Qualification of Fuzzy Statements Under Fuzzy Certainty

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Abstract. In many problems the information can be imprecise and uncertain simultaneously. Linguistic terms can be then used to represent each one of these aspects. In some applications it is desirable to combine imprecision and uncertainty into a single value which appropriately describes the original information. We propose a method to combine imprecision and uncertainty when they are expressed as trapezoidal fuzzy numbers and the final goal is to obtain a normalized fuzzy number. This property is very useful in several applications like flexible querying processes, where the linguistic label used in the query is always normalized.

1 Introduction

The aim of this work is to propose a solution to the problem of uncertainty qualification of fuzzy statements [4] when the certainty is expressed as a fuzzy number. In a previous paper [6] we proposed a method to solve this problem when the certainty is expressed as a real number. The main idea of this proposal was the following. Let us suppose we have a fuzzy value A understood as acting as a fuzzy restriction on the possible values of a variable X, and this value is affected by a certainty degree, say α. Then, the problem is to represent a qualified statement like "it is α-certain that X is A".

This situation can be formulated as a conditional expression, using the generalized modus ponens, in the following terms:

– *if the certainty level is 1, then the value is A.*
– *if the certainty level is $\alpha < 1$*, then the value is $T(A)$, where $T(A)$ is a transformation of the original fuzzy set A.

In this way, the qualified statements "it is α-certain that X is A" is represented as "X is T(A)".

Therefore, a natural way to solve the problem is to consider that the transformation we are handling is $T(A)$ defined as: $\mu_{T(A)}(x) = I(\alpha, \mu_A(x))$ where I is a material implication function which reflects the interpretation given to the compatibility degree.

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There exist in the literature two main ways of dealing with imprecise and uncertain data and can be interpreted as follows.

1. To truncate: if the datum is "A with certainty α", then $T(A)$ is defined by the membership function $\mu_{T(A)}(x) = min(\alpha, \mu_A(x))$ which directly implies that we are using Mamdani's implication in our reasoning.
2. To expand: if we assume that α is a necessity, then $T(A)$ is given by the membership function $\mu_{T(A)}(x) = max(1 - \alpha, \mu_A(x))$, which corresponds to Kleene-Dienes' implication as foundation of our reasoning.

These proposals can be useful in many applications, but they can also be inappropriate in many others. Thus, Mamdani's implication obliges us to work with non-normalized fuzzy values. Kleene-Dienes' implication obliges to assign the same possibility to all the points of the underlying domain independently from the distance to the support set of the fuzzy value. Therefore, the proposed solutions give rise to a series of inconveniences: the interpretability, in some cases, and those ones derived from the use of non-normalized or non-trapezoidal fuzzy sets.

As an alternative proposal, in [8,6] we proposed a certainty qualification method that consists in increasing the imprecision around the support set of value A depending on an uncertainty value, that is, the imprecision is distributed according to a metric which takes into account the nearness to the original information. This proposal is based on the use of information measures that allow us to transform the uncertainty of the fuzzy statement into imprecision. For example, when we have the information that *"X is black"* with certainty α, it is not very convenient to assign a positive possibility to color white but to colors near enough to black depending on value α.

Therefore, the process we proposed in [6] was to define $T(A)$ in two steps:

1. First, by considering that the height of a fuzzy number is the certainty degree associated to it [2,5], we use the certainty degree α associated to the fuzzy value A to truncate it at level α. After this operation, we obtain a non-normalized fuzzy set A^α. Nevertheless, the resulting fuzzy value remains trapezoidal.
2. Since, in many applications, non-normalized fuzzy sets give rise to a series of inconveniences, in a second step we normalize it. To do this, we assume that uncertainty is being translated into imprecision under certain conditions. The most important point to be considered is that the amount of information provided by the fuzzy number remains equal before and after the normalization process. $T_\alpha(A)$ will stand for the obtained normalized fuzzy value, whose imprecision is, obviously, larger than A^α imprecision, as it has been made completely true (its height is 1 again).

