a by-product of our

[1] A word $w \in \{0,1\}^d$ is *periodic* if there exists $1 \le p \le d-1$ such that $w_i = w_{i+p}$ for $1 \le i \le d-p$. A word is *aperiodic* if it is not periodic.

results about the conservation of other interesting properties like surjectivity, left (or right) closingness *etc.*

Any CA with memory $m \leq 0$ is well defined both on $A^{\mathbb{Z}}$ and on $A^{\mathbb{N}}$. In the $A^{\mathbb{N}}$ we prefer to use the slightly different notation Φ_m in order to avoid confusion with the $A^{\mathbb{Z}}$ case. The mapping $\Phi_m : A^{\mathbb{N}} \mapsto A^{\mathbb{N}}$ acts on any configuration $x \in A^{\mathbb{N}}$ as follows

$$\forall i \in \mathbb{N}, \quad \Phi_m(x)_i = f(x_{i-m}, \ldots, x_{i-m+d}) \ .$$

Along the same line of thoughts as before, one can wonder which properties are conserved when passing from $A^{\mathbb{N}}$ to $A^{\mathbb{Z}}$ using the same local rule (with memory $m \leq 0$). The opposite case, *i.e.*, when passing from $A^{\mathbb{Z}}$ to $A^{\mathbb{N}}$ is trivial.

In [3], Blanchard and Maass show a deep combinatorial characterization of expansive CA on $A^{\mathbb{N}}$. These results were successively extended by Boyle and Fiebig [5]. Unfortunately, both the constructions are not of help for the $A^{\mathbb{Z}}$ case.

In this paper we show that most of the interesting properties are conserved when passing from $A^{\mathbb{N}}$ to $A^{\mathbb{Z}}$. If the same holds for DPO is still an open question.

2 Topology and Dynamical Properties

In order to study the dynamical properties of CA, $A^{\mathbb{Z}}$ is usually equipped with the Thychonoff metric d defined as follows

$$\forall x, y \in A^{\mathbb{Z}},\ d(x,y) = 2^{-n},\ \text{where}\ n = \min\left\{i \geq 0 \,:\, x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\right\} \ .$$

Then $A^{\mathbb{Z}}$ is a compact, totally disconnected and perfect topological space. For any pair $i, j \in \mathbb{Z}$, with $i \leq j$, we denote by $x_{[i,j]}$ the word $x_i \cdots x_j \in A^{j-i+1}$, *i.e.*, the portion of the configuration $x \in A^{\mathbb{Z}}$ inside the integer interval $[i,j] = \{k \in \mathbb{Z} : i \leq k \leq j\}$. A *cylinder* of block $u \in A^k$ and position $i \in \mathbb{Z}$ is the set $C_i(u) = \{x \in A^{\mathbb{Z}} : x_{[i,i+k-1]} = u\}$. Cylinders are clopen sets *w.r.t.* the Thyconoff metric.

Given a CA F_m, a configuration $x \in A^{\mathbb{Z}}$ is a *periodic* point of F_m if there exists an integer $p > 0$ such that $F_m^p(x) = x$. The minimum p for which $F_m^p(x) = x$ holds is called *period* of x. If the set of all periodic points of F_m is dense in $A^{\mathbb{Z}}$, we say that the CA has the *denseness of periodic orbits* (DPO). A CA F_m has the *joint denseness of periodic orbits* (JDPO) if it has a dense set of points which are periodic both for F_m and σ.

The study of the chaotic behavior of CA (and more in general of discrete dynamical systems) is interesting and it captured the attention of researchers in the last twenty years. Although there is not a universally accepted definition of chaos, the notion introduced by Devaney is the most popular one [9]. It is characterized by three properties: sensitivity to the initial conditions, DPO and transitivity.

Recall that a CA F_m is *sensitive to the initial conditions* (or simply *sensitive*) if there exists a constant $\varepsilon > 0$ such that for any configuration $x \in A^{\mathbb{Z}}$ and any $\delta > 0$ there is a configuration $y \in A^{\mathbb{Z}}$ such that $d(y,x) < \delta$ and $d(F_m^n(y), F_m^n(x)) > \varepsilon$ for some $n \in \mathbb{N}$. A CA F_m is *(topologically) transitive* if for any pair of non-empty open sets $U, V \subseteq A^{\mathbb{Z}}$ there exists an integer $n \in \mathbb{N}$ such that $F_m^n(U) \cap V \neq \emptyset$. All the transitive CA are sensitive [8]. A CA is *(topologically) mixing* if for any pair of non-empty open sets $U, V \subseteq A^{\mathbb{Z}}$ there exists an integer $n \in \mathbb{N}$ such that for any $t \geq n$ we have $F_m^t(U) \cap V \neq \emptyset$. Trivially, any mixing CA is also transitive.

