On Possibilistic/Fuzzy Optimization

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Abstract. We focus on possibilistic/fuzzy optimality in the framework of mathematical programming problem with a possibilistic objective function. We observe the interaction between possibilistic objective function values. Two optimality concepts, possible and necessary optimalities are reviewed. The necessary soft optimality is investigated.

1 Introduction

Possibilistic programming treats ambiguous parameters in programming problems. Various approaches have been proposed to possibilistic programming problems (see, for example, [1]). As stochastic programming approaches are classified into chance constrained programming approaches, recourse problem (two-stage problem) approaches and distribution problem approaches, possibilistic programming approaches can be classified into modality constrained programming approaches, recourse problem approaches and optimization approaches. Many of possibilistic programming approaches can be regarded as special cases of modality constrained programming approaches. Therefore, modality constrained programming approaches have well-investigated by many authors, so far. Optimization approaches have more or less investigated and recourse problem approaches have little investigated. Many overviews of possibilistic/ fuzzy programming approaches are devoted for modality programming approaches which are more tractable than the others.

In this paper, we review and investigate optimization approaches. In optimization approaches, we analyze the range of optimal solutions with respect to the fluctuation of uncertain parameters. We restrict ourselves to linear programming problems with ambiguous objective function coefficients. First, we observe the induced interaction in the comparison between possibilistic objective function values. Then, possibly and necessarily optimal solutions are defined. Possible and necessary optimality tests are given. Finally, necessarily soft optimal solutions are investigated as a relaxed concept of necessarily optimal solutions.

2 Comparison of Possibilistic Objective Function Values

In this paper, we treat the following linear programming problem with ambiguous objective function coefficients:

$$\text{maximize } \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}, \quad \text{subject to } A\boldsymbol{x} \leq \boldsymbol{b}, \tag{1}$$

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where $A = (a_{ij})$ is an $m \times n$ matrix and $\boldsymbol{b} = (b_1, \ldots, b_m)^{\mathrm{T}}$. $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ is a decision variable vector. Moreover, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)^{\mathrm{T}}$ is a vector of ambiguous coefficients. We assume that we know $\boldsymbol{\gamma}$ realizes in a bounded n-dimensional fuzzy set Γ. Such a vector $\boldsymbol{\gamma}$ is said to be a possibilistic variable vector restricted in Γ. For the sake of simplicity, the feasible region of Problem (1) is denoted by X. The membership function μ_Γ of fuzzy set Γ is assumed to be upper semi-continuous. The boundedness of fuzzy set Γ is characterized by

$$\forall \varepsilon > 0, \exists r \in \mathbf{R}; \{\boldsymbol{c} \in \mathbf{R}^n : \mu_\Gamma(\boldsymbol{c}) \geq \varepsilon\} \subset \{\boldsymbol{c} \in \mathbf{R}^n : ||\boldsymbol{c}|| < r\}. \tag{2}$$

Given a solution $\boldsymbol{x} \neq \mathbf{0}$, by the extension principle, its objective function value is given as a fuzzy set $Y(\boldsymbol{x})$ having the following membership function:

$$\mu_{Y(\boldsymbol{x})}(y) = \sup_{\boldsymbol{c}} \{\mu_\Gamma(\boldsymbol{c}) : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = y\}. \tag{3}$$

There are various ways to compare two fuzzy numbers Z_1 and Z_2 For example, based on the possibility theory, the following two indices are obtained:

$$\mathrm{POS}(Z_1 \geq Z_2) = \sup_{r_1, r_2} \{\min(\mu_{Z_1}(r_1), \mu_{Z_2}(r_2)) : r_1 \geq r_2\}, \tag{4}$$

$$\mathrm{NES}(Z_1 \geq Z_2) = 1 - \sup_{r_1, r_2} \{\min(\mu_{Z_1}(r_1), \mu_{Z_2}(r_2)) : r_1 < r_2\}. \tag{5}$$

where μ_{Z_1} and μ_{Z_2} are membership functions of Z_1 and Z_2. Possibility degree $\mathrm{POS}(Z_1 \geq Z_2)$ shows to what extent Z_1 is possibly larger than or equal to Z_2. Similarly, Necessity degree $\mathrm{NES}(Z_1 \geq Z_2)$ shows to what extent Z_1 is necessarily larger than or equal to Z_2. When Z_1 and Z_2 are closed intervals $[z_1^{\mathrm{L}}, z_1^{\mathrm{R}}]$ and $[z_2^{\mathrm{L}}, z_2^{\mathrm{R}}]$, respectively, we have the following equivalences which show their meanings and difference remarkably:

$$\mathrm{POS}(Z_1 \geq Z_2) = 1 \Leftrightarrow z_1^{\mathrm{R}} \geq z_2^{\mathrm{L}}, \qquad \mathrm{NES}(Z_1 \geq Z_2) = 0 \Leftrightarrow z_1^{\mathrm{L}} < z_2^{\mathrm{R}}. \tag{6}$$