In fact, in the fuzzy querying process the linguistic labels used are always normalized what makes it necessary that the stored data are also normalized in order to carry out a semantically coherent matching computation.

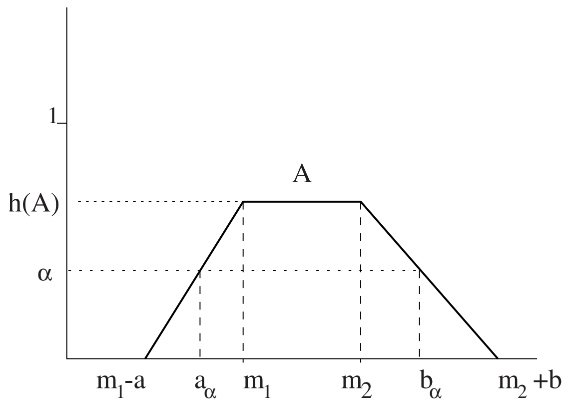

Fig. 1. Trapezoidal Fuzzy Number

2 Previous Results

A fuzzy value is a fuzzy representation about the real value of a property when it is not precisely known. We will use $\widetilde{I\!R}$ to denote the set of fuzzy numbers.

The interval $[a_\alpha, b_\alpha]$ (see figure 1) is called the α-cut of A. Therefore, fuzzy numbers are fuzzy quantities whose α-cuts are closed and bounded intervals: $A_\alpha = [a_\alpha, b_\alpha]$ with $\alpha \in (0, 1]$. The set $Supp(A) = \{x \in I\!R \mid A(x) > 0\}$ is called the *support set of* A[1] and $h(A)$ denotes the height of the fuzzy number A. If there is, at least, one point x verifying $A(x) = 1$ we say that A is a *normalized* fuzzy number.

Usually, a trapezoidal shape is used in order to represent fuzzy numbers. This representation is very useful as the fuzzy number is completely characterized by five parameters (m_1, m_2, a, b) and the height $h(A)$, as figure 1 shows. The interval $[m_1, m_2]$ (i.e, the set $\{x \in Supp(A) \mid \forall\, y \in I\!R,\; A(x) \geq A(y)\}$) will be called *modal set*. The values a and b are called left and right *spreads*, respectively.

The basic idea underlying this work is that when a fuzzy number is not normalized, the situation can be interpreted as a lack of confidence in the information provided by such a number [2,5]. In fact, the height of the fuzzy number could be considered as a certainty degree of the represented value, and this implies that normalized fuzzy numbers represent imprecise quantities on which we have complete certainty.

Since the first step in our proposal is to truncate, we can consider that the truncated fuzzy number represents the imprecise information and moreover it shows a certain level of uncertainty.

In [6], we show how uncertainty can be translated, using a suitable transformation, into imprecision, taking into account that to reduce the uncertainty about a fuzzy number implies to increase the imprecision of such number. This

[1] In the rest of the paper $A(x)$ will stand for $\mu_A(x)$.

transformation is made in such a way that the amount of information provided by the fuzzy number is the same before and after the modification.

Our idea is to transform the truncated fuzzy number in order to obtain a completely certain fuzzy number.

As pointed out in the previous section, we are going to translate fuzzy uncertainty into imprecision under given conditions. The most important of these conditions is that the amount of information provided by the fuzzy number remains equal before and after the transformation. Therefore, the first step is to define an information function for fuzzy numbers.

In [6], we propose an axiomatic definition of information, partially inspired in the theory of generalized information given by Kampé de Fériet [7] and that can be related to the precision indexes [3] and the specificity concept introduced by Yager in [11].

