Friction Model by Using Fuzzy Differential Equations

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Abstract. In the present paper we propose a novel approach for modeling friction, by using fuzzy differential equations under the strongly generalized differentiability concept. The key point is a continuous fuzzyfication of the signum function. The lack of the uniqueness for the solutions of a fuzzy differential equation allows us to choose the solution which better reflects the behavior of the modeled real-world system, so it allows us to incorporate expert's knowledge in our model. Numerical solutions of the fuzzy differential equations modeling dry friction are proposed. In order to show how the expert's knowledge can be incorporated in the system, we study the dry friction equation with different additional assumptions.

1 Introduction

The existing models of the friction forces show discontinuous variation at the zero transition of the velocity (see e.g. [5], [19], [17], [11], [12]). The effects of friction at low velocities are due to local properties of the materials and an accurate model of these phenomena is possible by taking into account properties both at the molecular and macroscopic level ([19], [16]). Since the information on the molecular level is usually unavailable we have uncertianties. These uncertainties are usually modeled by considering the friction force as a multivalued function and in this case the equations of motion are considered as differential inclusions. This approach is used in several works (see [1], [15], [8], etc.). The idea behind using differential inclusions is substituting the signum function by a multivalued function. This model often manifests chaotic behavior ([11], [10]).

The above discussion shows that the model of a system with friction is often subject of non-statistical uncertainties. So, in order to model the behavior of a system under the presence of the friction forces we have to take into account these uncertainties. In order to take into account these uncertainties we propose

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in the present paper an alternative fuzzy model based on fuzzy differential equations (FDEs). Surely other techniques can be easily imagined (such as interval methods) but these are subject of further research. Also, in our proposed method it is possible to incorporate expert knowledge about the system under study and this property can be turned into an advantage in future studies.

FDEs appear naturally as tools for modeling dynamical systems under uncertainty. Till now, they are rarely used in modeling real-world systems since their theory was developed relatively recently. Also, as it is shown in several recent papers, FDEs are not just an easy extension of the theory of ODEs to the fuzzy case (see e.g. [13], [14], [2]). This fact is also slowing down the extension of the applicability of FDEs. There are several different interpretations of the notion of a FDE (for a discussion about them please refer to [2]). In the present paper we will use the so called strongly generalized differentiability concept introduced recently as a method which solved some problems with the other FDE interpretations (H-derivative (see [18]) or fuzzy differential inclusions (see [7])).

Strongly generalized differentiability was introduced in [3]. The strongly generalized derivative is defined for a larger class of fuzzy-number-valued functions than the H-derivative and fuzzy differential equations can have solutions with decreasing length of their support (this was not the case for the H-derivatve). Also, contrary to the case of differential inclusions, the derivative concept for fuzzy-number-valued function is defined and this makes this method more appropriate for numerical computations. First order linear fuzzy differential equations are investigated in [4] and the behavior of their solutions motivate also the use of the above cited results in the present paper for building a novel friction model.

The key point in our discussion is how to fuzzify the classical model in order to get meaningful conclusions. The key role in this fuzzification is played by the frictional term in the equations of movement with friction. In the present paper, following [7], we fuzzify in a heuristic way the Signum function, by making this term also continuous fuzzy valued function. However we have gained the continuity of the frictional term, since it is a fuzzy one, we obtain a fuzzy solution for our model. The interpretation of this model is the fuzzy set of trajectories, attainable by the system ([7]). The lack of uniqueness of the solution of a fuzzy differential equation under the generalized differentiability concept at first sight could be seen as a disadvantage. But it is turned into an advantage in the present paper since we are able this way to include in our model knowledge based on observations of the modeled system. In the present paper we do not deal with the problem of control, this being subject of further research.

After a preliminary section we propose in Section 3 the heuristic fuzzy model for friction forces with a discussion on dry friction. In Section 4 we present also some preliminary results how friction is modeled by using the proposed approach. We end up with some conclusions and further research topics.

