Advances in the Geometrical Study of Rotation-Invariant T-Norms

Koen C. Maes and Bernard De Baets

Department of Applied Mathematics, Biometrics and Process Control
Ghent University, Coupure links 653, B-9000 Gent, Belgium
{Koen.Maes,Bernard.DeBaets}@UGent.be

Abstract. T-norm properties for left-continuous, increasing $[0,1]^2 \to [0,1]$ functions can be fully described in terms of contour lines. For a left-continuous t-norm T, the rotation-invariance property comes down to the continuity of its contour line C_0. However, contour lines are inadequate to investigate the geometrical structure of these rotation-invariant t-norms. Enforced with the companion and zooms it is possible to totally reconstruct T by means of its contour line C_0 and its β-zoom, with β the unique fixpoint of C_0.

Keywords: Rotation-invariant t-norm, contour line, companion, zoom, associativity.

1 Introduction

Originally, triangular norms were introduced in order to generalize the triangle inequality towards probabilistic metric spaces [15]. Nowadays, they are widely used in fuzzy set theory.

Definition 1. *A* triangular norm *or* t-norm T *is an associative, commutative, increasing* $[0,1]^2 \to [0,1]$ *function that has neutral element* 1*.*

So far only the class of continuous t-norms has been fully characterized (see e.g. [8]). In particular, this class comprises the three prototypical t-norms: the *minimum operator* $T_{\mathbf{M}}(x,y) = \min(x,y)$, the *algebraic product* $T_{\mathbf{P}}(x,y) = x\,y$ and the *Łukasiewicz t-norm* $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$.

A t-norm T is called left-continuous if all its partial functions $T(x,\bullet)$ (and hence also $T(\bullet,x)$) are left-continuous [8]. In most studies dealing with t-norms, it is required that the t-norms in question should be left-continuous. In monoidal t-norm based logic (MTL logic) for example, where the implication is defined as the residuum of the conjunction, left-continuous t-norms ensure the definability of the t-norm-based residual implicator [2].

The rotation-invariance of a left-continuous t-norm T is equivalent with the continuity and with the involutivity of its contour line C_0 that determines the intersection of T with the plane containing its domain $[0,1]^2$. In particular, this contour line coincides with the residual negator of T and, therefore, rotation-invariant t-norms are of great interest to people working on involutive monoidal t-norm based logic (IMTL logic) [1,10] and fuzzy type theory [14].

P. Melin et al. (Eds.): IFSA 2007, LNAI 4529, pp. 543–552, 2007.

2 Tools

Studying the structure of a (left-continuous) increasing $[0,1]^2 \to [0,1]$ function T, it is often worthwhile to observe this function from a different point of view. We present here three functions that describe T in an alternative way. They will prove to be indispensable for the decomposition and construction of rotation-invariant t-norms.

2.1 Contour Lines

Contour lines of an increasing $[0,1]^2 \to [0,1]$ function T are defined as the upper, lower, right or left limits of its horizontal cuts, *i.e.* the intersections of its graph by planes parallel to the domain $[0,1]^2$. Although there are four different types of contour lines, those determined by the upper limits of the horizontal cuts are of particular interest for the study of rotation-invariant t-norms [12].

Definition 2. [11] *Let* $a \in [0,1]$*. The* contour line C_a *of an increasing* $[0,1]^2 \to [0,1]$ *function* T *is the* $[0,1] \to [0,1]$ *function defined by*

$$C_a(x) = \sup\{t \in [0,1] \mid T(x,t) \leq a\}\,. \tag{1}$$

For a left-continuous t-norm T, the contour line C_a equals the partial function $I_T(\bullet, a)$ of the *residual implicator* I_T (see e.g. [4]). In particular, the contour line C_0 coincides with the *residual negator* N_T, defined by $N_T = I_T(\bullet, 0)$. Contour lines of a continuous t-norm T are also called *level functions* [9].

Property 1. [11,12] A contour line C_a, with $a \in [0,1]$, of an increasing $[0,1]^2 \to [0,1]$ function T satisfies the following properties:

1. C_a is decreasing.
2. $C_a \leq C_b$, for every $b \in [a,1]$.
3. If T is left-continuous, then C_a is left-continuous.