Let F_m be a CA. A configuration $x \in A^{\mathbb{Z}}$ is an *equicontinuous point* for F_m if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in A^{\mathbb{Z}}$, $d(y, x) < \delta$ implies that $\forall n \in \mathbb{N}$, $d(F_m^n(y), F_m^n(x)) < \varepsilon$. A CA is said to be *equicontinuous* if the set E of all its equicontinuous points is the whole $A^{\mathbb{Z}}$, while it is said to be *almost equicontinuous* if E is residual (*i.e.*, E can be obtained by a infinite intersection of dense open subsets). In [11], Kůrka proved that a CA is almost equicontinuous iff it is non-sensitive iff it admits a blocking word[2].

All the above definitions can be easily adapted to work on $A^{\mathbb{N}}$.

3 Shifting

Let $\langle A^{\mathbb{Z}}, F_m \rangle$ be a CA. For a fixed $h \in \mathbb{Z}$, we consider the CA $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$. Since $F_{m+h} = \sigma^h \circ F_m$, we say that the CA $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ is obtained by a *shifting operation* which move the memory of the originally given CA from m to $m + h$. In this section we study which properties are preserved by the shifting operation.

A CA $\langle A^{\mathbb{Z}}, F_m \rangle$ is *surjective* (resp., *injective*) if F_m is surjective (resp., injective). It is *right* (resp., *left*) *closing* iff $F_m(x) \neq F_m(y)$ for any pair $x, y \in A^{\mathbb{Z}}$ of distinct left (resp., right) asymptotic configurations, *i.e.*, $x_{(-\infty,n]} = y_{(-\infty,n]}$ (resp., $x_{[n,\infty)} = y_{[n,\infty)}$) for some $n \in \mathbb{Z}$, where $z_{(-\infty,n]}$ (resp., $z_{[n,\infty)}$) denotes the portion of a configuration z inside the infinite integer interval $(-\infty, n]$ (resp., $[n, \infty)$). A CA is said to be *closing* if it is either left or right closing. Recall that a closing CA has JDPO [6] and that a CA is open iff it is both left and right closing [6].

Proposition 1. *Let $\langle A^{\mathbb{Z}}, F_m \rangle$ be a CA. For any $h \in \mathbb{Z}$, $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ is surjective (resp., injective, right-closing, left-closing, has JDPO) iff $\langle A^{\mathbb{Z}}, F_m \rangle$ is surjective (resp., injective, right-closing, left-closing, has JDPO).*

Proof. All the statements follow immediately from the definition of F_{m+h}. □

The following theorem establishes the behavior of the shift operation *w.r.t.* sensitivity, equicontinuity and almost equicontinuity. Its proof is essentially contained in [13].

Theorem 1. *For any CA $\langle A^{\mathbb{Z}}, F_m \rangle$ one and only one of the following statements holds:*

$\mathcal{S}_0$: *the CA $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ is nilpotent[3] (and then equicontinuous) for any $h \in \mathbb{Z}$;*

[2] A word $u \in A^k$ is s-blocking ($s \leq k$) for a CA F_m if there exists an offset $j \in [0, k-s]$ such that for any $x, y \in C_0(u)$ and any $n \in \mathbb{N}$, $F_m(x)_{[j,j+s-1]} = F_m(y)_{[j,j+s-1]}$.

[3] A CA F_m is nilpotent if there is a symbol $a \in A$ and an integer $n > 0$ such that for any configuration $x \in A^{\mathbb{Z}}$ we have $F_m^n(x) = (a)^\infty$ (infinite concatenation of a with itself).