A comparison index between two fuzzy numbers is often applied to the comparison of possibilistic objective function values discarding their interaction in literature. Next example demonstrates the inadequacy caused by the desertion of the interaction.

Example 1. Let $n = 2$ and $\Gamma = [1, 2] \times [-2, -1]$. Namely, we consider a case when each objective function coefficient is given by a closed interval. Consider two feasible solutions $\boldsymbol{x}^1 = (2, 1)^{\mathrm{T}}$ and $\boldsymbol{x}^2 = (3, 1)^{\mathrm{T}}$. We have $Y(\boldsymbol{x}^1) = [0, 3]$ and $Y(\boldsymbol{x}^2) = [1, 5]$. Let us apply (6) discarding the interaction between $Z_1 = Y(\boldsymbol{x}^1)$ and $Z_2 = Y(\boldsymbol{x}^2)$. We obtain $\mathrm{POS}(Z_1 \geq Z_2)$= 1 which implies that the objective function value of $\boldsymbol{x}^1$ can be larger than or equal to that of $\boldsymbol{x}^2$. On the other hand, we have

$$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^1 = 2c_1 + c_2 < 3c_1 + c_2 = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^2, \ \ \forall c_1 \in [1, 2], \ \forall c_2 \in [-2, -1]. \tag{7}$$

This insists that the objective function value of $\boldsymbol{x}^1$ can never be larger than or equal to that of $\boldsymbol{x}^2$. Because the realized values of c_1 and c_2 are common independent of the selection of a feasible solution to Problem (1), the latter result is correct. Therefore, the direct application of index $\mathrm{POS}(Z_1 \geq Z_2)$ is not adequate for the problem setting.

Similarly, from (6), we obtain $\mathrm{NES}(Z_2 \geq Z_1) = 0$. This implies that there exists $(c_1, c_2)^{\mathrm{T}} \in \Gamma$ such that the objective function value of $\boldsymbol{x}^2$ is less than that of $\boldsymbol{x}^1$. However, this is neither true. As is shown in (7), for all $(c_1, c_2)^{\mathrm{T}} \in \Gamma$, the objective function value of $\boldsymbol{x}^2$ is larger than that of $\boldsymbol{x}^1$.

Now we emphasize the reason why indices defined by (4) and (5) do not work in Example 1. Let ζ_1 and ζ_2 be possibilistic variables restricted by Z_1 and Z_2. In (4) and (5), it is implicitly assumed that ζ_2 is independent of ζ_1 and vice versa.

In Example 1, we set $Z_1 = Y(\boldsymbol{x}^1)$ and $Z_2 = Y(\boldsymbol{x}^2)$. Namely, they are possible ranges of $\zeta_1 = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^1$ and $\zeta_2 = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^2$, respectively. Both ζ_1 and ζ_2 depend on the possibilistic variable vector $\boldsymbol{\gamma}$ restricted by $\Gamma = [1, 2] \times [-2, -1]$. Because of this fact, the implicit assumption in (4) and (5) does not hold. For example, when $\zeta_1 = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^1 = 0$, the possible values of $\gamma \in \Gamma$ are in

$$\{(c_1, c_2)^{\mathrm{T}} \in \mathbf{R}^2 : 2c_1 + c_2 = 0, 1 \leq c_1 \leq 2, -2 \leq c_2 \leq -1\} = \{(1, -2)\}.$$

Namely, from the information $\zeta_1 = 0$, in this case, we know that $\boldsymbol{\gamma}$ uniquely takes $(1, -2)^{\mathrm{T}}$. Therefore, the value ζ_2 takes is also uniquely known, i.e., $\zeta_2 = (1, -2)\boldsymbol{x}^1 = 1$. Generally, when $\zeta_1 = q$, the possible range of ζ_2 is given by

$$\{3c_1 + c_2 : 2c_1 + c_2 = q,\ 1 \leq c_1 \leq 2,\ -2 \leq c_2 \leq -1\}.$$

This range varies depending on ζ_1-value q. Therefore, ζ_2 interacts with ζ_1. Similarly, ζ_1 interacts with ζ_2.