The greatest merit of contour lines is that they can be used to express all t-norm properties in an alternative way. Further on, this will allow us to provide a geometrical interpretation of the associativity property. Dealing with contour lines of the type C_a the left-continuity of T is required.

Theorem 1. [11] *For a left-continuous, increasing* $[0,1]^2 \to [0,1]$ *function* T *having absorbing element* 0 *the following characterizations hold:*

1. *T has neutral element $e \in \,]0,1]$ if and only if $e \leq C_a(x) \Leftrightarrow x \leq a$ and $C_a(e) = a$ hold for every $(x,a) \in [0,1]^2$.*
2. *T is commutative if and only if $C_a(x) < y \Leftrightarrow C_a(y) < x$ holds for every $(x,y,a) \in [0,1]^3$.*
3. *T is associative if and only if $C_a(T(x,y)) = C_{C_a(x)}(y)$ holds for every $(x,y,a) \in [0,1]^3$.*

The characterization of the commutativity of T comes down to the **id**-orthosymmetry of its contour lines [13]. Taking into account the tight correspondence between contour lines and the residual implicator of a left-continuous t-norm T, the above theorem expresses the associativity of T by means of the *portation law* (*i.e.* $I_T(T(x,y),z) = I_T(x, I_T(y,z))$, for every $(x,y,z) \in [0,1]^3$) [5].

Corollary 1. [12] *For a left-continuous t-norm T it holds for every $(x,y,z,a) \in [0,1]^4$ that*

$$T(x,y) \leq C_a(z) \;\Leftrightarrow\; T(x,z) \leq C_a(y)\,. \tag{2}$$

Jenei [7] has recently shown that, for a commutative, left-continuous, increasing $[0,1]^2 \to [0,1]$ function T that has absorbing element 0, Eq. (2) is equivalent with the associativity of T. Note that for his characterization the commutativity of T is required, this in contrast to our characterization in Theorem 1. His result can also be easily retrieved from the last two characterizations in Theorem 1.

2.2 The Companion

A second useful tool to study an increasing $[0,1]^2 \to [0,1]$ function T is its companion Q.

Definition 3. [12] *The* companion Q *of an increasing* $[0,1]^2 \to [0,1]$ *function* T *is the* $[0,1]^2 \to [0,1]$ *function defined by*

$$Q(x,y) = \sup\{t \in [0,1] \mid C_t(x) \leq y\}\,.$$

The following properties provide better insight into the geometrical structure of Q.

Property 2. [13] The companion Q of an increasing $[0,1]^2 \to [0,1]$ function T satisfies the following properties:

1. Q is increasing in both arguments.
2. $Q(x,y) = \inf\{T(x,u) \mid u \in]y,1]\}$, with $\inf \emptyset = 1$.
3. $T(x,y) \leq Q(x,y)$, for every $(x,y) \in [0,1]^2$.
4. $Q(x,\bullet)$ is right-continuous for every $x \in [0,1]$.
5. If T has neutral element 1, then $Q(x,y) \leq T_{\mathbf{M}}(x,y)$, for every $(x,y) \in [0,1] \times [0,1[$.

The second property allows to straightforwardly construct the graph of Q (*i.e.* $\{(x,y,Q(x,y)) \mid (x,y) \in [0,1]^2\}$) from the graph of T (*i.e.* $\{(x,y,T(x,y)) \mid (x,y) \in [0,1]^2\}$). It suffices to convert the partial functions $T(x,\bullet)$ into right-continuous functions and to replace the set $\{(x,1,x) \mid x \in [0,1]\}$ by $\{(x,1,1) \mid x \in [0,1]\}$ as $Q(x,1) = 1$ must hold for every $x \in [0,1]$. Clearly, $Q(x,y) = T(x,y)$ whenever $T(x,\bullet)$ is right-continuous in $y \in [0,1[$. Every left-continuous increasing binary function T that has absorbing element 0 is totally determined by its companion Q. Note also that $Q(x,1) = 1$ and $Q(1,x) = x$ prevent Q from being commutative.