Definition 1. *Let* $\mathcal{D} \subseteq \widetilde{I\!R} \mid I\!R \subseteq \mathcal{D}$*; we say that* $I : \mathcal{D} \longrightarrow [0,1]$ *is an* **information function** *on* $\mathcal{D}$ *if it verifies:*

1. $I(A) = 1, \; \forall \; A \in I\!R$
2. $\forall \; A, B \in \mathcal{D} \;\mid\; h(A) = h(B)$ *and* $A \subseteq B \Longrightarrow I(B) \leq I(A)$.

The information about fuzzy numbers may depend on different factors, in particular, on imprecision and certainty. In this work, we focus on general types of information related only to these two factors.

Definition 2. *The* **imprecision** *[5] of a fuzzy number is defined as follows:*

$$\forall \; A \in \widetilde{I\!R}, \; imp(A) = \int_0^{h(A)} (b_\alpha - a_\alpha \;) d\alpha$$

With respect to the height (certainty) and the imprecision of a fuzzy value, we define the following general type of function [5]:

$$\forall \; A \in \widetilde{I\!R}, \; I(A) = \frac{h(A)}{k * imp(A) + 1}$$

where $h(A)$ is A height, imp(A) is the imprecision associated to A and $k \neq 0$ is a parameter which depends on the domain scale. This is the simplest function that verifies the mentioned properties of information functions.

Once we have an information function on fuzzy numbers, we can use it to define transformations which preserve the information amount it provides. The idea is to find an *equivalent* representation of the considered fuzzy number in such a way that we change uncertainty by imprecision keeping constant the relationship between them, which is determined by the information function.

The aim of the transformations we are proposing in this section is, basically, to be able to modify the height of a fuzzy number but keeping the information contained in it.

The definition of transformation will be obtained from the condition of equality in the information but, as a first step, we must establish what we understand for transformation of a fuzzy number on a subset of $\widetilde{I\!R}$.

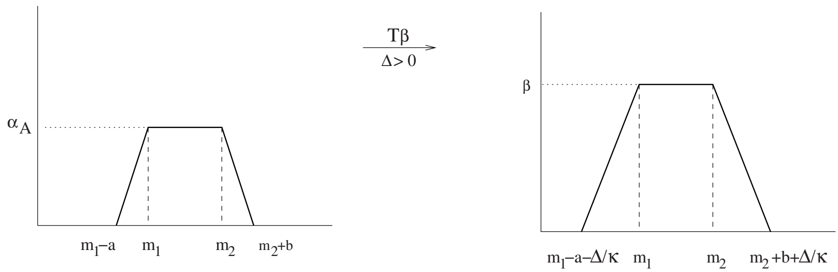

Fig. 2. Transformation that increases imprecision

Definition 3. *Let us consider $\alpha \in (0, 1]$ and the class of fuzzy numbers $D \subseteq \widetilde{\mathbb{R}}$. We say that*

$$T_\alpha : D \longrightarrow \widetilde{\mathbb{R}}$$

is a **transformation** *for an information function I on D, if it verifies that:*

1. $T_\alpha(A) \in D$
2. $h(T_\alpha(A)) = \alpha$
3. $I(T_\alpha(A)) = I(A), \ \forall \ A \in D$

We will note by τ the class of trapezoidal fuzzy numbers on $\mathbb{R}$. Given a fuzzy number $A \in \tau$, we are looking for the conditions that another fuzzy number B, with fixed height $\alpha \in (0, 1]$, must hold to have the same information amount as A. Assuming the following conditions:

1. modal imprecision is preserved,
2. the increase/decrease of imprecision is equally distributed in the right and left sides of the fuzzy number independently from its shape,

we proposed in [6] the following transformation:

Definition 4. *Let $A \in \tau$ such that*

$$A = \{(m_1, m_2, a, b), \alpha_A\}$$

where m_1, m_2, a and b are shown in figure 1 and α_A is the height of A.

Let $\alpha \in (0, 1]$ be. We will denote $\Delta(\alpha_A, \alpha) = \Delta$ and define

$$T_\alpha(A) = \{(m_1, m_2, a + \frac{\Delta}{k}, b + \frac{\Delta}{k}), \alpha\}$$

for those α in which the transformation makes sense.

