Cuts for Conic Mixed-Integer Programming

Alper Atamtürk and Vishnu Narayanan

Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720-1777 USA
atamturk@berkeley.edu, vishnu@ieor.berkeley.edu

Abstract. A conic integer program is an integer programming problem with conic constraints. Conic integer programming has important applications in finance, engineering, statistical learning, and probabilistic integer programming.

Here we study mixed-integer sets defined by second-order conic constraints. We describe general-purpose conic mixed-integer rounding cuts based on polyhedral conic substructures of second-order conic sets. These cuts can be readily incorporated in branch-and-bound algorithms that solve continuous conic programming relaxations at the nodes of the search tree. Our preliminary computational experiments with the new cuts show that they are quite effective in reducing the integrality gap of continuous relaxations of conic mixed-integer programs.

Keywords: Integer programming, conic programming, branch-and-cut.

1 Introduction

In the last two decades there have been major advances in our capability of solving linear integer programming problems. Strong cutting planes obtained through polyhedral analysis of problem structure contributed to this success substantially by strengthening linear programming relaxations of integer programming problems. Powerful cutting planes based on simpler substructures of problems have become standard features of leading optimization software packages. The use of such structural cuts has improved the performance of the linear integer programming solvers dramatically.

On another front, since late 1980's we have experienced significant advances in convex optimization, particularly in conic optimization. Starting with Nesterov and Nemirovski [22, 23, 24] polynomial interior point algorithms that have earlier been developed for linear programming have been extended to conic optimization problems such as convex quadratically constrained quadratic programs (QCQP's) and semidefinite programs (SDP's).

Availability of efficient algorithms and publicly available software (CDSP[9], DSDP[7], SDPA[33], SDPT3[32], SeDuMi[30]) for conic optimization spurred many optimization and control applications in diverse areas ranging from medical imaging to signal processing, from robust portfolio optimization to truss design. Commercial software vendors (e.g. ILOG, MOSEK, XPRESS-MP) have

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responded to the demand for solving (continuous) conic optimization problems by including stable solvers for second-order cone programming (SOCP) in their recent versions.

Unfortunately, the phenomenal advances in continuous conic programming and linear integer programming have so far not translated to improvements in conic integer programming, i.e., integer programs with conic constraints. Solution methods for conic integer programming are limited to branch-and-bound algorithms that solve continuous conic relaxations at the nodes of the search tree. In terms of development, conic integer programming today is where linear integer programming was before 1980's when solvers relied on pure branch-and-bound algorithms without the use of any cuts for improving the continuous relaxations at the nodes of the search tree.

Here we attempt to improve the solvability of conic integer programs. We develop general purpose cuts that can be incorporated into branch-and-bound solvers for conic integer programs. Toward this end, we describe valid cuts for the second-order conic mixed-integer constraints (defined in Section 2). The choice of second-order conic mixed-integer constraint is based on *(i)* the existence of many important applications modeled with such constraints, *(ii)* the availability of efficient and stable solvers for their continuous SOCP relaxations, and *(iii)* the fact that one can form SOCP relaxations for the more general conic programs, which make the cuts presented here widely applicable to conic integer programming.

1.1 Outline

In Section 2 we introduce conic integer mixed-programming, briefly review the relevant literature and explain our approach for generating valid cuts. In Section 3 we describe conic mixed-integer rounding cuts for second-order conic mixed-integer programming and in Section 4 we summarize our preliminary computational results with the cuts.

2 Conic Integer Programming

A conic integer program (CIP) is an integer program with conic constraints. We limit the presentation here to second-order conic integer programming. However, as one can relax more general conic programs to second-order conic programs [14] our results are indeed applicable more generally.