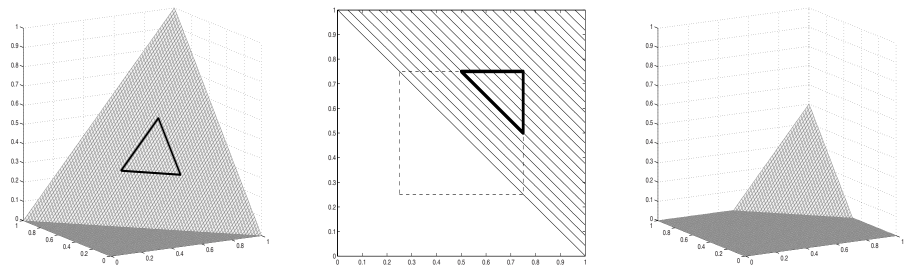
2.3 Zooms

Finally, every increasing $[0,1]^2 \to [0,1]$ function T is trivially described by its associated set of (a,b)-zooms.

Definition 4. *Let T be an increasing $[0,1]^2 \to [0,1]$ function and take $(a,b) \in [0,1]^2$ such that $a < b$ and $T(b,b) \le b$. Consider an $[a,b] \to [0,1]$ isomorphism σ. The (a,b)-zoom $T^{(a,b)}$ of T is the $[0,1]^2 \to [0,1]$ function defined by*

$$T^{(a,b)}(x,y) = \sigma\left[\max\left(a, T(\sigma^{-1}[x], \sigma^{-1}[y])\right)\right] .$$

If $b = 1$ we simply talk about the a-zoom T^a of T.



$T_{\mathbf{L}}$ Contour plot of $T_{\mathbf{L}}$ $T_{\mathbf{L}}^{(\frac{1}{4},\frac{3}{4})}$

Fig. 1. The $(\frac{1}{4},\frac{3}{4})$-zoom of $T_{\mathbf{L}}$

The graph of $T^{(a,b)}$ is determined by rescaling the set $\{(x,y,T(x,y)) \mid (x,y) \in [a,b]^2 \wedge a < T(x,y)\}$ (zoom in) into the unit cube (zoom out). Figure 1 illustrates this procedure for the Łukasiewicz t-norm $T_{\mathbf{L}}$, with $a = \frac{1}{4}$, $b = \frac{3}{4}$ and $\sigma = \varsigma$, where ς is the *linear rescaling* of $[a,b]$ into $[0,1]$ (*i.e.* $\varsigma(x) = (x-a)/(b-a)$, for every $x \in [a,b]$). In our examples we will always use this linear rescaling function.

Whenever $T(b,b) \le a$, the function $T^{(a,b)}$ is trivially constant: $T^{(a,b)}(x,y) = a$, for every $(x,y) \in [0,1]^2$. For $b = 1$ the boundary condition $T(1,1) \le 1$ is always true such that the a-zoom of T is defined for every $a < 1$. Note that $T^0 = T_{\sigma^{-1}}$, where $T_{\sigma^{-1}}$ denotes the σ^{-1}-transform of T (*i.e.* $T_{\sigma^{-1}}(x,y) := \sigma[T(\sigma^{-1}[x], \sigma^{-1}[y])]$).

Since the (a,b)-zoom $T^{(a,b)}$ of an arbitrary increasing function T is totally determined by $T|_{[a,b]^2}$, its contour lines and companion can be computed from the contour lines and companion of T. In case $T^{(a,b)}$ has neutral element 1, we obtain a straightforward relationship between its contour lines and those of the original function T.