$\mathcal{S}_1$: *there exists an integer* $\bar{h}$ *with* $\bar{h}+m \in [-d,d]$ *such that the CA* $\langle A^{\mathbb{Z}}, F_{m+h}\rangle$ *is equicontinuous for* $h=\bar{h}$ *and it is sensitive for any* $h \neq \bar{h}$*;*
$\mathcal{S}_2$: *there is a finite interval* $I \subset \mathbb{Z}$*, with* $I+m \subseteq [-d,d]$*, such that the CA* $\langle A^{\mathbb{Z}}, F_{m+h}\rangle$ *is strictly almost equicontinuous but not equicontinuous iff* $h \in I$ *(and then it is sensitive for any other* $h \in \mathbb{Z}\setminus I$*);*
$\mathcal{S}_3$: *the CA* $\langle A^{\mathbb{Z}}, F_{m+h}\rangle$ *is sensitive (*ever-sensitivity*) for any* $h \in \mathbb{Z}$*.*

In the case of surjective CA, Theorem 1 can be restated as follows.

Theorem 2. *For any surjective CA* $\langle A^{\mathbb{Z}}, F_m\rangle$ *one and only one of the following statements holds:*

$\mathcal{S}'_1$: *there exists an integer* h'*, with* $h'+m \in [-d,d]$*, such that the CA* F_{m+h} *is equicontinuous for* $h=h'$ *and it is mixing for* $h \neq h'$*;*
$\mathcal{S}'_2$: *there exists an integer* h'*, with* $h'+m \in [-d,d]$*, such that the CA* F_{m+h} *is strictly almost equicontinuous but not equicontinuous for* $h=h'$ *and it is mixing for* $h \neq h'$*;*
$\mathcal{S}'_3$: *there is at most a finite set* $I \subset \mathbb{Z}$*, with* $I+m \subseteq [-d,d]$*, such that if* $h \in I$ *then the CA* F_{m+h} *is sensitive but not mixing, while it is mixing if* $h \in \mathbb{Z}\setminus I$*.*

Proof. By Theorem 1, it is enough to prove that if a surjective CA F_m is almost equicontinuous then for any $h \neq 0$ the CA F_{m+h} is mixing. We give the proof for $h>0$, the other case is similar. Let $u \in A^k$ and $v \in A^q$ be two arbitrary blocks and let $w \in A^s$ be a r-blocking word with offset j where r is the radius of the CA. The word wvw is a l-blocking ($l=s+q+r$) with offset j and the configuration $y=(wv)^{\infty}$ is periodic for F_m (see, for instance, the proof of Theorem 5.24 in [12]). Let $p>0$ be the period of y. Then for any configuration $x \in C_0(wvw)$ and any $n \in \mathbb{N}$ we have that $F_m^{p+n}(x)_{[j,j+l-1]} = F_m^n(x)_{[j,j+l-1]}$, in particular $F_m^p(x)_{[s,s+q-1]} = x_{[s,s+q-1]} = v$. Let $t_0>0$ be a multiple of p such that $ht_0 - s \geq k$ and $ht_0 - s + (p-1)(h-r) \geq k$. For any integer $t \geq t_0$, let us consider a configuration $z \in C_0(u) \cap C_{ht-s+a(h-r)}(v')$ where $a = t \mod p$ and $v' \in f^{-a}(wvw)$ is an a-preimage block of wvw. In this way we are sure that $F_{m+h}^t(z) \in C_0(v)$ and thus the CA F_{m+h} is mixing. □

We recall that a CA F_m is *positively expansive* if there exists a constant $\varepsilon>0$ such that for any pair of distinct configurations x,y we have $d(F_m^n(y),F_m^n(x)) \geq \varepsilon$ for some $n \in \mathbb{N}$. The next proposition assures that (positively) expansive CA are in class $\mathcal{S}'_3$, in particular they are ever-sensitive.

Proposition 2. *If* $\langle A^{\mathbb{Z}}, F_m\rangle$ *is a positively expansive CA, then for any* $h \in \mathbb{Z}$ *the CA* $\langle A^{\mathbb{Z}}, F_{m+h}\rangle$ *is sensitive.*