A *second-order conic mixed-integer program* is an optimization problem of the form

$$\text{(SOCMIP)} \quad \begin{aligned} &\min \; cx + ry \\ &\text{s.t.} \; \| A_i x + G_i y - b_i \| \leq d_i x + e_i y - h_i, \quad i = 1, 2, \ldots, k \\ &\quad\;\; x \in \mathbb{Z}^n, \; y \in \mathbb{R}^p \; . \end{aligned}$$

Here $\| \cdot \|$ is the Euclidean norm, A_i, G_i, b are rational matrices with m_i rows, and c, r, d_i, e_i are rational row vectors of appropriate dimension, and h_i is a rational scalar. Each constraint of SOCMIP can be equivalently stated as $(A_i x + G_i y - b_i, d_i x + e_i y - h) \in \mathcal{Q}^{m_i+1}$, where

$$\mathcal{Q}^{m_i+1} := \{(t, t_o) \in \mathbb{R}^{m_i} \times \mathbb{R} : \| t \| \leq t_o\} \ .$$

For $n = 0$, SOCMIP reduces to SOCP, which is a generalization of linear programming as well as convex quadratically constrained quadratic programming. If $G_i = 0$ for all i, then SOCP reduces to linear programming. If $e_i = 0$ for all i, then it reduces to QCQP after squaring the constraints. In addition, convex optimization problems with more general norms, fractional quadratic functions, hyperbolic functions and others can be formulated as an SOCP. We refer the reader to [2, 6, 10, 18, 25] for a detailed exposure to conic optimization and many applications of SOCP.

2.1 Relevant Literature

There has been significant work on deriving conic (in particular SDP) relaxations for (linear) combinatorial optimization problems [1, 13, 19] for obtaining stronger bounds for such problems than the ones given by their natural linear programming relaxations. We refer the reader to Goemans [12] for a survey on this topic. However, our interest here is not to find conic relaxations for linear integer problems, but for conic integer problems.

Clearly any method for general nonlinear integer programming applies to conic integer programming as well. Reformulation-Linearization Technique (RLT) of Sherali and Adams [27] initially developed for linear 0-1 programming has been extended to nonconvex optimization problems [28]. Stubbs and Mehrotra [29] generalize the lift-and-project method [5] of Balas et. al for 0-1 mixed convex programming. See also Balas [4] and Sherali and Shetti [26] on disjunctive programming methods. Kojima and Tunçel [15] give successive semidefinite relaxations converging to the convex hull of a nonconvex set defined by quadratic functions. Lasserre [16] describes a hierarchy of semidefinite relaxations nonlinear 0-1 programs. Common to all of these general approaches is a hierarchy of convex relaxations in higher dimensional spaces whose size grows exponentially with the size of the original formulation. Therefore using such convex relaxations in higher dimensions is impractical except for very small instances. On the other hand, projecting these formulations to the original space of variables is also very difficult except for certain special cases.

Another stream of more practically applicable research is the development of branch-and-bound algorithms for nonlinear integer programming based on linear outer approximations [8, 17, 31]. The advantage of linear approximations is that they can be solved fast; however, the bounds from linear approximations may not be strong. However, in the case of conic programming, and in particular second-order cone programming, existence of efficient algorithms permits the use of continuous conic relaxations at the nodes of the branch-and-bound tree.

The only study that we are aware of on developing valid inequalities for conic integer sets directly is due to Çezik and Iyengar [11]. For a pointed, closed, convex cone $\mathcal{K} \subseteq \mathbb{R}^m$ with nonempty interior, given $S = \{x \in \mathbb{Z}^n : b - Ax \in \mathcal{K}\}$, their approach is to write a linear aggregation

$$\lambda' Ax \leq \lambda' b \ \text{ for some fixed } \ \lambda \in \mathcal{K}^*, \tag{1}$$

where $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$ and then apply the Chvátal-Gomory (CG) integer rounding cuts [20] to this linear inequality. Hence, the resulting cuts are linear in x as well. For the mixed-integer case as the convex hull feasible points is not polyhedral and has curved boundary (see Figure 2 in Section 3). Therefore, nonlinear inequalities may be more effective for describing or approximating the convex hull of solutions.

