An Approach to Theory of Fuzzy Discrete Signals

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Abstract. The paper presents an approach to description of fuzzy discrete functions and Fourier transform of such functions taking in consideration their uncertainty. Conventional approach to uncertainty employs probabilistic description. Here, fuzzy logic theory is applied to describe this uncertainty. A definition of transform, called later Discrete Fuzzy Fourier Transform and definition of Inverse Discrete Fuzzy Fourier Transform are proposed. Some properties of such transformations and examples of applications and comparison with conventional approach are shown.

1 Introduction

Discrete signals play an important part in the theory of many branches of science. Such signals contain fuzziness included in the function itself or in the parameters. For example sampled and quantified signal contains fuzziness itself because exact value of a sample is unknown after quantization process. Such situation occurs also in image transformations by optical systems. Original image is crisp but image on the photographic plate is blurred. Dubois and Prade [8] call this type of ill-known functions fuzzifying functions. Fuzzifying functions have been studied by Sugeno [12] under the name of fuzzy correspondences. These concepts of fuzzy function is mathematically equivalent to fuzzy relation. Conventional approach in such cases of imprecise data uses probability description.

Very often there are no possibility to gather many experimental data and information about situation concerning similar circumstances, occurrences or phenomena. Moreover, we use sometimes our intuition to build the model of phenomenon. For such kind of data, fuzzy description is more justifiable then probabilistic description.

In the paper, an approach based on uncertainty description using fuzzy logic is proposed. Uncertainty can be introduced in two ways. Function can be treated as function of discrete crisp variables and fuzzy parameters or fuzziness can be included in the function itself not in the parameters.

The idea of fuzzy approach to signal processing was proposed by Kosko [9], but he considered signals as crisp with random impulsive noise similarly as in conventional approach. Only during signal processing fuzzy algorithms was used. The concept leads to fuzzy filtering. The concept of fuzzy filters and other fuzzy techniques was developed also by other researchers especially for image processing [1],[10].

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The name of fuzzy transform was used also in [11]. However the approach is quite different. It is based on discrete partition of time space on intervals and fuzzy description by membership functions (called basic functions) defined on each interval. It can be used also for description of fuzzy discrete signal.

Notions of fuzzy signal, fuzzy Fourier transform and fuzzy correlation proposed by the author are based on the concept of norms and scalar products as in conventional definitions to preserve similar properties. The first approach to analog fuzzy signals and fuzzy Fourier transform was published by author in [2]. After the definition was enlarged also on fuzzifying functions [3]. Two-dimensional fuzzy Fourier transform and fuzzy convolution were proposed in [4] and [6] with application to image processing.

In this paper an approach to discrete fuzzy function is proposed.

2 Concept of Fuzzy Discrete Signal

In theoretical conventional approach the space of discrete signals is defined as follows.

Definition 1. *(Class l^2) Discrete function $x[n]$ of integer argument n belongs to the class $l^2(n_1, n_2)$ where integers $n_1, n_2 \in (-\infty, \infty)$ if, and only if the sum*

$$E_x = \sum_{n=n_1}^{n_2} |x[n]|^2 \tag{1}$$

is finite.

The value E_x represents energy of signal $x[n]$ and l^2 is called *class of discrete signals with finite energy*. Square root of E_x plays role of a norm in the space l^2, $\| x[n] \| = \sqrt{E_x}$.

This conventional definition can be enlarged for fuzzy discrete signals in following way.

Definition 2. *(Class l_f^2) Fuzzy discrete function $\mathsf{x}[n, \alpha]$ where n is integer and α is a fuzzy argument, being a fuzzy normal set or vector of fuzzy arguments, belongs to the class $l_f^2(n_1, n_2)$ where $n_1, n_2 \in (-\infty, \infty)$ are integers if, and only if, for any $a \in supp(\alpha)$ the sum*

$$E_x(\alpha) = \sum_{n=n_1}^{n_2} |x[n, a]|^2 \tag{2}$$

is finite.