Since the implicit assumption of (4) and (5) does not hold, indices defined by (4) and (5) cannot be applied without any modification. For the comparison between possibilistic objective function values, the following modified indices [2] are adequate:

$$\mathrm{POS}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^1 \geq \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^2) = \sup_{\boldsymbol{c}}\{\mu_\Gamma(\boldsymbol{c}) : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^1 \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^2\}, \tag{8}$$

$$\mathrm{NES}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^1 \geq \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^2) = 1 - \sup_{\boldsymbol{c}}\{\mu_\Gamma(\boldsymbol{c}) : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^1 < \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^2\}. \tag{9}$$

In literature, the desertion exemplified in Example 1 often appears when possibilistic objective function values are compared. Moreover, we note that under other interpretations of fuzzy coefficients, the discussion about the inadequacy is not valid. For example, when a fuzzy objective function is regarded as a collection of objective functions, e.g., a collection of utility functions of many decision makers, the above discussion does not make sense.

3 Possibly and Necessarily Optimal Solutions

We define an optimal solution set $S(\boldsymbol{c})$ of Problem (1) with respect to $\boldsymbol{\gamma} = \boldsymbol{c}$ by

$$S(\boldsymbol{c}) = \left\{\boldsymbol{x} \in X : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \max_{\boldsymbol{y} \in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}\right\}. \tag{10}$$

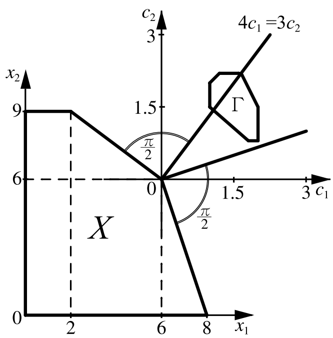

Fig. 1. A possibly optimal solution

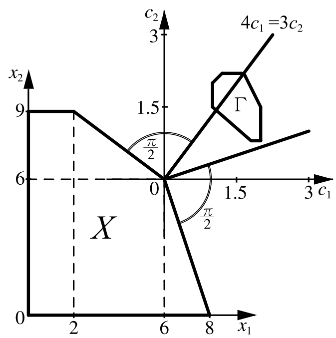

Fig. 2. A necessarily optimal solution

Using $S(\boldsymbol{c})$, we can define the following two optimal solution sets ΠS and NS to Problem (1) when Γ is a crisp set:

$$\Pi S = \bigcup_{\boldsymbol{c} \in \Gamma} S(\boldsymbol{c}), \qquad NS = \bigcap_{\boldsymbol{c} \in \Gamma} S(\boldsymbol{c}). \tag{11}$$

$\boldsymbol{x} \in \Pi S$ implies that there exists $\boldsymbol{c} \in \Gamma$ to which $\boldsymbol{x}$ is an optimal solution. Namely, $\boldsymbol{x} \in \Pi S$ can be optimal for at least one possible realization of $\boldsymbol{\gamma}$, and then, it is called a possibly optimal solution. On the other hand, $\boldsymbol{x} \in NS$ implies that for all $\boldsymbol{c} \in \Gamma$, $\boldsymbol{x}$ is optimal. Namely, $\boldsymbol{x} \in NS$ is always optimal for all possible realizations of $\boldsymbol{\gamma}$, and then it is called a necessarily optimal solution. We have $NS \subseteq \Pi S$.

Example 2. Consider Problem (1) where A and $\boldsymbol{b}$ are defined by

$$A = \begin{pmatrix} 3 & 3 & 0 & -1 & 0 \\ 4 & 1 & 1 & 0 & -1 \end{pmatrix}^{\mathrm{T}}, \qquad \boldsymbol{b} = (42, 24, 9, 0, 0)^{\mathrm{T}}, \tag{12}$$

and Γ is given by

$$\begin{aligned} \Gamma = \{(c_1, c_2)^{\mathrm{T}} : {} & 3.5 \le 2c_1 + c_2 \le 5.5,\ 3.4 \le c_1 + 2c_2 \le 6, \\ & -1 \le c_1 - c_2 \le 1.3,\ 1 \le c_1 \le 2,\ 0.8 \le c_2 \le 2.2\}. \end{aligned} \tag{13}$$