Property 3. Consider an increasing $[0,1]^2 \to [0,1]$ function T. Take $(a,b) \in [0,1]^2$, such that $a < b$ and $T(b,b) \le b$. Let σ be an arbitrary $[a,b] \to [0,1]$ isomorphism. If the (a,b)-zoom $T^{(a,b)}$ has contour lines $C_d^{(a,b)}$ and companion $Q^{(a,b)}$ then the following properties hold:

1. $T^{(a,b)}$ is increasing in both arguments.
2. $Q^{(a,b)}(x,y) = \sigma[Q(\sigma^{-1}[x], \sigma^{-1}[y])]$, for every $(x,y) \in [0,1]^2$ s.t. $C_0^{(a,b)}(x) \leq y < 1$.
3. If T is left-continuous, then $T^{(a,b)}$ is left-continuous.
4. $C_d^{(a,b)}(x) = \sigma[C_{\sigma^{-1}[d]}(\sigma^{-1}[x])]$ holds if
   (a) $b = 1$, $T(1,a) \leq a$ and $(x,d) \in [0,1]^2$;
   (b) $T^{(a,b)}$ has neutral element 1 and $(x,d) \in [0,1]^2$ s.t. $d < x$.
5. If T is associative and $\max(T(a,b), T(b,a)) \leq a$, then $T^{(a,b)}$ is also associative.

In accordance to Definition 4 we will usually denote the contour lines of $T^a(= T^{(a,1)})$ as $C^a(= C^{(a,1)})$ and its companion as $Q^a(= Q^{(a,1)})$. Zooms are extremely suited to study an increasing function T that satisfies $T \leq T_{\mathbf{M}}$. The restrictions $T(b,b) \leq b$ (Definition 4), $T(1,a) \leq a$ and $\max(T(a,b), T(b,a)) \leq a$ (Property 3) then trivially hold.

Definition 5. [6] *A* t-subnorm T *is an associative, commutative, increasing* $[0,1]^2 \to [0,1]$ *function that satisfies* $T \leq T_{\mathbf{M}}$.

Clearly, all t-norms are t-subnorms. Due to its boundary condition we can construct all (a,b)-zooms $(a < b)$ of every t-subnorm. Moreover, all these (a,b)-zooms are t-subnorms ass well.

Corollary 2. *Consider* $(a,b) \in [0,1]^2$ *such that* $a < b$. *Then the* (a,b)*-zoom of a t-subnorm is a t-subnorm and the a-zoom of a t-norm is a t-norm.*

The $(\frac{1}{4}, \frac{3}{4})$-zoom in Fig. 1 is a t-subnorm but not a t-norm. No (a,b)-zoom, with $b < 1$, of the Łukasiewicz t-norm $T_{\mathbf{L}}$ can be a t-norm. The latter follows from the observation that $T^{(a,b)}$ has neutral element 1 whenever $T(x,b) = T(b,x) = x$, for every $x \in \,]a,b]$. Dealing with $T_{\mathbf{L}}$ this only occurs for $b = 1$. Otherwise, every (a,b)-zoom of the minimum operator $T_{\mathbf{M}}$ equals $T_{\mathbf{M}}$ itself.

3 Rotation-Invariant T-Norms

3.1 A Continuous Contour Line

Definition 6. [5] *Let* N *be an involutive negator (*i.e. *an involutive decreasing* $[0,1] \to [0,1]$ *function). An increasing* $[0,1]^2 \to [0,1]$ *function* T *is called* rotation invariant w.r.t. an involutive negator N *if for every* $(x,y,z) \in [0,1]^3$ *it holds that*

$$T(x,y) \leq z \;\Leftrightarrow\; T(y, z^N) \leq x^N \,. \tag{3}$$

This property was first described by Fodor [3]. Jenei [5] emphasized its geometrical interpretation by referring to it as the rotation-invariance of T w.r.t. N. Recently, Jenei [7] used Eq. (2) to define the (algebraical) rotation invariance property. However, as pointed out before, Eq. (2) merely expresses the associativity of T. As will be shown later on, a geometrical notion of rotation can be attributed to Eq. (3) but in general not to Eq. (2)

Theorem 2. [12] *For a left-continuous t-norm T, the following assertions are equivalent:*

1. C_a *is continuous.*
2. C_a *is involutive on* $[a, 1]$.
3. $T(x, y) = C_a(C_{C_a(x)}(y))$, *for every* $(x, y) \in [0, 1]^2$ *s.t.* $C_a(x) < y$.
4. $T(x, y) \leq z \Leftrightarrow T(x, C_a(z)) \leq C_a(y)$, *for every* $(x, y, z) \in [a, 1]^3$.
5. $Q(x, y) < C_a(z) \Leftrightarrow Q(x, z) < C_a(y)$, *for every* $(x, y, z) \in [a, 1] \times [a, 1[^2$.

