# The Role of Entropy in Intuitionistic Fuzzy Contrast Enhancement

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**Abstract.** In this paper we study the impact of selecting different entropy measures in the framework of intuitionistic fuzzy image processing and especially in the process of intuitionistic fuzzification of images. Different notions of entropy characterized by different properties are reviewed and their behavior is thoroughly studied under the scope of performing contrast enhancement. Finally, experimental results using gray-scale images reveal the characteristics of the aforementioned measures.

## 1 Introduction

Entropy, a measure of information carried by a system, is a fundamental concept in digital image processing. Therefore, it is not surprising that new theories, such as fuzzy sets (FSs) theory, as well as their extensions, seek intuitive ways to adopt and express the notion of entropy in their particular context.

Intuitionistic fuzzy image processing (IFIP), recently introduced in [1] and [2], provides a flexible, yet solid, mathematical framework for dealing with the vagueness present in a digital image. This is carried out by modelling the hesitancy characterizing image pixels, using Atanassov's intuitionistic fuzzy sets (A–IFSs) theory [3,4]. A–IFSs constitute a generalization of Zadeh's fuzzy sets (FSs) [5], by considering also corresponding degrees of hesitancy. It is this additional degree of freedom that allows for the flexible modelling of imprecise or/and imperfect information often present in images.

In this paper the different concepts of intuitionistic fuzzy entropy are reviewed and their behavior is studied in the context of IFIP for performing contrast enhancement. Evaluation of these measures using real-world images reveal their particular characteristics that are to be exploited for different applications of contrast enhancement in the context of IFIP.

This paper is organized as follows. In Sect. 2 elements of A–IFSs are presented and their geometrical representation of in two- and three-dimensional spaces is discussed. Sect. 3 reviews different concepts of entropy in the intuitionistic fuzzy setting. An overview of the IFIP framework is presented in Sect. 4. Finally, experimental results are given in Sect. 5, while conclusions are drawn in Sect. 6.

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## 2 Intuitionistic Fuzzy Sets

In this section, we briefly review the basic notions, concepts, and definitions of A–IFSs, as well as their geometrical representations in two- and three-dimensional spaces.

### 2.1 Elements of Intuitionistic Fuzzy Sets Theory

**Definition 1.** *An FS $\tilde{A}$ defined on a universe $X$ may be given as [5]*

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x)\rangle | x \in X\} \ , \tag{1}$$

*where $\mu_{\tilde{A}}(x) : X \to [0, 1]$ is the membership function of $\tilde{A}$.*

The membership function of $\tilde{A}$ describes the *degree of belongingness* of $x \in X$ in $\tilde{A}$.

**Definition 2.** *An A–IFS $A$ defined on a universe $X$ is given by [3,4]*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \ , \tag{2}$$

*where*

$$\mu_A(x) : X \to [0, 1] \qquad \textit{and} \qquad \nu_A(x) : X \to [0, 1] \ ,$$

*with the condition*

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1 \ , \tag{3}$$

*for all $x \in X$.*

The values of $\mu_A(x)$ and $\nu_A(x)$ denote the *degree of belongingness* and the *degree of non-belongingness* of $x$ to $A$, respectively. For an A–IFS $A$ in $X$ we call the *intuitionistic index* of an element $x \in X$ in $A$ the following expression

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \ . \tag{4}$$

We can consider $\pi_A(x)$ as a *hesitancy degree* of $x$ to $A$ [3,4]. From (4) it is evident that

$$0 \leqslant \pi_A(x) \leqslant 1 \tag{5}$$

for all $x \in X$.

FSs can also be represented using the notation of A–IFSs. An FS $\tilde{A}$ defined on $X$ can be represented as the following A–IFS

$$A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in X\} \ , \tag{6}$$

with $\pi_A(x) = 0$ for all $x \in X$.