2.2 A New Approach

Our approach for deriving valid inequalities for SOCMIP is to decompose the second-order conic constraint into simpler polyhedral sets and analyze each of these sets. Specifically, given a second-order conic constraint

$$\| Ax + Gy - b \| \leq dx + ey - h \tag{2}$$

and the corresponding second-order conic mixed-integer set

$$C := \left\{ x \in \mathbb{Z}^n_+, \ y \in \mathbb{R}^p_+ : (x, y) \text{ satisfies (2)} \right\},$$

by introducing auxiliary variables $(t, t_o) \in \mathbb{R}^{m+1}$, we reformulate (2) as

$$t_o \leq dx + ey - h \tag{3}$$

$$t_i \geq |a_i x + g_i y - b_i| \,, \ i = 1, \ldots, m \tag{4}$$

$$t_o \geq \| t \|, \tag{5}$$

where a_i and g_i denote the ith rows of matrices A and G, respectively. Observe that each constraint (4) is indeed a second-order conic constraint as $(a_i x + g_i y - b_i, t_i) \in \mathcal{Q}^{1+1}$, yet polyhedral. Consequently, we refer to a constraint of the form (4) as a *polyhedral second-order conic constraint.*

Breaking (2) into polyhedral conic constraints allows us to exploit the implicit polyhedral set for each term in a second-order cone constraint. Cuts obtained for C in this way are linear in (x, y, t); however, they are nonlinear in the original space of (x, y).

Our approach extends the successful polyhedral method for linear integer programming where one studies the facial structure of simpler building blocks to second-order conic integer programming. To the best of our knowledge such an analysis for second-order conic mixed-integer sets has not been done before.

3 Conic Mixed-Integer Rounding

For a mixed integer set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$, we use relax$(X)$ to denote its continuous relaxation in $\mathbb{R}^n \times \mathbb{R}^p$ obtained by dropping the integrality restrictions and conv(X) to denote the convex hull of X. In this section we will describe the cuts for conic mixed-integer programming, first on a simple case with a single integer variable. Subsequently we will present the general inequalities.

3.1 The Simple Case

Let us first consider the mixed-integer set

$$S_0 := \{(x, y, w, t) \in \mathbb{Z} \times \mathbb{R}^3_+ \; : | \, x + y - w - b \, | \le t\} \tag{6}$$

defined by a simple, yet non-trivial polyhedral second-order conic constraint with one integer variable. The continuous relaxation relax(S_0) has four extreme rays: $(1, 0, 0, 1)$, $(-1, 0, 0, 1)$, $(1, 0, 1, 0)$, and $(-1, 1, 0, 0)$, and one extreme point: $(b, 0, 0, 0)$, which is infeasible for S_0 if $f := b - \lfloor b \rfloor > 0$. It is easy to see that if $f > 0$, conv(S_0) has four extreme points: $(\lfloor b \rfloor, f, 0, 0)$, $(\lfloor b \rfloor, 0, 0, f)$, $(\lceil b \rceil, 0, 1 - f, 0)$ and $(\lceil b \rceil, 0, 0, 1 - f)$. Figure 1 illustrates S_0 for the restriction $y = w = 0$.

Proposition 1. *The simple conic mixed-integer rounding inequality*

$$(1 - 2f)(x - \lfloor b \rfloor) + f \le t + y + w \tag{7}$$

cuts off all points in relax$(S_0) \setminus$ conv(S_0).

Observe that inequality (7) is satisfied at equality at all extreme points of conv(S_0). Proposition 1 can be proved by simply checking that every intersection of the hyperplanes defining S_0 and (7) is one of the four extreme points of conv(S_0) listed above.