The third assertion can be seen as an adjustment of the portation law. The fourth and fifth assertion are closely related to Eq. (2). If $a = 0$ then the additional restriction $C_0(x) < y$ in assertion 3 can be omitted. Furthermore, taking into account that $C_0 = N_T$, one can then recognize in assertion 4 the rotation-invariance of T w.r.t. its residual negator N_T. Jenei has proven that every t-norm T that is rotation-invariant w.r.t. an involutive negator N is necessarily left-continuous and $N_T = N$ [5]. Therefore, it becomes superfluous to mention the negator N explicitly. For a left-continuous t-norm T, its rotation-invariance is also equivalent with the continuity of its contour line C_0 (Theorem 2). Herein lies the true meaning of the rotation-invariance property. We briefly call a t-norm *rotation invariant* if it is left-continuous and has a continuous contour line C_0. Note that the continuity of C_0 does not necessarily imply the left-continuity of T [12].

Theorem 3. *Consider a left-continuous t-norm T and take $a \in [0, 1]$ such that $a < \alpha := \inf\{t \in [0, 1] \mid C_a(t) = a\}$. Then the following assertions are equivalent:*

1. C_a *is continuous on* $]a, 1]$.
2. C_a *is involutive on* $]a, \alpha[$.
3. $C_a(]a, \alpha[) =]a, \alpha[$.
4. $T^{(a,\alpha)}$ *is a rotation-invariant t-norm.*

In particular, if C_a is continuous then $\alpha = 1$.

To better comprehend the structure of t-norms that have a (partially) continuous contour line C_a we thus need to focus first on the structure of rotation-invariant t-norms. Studying these t-norms, Jenei provided a real breakthrough by introducing his rotation and rotation-annihilation construction [6]. Unfortunately, his decompositions and constructions were not able to describe all rotation-invariant t-norms [12,13]. The Łukasiewicz t-norm $T_{\mathbf{L}}$, for example, did not fit into his framework. We will present an alternative approach.

3.2 Decomposition Revisited

Let T be a rotation-invariant t-norm and β be the unique fixpoint of C_0. As depicted in Fig. 2, we partition area $\mathcal{D} = \{(x, y) \in [0, 1]^2 \mid C_0(x) < y\}$ into four parts:

$$\begin{aligned}
\mathcal{D}_{\mathrm{I}} &= \{(x, y) \in]\beta, 1]^2 \mid C_\beta(x) < y\}, \\
\mathcal{D}_{\mathrm{II}} &= \{(x, y) \in]0, \beta] \times]\beta, 1] \mid C_0(x) < y\}, \\
\mathcal{D}_{\mathrm{III}} &= \{(x, y) \in]\beta, 1] \times]0, \beta] \mid C_0(x) < y\}, \\
\mathcal{D}_{\mathrm{IV}} &= \{(x, y) \in]\beta, 1[^2 \mid y \leq C_\beta(x)\}.
\end{aligned}$$

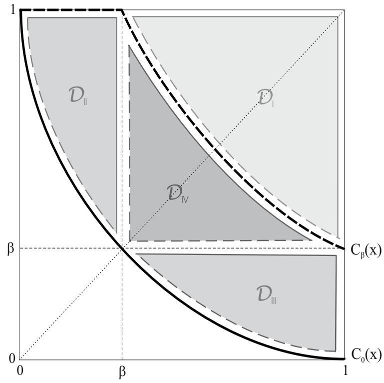


Fig. 2. The partition $\mathcal{D} = \mathcal{D}_{\text{I}} \cup \mathcal{D}_{\text{II}} \cup \mathcal{D}_{\text{III}} \cup \mathcal{D}_{\text{IV}}$

Due to the left-continuity of T it is obvious that $T(x, y) = 0$ holds for every $(x, y) \notin \mathcal{D}$.