**Definition 3.** *The complementary set $A^c$ of $A$ is defined as*

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\} \ . \tag{7}$$

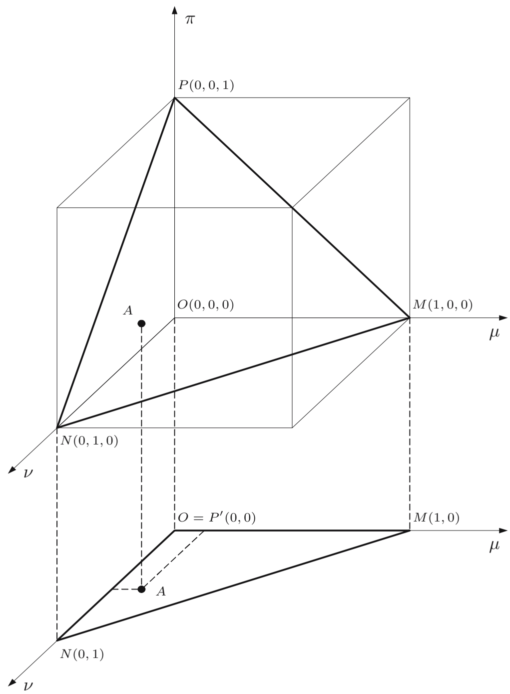


**Fig. 1.** Geometrical representation of an A–IFS $A$ in $X = \{x\}$. (Adopted from [6].)

Throughout this paper by $\mathscr{IFS}(X)$ we denote the set of all A–IFSs defined on $X$. Correspondingly, $\mathscr{FS}(X)$ is the set of all FSs on $X$, while $2^X$ denotes the set of all crisp sets.

Finally, Atanassov [4] proposed an operator, namely the *Atanassov's operator*, to de-construct an A–IFS into an FS.

**Definition 4.** *If* $A \in \mathscr{IFS}(X)$*, then* $D_\alpha : \mathscr{IFS}(X) \to \mathscr{FS}(X)$*, where*

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle | x \in X\} \ , \tag{8}$$

*with* $\alpha \in [0,1]$.

## 2.2 Geometrical Representation of A–IFSs

Generalizing Kosko's [7] geometrical representation of FSs, Atanassov [4] proposed a similar interpretation of A–IFSs in a Euclidean plane with Cartesian coordinates. Szmidt and Kacprzyk [8] extended Atanassov's approach by considering all three parameters of A–IFSs and proposed the geometrical representation of A–IFSs as a mapping into a simplex in the unit cube. Moreover, they demonstrated that Atanassov's interpretation is simply the orthogonal projection of the simplex of their definition into the Euclidean plane. Both representations are illustrated in Fig. 1 for an A–IFS $A$ in $X = \{x\}$.

## 3 Notions of Entropy in the Intuitionistic Fuzzy Setting

Entropy plays an important role in digital image processing. Therefore, it comes as no surprise that the notion of entropy constituted a fundamental aspect from the beginning of the development of FSs theory. De Luca and Termini [9] were the first to introduce an axiomatic skeleton of a nonprobabilistic entropy in the setting of FSs theory that captured our intuition regarding the very essence of fuzzy entropy.

As a natural consequence, the quest for entropy measures in the context of A–IFSs was a very interesting topic that intrigued many researchers working in this field. Burillo and Bustince [10] were the first to state and propose an axiomatic skeleton of entropy for A–IFSs and interval-valued fuzzy sets.

**Definition 5 (Burillo and Bustince [10]).** *A real function $E : \mathscr{IFS}(X) \to \mathbb{R}^+$ is called an entropy on $\mathscr{IFS}(X)$, if $E$ has the following properties*

**(E1)** $E(A) = 0$ *if and only if* $A \in \mathscr{FS}(X)$,
**(E2)** $E(A) = Cardinal(X)$ *if and only if* $\mu_A(x) = \nu_A(x) = 0$ *for all* $x \in X$,
**(E3)** $E(A) = E(A^c)$ *for all* $A \in \mathscr{IFS}(X)$,
**(E4)** $E(A) \geqslant E(B)$ *if* $\mu_A(x) \leqslant \mu_B(x)$ *and* $\nu_A(x) \leqslant \nu_B(x)$ *for all* $x \in X$.

Motivated by De Luca and Termini's set of axiomatic requirements, Szmidt and Kacprzyk [8] proposed an alternative interpretation of entropy, accompanied by a different set of axioms.