As shown in Fig. 1, $(x_1, x_2)^{\mathrm{T}} = (6, 6)^{\mathrm{T}}$ is optimal for $(c_1, c_2)^{\mathrm{T}} \in \Gamma$ such that $3c_2 \le 4c_1$, and $(x_1, x_2)^{\mathrm{T}} = (2, 9)^{\mathrm{T}}$ is optimal for $(c_1, c_2)^{\mathrm{T}} \in \Gamma$ such that $3c_2 \ge 4c_1$. Moreover, all solutions on line segment between those solutions are optimal for $(c_1, c_2)^{\mathrm{T}} \in \Gamma$ such that $3c_2 = 4c_1$. Therefore, we have infinitely many possibly optimal solutions on the line segment. However, we have no necessarily optimal solution.

On the other hand, we consider Γ defined by

$$\Gamma = \{(c_1, c_2)^{\mathrm{T}} : c_1 + c_2 \ge 3,\ c_1 \ge c_2,\ c_1 \le 2c_2,\ c_1 \le 2.5,\ c_2 \le 2\}. \tag{14}$$

In this case, as shown in Fig. 2, $(x_1, x_2)^{\mathrm{T}} = (6, 6)^{\mathrm{T}}$ is optimal for all $\boldsymbol{c} \in \Gamma$. Namely, the solution is a necessarily optimal solution. From $NS \subseteq \Pi S$, $(x_1, x_2)^{\mathrm{T}} = (6, 6)^{\mathrm{T}}$ is also a possibly optimal solution.

As shown in Example 2, it is possible that infinitely many possibly optimal solutions exist and that no necessarily optimal solution exists.

In order to show the need of possible and necessary optimality, we continue to treat a case when Γ is crisp. In modality constrained programming problems [1], various treatments of possibilistic objective functions are proposed. In crisp case, lower and upper bounds and the center values are often optimized and the width of the possible range of objective function values is minimized. Many approaches may regard a complete optimal solution to the following biobjective programming problem maximizing lower and upper bounds of objective function value as the most reasonable solution, if it exists:

$$\underset{\boldsymbol{x} \in X}{\text{maximize}} \quad \left(\min_{\boldsymbol{c} \in \Gamma} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x}, \ \max_{\boldsymbol{c} \in \Gamma} \boldsymbol{c}^{\mathrm{T}} \boldsymbol{x} \right) \tag{15}$$

Next example shows that, even if a complete optimal solution to Problem (15) exists, it is not always the most reasonable solution to Problem (1).

Example 3. Consider Problem (1) with the following A and $\boldsymbol{b}$:

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & -1 \end{pmatrix}^{\mathrm{T}}, \qquad \boldsymbol{b} = (12, 24, 9, 0, 0)^{\mathrm{T}} \tag{16}$$

Moreover, Γ is defined by

$$\Gamma = \{(c_1, c_2)^{\mathrm{T}} : 7c_1 - 5c_2 \le 4, \ c_2 \le 2, \ -3c_1 + 5c_2 \ge 2, \ c_1 \ge 1\}. \tag{17}$$

Consider $(1, 1)^{\mathrm{T}}$ and $(3, 3)^{\mathrm{T}} \in \Gamma$. For all $\boldsymbol{c} \in \Gamma$, we have $(1, 1)^{\mathrm{T}} \le \boldsymbol{c} \le (3, 3)^{\mathrm{T}}$. Therefore, Problem (15) becomes

$$\underset{\boldsymbol{x} \in X}{\text{maximize}} \ (x_1 + x_2, 3x_1 + 3x_2). \tag{18}$$

As shown in Fig. 3, $\boldsymbol{x}^0 = (6, 6)^{\mathrm{T}}$ is a complete optimal solution to Problem (15). However, in Fig. 3, the shaded region of Γ where $\boldsymbol{x}^0$ becomes optimal is much smaller than the other region of Γ. The possibly optimal solution set in this problem is shown as the line segment between points $(6, 6)^{\mathrm{T}}$ and $(3, 9)^{\mathrm{T}}$. Then, $\boldsymbol{x}^0$ is even extreme in the possibly optimal solution set. From these points of view, $\boldsymbol{x}^0$ is not necessarily the most reasonable solution.

As shown in Example 3, a complete optimal solution to Problem (15) is not always the most reasonable solution. When a necessarily optimal solution exists, it is the most reasonable solution since it is optimal for all possible realizations of $\boldsymbol{\gamma}$. On the other hand, a possibly optimal solution is a solution optimal for at least one possible realization of $\boldsymbol{\gamma}$, therefore, it can be regarded as a solution with minimum rationality. To sum up, possible optimality is the minimum requirement for the optimal solution to Problem (1) while necessary optimality is the ideal.