Theorem 4. [12] *Consider a rotation-invariant t-norm T. Let σ be an arbitrary $[\beta, 1] \to [0, 1]$ isomorphism with β the fixpoint of C_0. Then there exists a left-continuous t-norm $\widehat{T}$ (with contour lines $\widehat{C}_a$) such that*

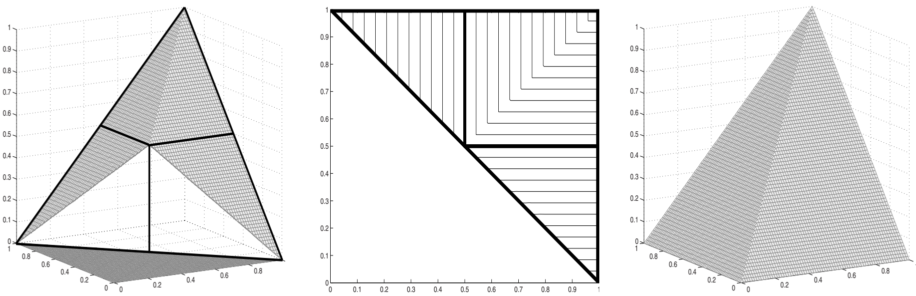
$$T(x,y) = \begin{cases} \sigma^{-1}\left[\widehat{T}(\sigma[x], \sigma[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\text{I}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\text{II}}\,, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(y)]}(\sigma[x])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\text{III}}\,, \\ 0, & \text{if } (x,y) \notin \mathcal{D}\,. \end{cases} \tag{4}$$

In particular, $\widehat{T} = T^\beta$.

Note that the isomorphism σ is used to compute the β-zoom T^β of T. Geometrically, $T|_{\mathcal{D}_{\text{I}}}$ is a rescaled version of $T^\beta|_{\mathcal{D}^\beta}$, where $\mathcal{D}^\beta = \{(x, y) \in [0, 1]^2 \mid 0 < T^\beta(x, y)\}$. $T|_{\mathcal{D}_{\text{II}}}$ is obtained by rotating $T|_{\mathcal{D}_{\text{I}}}$ 120 degrees to the left around the axis $\{(x, y, z) \in [0, 1]^2 \mid y = x \ \wedge \ z = 1 - x\}$. Similarly, rotating $T|_{\mathcal{D}_{\text{I}}}$ 120 degrees to the right around this axis determines $T|_{\mathcal{D}_{\text{III}}}$. As illustrated in [12], these rotations sometimes have to be reshaped to fit into the areas $\mathcal{D}_{\text{II}}$ and $\mathcal{D}_{\text{III}}$, respectively. The contour lines C_0 and C_β cause this reshaping. Solely the continuity of the contour line C_0 is responsible for the existence of the geometrical (transformed) rotations. T-norms such as the minimum operator $T_{\mathbf{M}}$ that do not have a continuous contour line do not have such geometrical symmetries. Therefore, only Eq. (3) and not Eq. (2) (see [7]) can be understood as the rotation-invariance property.

If T^β has no zero divisors, then $\mathcal{D}_{\text{IV}}$ is empty and Eq. (4) totally determines T. These particular t-norms have also been (alternatively) described by Jenei [6].

Figure 3 depicts the decomposition of the *nilpotent minimum* $T^{\mathbf{nM}}$ ($T^{\mathbf{nM}}(x,y) = 0$ whenever $x+y \leq 1$ and $T^{\mathbf{nM}}(x,y) = \min(x,y)$ elsewhere). The bold black lines in the figures indicate the partition $\mathcal{D} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}}$ (for the nilpotent minimum $\mathcal{D}_{\mathrm{IV}} = \emptyset$).



(a) $T^{\mathbf{nM}}$ (b) Contour plot of $T^{\mathbf{nM}}$ (c) $(T^{\mathbf{nM}})^{\frac{1}{2}}$

Fig. 3. Decomposition of the nilpotent minimum $T^{\mathbf{nM}}$

As illustrated in [12], $T|_{\mathcal{D}_{\mathrm{IV}}}$ is in general not uniquely determined by C_0 and T^{β}. Examining numerous examples, we noticed that the filling-in of area $\mathcal{D}_{\mathrm{IV}}$ is uniquely fixed whenever both C_0 and C_β are continuous. Invoking Theorem 3 we can generalize our decomposition from [12] in the following way.