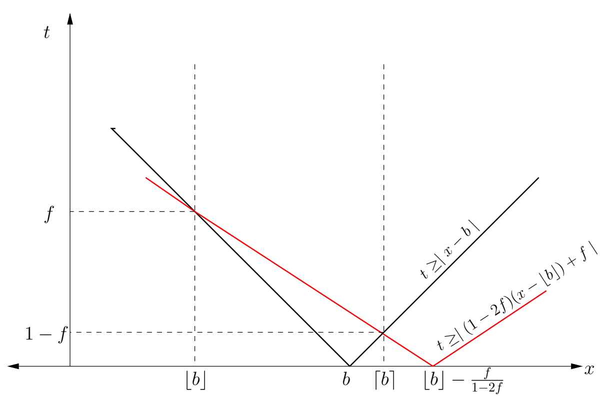

Fig. 1. Simple conic mixed-integer rounding cut

The simple conic mixed-integer rounding inequality (7) can be used to derive nonlinear conic mixed-integer inequalities for nonlinear conic mixed-integer sets. The first observation useful in this direction is that the piecewise-linear conic inequality

$$|(1-2f)(x-\lfloor b\rfloor)+f| \le t+y+w \tag{8}$$

is valid for S_0. See Figure 1 for the restriction $y = w = 0$.

In order to illustrate the nonlinear conic cuts, based on cuts for the polyhedral second-order conic constraints (4), let us now consider the simplest nonlinear second-order conic mixed-integer set

$$T_0 := \left\{(x,y,t) \in \mathbb{Z}\times\mathbb{R}\times\mathbb{R} \;:\; \sqrt{(x-b)^2+y^2} \le t\right\} . \tag{9}$$

The continuous relaxation relax(T_0) has exactly one extreme point $(x, y, t) = (b, 0, 0)$, which is infeasible for T_0 if $b \notin \mathbb{Z}$. Formulating T_0 as

$$t_1 \ge |x-b| \tag{10}$$

$$t \ge \sqrt{t_1^2+y^2}, \tag{11}$$

we write the piecewise-linear conic inequality (8) for (10). Substituting out the auxiliary variable t_1, we obtain the *simple nonlinear conic mixed-integer rounding inequality*

$$\sqrt{((1-2f)(x-\lfloor b\rfloor)+f)^2+y^2} \le t, \tag{12}$$

which is valid for T_0.

Proposition 2. *The simple nonlinear conic mixed-integer rounding inequality* (12) *cuts off all points in* relax$(T_0) \setminus$ conv(T_0).

Proof. First, observe that for $x = \lfloor b\rfloor - \delta$, the constraint in (9) becomes $t \ge \sqrt{(\delta+f)^2+y^2}$, and (12) becomes $t \ge \sqrt{(f-(1-2f)\delta)^2+y^2}$. Since $(\delta+f)^2 - (f-(1-2f)\delta)^2 = 4f\delta(1+\delta)(1-f) \ge 0$ for $\delta \ge 0$ and for $\delta \le -1$, we see that (12) is dominated by relax(T_0) unless $\lfloor b\rfloor < x < \lceil b\rceil$. When $-1 < \delta < 0$ (i.e., $x \in (\lfloor b\rfloor, \lceil b\rceil)$), $4f\delta(1+\delta)(1-f) < 0$, implying that (12) dominates the constraint in (9).

We now show that if $(x_1, y_1, t_1) \in$ relax(T_0) and satisfies (12), then $(x_1, y_1, t_1) \in$ conv(T_0). If $x_1 \notin (\lfloor b\rfloor, \lceil b\rceil)$, it is sufficient to consider $(x_1, y_1, t_1) \in$ relax(T_0) as (12) is dominated by relax(T_0) in this case. Now, the ray $R_1 := \{(b, 0, 0) + \alpha(x_1 - b, y_1, t_1) \;:\; \alpha \in \mathbb{R}_+\} \subseteq$ relax(T_0). Let the intersections of R_1 with the hyperplanes $x = \lfloor x_1\rfloor$ and $x = \lceil x_1\rceil$ be $(\lfloor x_1\rfloor, \bar{y}_1, \bar{t}_1)$, $(\lceil x_1\rceil, \hat{y}_1, \hat{t}_1)$, which belong to T_0. Then (x_1, y_1, t_1) can be written as a convex combination of points $(\lfloor x_1\rfloor, \bar{y}_1, \bar{t}_1)$, $(\lceil x_1\rceil, \hat{y}_1, \hat{t}_1)$; hence $(x_1, y_1, t_1) \in$ conv(T_0).