In figure 2 it is shown how an increment of height produces an increment of imprecision.

In the proposed transformation, the relation between certainty and imprecision is the following:

– An increase of certainty means an increase of imprecision.
– A decrease of imprecision means a decrease of certainty.

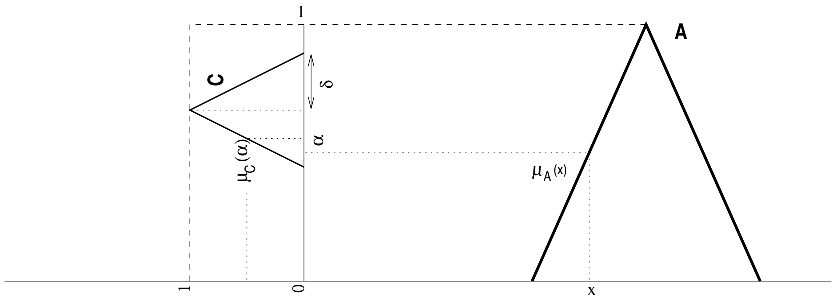

Fig. 3. Fuzzy Certainty on a Fuzzy Value: The scale used in both axels is not the same for the sake of clarity

3 Fuzzy Uncertainty

Once, we know how to solve the qualification problem when the uncertainty is represented as a real value, now the problem is to extend this process when the uncertainty is represented as a fuzzy value. Thus, we want to translate the information "X is A is C", when C is a fuzzy trapezoidal number, into "X is $T_C(A)$".

The difficulty is now to give a suitable procedure for computing $T_C(A)$ since now C is a trapezoidal fuzzy number. To do this, we will consider that, for any possible truncation level α, the membership function of the linguistic label modifies in a certain way the certainty level. In fact we can assume that:

$$(X\ is\ A)\ is\ C \longleftrightarrow \forall\alpha \in [0,1],\ X\ is\ A\ to\ a\ degree\ C(\alpha),\ \alpha \in [0,1]$$

Figure 3 depicts the general problem we are trying to explain.

A possible way to solve this problem is to define $T_C(A)$ in such a way that it summarizes the right side of the above sentence by means of some average. It should be remarked that the membership function $C(.)$ induces two fuzzy measures (possibility/necessity) on the [0,1] interval and that the membership function of any fuzzy number transformed at certainty level α can be considered as a function depending on both $\alpha \in [0,1]$ and x, which ranges on another real interval. A method that allows the use of such average is the Sugeno's integral.

In [10], Sugeno introduced the concept of fuzzy integral of a fuzzy measure as a way to compute some kind of average value of a function in terms of the underlying fuzzy measure. Obviously, fuzzy measures formally include possibility/necessity measures as special cases. Fuzzy integrals are interpreted as subjective evaluations of objects where subjectivity is represented by means of fuzzy measures.

The fuzzy integral over a referential set X of a function $f(x)$ with respect to a fuzzy measure g is defined as follows:

$$\int_X f(x) \circ g(.) = sup_{\alpha\in[0,1]}\{\alpha \wedge g(F_\alpha)\}$$

where $F_\alpha = \{x | f(x) \geq \alpha\}$.

In the case that the measure g is a possibility defined by means of the membership function of a fuzzy set $\mu(x)$ with referential X, the Sugeno's integral has the following expression [9]:

$$\int_X f(x) \circ g(.) = sup_{x \in X}(f(x) \wedge \mu(x)).$$

On the other hand, if we assume the considered fuzzy measure g is a necessity induced by the fuzzy set $\mu(x)$, then we have the following expression [9]:

$$\int_X f(x) \circ g(.) = inf_{x \in X}(f(x) \vee (1 - \mu(x))).$$

As we have stated above, the basic idea of our approaches is to use the fuzzy measures (possibility, necessity) induced by the membership function $C(.)$ of the linguistic evaluation of certainty, to compute the *average* of the transformed fuzzy number, by means of Sugeno's integral.