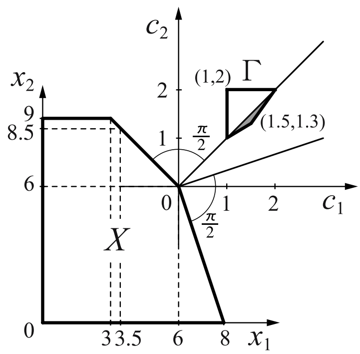

Fig. 3. Solution $(6,6)^{\mathrm{T}}$ in Example 3

Now let us describe possibly and necessarily optimal solutions when Γ is a fuzzy set. In this case, both possibly optimal solution set ΠS and necessarily optimal solution set NS become fuzzy sets define by the following membership functions (see [3]):

$$\mu_{\Pi S}(\boldsymbol{x}) = \begin{cases} \sup_{\boldsymbol{c}} \{\mu_\Gamma(\boldsymbol{c}) : \boldsymbol{x} \in S(\boldsymbol{c})\}, & \text{if } \boldsymbol{x} \in X, \\ 0, & \text{if } \boldsymbol{x} \notin X, \end{cases} \tag{19}$$

$$\mu_{NS}(\boldsymbol{x}) = \begin{cases} \inf_{\boldsymbol{c}} \{1 - \mu_\Gamma(\boldsymbol{c}) : \boldsymbol{x} \notin S(\boldsymbol{c})\}, & \text{if } \boldsymbol{x} \in X, \\ 0, & \text{if } \boldsymbol{x} \notin X. \end{cases} \tag{20}$$

Obviously, we have $\mu_{NS}(\boldsymbol{x}) \leq \mu_{\Pi S}(\boldsymbol{x})$, i.e., $NS \subseteq \Pi S$. We also obtain the following stronger relation:

$$\mu_{NS}(\boldsymbol{x}) > 0 \Rightarrow \mu_{\Pi S}(\boldsymbol{x}) = 1. \tag{21}$$

This comes from a relation between possibility and necessity, i.e., if an event is somehow necessary, it should be totally possible.

An optimal solution to a linear programming problem with an objective function $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ is a feasible solution satisfying $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}$ for all $\boldsymbol{y} \in X$, in other words, there is no feasible solution $\boldsymbol{y} \in X$ such that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} > \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. Applying (8) and (9), for $\boldsymbol{x} \in X$, we have the following properties which corresponds to the property of the optimal solution mentioned above:

$$\mu_{\Pi S}(\boldsymbol{x}) = \inf_{\boldsymbol{y} \in X} \mathrm{POS}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{y}) = 1 - \sup_{\boldsymbol{y} \in X} \mathrm{NES}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{y} > \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}), \tag{22}$$

$$\mu_{NS}(\boldsymbol{x}) = \inf_{\boldsymbol{y} \in X} \mathrm{NES}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{y}) = 1 - \sup_{\boldsymbol{y} \in X} \mathrm{POS}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{y} > \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}), \tag{23}$$

where $\mathrm{POS}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^1 > \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^2)$ and $\mathrm{NES}(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^1 > \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}^2)$ are defined by (8) and (9) with replacements of '$\geq$' and '$\leq$' with '$>$' and '$<$', respectively.

4 Possible and Necessary Optimality Tests

Possible and necessary optimalities are fundamental criteria for the solution selection as described earlier. It is worthwhile to calculate degrees of possible and necessary optimalities $\mu_{\Pi S}(\boldsymbol{x})$ and $\mu_{NS}(\boldsymbol{x})$ for justifying the selection of a solution $\boldsymbol{x} \in X$.

Let $A_{j\cdot}$ be the j-th row of A. Using the optimality condition of the linear programming problem, the necessary and sufficient condition for $\boldsymbol{x} \in X$ to be $\boldsymbol{x} \in S(\boldsymbol{c})$ is given as (see [3])

(a) There exists $j \in \{1, \ldots, m\}$ such that $A_{j\cdot}\boldsymbol{x} - b_j = 0$.
(b) There exists $\boldsymbol{v} \geq \mathbf{0}$ such that $A_0^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{c}$, where A_0 is a submatrix of A composed all rows $A_{j\cdot}$ such that $A_{j\cdot}\boldsymbol{x} - b_j = 0$.