It will be shown that $\sqrt{E_x(\alpha)}$ is a norm $\|\mathsf{x}[n, \alpha]\|$ of $\mathsf{x}[n, \alpha]$ in l_f^2. It satisfies the axioms:

(1) $\|\mathsf{x}[n, \alpha]\| \geq 0$ for all $x \in l_f^2$
(2) $\|\mathsf{x}[n, \alpha]\| = 0 \Leftrightarrow \mathsf{x}[n, \alpha] = 0$, i.e. all $x[n, a] = 0$ where $a \in supp(\alpha)$
(3) $\|\lambda \mathsf{x}[n, \alpha]\| = |\lambda| \|\mathsf{x}[n, \alpha]\|$ where λ is real constant
(4) $\|\mathsf{x}_1[n, \alpha] + \mathsf{x}_2[n, \beta]\|^2 \leq \|\mathsf{x}_1[n, \alpha]\| + \|\mathsf{x}_2[n, \beta]\|$

The first three properties are obvious. The triangle inequality (4) is proved using inequality for functions in conventional space $l^p(n_1, n_2)$

$$\{\sum_{n=n_1}^{n_2}(x[n]+y[n])^p\}^{1/p} \leq \{\sum_{n=n_1}^{n_2}(x[n]]^p\}^{1/p} + \{\sum_{n=n_1}^{n_2}(y[n])^p dt\}^{1/p}$$

where $p \geq 1$. The inequality is applied for any $a \in supp(\alpha)$ and $b \in supp(\beta)$

Example 1 (Fuzzy discrete signal). Let function takes fuzzy values $x[n] = 2k + \alpha$ where $k = -4, -3, ..., 4$ and α is fuzzy number described by membership functions of trapezoidal shape $trapeze(-1.5, -0.5, 0.5, 1.5)$. Signal $x[n]$ is shown in the Fig. 1 (left).

Example 2 (Fuzzy linguistic signal). In one of his papers [6] the author introduced a concept of linguistic signal. It is a discrete fuzzy function $x[n, \alpha]$ taking discrete values $x_l[n]$ where for any discrete level l is attached a linguistic term v_l. Let discrete levels x_l for any n can take only one of $l = 1, ..., 9$ different linguistic values v_l called NV, NL,..., PV. Let uncertainty of each value $x_l[n]$ is described by the similar fuzzy set α with membership function $\mu_\alpha(a)$ with trapezoidal shape. The linguistic signal corresponding to fuzzy signal from Example 1 is presented in the Fig. 1 (right).

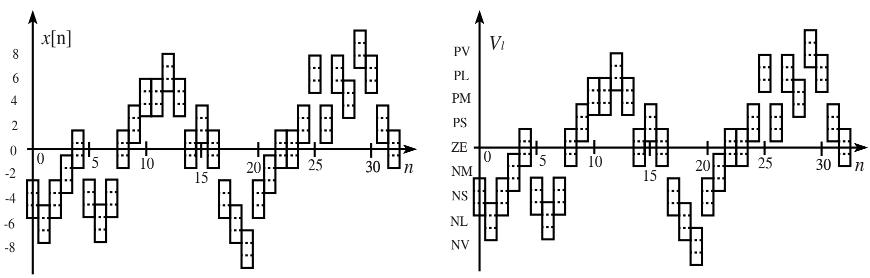

Fig. 1. Example of discrete fuzzy signal (left) and linguistic signal (right)

Such situation occurs in practice where analog-digital (A/D) converter can introduce an error, for example the value of last bit is not sure.

3 Concept of Fuzzy Discrete Fourier Transform

In conventional approach Fourier transform of discrete set of real or complex numbers $x[n]$, $n \in Z$ (integers) is defined as follows

$$X(f) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i2\pi nf} \tag{3}$$

For practical reasons a finite-length sequence $x[n]$ is obviously needed, $n = 0, ...,$ $N-1$. The transform is periodic. Moreover, $X(f)$ is evaluated at an arbitrary number M of uniformly-spaced frequencies f_m across one period $f_m = m/M$, $m = 0, ..., M-1$. Hence, discrete Fourier transform is calculated as following

$$X[m] = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi nm/M} \tag{4}$$

In Discrete Fourier Transform (DFT) procedures used for calculation the number $M = N$ is assumed. Thus follow definition is used

$$X[m] = \sum_{n=0}^{N-1} x[n]\, e^{-i2\pi nm/N} \tag{5}$$

Inverse transform has a form

$$x[n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m]\, e^{-i2\pi nm/N} \tag{6}$$

Sometimes in (5) and (6) the term $\sqrt{N}$ is used for symmetry.