At this point, it is necessary to remark that the transformation process of any fuzzy number $A(.)$ with *crisp certainty* value α has two steps:

(i) Truncating the fuzzy number at the level α, obtaining an non-normalized fuzzy number $A^\alpha(.)$.
(ii) Transforming $A^\alpha(.)$ into a normalized fuzzy number $T(A)$.

The idea is the following. In a first step, we apply the Sugeno's integral to the function $f(\alpha, x) = A^\alpha(x)$ with respect to the α variable, obtaining a possibly non-normalized fuzzy number. This fuzzy number will be transformed into a normalized one in the step ii. This process can be done in two different ways depending on wether we use the possibility or the necessity measures to perform the integral.

Thus, let $\Pi_C(.)$ stand for the possibility measure induced by C and $T_p(.)$ stand for the mean of the truncated fuzzy numbers. Then we have:

$$T_p(x) = \int_{[0,1]} A^\alpha(x) \circ \Pi_C(\alpha) = sup_{\alpha \in [0,1]}(A^\alpha(x) \wedge C(\alpha)) =$$
$$= sup_{\alpha \in [0,1]}(A(x) \wedge \alpha \wedge C(\alpha)) = A(x) \wedge sup_{\alpha \in [0,1]}(\alpha \wedge C(\alpha))$$

If $C_p = sup_{\alpha \in [0,1]}(\alpha \wedge C(\alpha))$, then we finally have:

$$T_p(x) = A(x) \wedge C_p$$

which indicates that, in the case of the possibility measure, the mean of truncated values is the result of truncating with an specific value which only depends on the linguistic label $C(.)$.

Alternatively, let $N_C(.)$ stand for the necessity measure induced by C and $T_n(.)$ stand for the mean of the truncated fuzzy numbers. Using expression in section 3, we have:

$$T_n(x) = \int_{[0,1]} A^\alpha(x) \circ N_C(\alpha) = inf_{\alpha \in [0,1]}(A^\alpha(x) \vee (1 - C(\alpha))) =$$
$$= inf_{\alpha \in [0,1]}(A(x) \wedge \alpha \vee (1 - C(\alpha))) = A(x) \wedge inf_{\alpha \in [0,1]}(\alpha \vee (1 - C(\alpha)))$$

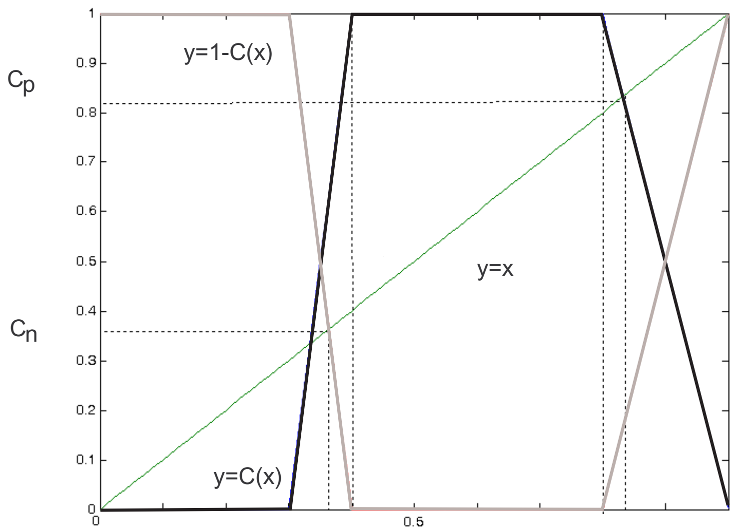

Fig. 4. Upper and lower measures

If $C_n = inf_{\alpha\in[0,1]}(\alpha \vee (1 - C(\alpha)))$, then we finally have:

$$T_n(x) = A(x) \wedge C_n$$

which indicates that, also in the case of the necessity measure, the mean of truncated values is the result of truncating with an specific value which only depends on the linguistic label $C(.)$

With the previous expression we have got two proposals for making this truncation or, what is the same, we obtain two different fuzzy values $T_p(x)$ and $T_n(x)$. In this first step we have integrated the fuzzy uncertainty C in the truncation process.