In what follows, we consider $\boldsymbol{x} \in X$ satisfying (a). From (b), we have

$$\mu_{\Pi S}(\boldsymbol{x}) = \sup_{\boldsymbol{v}}\{\mu_{\Gamma}\left(A_0^{\mathrm{T}}\boldsymbol{v}\right) : \boldsymbol{v} \geq \mathbf{0}\}, \tag{24}$$

$$\mu_{NS}(\boldsymbol{x}) = \inf_{\boldsymbol{c}}\{1 - \mu_{\Gamma}(\boldsymbol{c}) : \forall \boldsymbol{v} \geq \mathbf{0}, A_0^{\mathrm{T}}\boldsymbol{v} \neq \boldsymbol{c}\}. \tag{25}$$

From the boundedness of Γ, the upper semi-continuity of μ_{Γ} and (24), $\mu_{\Pi S}(\boldsymbol{x})$ can be obtained as the optimal value of

$$\text{maximize} \quad h, \quad \text{subject to} \quad A_0^{\mathrm{T}}\boldsymbol{v} \in [\Gamma]_h, \ \boldsymbol{v} \geq \mathbf{0}. \tag{26}$$

We consider a special case when Γ is characterized by membership function,

$$\mu_{\Gamma}(\boldsymbol{c}) = \min_{k=1,\ldots,p} \varphi(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{c}). \tag{27}$$

where $p > n$ and $\varphi : \mathbf{R} \to [0, 1]$ is an upper semi-continuous non-increasing function satisfying $\lim_{r \to +\infty} \varphi(r) = 0$. We define a pseudo-inverse of φ, $\varphi^* : [0, 1] \to \mathbf{R}^n \cup \{+\infty\}$ by $\varphi^*(h) = \sup_r\{r : \varphi(r) \geq h\}$. Then, we have

$$\boldsymbol{c} \in [\Gamma]_h \Leftrightarrow \boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{c} \leq \varphi^*(h), \ k = 1, \ldots, p \tag{28}$$

Hence, we obtain the optimal value of Problem (26) as $\varphi(\hat{s})$ by calculating the optimal value $\hat{s}$ of the following linear programming problem:

$$\text{minimize} \quad s, \quad \text{subject to} \quad \boldsymbol{d}_k^{\mathrm{T}}A_0^{\mathrm{T}}\boldsymbol{v} \leq s, \ k = 1, \ldots, p, \ \boldsymbol{v} \geq \mathbf{0}. \tag{29}$$

On the other hand, from (25), we have

$$\mu_{NS}(\boldsymbol{x}) \geq h \Leftrightarrow \left(\mu_{\Gamma}(\boldsymbol{c}) > 1 - h \Rightarrow \exists \boldsymbol{v} \geq \mathbf{0}; A_0^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{c}\right). \tag{30}$$

Let $(\Gamma)_{1-h} = \{\boldsymbol{c} : \mu_{\Gamma}(\boldsymbol{c}) > 1 - h\}$. Then, from (30), we have

$$\begin{aligned}\mu_{NS}(\boldsymbol{x}) &= \sup\left\{h : \sup_{\boldsymbol{c} \in (\Gamma)_{1-h}} \inf_{\boldsymbol{v} \geq \mathbf{0}} |A_0^{\mathrm{T}}\boldsymbol{v} - \boldsymbol{c}| \leq 0\right\} \\ &= \max\left\{h : \max_{\boldsymbol{c} \in \mathrm{cl}(\Gamma)_{1-h}} \min_{\boldsymbol{v} \geq \mathbf{0}} |A_0^{\mathrm{T}}\boldsymbol{v} - \boldsymbol{c}| \leq 0\right\},\end{aligned} \tag{31}$$

where $\mathrm{cl}(\Gamma)_{1-h}$ is the closure of $(\Gamma)_{1-h}$. From the boundedness of $(\Gamma)_{1-h}$ and the continuity of $|A_0^{\mathrm{T}}\boldsymbol{v}-\boldsymbol{c}|$, we replaced '$(\Gamma)_{1-h}$', 'sup' and 'inf' with '$\mathrm{cl}(\Gamma)_{1-h}$', 'max' and 'min', respectively.