**Definition 6 (Szmidt and Kacprzyk [8]).** *A real function $E' : \mathscr{IFS}(X) \to \mathbb{R}^+$ is called an entropy on $\mathscr{IFS}(X)$, if $E$ has the following properties*

**(E5)** $E'(A) = 0$ *if and only if* $A \in 2^X$,
**(E6)** $E'(A) = 1$ *if and only if* $\mu_A(x) = \nu_A(x)$ *for all* $x \in X$,
**(E7)** $E'(A) = E'(A^c)$ *for all* $A \in \mathscr{IFS}(X)$,
**(E8)** $E'(A) \leqslant E'(B)$ *if*
$\mu_A(x) \leqslant \mu_B(x)$ *and* $\nu_A(x) \geqslant \nu_B(x)$ *for* $\mu_B(x) \leqslant \nu_B(x)$
*or*
$\mu_A(x) \geqslant \mu_B(x)$ *and* $\nu_A(x) \leqslant \nu_B(x)$ *for* $\mu_B(x) \geqslant \nu_B(x)$ *for all* $x \in X$.

The aforementioned definition degenerates to De Luca and Termini's definition when FSs are considered. A generalized framework of Definition 6 was introduced in [11]. Finally, it should be mentioned that a connection between the different concepts of entropy for A–IFSs was explored and proved in [12].

### 3.1 Review of Intuitionistic Fuzzy Entropy Measures

Based on the aforementioned notions and definitions of entropy, different entropy measures for A–IFSs were proposed in the literature. Along with their definition of intuitionistic fuzzy entropy, Burillo and Bustince [10] proposed the following entropy

$$E_1(A) = \frac{1}{n}\sum_{i=1}^{n} \pi_A(x_i) \ , \tag{9}$$

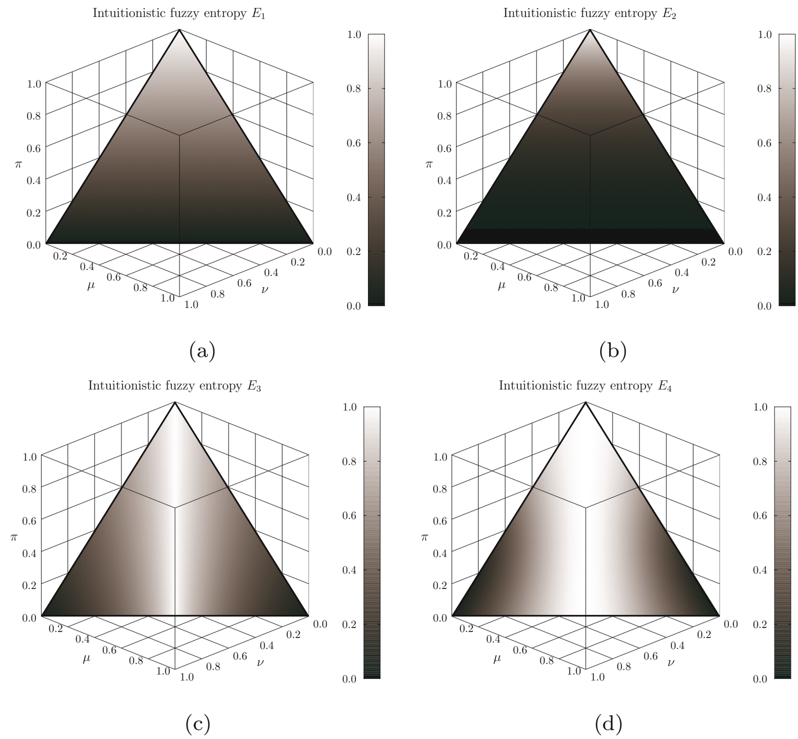


**Fig. 2.** Plots of entropy measures (a) $E_1$, (b) $E_2$, (c) $E_3$, and (d) $E_4$ for an A–IFS defined in $X = \{x\}$

which satisfies the axiomatic requirements **E1**–**E4** and expresses the *degree of intuitionism* of the set $A$. In (9), the normalization factor $\frac{1}{n}$ has been added, in order for the entropy $E_1$ to lie in the $[0, 1]$ interval. Additionally, Burillo and Bustince proposed an alternative entropy measure given by

$$E_2(A) = \frac{1}{n} \sum_{i=1}^{n} \left(1 - (\mu_A(x_i) + \nu_A(x_i))\, e^{1-(\mu_A(x_i)+\nu_A(x_i))}\right) . \tag{10}$$