Proof. For $h=0$ the thesis immediately follows by perfectness of $A^{\mathbb{Z}}$. We give the proof for $h>0$, the case $h<0$ is similar. Let $q=\max\{r,h,s\}+1$, where r is the radius of F_m and $s \in \mathbb{N}$ is an integer such that $\frac{1}{2^s}$ is less than the expansivity constant of the given CA. We show that $\langle A^{\mathbb{Z}}, F_{m+h}\rangle$ is sensitive with sensitivity constant $\epsilon = \frac{1}{2^q}$. Chosen an arbitrary $k \in \mathbb{N}$ and a configuration $x \in A^{\mathbb{Z}}$, consider a configuration $y \in A^{\mathbb{Z}}$ such that $y_{(-\infty,k]} = x_{(-\infty,k]}$ with $y_{k+1} \neq x_{k+1}$. By the expansivity of F_m, the sequence $\{j_n\}_{n\in\mathbb{N}} = \min\{i \in \mathbb{Z} : F_m^n(y)_i \neq F_m^n(x)_i\}$ is well-defined and for any $n \in \mathbb{N}$ we have $j_{n+1} - j_n \geq -r$. We now prove the existence of an integer $t \in \mathbb{N}$ such that $F_m^t(y)_{[-q+ht,q+ht]} \neq F_m^t(x)_{[-q+ht,q+ht]}$ (equivalent to $d(F_{m+h}^t(y), F_{m+h}^t(x)) \geq \epsilon$). By contradiction, assume that no integer satisfies this condition. Thus, for any $t \in \mathbb{N}$, we have $j_t \notin [-q+ht, q+ht]$. If for all $t \in \mathbb{N}$, $j_t > q + ht$ we obtain a contradiction since the original CA is positively expansive. Otherwise, there is an integer $t \in \mathbb{N}$, such that $j_t < -q+ht$ and $j_{t-1} > q+h(t-1)$. In this way, we have $j_t - j_{t-1} < -2q+h < -r$, obtaining again a contradiction. □

The following is a long-standing conjecture in CA theory which dates back at least to [4].

Conjecture 1. Any Surjective CA has DPO.

By Proposition 1, we have that the shift operation conserves surjectivity. Therefore, Conjecture 1 leads naturally to the following.

Conjecture 2. For any CA $\langle A^{\mathbb{Z}}, F_m \rangle$ and any $h \in \mathbb{Z}$, $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ has DPO iff $\langle A^{\mathbb{Z}}, F_m \rangle$ has DPO.

Recall that surjective almost equicontinuous CA have JDPO [4] and that closing CA have JDPO [6]. By Proposition 1, JDPO is preserved by the shifting operation, so all the CA in the classes $\mathcal{S}_1'$ and $\mathcal{S}_2'$ have JDPO. We conjecture that the same holds for (non closing) CA in $\mathcal{S}_3'$:

Conjecture 3. A CA has DPO if it has JDPO.

We want show the equivalence between Conjectures 2 and 3 but before we need the following notion. A CA $\langle A^{\mathbb{Z}}, F_m \rangle$ is *strictly right* (resp., *strictly left*) if $m < 0$ (resp., $d - m < 0$).

Proposition 3. *A surjective strictly right (or strictly left) CA is mixing.*

Proof. We give the proof for a strictly right CA, the case of strictly left CA is similar. Consider a strictly right CA with memory m and diameter d. For any $u, v \in A^\star$ and $i, j \in \mathbb{Z}$, consider the two cylinders $C_i(u)$ and $C_j(v)$. Fix $t \in \mathbb{N}$ such that $i - (d-m)t + |u| < j$. Let $x \in C_i(u)$. The value of x in $[i, i+|u|]$ depends only on the value of $F_m^{-t}(x)$ in $[i - mt, i - (d-m)t + |u| - 1]$. Therefore build $y \in A^{\mathbb{Z}}$ such that $\forall k \in \mathbb{Z}, y_k = v_{k-j+1}$ if $j \leq k \leq j + |v|$ and $y_k = x_k$, otherwise. Then $F_m^t(y) \in C_i(u)$ and $y \in C_j(v)$. □

Proposition 4 (Theorem 3.2 in [7]). *Let $\langle A^{\mathbb{Z}}, F_m \rangle$ be a strictly right CA. Any periodic configuration for the CA is also periodic for σ.*

The following corollary is a trivial consequence of the previous proposition.

Corollary 1. *Consider a strictly right CA. If it has DPO then it has JDPO too.*

Proposition 5. *The following statements are equivalent:*

1. *for any $h \in \mathbb{Z}$, $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ has DPO iff $\langle A^{\mathbb{Z}}, F_m \rangle$ has DPO (Conjecture 2).*
2. *if $\langle A^{\mathbb{Z}}, F_m \rangle$ has DPO, then it also has JDPO (Conjecture 3).*

Proof. (1 ⇒ 2). Let $\langle A^{\mathbb{Z}}, F_m \rangle$ be a CA with DPO. There exists an integer h such that the $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ is a strictly right CA. Then, by the hypothesis, it has DPO. Corollary 1 and Proposition 1 assure that $\langle A^{\mathbb{Z}}, F_m \rangle$ has JDPO. (2 ⇒ 1). By the hypothesis, a CA $\langle A^{\mathbb{Z}}, F_m \rangle$ has DPO iff it has JDPO. Proposition 1 concludes the proof. □

As a by-product of our investigations we have the following result.