Consider Γ defined by (27). We have

$$\boldsymbol{c} \in \mathrm{cl}(\Gamma)_{1-h} \Leftrightarrow \boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{c} \leq \bar{\varphi}^*(1-h),\ k=1,\dots,p, \tag{32}$$

where we define $\bar{\varphi}^* : [0,1] \to \mathbf{R}^n \cup \{+\infty\}$ by

$$\bar{\varphi}^*(h) = \begin{cases} \sup_r\{r : \varphi(r) > h\}, & \text{if } h < 1 \\ -\infty & \text{if } h = 1. \end{cases} \tag{33}$$

Since Γ is bounded, $\mathrm{cl}(\Gamma)_{1-h}$ becomes a polytope. Let $V(\mathrm{cl}(\Gamma)_{1-h})$ be a set of vertices of $\mathrm{cl}(\Gamma)_{1-h}$. Then any point in $\mathrm{cl}(\Gamma)_{1-h}$ can be expressed as a convex combination of points in $V(\mathrm{cl}(\Gamma)_{1-h})$. Therefore, for any Q such that $V(\mathrm{cl}(\Gamma)_{1-h}) \subseteq Q(h) \subseteq \mathrm{cl}(\Gamma)_{1-h}$, we have

$$\max_{\boldsymbol{c}\in\mathrm{cl}(\Gamma)_{1-h}} \min_{\boldsymbol{v}\geq\mathbf{0}} |A_0^{\mathrm{T}}\boldsymbol{v}-\boldsymbol{c}| \leq 0 \Leftrightarrow \max_{\boldsymbol{c}\in Q(h)} \min_{\boldsymbol{v}\geq\mathbf{0}} |A_0^{\mathrm{T}}\boldsymbol{v}-\boldsymbol{c}| \leq 0 \tag{34}$$

If there is a set mapping $Q(h) = \{\boldsymbol{c}^j(h),\ j=1,\dots,u\}$ such that $V(\mathrm{cl}(\Gamma)_{1-h}) \subseteq Q(h) \subseteq \mathrm{cl}(\Gamma)_{1-h}$ for any $h \in (0,1]$, we have $\mu_{NS}(\boldsymbol{x}) = \min_{j=1,\dots,u} h^j$, where h^j is the optimal value of the following linear programming problem:

$$\text{maximize} \ \ h, \quad \text{subject to} \ \ A_0^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{c}^j(h),\ \boldsymbol{v} \geq \mathbf{0}. \tag{35}$$

We may define $\boldsymbol{c}^j(h)$ of $Q(h)$ as the $\boldsymbol{c}$-value of an optimal solution to the following linear programming problem with respect to P_j composed of n elements from $\{1,\dots,p\}$:

$$\text{minimize} \sum_{k\in P_j} s_k, \quad \text{subject to} \ \ \boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{c}+s_k = \bar{\varphi}^*(1-h),\ s_k \geq 0,\ k=1,\dots,p. \tag{36}$$

We have $\binom{n}{p}$ P_j's. Therefore, $Q(h)$ is a finite set with at most $u = \binom{n}{p}$ elements $\boldsymbol{c}^j(h)$ for each $h \in (0,1]$.

Since maximizing h is equivalent to maximizing $\bar{\varphi}^*(1-h)$, introducing Problem (36) into Problem (35), we obtain the following two-phase linear programming problem with a sufficiently small positive number ϵ:

$$\text{maximize} \ -\sum_{k\in P_j} s_k + \epsilon s, \quad \text{subject to} \begin{cases} A_0^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{c},\ \boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{c} + s_k = s, \\ \boldsymbol{v} \geq \mathbf{0},\ s_k \geq 0,\ k=1,2,\dots,p \end{cases} \tag{37}$$

Let $s(P_j)$ be s-value at an optimal solution to Problem (37). Then we have $\mu_{NS}(\boldsymbol{x}) = \min_{j=1,2,\dots,u}(1-\varphi(s(P_j)))$.

As shown above, $\mu_{NS}(\boldsymbol{x})$ can be obtained by solving multiple linear programming problems. However, it requires to solve $\binom{n}{p}$ problems and thus, it will not be very efficient. A more efficient method can be designed based on global optimization techniques.

We have described computation methods for degrees of possible and necessary optimalities. It is also interesting to obtain all possibly and necessarily optimal solutions. In [4], an enumeration method for possibly optimal extreme pints is proposed.

5 Necessarily Soft Optimal Solution

As described before, there is no guarantee that a necessarily optimal solution $\boldsymbol{x}$ such that $\mu_{NS}(\boldsymbol{x}) > 0$ while it is the most reasonable solution. Even if it exists, $\mu_{NS}(\boldsymbol{x})$ is often small. This is because the requirement for the necessary optimality is very strong.