The first measure of entropy satisfying axioms **E5**–**E8** was introduced by Szmidt and Kacprzyk [8] as a ratio of distances between an A–IFS and its nearest and farthest crisp sets, respectively. The aforementioned entropy is given by

$$E_3(A) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\max Count\,(A_i \cap A_i^c)}{\max Count\,(A_i \cup A_i^c)} \right) , \tag{11}$$

where $\max Count$ is the *biggest cardinality* of an A–IFS calculated using the following formula

$$\max Count(A) = \sum_{i=1}^{n} (\mu_A(x_i) + \pi_A(x_i)) \tag{12}$$

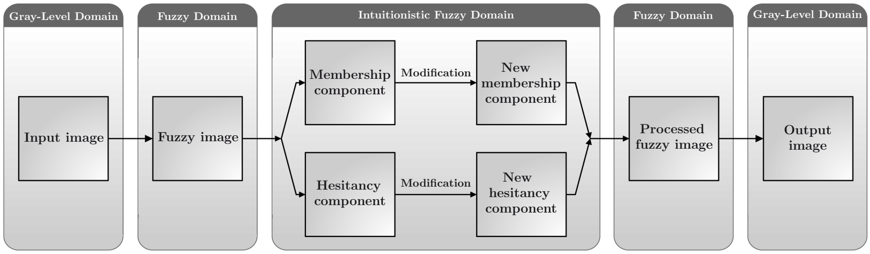


**Fig. 3.** Overview of the IFIP framework

and $A_i$ denotes the single-element A–IFS corresponding to the $i$-the element $x_i$ of the universe $X$, described as $A_i = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle\}$; i.e. $A_i$ is the $i$-th "component" of $A$.

An inner product-based entropy measure, also satisfying the axiomatic requirements of **E5**–**E8**, was introduced in [13] and is given by

$$E_4(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{2\mu_A(x_i)\nu_A(x_i) + \pi_A^2(x_i)}{\pi_A^2(x_i) + \mu_A^2(x_i) + \nu_A^2(x_i)} \ . \tag{13}$$

Fig. 2 illustrates the aforementioned entropy measures using the geometrical representation of A–IFSs described in Sect. 2.2 for a singe-element universe $X = \{x\}$. The gray level of each point $(\mu_A(x), \nu_A(x), \pi_A(x))$ on the simplex denotes the entropy value of the set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ corresponding to that point.

Finally, the entropy measures listed in this section, satisfying and expressing different concepts of intuitionistic fuzzy entropy, will be evaluated in order to assess their behavior, under the scope of performing contrast enhancement, using the IFIP framework.

# 4 From Images to A–IFSs: Entropy Optimization

## 4.1 The Intuitionistic Fuzzy Image Processing Framework

Intuitionistic fuzzy image processing (IFIP) [1,2], involves in general a set of operations carried out using the concepts and elements of A–IFSs theory. Fig. 3 shows an overview of the IFIP framework. In the first stage the image is transferred into the fuzzy domain and sequentially into the intuitionistic fuzzy domain, where the main processing is performed. The inverse procedure is carried out in order to obtain the processed image in the gray-level domain. In this paper we focus on the role of intuitionistic fuzzy entropy measures in the stage of analyzing the image into its intuitionistic fuzzy components; i.e. the stage of "*intuitionistic fuzzification*".

## 4.2 Intuitionistic Fuzzification

In [2], an intuitionistic fuzzification scheme for constructing the A–IFS corresponding to a gray-scale image was proposed, based on the optimization of its intuitionistic fuzzy entropy. In this section, we briefly describe the aforementioned approach.

Let us consider an image $A$ of size $M \times N$ pixels having $L$ gray levels $g$ ranging between 0 and $L-1$. The image can be considered as an array of fuzzy singletons [14,15], with each element of the array denoting the membership value of the corresponding pixel, with respect to an image property. For the task of contrast enhancement we consider the property "*brightness*" of the intensity levels. Therefore, the image in the fuzzy domain can be represented as the FS

$$\tilde{A} = \{\langle g_{ij}, \mu_{\tilde{A}}(g_{ij})\rangle | g_{ij} \in \{0, \ldots, L-1\}\} \ , \tag{14}$$

with $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$.