Fuzzy approach, proposed here, is quite different. Firstly it can be noticed that fuzziness is very often included in the function itself not in the parameters. Therefore, the parameter α in $\mathsf{x}[n, \alpha]$ will be omitted for generality. Next, the concept of discrete pseudo α-level curves is introduced. The term "discrete curves" is used here for simplicity and understand as series of values $x[n]$ for $n = 0, ..., N-1$.

Definition 3. *(Discrete pseudo α-level curves) Let value of fuzzifying function $\mathsf{x}[n]$ of crisp discrete integer variable n for any $n \in (-\infty, \infty)$ is a fuzzy number (convex normal fuzzy set). Let $\mu_{\mathsf{x}}(x[n])$ be membership function of $\mathsf{x}[n]$. Let for any $\alpha \in (0, 1)$ the equation $\mu_{\mathsf{x}}(y[n]) = \alpha$ has two and only two solutions $y[n] = x^{\mp}_{\alpha}[n]$ and $y[n] = x^{\pm}_{\alpha}[n]$, and only one solution $y[n] = x[n]$ for $\alpha = 1$. The solutions $x^{\mp}_{\alpha}[n]$ $x^{\pm}_{\alpha}[n]$, will be called* pseudo α-level curves *of the* $\mathsf{x}[n]$.

This concept is something similar to α-level curves $x^{-}_{\alpha}(t)$ $x^{+}_{\alpha}(t)$ introduced by Dubois and Prade [8] and after in [5]. However, here functions are discrete and the order of the $x^{\mp}_{\alpha}[n]$ $x^{\pm}_{\alpha}[n]$ can change with n and they may do not fulfill inequality as $x^{-}_{\alpha}[n] \leq x^{+}_{\alpha}[n]$. Sometimes $x^{\mp}_{\alpha}[n]$ can be grater than $x^{\pm}_{\alpha}[n]$. For instance, such situation occurs for fuzzy function $\Pi(t)\, cos(2\pi f_0 t)$ where $\Pi(t)$ is crisp function with rectangular shape and f_0 is fuzzy number with membership function with triangular shape $\mu_{f_0}(f) = triangle(0.9, 1, 1.1)$.

Definition 4. *(Discrete Fuzzy Fourier Transform) Let value of discrete fuzzifying function $\mathsf{x}[n]$ of crisp integer variable n for any $n \in [0, N-1]$ is a fuzzy number (convex normal fuzzy set). Let pseudo α-level curves $x^{\pm}_{\alpha}[n]$ $x^{\mp}_{\alpha}[n]$ of the $\mathsf{x}[n]$, are continuous with respect to α and for $\alpha \to 1$ limit $x^{\mp}_{1}[n] \to x[n]$ and $x^{\pm}_{1}[n] \to x[n]$. Let all the pseudo α-level curves, $\alpha \in (0, 1]$, fulfill condition (1). Then, it is possible to introduce two conventional sums*

$$X_\alpha^{\mp}[m] = \sum_{n=0}^{N-1} x_\alpha^{\mp}[n]\, e^{-i2\pi nm/N} \tag{7}$$

$$X_\alpha^{\pm}[m] = \sum_{n=0}^{N-1} x_\alpha^{\pm}[n]\, e^{-i2\pi nm/N} \tag{8}$$

which will be called pseudo α-level curves *of DFuzFT. The set* $\mathsf{X}[m]$ *of all pseudo* α*-level curves,* $\alpha \in (0,1]$*, will be called* Discrete Fuzzy Fourier Transform *(DFuzFT) and written in the form of sum*