As it happens with all dual measures, the expert can choose either to work with both of them or to decide which one is the most suitable for the purpose of the system. In figure 4 we graphically show the results obtained considering that the linguistic label C has a trapezoidal membership function.

After the truncation, it is necessary to perform the corresponding transformations in order to obtain a normalized fuzzy number. $T_N(.), T_P(.)$ will stand for the transformed $T_n(.)$ and $T_p(.)$, respectively. They can be directly obtained by the process described in section 2.

Moreover, we can conclude that $T_N(.)$ offers us a more imprecise transformed fuzzy number that $T_P(.)$ since

$$T_P(.) \subseteq T_N(.).$$

4 Conclusions

We have addressed the problem of dealing with linguistic uncertainty associated with a fuzzy quantity. With the basic idea of transforming uncertainty into imprecision, two possible approaches have been presented; all of them give transformations of the initial fuzzy number that lead to normalized fuzzy numbers. Explicit expressions of such transformed fuzzy numbers have also been obtained. This is a particularly useful property from the storage point of view (e.g. within the databases world or in a data warehousing context), since it provides us with a simple and unified representation for both certain and uncertain fuzzy values.

References

1. Bordogna G., Pasi G. *Modeling Linguistic Qualifiers of Uncertainty in a Fuzzy Database*. Int. Journ. of Intelligent Systems, Vol. 15, pp. 995-1014. (2000)
2. Dubois, D. *Modeles mathematiques de l'imprecis et de l'incertain en vue d'applications aux techniques d'aide a la decision*. Ph.D Thesis. Université Scientifique et Médicale de Grenoble. (1983)
3. Dubois D., Prade H. *Possibility Theory*. Plenum Press. (1985)
4. Dubois D., Prade H., Yager R. *Merging Fuzzy Information*. In *Fuzzy Sets in Approximate Reasoning and Information Systems*, pp- 335-401. Eds. J.C. Bezdek, D. Dubois, H. Prade. Kluwer Academic Publishers. (1999)
5. Gonzalez, A. *Métodos Subjetivos para la Comparación de Números Difusos*. PhD Thesis. Granada University. (1987)
6. Gonzalez A., Pons O., Vila M.A. *Dealing with uncertainty and imprecision by means of fuzzy numbers*. Int. Journal of Approximate Reasoning, vol. 21, pp. 233-256. (1999)
7. Kampé de Feriét, J. *La theorie generaliseé de l'information et le mesure subjetive de l'information*. Lecture notes in Math: Théories de l'information. Eds. Kampé de Fèriet and C.F. Picard. Springer-Verlag, vol. 398, pp. 1-28. (1973)
8. Pons O., Cubero J.C., Gonzalez A., Vila M.A. *Uncertain Fuzzy Values still in the Framework of First Order Logic*. Proc. of Second Int. Workshop on Logic Programming and Soft Computing: Theory and Applications. Manchester (U.K.). (1989)
9. Suárez García F. and Gil Alvarez P., *Two families of fuzzy integrals*. Fuzzy Sets Syst., vol. 18 (1), pp. 67–81. (1986)
10. Sugeno M. *Theory of fuzzy integrals and its applications*. Thesis. Tokio Inst. of Technology. (1974)
11. Yager, R.R. *Measurement of Properties on Fuzzy Sets and Possibility Distributions*. E.P. Klement Ed. Proc. 3rd Intern. Seminar on Fuzzy Set Theory. Johannes Keplas Univ. Liuz, pp. 211-222. (1981)