2 Preliminaries

We denote by $\mathbf{R}_F$ the space of fuzzy numbers, i.e., fuzzy subsets of the real line $u : \mathbf{R} \to [0,1]$, satisfying the following properties:

(i) u is normal i.e. $\exists x_u \in \mathbf{R}$ with $u(x_u) = 1$;

(ii) u is convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1]$, $x, y \in \mathbf{R}$);

(iii) u is upper semi-continuous on $\mathbf{R}$;

(iv) $\overline{\{x \in \mathbf{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

For $0 < r \leq 1$, denote $[u]^r = \{x \in \mathbf{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbf{R};\ u(x) > 0\}}$. Then it is well-known that for any $r \in [0,1]$, $[u]^r$ is a bounded closed interval. For $u, v \in \mathbf{R}_F$, and $\lambda \in \mathbf{R}$, the sum $u + v$ and the product $\lambda \cdot u$ are defined by $[u+v]^r = [u]^r + [v]^r$, $[\lambda \cdot u]^r = \lambda [u]^r$, $\forall r \in [0,1]$.

Let $D : \mathbf{R}_F \times \mathbf{R}_F \to \mathbf{R}_+ \cup \{0\}$, $D(u,v) = \sup_{r\in[0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$, be the Hausdorff distance between fuzzy numbers, where $[u]^r = [u_-^r, u_+^r]$, $[v]^r = [v_-^r, v_+^r]$. In this case $(\mathbf{R}_F, D)$ is a complete metric space. The above operations and the metric space structure allows us to build a mathematical analysis over the space of fuzzy numbers, however some problems appear due to the lack of some properties.

The so called H-difference or Hukuhara difference will play a key role in the present paper. Let us recall its definition.

Definition 1. *(see e.g. [18]). Let $x, y \in \mathbf{R}_F$. If there exists $z \in \mathbf{R}_F$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x \ominus y$.*

In this paper the "$\ominus$" sign stands always for H-difference and let us remark that $x \ominus y \neq x + (-1)y$. We will denote for simplicity $x + (-1)y$ by $x - y$.

Let us recall the definition of strongly generalized differentiability proposed in [3].

Definition 2. *Let $f : (a,b) \to \mathbf{R}_F$ and $x_0 \in (a,b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbf{R}_F$, such that*

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\searrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0),$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $f(x_0 - h) \ominus f(x_0)$ and the limits

$$\lim_{h\searrow 0} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h\searrow 0} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $f(x_0 - h) \ominus f(x_0)$ and the limits

$$\lim_{h\searrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\searrow 0} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h)$, $f(x_0) \ominus f(x_0-h)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0).$$

(division by h and $(-h)$ is understood as the multiplication of a fuzzy number by the scalars $\frac{1}{h}$ and $-\frac{1}{h}$, respectively).

We say that a function is (i)-differentiable if it is differentiable as in the previous Definition 2, (i), etc.

Concerning the existence of solutions of a fuzzy initial value problem under generalized differentiability in [3] we have proved that under some relaxed conditions (for which the reader is asked to consult [3]) the fuzzy initial value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

has two solutions (one (i)-differentiable and the other one (ii)- differentiable) $y, \overline{y} : [x_0, x_0 + r] \to B(y_0, q)$ and the successive iterations

$$y_0(x) = y_0$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt, \tag{1}$$

and

$$\overline{y}_0(x) = y_0$$

$$\overline{y}_{n+1}(x) = y_0 \ominus (-1) \cdot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt \tag{2}$$

converge to these two solutions respectively.

The FDEs will have in the present paper will have input data trapezoidal fuzzy numbers. We recall that for $a < b < c < d$, $a, b, c, d \in \mathbf{R}$, the trapezoidal fuzzy number $u = (a, b, c, d)$ determined by a, b, c and d is given such that $u_-^r = a + (b-a)r$ and $u_+^r = d - (d-c)r$ are the endpoints of the $r-$level sets, for all $r \in [0, 1]$.