Theorem 3. *In the CA settings, the following statements are equivalent*

1. *surjectivity implies DPO;*
2. *surjectivity implies JDPO;*
3. *for strictly right CA, topological mixing implies DPO.*

Proof. (1. ⇔ 3.) It is obvious that 1. implies 3. For the opposite implication assume that 3 is true. Let $\langle A^{\mathbb{Z}}, F_m \rangle$ be a surjective CA. There exists an integer h such that $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ is strictly right. By Propositions 1 and 3, $\langle A^{\mathbb{Z}}, F_{m+h} \rangle$ is a surjective and topologically mixing CA. Then, by the hypothesis, it has DPO. Corollary 1 assures that it also has JDPO. Using Proposition 1 again we conclude the proof. The proof for (1. ⇔ 2.) can be obtained in a similar way. □

Theorem 3 tells that in order to prove Conjecture 1 one can focus on mixing strictly right CA. Remark that all known examples of topologically mixing CA have DPO. We want to present a result which furthermore support the common feeling that Conjecture 1 is true. First, we need a technical lemma.

Lemma 1. *Any configuration of a CA (on $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$) which is σ-periodic of period p has a CA image which is σ-periodic with period p' that divides p.*

Proof. Consider a CA F_m on $A^{\mathbb{Z}}$. Let x be a periodic configuration of σ with period p. By the definition of CA, $F_m(x)$ is still a periodic configuration of period q for σ but, in general, $q \leq p$. If $q = p$ then we are done. Otherwise, let m be the largest integer such that $mq < p$. By the Hedlund's theorem we have

$$F_m \circ \sigma^p(x) = \sigma^p \circ F_m(x) \ . \tag{1}$$

Compose both members of (1) with σ^{-mq}. The left-hand side gives $\sigma^{-mq} \circ F_m \circ \sigma^p(x) = \sigma^{-mq} \circ F_m(x) = F_m(\sigma^{-mq}(x)) = F_m(x)$ since x (resp., $F(x)$) is periodic of period p (resp., q) for σ. The right-hand side gives simply $\sigma^{p-mq}(F_m(x))$. And hence (1) can be rewritten as $F(x) = \sigma^{p-mq}(F_m(x))$ which implies that $p - mq = q$ since q is the period of $F(x)$ according to σ. We conclude that $p = (m+1)q$. □

Proposition 6. *Any surjective CA (both on $A^{\mathbb{Z}}$ and on $A^{\mathbb{N}}$) has an infinite set of points which are jointly periodic for the CA and σ.*

Proof. Consider a CA F_m on $A^{\mathbb{Z}}$. By Lemma 1, given a σ-periodic configuration x of period p, $F_m(x)$ is periodic for σ with period p' which divides p. Let Π_n be the set of periodic points of σ of period n. Hence, if p is a prime number and $x \in \Pi_p$ we have that $F_m(x)$ belongs either to Π_p or to Π_1. By a result of Hedlund [10], we know that for surjective CA, each configuration has a finite number of pre-images. In particular, each Π_p has a finite number of pre-images by F_m. Hence, if p is a big enough prime number we have that $F_m(\Pi_p) \cap \Pi_p \neq \emptyset$. This concludes the proof since for any prime number $p' > p$ we must have $F_m(\Pi_p) \subseteq \Pi_p$. □

4 Lifting

For a fixed local rule f, consider the two one-sided CA $\langle A^{\mathbb{Z}}, F_m \rangle$ and $\langle A^{\mathbb{N}}, \Phi_m \rangle$ on $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$, respectively. They share the same local rule f and the same memory $m \leq 0$. In this section we study the properties that are conserved when passing from $A^{\mathbb{Z}}$ to $A^{\mathbb{N}}$ and *vice-versa*.