$$\mathsf{X}[m] = \sum_{n=0}^{N-1} \mathsf{x}[n]\, e^{-i2\pi nm/N} \tag{9}$$

Definition 5. *(Inverse Discrete Fuzzy Fourier Transform) Let* $X_\alpha^{\mp}[m]$ *and* $X_\alpha^{\pm}[m]$ *be pseudo* α*-level curves describing DFuzFT* $\mathsf{X}[m]$*. Then set of sums*

$$x_\alpha^{\mp}[n] = \frac{1}{N} \sum_{m=0}^{N-1} X_\alpha^{\mp}[m]\, e^{i2\pi nm/N} \tag{10}$$

$$x_\alpha^{\pm}[n] = \frac{1}{N} \sum_{m=0}^{N-1} X_\alpha^{\pm}[m]\, e^{i2\pi nm/N} \tag{11}$$

for all values $\alpha \in (0,1]$ *will be called* Inverse Discrete Fuzzy Fourier Transform *(IDFuzFT) and write in the form of sum*

$$\mathsf{x}[n] = \frac{1}{N} \sum_{m=0}^{N-1} \mathsf{X}[m]\, e^{i2\pi nm/N} \tag{12}$$

If α is a parameter with membership function $\mu_\alpha(a)$ and $supp(\alpha) = [a_1, a_2]$, then fuzzy signal can be written in the form of union

$$\mathsf{x}[n] = \int \mu_\alpha(a)/x[n] \tag{13}$$

Let and both transforms DFuzFT and IDFuzFT exist. Transformation $x_\alpha^{\mp}[n] \Leftrightarrow X_\alpha^{\mp}[m]$ is one to one. Thus, $X_\alpha^{\mp}[m]$ conserves the same α-level as $x_\alpha^{\mp}[n]$. Therefore, DFuzFT can be written in the form of union

$$\mathsf{X}[m] = \int \mu_\alpha(a)/X[m] \tag{14}$$

Now arise a question what membership function has DFuzFT? Real value of membership can be found as set of all values

$$X_\alpha^{-}([m]) = \min(X_\alpha^{\mp}[m], X_\alpha^{\pm}[m]) \tag{15}$$

$$X_\alpha^{+}([m]) = \max(X_\alpha^{\mp}[m], X_\alpha^{\pm}[m]) \tag{16}$$

Sometimes it will be useful to obtain crisp result for transform. It is possible to find weighted average of $\mathsf{X}[m]$

$$\overline{\mathsf{X}[m]} = \frac{\int_{a_1}^{a_2} \mu_\alpha(a) X[m,a] da}{\int_{a_1}^{a_2} \mu_\alpha(a) da} \tag{17}$$

It will be called *Discrete Fourier Transform of Fuzzy Function* (DFTFF).

Comments. One remark must be added. Both operations of transformation, i.e. DFuzFT and DFTFF, are not fuzzy itself. They are only performed on fuzzy functions. The name "fuzzy Fourier" is used to accentuate that result of DFuzFT is fuzzy function in contradiction to DFTFF where result is crisp function.

4 Some Properties of Discrete Fuzzy Fourier Transform

It can be shown that many properties of DFuzFT are similar as for conventional DFT. Let $\mathsf{X}[m]$ be DFuzFT of $\mathsf{x}[n]$ written as $\mathsf{x}[n] \leftrightarrow \mathsf{X}[m]$. For instance consider shift, addition, and modulation property.

Shift. Let $\mathsf{x}[n] \leftrightarrow \mathsf{X}[m]$ then

$$\mathsf{x}[n-k] \leftrightarrow \mathsf{X}[m] e^{i2\pi km/N} \tag{18}$$

Proof. Any $x_\alpha^\mp[n]$ is crisp. Therefore, from conventional shift property it follows $x_\alpha^\mp[n-k] \leftrightarrow X[m]_\alpha^\mp e^{i2\pi km/N}$. Similarly for $x_\alpha^\pm[n]$. Thus, (18) is true.

Addition. Let $\mathsf{x}[n] \leftrightarrow \mathsf{X}[m]$ and $\mathsf{y}[n] \leftrightarrow \mathsf{Y}[m]$ then

$$\mathsf{x}[n] + \mathsf{y}[n] \leftrightarrow \mathsf{X}[m] + \mathsf{Y}[m] \tag{19}$$

Proof. It was shown in [8] that sum of α-level curves conserves α-level. Of course, it is true also for discrete case. Then sum $x_\alpha^\mp[n] + y_\alpha^\mp[n]$ has also the same pseudo α-level. Thus (19) is true.