A basic procedure of IFIP is the derivation of a combination of membership and non-membership functions that model the gray levels of the image in an optimal way. The optimality is considered under the scope of maximizing the intuitionistic fuzzy entropy of the image and thus it is called "*maximum intuitionistic fuzzy entropy principle*" [2]. The family of parametric membership and non-membership functions, used for optimization, is given respectively by

$$\mu_A(g; \lambda) = 1 - (1 - \mu_{\tilde{A}}(g))^{\lambda} \tag{15}$$

and

$$\nu_A(g; \lambda) = (1 - \mu_{\tilde{A}}(g))^{\lambda(\lambda+1)} \ , \tag{16}$$

with $\lambda \geqslant 0$, where the membership function $\mu_{\tilde{A}}(g)$ of the fuzzified image is given by

$$\mu_{\tilde{A}}(g) = \frac{g - g_{min}}{g_{max} - g_{min}} \ . \tag{17}$$

Moreover, the optimization criterion involved can be formulated as follows

$$\lambda_{opt} = \arg \max_{\lambda \geqslant 1} \{E(A; \lambda)\} \ , \tag{18}$$

where $E$ is an entropy measure.

After obtaining the optimal parameter $\lambda_{opt}$, the image is represented as the following A–IFS

$$A_{opt} = \{\langle g, \mu_A(g; \lambda_{opt}), \nu_A(g; \lambda_{opt})\rangle | g \in \{0, \ldots, L-1\}\} \ . \tag{19}$$

By applying Atanassov's operator to the A–IFS $A_{opt}$, we obtain different representations of the image in the fuzzy domain, depending on the parameter $\alpha$ selected. The "*maximum index of fuzziness intuitionistic defuzzification*" procedure was proposed in [12] for selecting the optimal parameter $\alpha_{opt}$, according to the following scheme

$$\alpha_{opt} = \begin{cases} 0 \ , & \text{if } \alpha'_{opt} < 0 \\ \alpha'_{opt} \ , & \text{if } 0 \leqslant \alpha'_{opt} \leqslant 1 \\ 1 \ , & \text{if } \alpha'_{opt} > 1 \end{cases} \ , \tag{20}$$

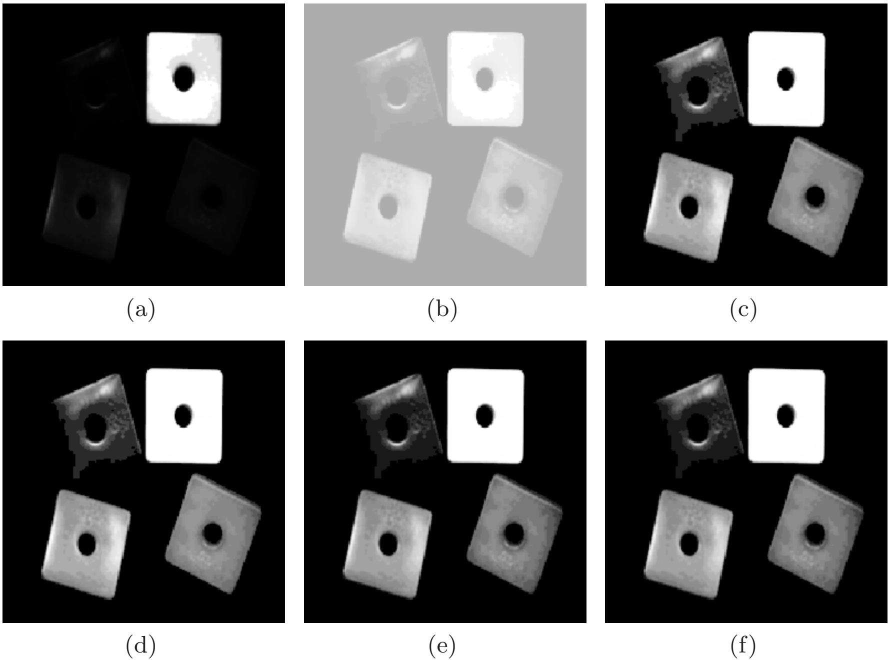

**Fig. 4.** (a) Under-exposed gray-scale image and images obtained using (b) the histogram equalization technique and the IFIP framework employing entropy (c) $E_1$ ($\lambda_{opt} = 6.71$), (d) $E_2$ ($\lambda_{opt} = 6.87$), (e) $E_3$ ($\lambda_{opt} = 11.30$), and (f) $E_4$ ($\lambda_{opt} = 11.05$)