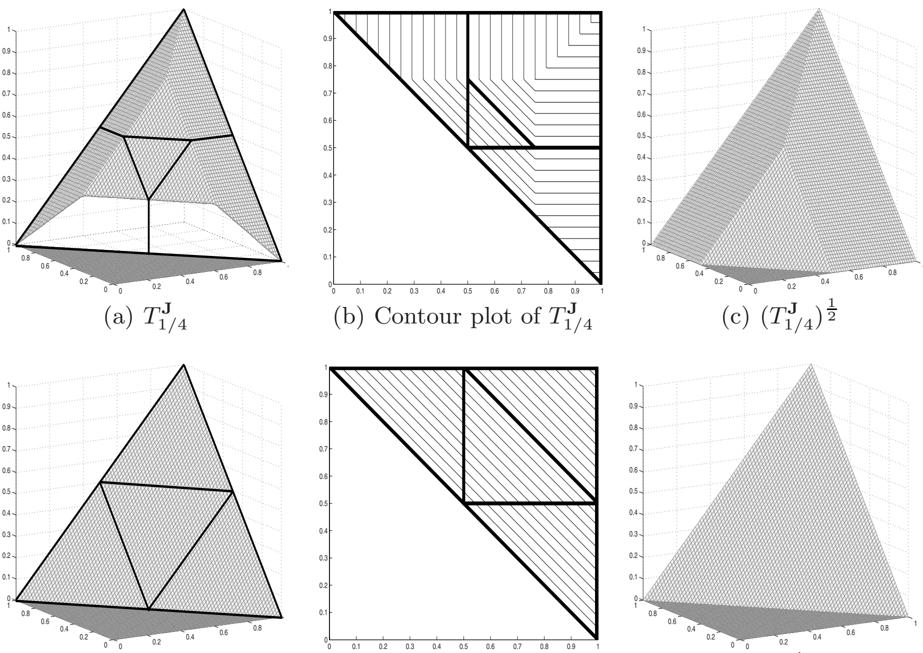
Theorem 5. *Consider a rotation-invariant t-norm T for which C_β is continuous on $]\beta,1]$, with β the unique fixpoint of C_0. Let σ be an arbitrary $[\beta,1] \to [0,1]$ isomorphism. Then there exists a left-continuous t-norm $\widehat{T}$ (with contour lines $\widehat{C}_a$ and companion $\widehat{Q}$) such that $\widehat{C}_0$ is continuous on $]0,1]$ and*

$$T(x,y) = \begin{cases} \sigma^{-1}\left[\widehat{T}(\sigma[x],\sigma[y])\right], & \text{if } (x,y) \in \mathcal{D}_{\mathrm{I}}, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(x)]}(\sigma[y])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{II}}, \\ C_0\left(\sigma^{-1}\left[\widehat{C}_{\sigma[C_0(y)]}(\sigma[x])\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{III}}, \\ C_0\left(\sigma^{-1}\left[\widehat{Q}(\widehat{C}_0(\sigma[x]),\widehat{C}_0(\sigma[y]))\right]\right), & \text{if } (x,y) \in \mathcal{D}_{\mathrm{IV}}, \\ 0, & \text{if } (x,y) \notin \mathcal{D}. \end{cases} \tag{5}$$

In particular, $\widehat{T} = T^{\beta}$ and $\widehat{Q}$ must be commutative on $[0,\hat{\alpha}[^2$, with $\hat{\alpha} = \inf\{t \in [0,1] \mid \widehat{C}_0(t) = 0\}$.

Geometrically, the filling-in of area $\mathcal{D}_{\mathrm{IV}}$ is obtained by rotating $T|_{\mathcal{D}_{\mathrm{I}} \cap [\beta,\sigma^{-1}(\hat{\alpha})]^2}$ 180 degrees to the front around the axis $\{(x,y,z) \in [0,1]^3 \mid x+y = \beta + \sigma^{-1}[\hat{\alpha}] \ \wedge \ z = \beta\}$. In case C_β is continuous it holds that $\hat{\alpha} = 1$ and the latter comes down to a 180 degree front-rotation of $T|_{\mathcal{D}_{\mathrm{I}}}$ around the axis $\{(x, y, z) \in [0,1]^3 \mid x + y = \beta + 1 \ \wedge \ z = \beta\}$. Again, the contour lines C_0 and C_β can cause some additional reshaping.