Consider the *projection* $P : A^{\mathbb{Z}} \to A^{\mathbb{N}}$ defined as follows: $\forall x \in A^{\mathbb{Z}}, \forall i \in \mathbb{N}, \quad P(x)_i = x_i$. Then, P is a continuous, open, and surjective function. Moreover, $\Phi_m \circ P = P \circ F_m$. Therefore, the CA $\langle A^{\mathbb{N}}, \Phi_m \rangle$ is a factor of $\langle A^{\mathbb{Z}}, F_m \rangle$. For these reasons, we also say that the CA on $A^{\mathbb{Z}}$ is obtained by a *lifting (up) operation* from the CA on $A^{\mathbb{N}}$ (having the same rule and memory). As an immediate consequence of the fact that $\Phi_m \circ P = P \circ F_m$, the CA on $A^{\mathbb{N}}$ inherits from the CA on $A^{\mathbb{Z}}$ several properties such as surjectivity, left closingness, openess, DPO, transitivity, mixing.

The following proposition shows that the injectivity property is lifted down only under special conditions. Proposition 8 proves that the opposite case (lift up) is verified without further hypothesis.

Proposition 7. *Let $\langle A^{\mathbb{Z}}, F_m \rangle$ be an injective one-sided CA. The CA $\langle A^{\mathbb{Z}}, \Phi_m \rangle$ is injective if and only if $m = 0$ and the left Welch index $L(f) = 1$*[4].

Proof. If Φ_m is injective (and then also surjective) we necessarily have $m = 0$. By contradiction, if $L(f) \geq 2$ there exist blocks u, w, w', v, with $w \neq w'$, such that $f(wu) = f(w'u) = v$. So $\Phi_0(x) = \Phi_0(y)$ for two suitable right-asymptotic configurations x, y obtained by extending to the right the words wu and $w'u$, respectively. Conversely let us assume that $L(f) = 1$ and $m = 0$. If, by contradiction, Φ_0 is not injective then there exist two distinct configurations $x, y \in A^{\mathbb{N}}$ having the same image $z \in A^{\mathbb{N}}$. By surjectivity, we have two cases to trait. In the first one, x and y are right asymptotic. So there is a block with two distinct right extensions which collapse by f in the same word, contrary to the fact that $L(f) = 1$. In the second one, for any i large enough we have $x_{[i,i+d-1]} \neq y_{[i,i+d-1]}$. By a result in [10], $L(f) = 1$ implies that f is leftmost permutive[5]. Thus we

[4] The left Welch index $L(f)$ is an un upper bound for the number of the left possible extensions of a block u which collapse by f in the same word. For a formal definition, see for instance [10]).

[5] A rule $f : A^{d+1} \to A$ is leftmost permutive iff for any $u \in A^d$, $b \in A$ there exists $a \in A$ such that $f(au) = b$.

are able to build two distinct configurations $x', y' \in A^{\mathbb{Z}}$, with $x = P(x')$ and $y = P(y')$, such that $F_0(x') = F_0(y')$, contrary to the hypothesis. □

Proposition 8. *If* $\langle A^{\mathbb{N}}, \Phi_m \rangle$ *is an injective (resp., surjective) CA, then the lifted CA* $\langle A^{\mathbb{Z}}, F_m \rangle$ *is injective (resp., surjective).*

Proof. Assume that F_m is not injective. There exist two configurations $x, y \in A^{\mathbb{Z}}$, with $x_i \neq y_i$ for some $i \geq 0$, such that $F_m(x) = F_m(y)$. Therefore $x' = P(x)$ and $y' = P(y)$ are two different configurations in $A^{\mathbb{N}}$ such that $\Phi_m(x') = \Phi_m(y')$.

By a theorem of Hedlund [10], we have that a CA is surjective (on $A^{\mathbb{N}}$ or on $A^{\mathbb{Z}}$) iff for any $u \in A^+, |f^{-1}(u)| = A^d$. □

The following result can be obtained immediately from the definitions.

Proposition 9. *Left-closingness is conserved by the lifting up operation.*

The following results is obtained immediately from the fact that a word is blocking for a CA on $A^{\mathbb{N}}$ iff it is blocking for its lifted CA.

Proposition 10. *A CA* $\langle A^{\mathbb{N}}, \Phi_m \rangle$ *is equicontinous (resp., almost equicontinuous) (resp., sensitive) iff the CA* $\langle A^{\mathbb{N}}, F_m \rangle$ *is equicontinous (resp., almost equicontinous) (resp., sensitive).*

Positive expansivity is not preserved by the lifting operation since, by Proposition 11, there are no positively expansive one-sided CA on $A^{\mathbb{Z}}$.