where

$$\alpha'_{opt} = \frac{\sum_{g=0}^{L-1} h_{\tilde{A}}(g)\pi_A(g;\lambda_{opt})\left(1 - 2\mu_A(g;\lambda_{opt})\right)}{2\sum_{g=0}^{L-1} h_{\tilde{A}}(g)\pi_A^2(g;\lambda_{opt})} , \tag{21}$$

with $h_{\tilde{A}}$ being the histogram of the fuzzified image $\tilde{A}$.

Finally, the image in the gray-level domain is obtained as

$$g' = (L-1)\mu_{D_{\alpha_{opt}}(A_{opt})}(g) , \tag{22}$$

where

$$\mu_{D_{\alpha_{opt}}(A_{opt})}(g) = \alpha_{opt} + (1 - \alpha_{opt})\mu_A(g;\lambda_{opt}) - \alpha_{opt}\nu_A(g;\lambda_{opt}) , \tag{23}$$

and $g'$, $g$ are the new and initial intensity levels, respectively.

## 5 Experimental Results

The main purpose of this work is to explore the role of intuitionistic fuzzy entropy in the process of intuitionistic fuzzification of images. Therefore, the aforementioned intuitionistic fuzzy entropy measures, were applied to low-contrasted images in order to perform contrast enhancement.

**Fig. 5.** (a) Over-exposed gray-scale image and images obtained using (b) the histogram equalization technique and the IFIP framework employing entropy (c) $E_1$ ($\lambda_{opt} = 1.60$), (d) $E_2$ ($\lambda_{opt} = 2.72$), (e) $E_3$ ($\lambda_{opt} = 0.59$), and (f) $E_4$ ($\lambda_{opt} = 0.45$)

Figs. 4(a) and 4(b) depict an under-exposed image along with its histogram equalized version. Figs. 4(c)–4(f) illustrate images processed using the IFIP framework employing the intuitionistic fuzzy entropy measures $E_1$, $E_2$, $E_3$, and $E_4$, respectively. One may observe that the images obtained using the IFIP framework have been drastically enhanced, revealing high-frequency edges and constant-intensity regions initially not visible due to the low contrast. Moreover, employing the entropy measures $E_1$ and $E_2$, results in a more radical enhancement of the initial image, with $E_2$ exhibiting a slightly better performance. Compared to the histogram-equalized image of Fig. 4(b) the IFIP framework delivers better results for contrast enhancement.

On the other hand, for the over-exposed image of Fig. 5 one may observe that even though entropies $E_1$ and $E_2$ enhance the initial image, the results are not satisfactory compared to the ones obtained using entropy measures $E_3$ and $E_4$ or to the image derived by the histogram equalization technique. However, the IFIP framework equipped with the entropies $E_3$ and $E_4$ yields images exhibiting an overall drastic, yet smooth, enhancement, in contrast with the histogram-equalized image of Fig. 5(b), which appears to be somewhat not natural, possessing regions that have been over-enhanced.

As a final remark, we can outline that the performance of an entropy measure depends more to the set of axioms that it conforms with, than to the form of the measure itself. By examining the corresponding images, as well as the values of $\lambda_{opt}$, entropy $E_3$ performs better for dark low-contrasted images, while $E_4$ for brighter ones. Finally, entropies satisfying properties **E5**–**E8** exhibit in general a better performance for any type of low-contrasted image.

## 6 Conclusions

In this paper we explored the role of entropy in the context of intuitionistic fuzzy image processing. Different entropy measures for A–IFSs with different characteristics were evaluated and their behavior to contrast enhancement of low-contrasted images was examined. Finally, experimental results to real-world images demonstrated that the different notions of intuitionistic fuzzy entropy treat images in different ways, thus making the selection of the appropriate entropy measure to be application-dependent.

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