Modulation. Let $\mathsf{x}[n] \leftrightarrow \mathsf{X}[m]$ then

$$\mathsf{x}[n]\, cos(2\pi k/N) \leftrightarrow \frac{1}{2}(\mathsf{X}[m-k] + \mathsf{X}[m+k]) \tag{20}$$

Proof. From conventional modulation property it follows that $x_\alpha^\mp[n] e^{i2\pi nk/N} \leftrightarrow X_\alpha^\mp[m-k]$. Similarly for $x_\alpha^\pm[n]$. Thus, from Euler formula $e^{i\vartheta} = cos\vartheta + isin\vartheta$ it follows that (20) is true.

It is known that Parseval's theorem play an important part in signal theory. Here we start from Carleman's formulation [7] of Parseval's theorem. The theorem can be proved also for discrete signals.

Theorem 1 (discrete Carleman). *If the sum*

$$\sum_{n=0}^{N-1} |x[n]|^2 \tag{21}$$

is finite then following sum with discrete Fourier transforms of $x[n]$

$$\frac{1}{N}\sum_{m=0}^{N-1} |X[m]|^2 \tag{22}$$

is finite and they are equals

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |X[m]|^2 \tag{23}$$

Fuzzy version of this theorem is presented below.

Theorem 2 (fuzzy signal energy). *Let* $\mathsf{x}[n]$ *belongs to* l_f^2 *and* $\mathsf{X}[m]$ *be discrete fuzzy Fourier transform of* $\mathsf{x}[n]$. *Then*

$$\sum_{n=0}^{N-1} |x_\alpha^\mp[n]|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |X_\alpha^\mp[m]|^2 \tag{24}$$

and

$$\sum_{n=0}^{N-1} |x_\alpha^\pm[n]|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |X_\alpha^\pm[m]|^2 \tag{25}$$

It will be written as

$$\sum_{n=0}^{N-1} |\mathsf{x}[n]|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\mathsf{X}[m]|^2 \tag{26}$$

Proof. Functions $x_\alpha^\mp[n]$, $x_\alpha^\pm[n]$ and their Fourier transforms $X_\alpha^\mp[m]$, $X_\alpha^\pm[m]$ are crisp. Assumption of the theorem assures that $x_\alpha^\mp[n]$ and $x_\alpha^\pm[n]$ belong to class l^2. Thus, conventional theorem can be applied to these functions for any $a \in supp(\alpha)$. Therefore, the equality (26) is consistent.

It must be noted that equivalence (26) concerns fuzzy functions. Such definition of signal energy leads to concept of fuzzy energy. Of course, crisp value of energy can be obtain as weighted average

$$E_{\mathsf{x}} = \frac{\int_{a_1}^{a_2} \mu_\alpha(a) \sum_{n=0}^{N-1} |x[n,a]|^2 da}{\int_{a_1}^{a_2} \mu_\alpha(a) da} \tag{27}$$

Example 3 (DFuzFT). Let vector of values equals $x[n]$=[-2 -3 -2 -1 0 -2 -4 -2 0 1 3 2 5 2 0 1 0 -2 -3 -5 -2 -1 0 0 1 3 1 3 2 1 3 1 0], but this values are not certain. Let uncertainty of $\mathsf{x}[n]$ is described by the set of membership functions $\mu_{\mathsf{x}}(x[n])$

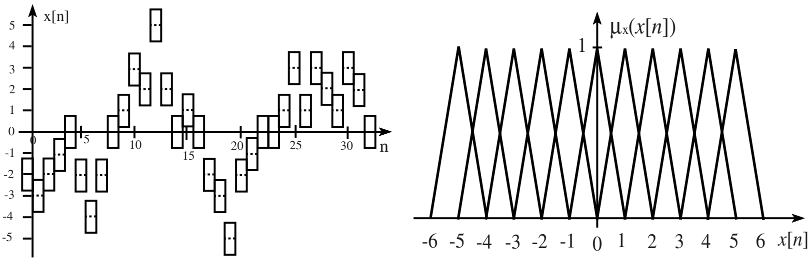

Fig. 2. Example of discrete fuzzy signal (left) and its membership function (right)