3 The Heuristic Fuzzy Model of Friction

In this section we propose a fuzzy differential equation modeling dry friction, model which is similar to the multivalued models in [1], [15] and the fuzzy model in [7]. In our proposed model we will fuzzify the signum function similarly to, etc. but simultaneously we transform it into a continuous term. As a consequence, the signum function will be in our model continuous fuzzy-valued function and the friction force as well. The velocity and position will be solutions of a system of FDEs and so, these are fuzzy terms.

The fuzzy differential equation modeling dry friction is

$$y'' + \alpha y' + \mu \cdot Sgn(y') + y = u(t), \tag{3}$$

where $\alpha, \mu \in \mathbf{R}$ are positive constants, $u(t)$ is a control signal and the signum function $Sgn(y')$ is given by (4) in our model (for simplicity we do not show the parameters ε, δ at each time they occur. The coordinate $y : \mathbf{R} \to \mathbf{R}_F$ is considered to be a trapezoidal fuzzy valued function. The initial conditions are considered to be crisp values.

The trapezoidal-valued signum function is

$$Sgn_{\varepsilon,\delta}(v) = \begin{cases} -1, \text{ if } \overline{v} \leq -\varepsilon \\ (-1, -1+\delta, 1-\delta, 1) \ominus \\ \ominus \left(-\frac{2}{\varepsilon}, -\frac{2+\delta}{\varepsilon}, -\frac{\delta}{\varepsilon}, 0\right) \cdot \overline{v}, \text{ if } |\overline{v}| < \varepsilon \\ 1, \text{ if } \overline{v} > \varepsilon \end{cases}. \tag{4}$$

It is easy to see that

$$\lim_{\varepsilon,\delta \to 0} Sgn_{\varepsilon,\delta}(v) = \begin{cases} -1, \text{ if } \overline{v} < 0 \\ [-1, 1], \text{ if } |\overline{v}| = 0 \\ 1, \text{ if } \overline{v} > 0 \end{cases},$$

which coincides with the interval-valued signum function proposed in [7] (the convergence is understood surely only pointwise).

In order to solve the equation we rewrite it as a system of first order FDEs as follows

$$\begin{cases} y' = v \\ v' = u(t) - \alpha v - \mu \cdot Sgn(v) - y \end{cases},$$

with the initial conditions $y(0) = y_0$ and $v(0) = v_0$. Analogously to the proof of the existence result in [3] a similar theorem can be proved for systems of equations. As a conclusion, the above system may have locally several solutions

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} v_n dt, \text{ or} \tag{5}$$

$$\overline{y}_{n+1}(t) = y_0 \ominus (-1) \int_{t_0}^{t} v_n dt, \tag{6}$$

and

$$v_{n+1}(t) = v_0 + \int_{t_0}^{t} (u(t) - \alpha v_n - \mu \cdot Sgn(v_n) - y_n) dt \text{ or} \tag{7}$$

$$v_{n+1}(t) = v_0 \ominus (-1) \int_{t_0}^{t} (u(t) - \alpha v_n - \mu \cdot Sgn(v_n) - y_n) dt. \tag{8}$$

In order to solve the problem we will employ a numerical method based on the classical Euler method. We consider the approximation given by this method sufficient for our purposes. Surely theoretical study and implementation of more sophisticated methods is subject of future research.

One step of the Euler's method in our case is given by

$$y(t+h) = y(t) + hv(t), \text{ or} \tag{9}$$
$$y(t+h) = y(t) \ominus (-1)hv(t) \tag{10}$$

and

$$v(t+h) = v(t) + h(u(t) - \alpha v(t) - \mu \cdot Sgn(v(t)) - y(t)) \text{ or} \tag{11}$$
$$v(t+h) = v(t) \ominus (-1)h(u(t) - \alpha v(t) - \mu \cdot Sgn(v(t)) - y(t)), \tag{12}$$

$h \in \mathbf{R}$ being the step size.