Figure 4 depicts the decomposition of the *Jenei t-norm* $T^{\mathbf{J}}_{1/4}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$. $T^{\mathbf{J}}_{1/4}$ can be created from the nilpotent minimum by lowering its values on $[\frac{1}{4}, \frac{3}{4}]^2$ in such a way that its $(\frac{1}{4}, \frac{3}{4})$-zoom equals $T_{\mathbf{L}}$. The bold black lines in the figures indicate the partition $\mathcal{D} = \mathcal{D}_{\mathrm{I}} \cup \mathcal{D}_{\mathrm{II}} \cup \mathcal{D}_{\mathrm{III}} \cup \mathcal{D}_{\mathrm{IV}}$.



(a) $T^{\mathbf{J}}_{1/4}$ (b) Contour plot of $T^{\mathbf{J}}_{1/4}$ (c) $(T^{\mathbf{J}}_{1/4})^{\frac{1}{2}}$

(d) $T_{\mathbf{L}}$ (e) Contour plot of $T_{\mathbf{L}}$ (f) $T_{\mathbf{L}}^{\frac{1}{2}}$

Fig. 4. Decomposition of the Jenei t-norm $T^{\mathbf{J}}_{1/4}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$

The geometrical symmetries of a rotation-invariant t-norm T establish in fact its associativity. In this respect Eq. (5) can also be used to construct rotation-invariant t-norms. Inspired by the geometrical interpretation of Eq. (5), we have called this construction the *triple rotation method* [13]. As the construction of t-norms falls outside the scope of this paper, we will not go into detail here.

References

1. F. Esteva, J. Gispert, L. Godo, and F. Montagna, *On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic*, Studia Logica **71** (2002), 199–226.

2. F. Esteva and L. Godo, *Monoidal t-norm based fuzzy logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems **124** (2001), 271–288.
3. J. Fodor, *A new look at fuzzy connectives*, Fuzzy Sets and Systems **57** (1993), 141–148.
4. J. Fodor and M. Roubens, *Fuzzy preference modelling and multicriteria decision support*, Kluwer Academic Publishers, 1994.
5. S. Jenei, *Geometry of left-continuous t-norms with strong induced negations*, Belg. J. Oper. Res. Statist. Comput. Sci. **38** (1998), 5–16.
6. S. Jenei, *How to construct left-continuous triangular norms - state of the art*, Fuzzy Sets and Systems **143** (2004), 27–45.
7. S. Jenei, *On the geometry of associativity*, Semigroup Forum, to appear.
8. E.P. Klement, R. Mesiar, and E. Pap, *Triangular norms*, Trends in Logic, Vol. 8, Kluwer Academic Publishers, 2000.
9. E.P. Klement, R. Mesiar, and E. Pap, *Different types of continuity of triangular norms revisited*, New Mathematics and Natural Computation **1** (2005), 195–211.
10. L. Lianzhen and L. Kaitai, *Involutive monoidal t-norm based logic and R_0 logic*, Internat. J. Intell. Systems **19** (2004), 491–497.
11. K.C. Maes and B. De Baets, *A contour view on uninorm properties*, Kybernetika **42** (3) (2006), 303–318.
12. K.C. Maes and B. De Baets, *On the structure of left-continuous t-norms that have a continuous contour line*, Fuzzy Sets and Systems, in press (DOI:10.1016/j.fss.2006.09.016).
13. K.C. Maes and B. De Baets, *The triple rotation method for constructing rotation-invariant t-norms*, Fuzzy Sets and Systems, submitted.
14. V. Novák, *On fuzzy type theory*, Fuzzy Sets and Systems **149** (2005), 235–273.
15. B. Schweizer and A. Sklar, *Probabilistic metric spaces*, Elsevier Science, New York, 1983.