On the other hand, if $x_1 \in (\lfloor b\rfloor, \lceil b\rceil)$, it is sufficient to consider (x_1, y_1, t_1) that satisfies (12), since (12) dominates the constraint in (9) for $x \in [\lfloor b\rfloor, \lceil b\rceil]$. If $f = 1/2$, (x_1, y_1, t_1) is a convex combination of $(\lfloor b\rfloor, y_1, t_1)$ and $(\lceil b\rceil, y_1, t_1)$. Otherwise, all points on the ray $R_2 := \{(x_0, 0, 0) + \alpha(x_1 - x_0, y_1, t_1) \;:\; \alpha \in \mathbb{R}_+\}$, where $x_0 = \lfloor b\rfloor - \frac{f}{1-2f}$, satisfy (12). Let the intersections of R_2 with the hyperplanes $x = \lfloor b \rfloor$ and $x = \lceil b \rceil$ be $(\lfloor b \rfloor, \bar{y}_1, \bar{t}_1)$, $(\lceil b \rceil, \hat{y}_1, \hat{t}_1)$, which belong to T_0. Note that the intersections are nonempty because $x_0 \notin [\lfloor b \rfloor, \lceil b \rceil]$. Then we see that (x_1, y_1, t_1) can be written as a convex combination of $(\lfloor b \rfloor, \bar{y}, \bar{t})$ and $(\lceil b \rceil, \hat{y}, \hat{t})$. Hence, $(x_1, y_1, t_1) \in \text{conv}(T_0)$ in this case as well. □

Proposition 2 shows that the curved convex hull of T_0 can be described using only two second-order conic constraints. The following example illustrates Proposition 2.

Example 1. Consider the second-order conic set given as

$$T_0 = \left\{ (x, y, t) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \ : \ \sqrt{\left(x - \frac{4}{3}\right)^2 + (y-1)^2} \leq t \right\} .$$

The unique extreme point of relax(T_0) $(\frac{4}{3}, 1, 0)$ is fractional. Here $\lfloor b \rfloor = 1$ and $f = \frac{1}{3}$; therefore,

$$\text{conv}(T_0) = \left\{ (x, y, t) \in \mathbb{R}^3 : \sqrt{\left(x - \frac{4}{3}\right)^2 + (y-1)^2} \leq t, \ \sqrt{\frac{1}{9}x^2 + (y-1)^2} \leq t \right\} .$$

We show the inequality $\sqrt{\frac{1}{9}x^2 + (y-1)^2} \leq t$ and the region it cuts off in Figure 2. Observe that the function values are equal at $x = 1$ and $x = 2$ and the cut eliminates the points between them.

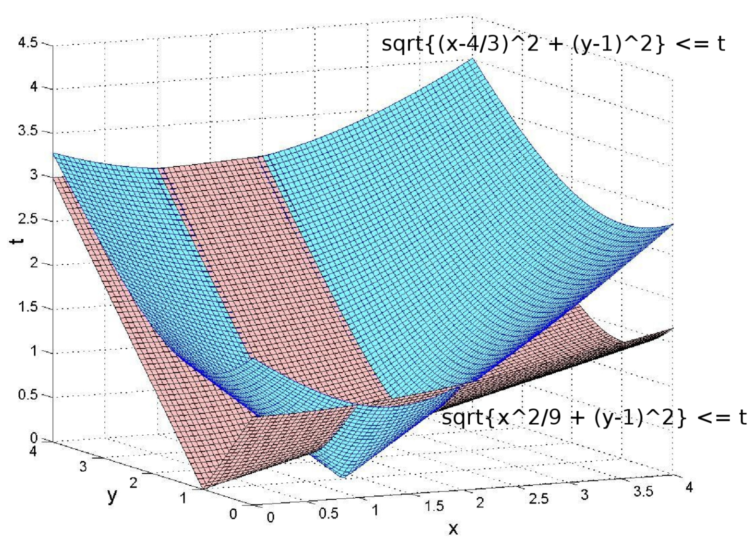

Fig. 2. Nonlinear conic integer rounding cut

3.2 The General Case

In this section we present valid inequalities for the mixed-integer sets defined by general polyhedral second-order conic constraints (4). Toward this end, let

$$S := \{x \in \mathbb{Z}^n_+, y \in \mathbb{R}^p_+,\ t \in \mathbb{R} :\ t \geq |ax + gy - b|\}\ .$$

We refer to the inequalities used in describing S as the trivial inequalities. The following result simplifies the presentation.