Since there may exist locally two solutions, if both of them exist we have to chose locally the one which better reflects the behavior of the real-world system modeled by the given equation. The possibility of this choice, allows us to incorporate further assumptions or observations about the behavior of the system.

4 Experimental Results

In the present section we will examine the above proposed model. The lack of uniqueness allows us to introduce is the system additional assumptions and based on these assumptions we chose locally the solution according to a choice function. As a measure of the uncertainty we have used the length of the 0-level set. So, if we say increasing uncertainty we understand increasing length of the 0-level set. Surely several other measures of the uncertainty exist in the literature.

We propose to use and compare experimentally several choice functions in two experimental settings. These are as follows: In the first experimental setting we have put $u_1(t) = \sin(t)$, $\alpha_1 = 1$, $\mu_1 = 0.4$, $\varepsilon_1 = 0.0001$, $\delta_1 = 0.6$ and in the second one

$$u_2(t) = \begin{cases} 5 \text{ if } 0 \le t < 2 \\ -5 \text{ if } 2 \le t < 10 \\ 4 \text{ if } 10 \le t \le 15 \end{cases},$$

$\alpha_2 = 2$, $\mu_2 = 1.4$, $\varepsilon_2 = 0.01$, and $\delta_2 = 0.7$.

In each of the Figures presented in the present paper, the upper graph represents the coordinate, while the lower graph will represents the velocity.

The choice functions which were tested in these experiments are described as follows.

- First is choosing always the "old" Hukuhara differentiable solution. Surely this is the most inconvenient choice, since uncertainty cannot be decreasing decrease under the Hukuhara differentiability concept ([7]). The experimental results show this behavior expected from the theory.

- Second is choosing solutions with increasing support if the "core", i.e. midpoint of the 1-level set is increasing in absolute value (this choice is based on the

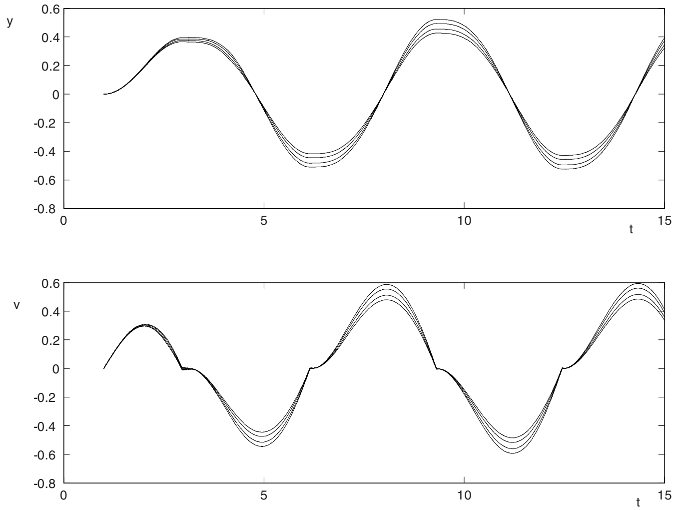

Fig. 1. Solution under the second choice function, first experiment

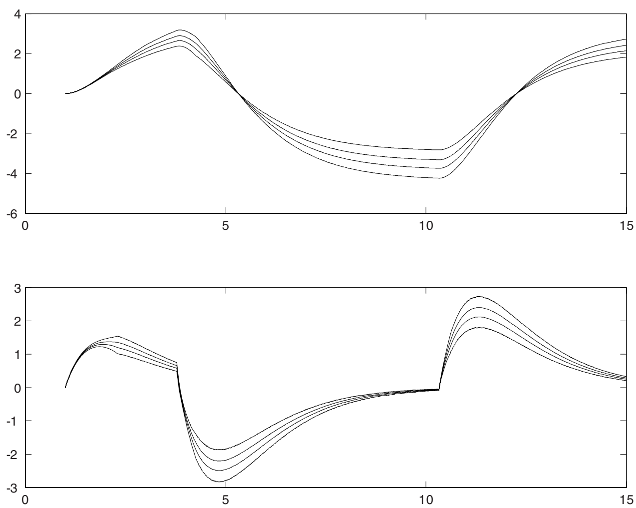

Fig. 2. Solution under the second choice function, second experiment

hypothesis that the uncertainty increases together with the value). In our model this is not consistent with the usual real behavior of the velocity. That is the static friction appears at velocity 0 and in this case around zero the uncertainty should increase (see Figs. 1, 2).