In real world problems, suboptimal solutions are often sufficiently good. Based on this idea, we proposed necessarily soft optimal solutions [5] in which the optimality of necessarily optimal solutions is relaxed to the suboptimality.

Let $\tilde{S}(\boldsymbol{c})$ be a fuzzy optimal solution set to linear programming problem with objective function coefficients $\boldsymbol{c}$. Its membership function can be defined by

$$\mu_{\tilde{S}(\boldsymbol{c})}(\boldsymbol{x}) = \begin{cases} \mu_{Dif}\left(\max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\right), & \text{if } \boldsymbol{x} \in X, \\ 0, & \text{otherwise,} \end{cases} \tag{38}$$

where $\mu_{Dif} : \mathbf{R} \to [0,1]$ is an upper semi-continuous non-increasing function. Equation (38) is based on the difference from the optimal value. We may have a similar approach based on the ratio to the optimal value. When $\forall \boldsymbol{c} \in \Gamma;\ \max_{\boldsymbol{x}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} > 0$, we may define

$$\mu_{\tilde{S}(\boldsymbol{c})}(\boldsymbol{x}) = \begin{cases} \mu_{Rat}\left(\dfrac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}}{\max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}}\right), & \text{if } \boldsymbol{x} \in X, \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

where $\mu_{Rat} : (-\infty, 1] \to [0,1]$ is upper semi-continuous non-decreasing function.

Using a fuzzy optimal solution set $\tilde{S}(\boldsymbol{c})$, a necessarily soft optimal solution set [5] $\widetilde{NS}$ is defined by the following membership function:

$$\mu_{\widetilde{NS}}(\boldsymbol{x}) = \inf_{\boldsymbol{c}} \max\left(1 - \mu_{\Gamma}(\boldsymbol{c}), \mu_{\tilde{S}(\boldsymbol{c})}(\boldsymbol{x})\right), \tag{40}$$

where when $\tilde{S}(\boldsymbol{c})$ is defined by (39), we assume for all $\boldsymbol{c}$ such that $\mu_{\Gamma}(\boldsymbol{c}) > 0$ $\max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} > 0$.

Based on the necessary soft optimality, the best solution can be an optimal solution to the following problem:

$$\underset{\boldsymbol{x}\in X}{\text{maximize}} \quad \mu_{\widetilde{NS}}(\boldsymbol{x}) \tag{41}$$

The solution is called a best necessarily soft optimal solution.

Now let us consider a case when Γ is a crisp set. In this case, for any μ_{Dif} and μ_{Rat}, Problems (41) with (38) and (39) are reduced to the following problems, respectively:

$$\underset{\boldsymbol{x}\in X}{\text{minimize}}\, R(\boldsymbol{x}) = \max_{\boldsymbol{c}\in\Gamma,\ \boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \quad \underset{\boldsymbol{x}\in X}{\text{maximize}}\, F(\boldsymbol{x}) = \min_{\boldsymbol{c}\in\Gamma} \frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}}{\max_{\boldsymbol{y}\in X} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}}. \tag{42}$$

Those problems are called a minimax regret problem and a maximin regret ratio problem. Those problems have the following good properties:

(a) $R(\boldsymbol{x}) = 0$ ($F(\boldsymbol{x}) = 1$) if and only if $\boldsymbol{x}$ is a necessarily optimal solution.
(b) $R(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in X$ ($F(\boldsymbol{x}) \leq 1, \forall \boldsymbol{x} \in X$).
(c) Any optimal solution is a possibly optimal solution.

From those properties, optimal solutions to those problems are regarded as possibly optimal solutions minimizing the deviation from necessary optimality.

The minimax regret problem and maximin regret ratio problem include nonconvex programs as their subproblems so that they are not very tractable. However, a solution algorithms based on a relaxation procedure has already proposed. To solve Problem (41), we further introduce the idea of bisection method to a solution method for (42). A solution algorithm converges a relaxation procedure and a bisection method simultaneously has proposed (see [5,6]).

6 Concluding Remarks

In this paper, we have reviewed and investigated the optimization approach to possibilistic/fuzzy programming problems. The formulated problems in this approach often include nonconvex subproblems so that applications of global optimization techniques are promising. On the other hand, by the development of interior point method, the range of tractable problems is enlarged. Solution methods for possibilistic/fuzzy optimization problems can also be developed by the introduction of new solution approaches.

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