(Fig. 2). The sets (7) (8) of all pseudo α-level curves for DFuzFT were calculated. The result, three series of points $X_0^-[m]$, $X_0^+[m]$ found using (16)(16) and $X_1[m]$, was shown in the Fig. 3 (left) as dashed and continuous lines, for DFuzFT with $\alpha = 0$ and 1. Of course, such result can be presented in similar way as in the Fig. 2 with rectangles showing possible changes of spectrum $X[m]$. Here lines are used for simplicity. In the Fig. 4 conventional FFT and DFTFF are compared.

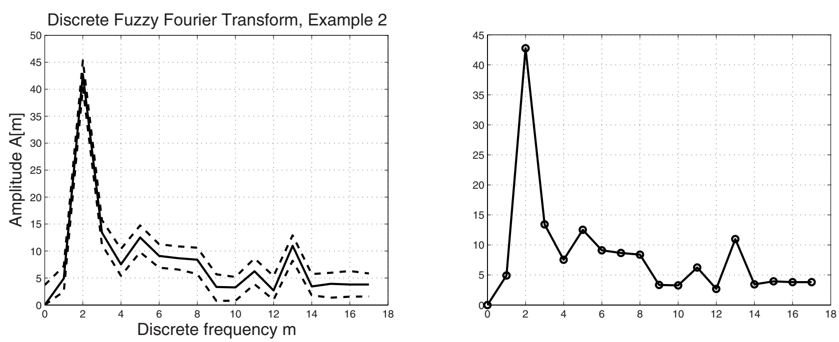

Fig. 3. Example 2. Discrete fuzzy Fourier transform (left). Solid line shows $X_1[m]$ where $\alpha = 1$, dashed lines show $X_\alpha^-[m]$ and $X_\alpha^+[m]$ for $\alpha = 0$. Comparison of $X_1[m]$ and conventional FFT (right), $X_1[m]$ - solid line, FFT - points.

Remark. Example 2 shows that values of $X_1[m]$ are identical with the result obtained by conventional FFT, see Fig. 3 (right). However, the idea of DFuzFT can be enlarged on the cases where the values of $x_1[n]$ are not unique. It is a case when membership functions of $x[n]$ have for example trapezoidal shapes. In such cases the value $x_1[n]$ and $X_1[m]$ are not unique. Therefore, DFuzFT concept must not be trivial extension of the FFT procedure. Such modification of the DFuzFT concept does not shown in this paper. The author in [6] presents one of possible solution proposed for such discrete fuzzy functions, called linguistic signals, where a concept of linguistic Fourier transform is introduced. In this case fuzzy signal as well as its Fourier transform have linguistic forms.

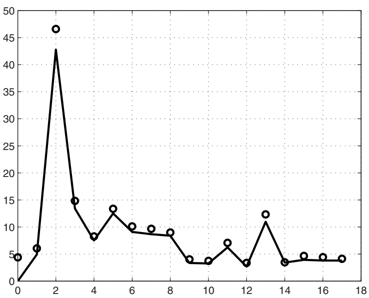

Fig. 4. Example 2. Comparison of DFTFF and conventional FFT, DFTFF - points, FFT - solid line.

5 Conclusions

In the paper new approach to Fourier transform is proposed based on fuzzy logic. It was shown that fuzzy transforms contain information about uncertainty. It is not the case in the conventional probabilistic approach where expected values are used. This information can be obtained in sufficiently easy way using conventional discrete Fourier transform. Effective numerical procedure can be build using popular programming tools as FFT. New definitions were introduced using concept of norms, i.e. in such a way to conserve general properties of conventional continuous and discrete Fourier transforms. It was shown that principal properties as shift, addition, modulation and many others, and also Parseval's theorem can be generalized for fuzzy case. Crisp result for transform (DFTFF) can be obtained also using average mean or other defuzzification procedure.

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