- The last choice is based on the expert opinion that when velocity is small the uncertainty is increasing. According to this choice function we set a threshold value for the velocity, under which we assume that the uncertainty increases.

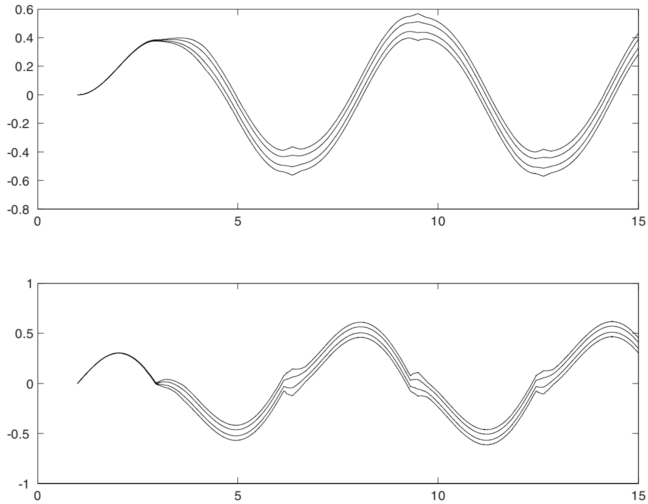

Fig. 3. Solution with the assumption that small velocity implies increasing uncertainty, first experiment

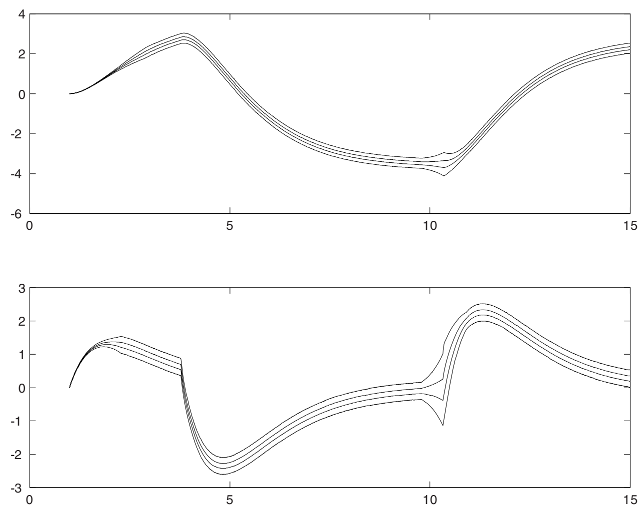

Fig. 4. Solution with the assumption that small velocity implies increasing uncertainty, second experiment

Otherwise we allow uncertainty to decrease. This choice is the most well motivated by the physical properties of the system since the principal source of uncertainty is the interaction at low velocities ([5], [12]). See Figs. 3, 4) for numerical results in this case. Surely an experimental comparison will be necessary in order to decide which choice function reflects better the real phenomena, but this is subject of future research.

5 Concluding Remarks

We have proposed a fuzzy model for dry friction and we have performed numerical experiments on it. Surely a more accurate comparison with the available experimental data and existing models is a subject of further research.

In the numerical experiments proposed in the present paper we have tested several choice functions based on different assumptions. These assumptions were crisp ones in this paper. As a next step in this research, we propose the use of fuzzy rules in the choice functions together with the fuzzy differential equations to build up a fuzzy model with the expert knowledge incorporated.

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