Proposition 11. *No one-sided CA* $\langle A^{\mathbb{Z}}, F_m \rangle$ *is positively expansive.*

Proof. For the sake of argument let us assume that the CA $\langle A^{\mathbb{Z}}, F_m \rangle$ be a positively expansive one-sided CA with expansivity constant $\epsilon > 0$. Let k be an integer such that $\frac{1}{2^k} < \epsilon$ and let $x, y \in A^{\mathbb{Z}}$ be two different configurations with $x_{[-k,\infty)} = y_{[-k,\infty)}$. We have that for any $t \in \mathbb{N}$, $d(F_m^t(x), F_m^t(y)) < \epsilon$. □

Proposition 12. *If* $\langle A^{\mathbb{N}}, \Phi_m \rangle$ *is mixing (resp., transitive), then its lifted CA* $\langle A^{\mathbb{Z}}, F_m \rangle$ *is mixing (resp., transitive).*

Proof. We prove the thesis for a topologically mixing CA. By Proposition 3, it is sufficient to consider the case $m = 0$. Let $u \in A^{2k+1}$, $v \in A^{2h+1}$ be two arbitrary blocks. There exist a sequence of one-sided configurations $x^{(n)} \in C_{l-k}(u)$ and a time $t_0 \in \mathbb{N}$ such that for any $t \geq t_0$, $\Phi_0^t(x^{(t-t_0)}) \in C_{l-h}(v)$ where $l = \max\{h, k\}$. Let $z^{(n)} \in A^{\mathbb{Z}}$ be a sequence of two-sided configurations such that $z^{(n)}_{[-k,\infty)} = x^{(n)}$. We have that for any $t \geq t_0$, $F_0^t(z^{(t-t_0)}) \in C_{-h}(v)$. □

The lifting (up) of DPO remains an open problem even if on the basis of the results of Section 3 we conjecture that it holds.

5 Conclusions and Future Works

In this paper we studied the behavior of two operations on the rule space of CA, namely, the shifting and lifting operations. These investigations helped to shape out a new scenario for the old- standing conjecture about the equivalence between surjectivity and DPO for CA: the study can be restricted to topologically mixing strictly right CA. This enhances a former idea of Blanchard [1]. The work can be continued along several directions: transitivity and stronger variants, languages generated by the involved CA, and attractors. Moreover a generalization of the obtained results and of the forthcoming studies can be considered in terms of directional dynamics introduced in [13]. The authors are currently investigating these subjects.

References

[1] Blanchard, F.: Dense periodic points in cellular automata, http://www.math.iupui.edu/∼mmisiure/open/

[2] Blanchard, F., Maass, A.: Dynamical behavior of Coven's aperiodic cellular automata. Theoretical Computer Science 163, 291–302 (1996)

[3] Blanchard, F., Maass, A.: Dynamical properties of expansive one-sided cellular automata, Israel Journal of Mathematics 99 pp. 149–174 (1997)

[4] Blanchard, F., Tisseur, P.: Some properties of cellular automata with equicontinuity points. Ann. Inst. Henri Poincaré, Probabilité et Statistiques 36, 569–582 (2000)

[5] Boyle, M., Fiebig, D., Fiebig, U.: A dimension group for local homeomorphisms and endomorphisms of onesided shifts of finite type. Journal f ur die Reine und Angewandte Mathematik 487, 27–59 (1997)

[6] Boyle, M., Kitchens, B.: Periodic points for cellular automata. Indag. Math. 10, 483–493 (1999)

[7] Cattaneo, G., Finelli, M., Margara, L.: Investigating topological chaos by elementary cellular automata dynamics. Theoretical Computer Science 244, 219–241 (2000)

[8] Codenotti, B., Margara, L.: Transitive cellular automata are sensitive. American Mathematical Monthly 103, 58–62 (1996)

[9] Devaney, R.L.: An introduction to chaotic dynamical systems, 2nd edn. Addison-Wesley, London (1989)

[10] Hedlund, G.A.: Endomorphism and automorphism of the shift dynamical system. Mathematical System Theory 3, 320–375 (1969)

[11] Kůrka, P.: Languages, equicontinuity and attractors in cellular automata. Ergod. Th. & Dynam. Sys. 17, 417–433 (1997)

[12] Kůrka, P.: Topological symbolic dynamics, Volume 11 of Cours Spécialisés, Société Mathématique de France (2004)

[13] Sablik, M.: Directional dynamics for cellular automata. a sensitivity to the initial conditions approach. Preprint (2006)