Proposition 3. *Any non-trivial facet-defining inequality for* $\operatorname{conv}(S)$ *is of the form* $\gamma x + \pi y \leq \pi_0 + t$. *Moreover, the following statements hold:*

1. $\pi_i < 0$ *for all* $i = 1, \ldots, p$;
2. $\frac{\pi_i}{\pi_j} = \left|\frac{g_i}{g_j}\right|$ *for all* $i, j = 1, \ldots, p$.

Hence it is sufficient to consider the polyhedral second-order conic constraint

$$\left|ax + y^+ - y^- - b\right| \leq t, \tag{13}$$

where all continuous variables with positive coefficients are aggregated into $y^+ \in \mathbb{R}_+$ and those with negative coefficients are aggregated into $y^- \in \mathbb{R}_+$ to represent a general polyhedral conic constraint of the form (4).

Definition 1. *For* $0 \leq f < 1$ *let the conic mixed-integer rounding function* $\varphi_f : \mathbb{R} \to \mathbb{R}$ *be*

$$\varphi_f(v) = \begin{cases} (1-2f)n - (v-n), & \text{if } n \leq v < n+f, \\ (1-2f)n + (v-n) - 2f, & \text{if } n+f \leq v < n+1\ . \end{cases} \quad n \in \mathbb{Z} \tag{14}$$

The conic mixed-integer rounding function φ_f is piecewise linear and continuous. Figure 3 illustrates φ_f.

Lemma 1. *The conic mixed-integer rounding function* φ_f *is superadditive on* $\mathbb{R}$.

Theorem 1. *For any* $\alpha \neq 0$ *the conic mixed-integer rounding inequality*

$$\sum_{j=1}^{n} \varphi_{f_\alpha}(a_j/\alpha)x_j - \varphi_{f_\alpha}(b/\alpha) \leq (t + y^+ + y^-)/|\alpha|, \tag{15}$$

where $f_\alpha = b/\alpha - \lfloor b/\alpha \rfloor$, *is valid for* S. *Moreover, if* $\alpha = a_j$ *and* $b/a_j > 0$ *for some* $j \in \{1, \ldots, n\}$, *then* (15) *is facet-defining for* $\operatorname{conv}(S)$.

Proof. (Sketch) It can be shown that $\varphi_{f_{a_j}}$ is the lifting function of inequality

$$(1-2f)(x - \lfloor b \rfloor) + f \leq (t + y^+ + y^-)/|a_j| \tag{16}$$

for the restriction

$$\left|a_j x_j + y^+ - y^- - b\right| \leq t$$

of (13) with $x_i = 0$ for $i \neq j$. Then the validity as well as the facet claim follows from superadditive lifting [3] of (16) with x_i for $i \neq j$. For $\alpha \neq 0$ validity follows by introducing an auxiliary integer variable x_o with coefficient α and lifting inequality

$$\left|\alpha x_o + y^+ - y^- - b\right| \leq t$$

with all $x_i,\ i = 1, \ldots, n$ and then setting $x_o = 0$. □

Remark 1. The continuous relaxation relax(S) has at most n fractional extreme points $(x^j, 0, 0, 0)$ of the form $x^j_j = b/a_j > 0$, and $x^j_i = 0$ for all $i \neq j$, which are infeasible if $b/a_j \notin \mathbb{Z}$. It is easy to check that conic mixed-integer rounding inequalities with $\alpha = a_j$ are sufficient to cut off all fractional extreme points $(x^j, 0, 0, 0)$ of relax(S) as for $x^j_i = 0$ inequality (15) reduces to (7).

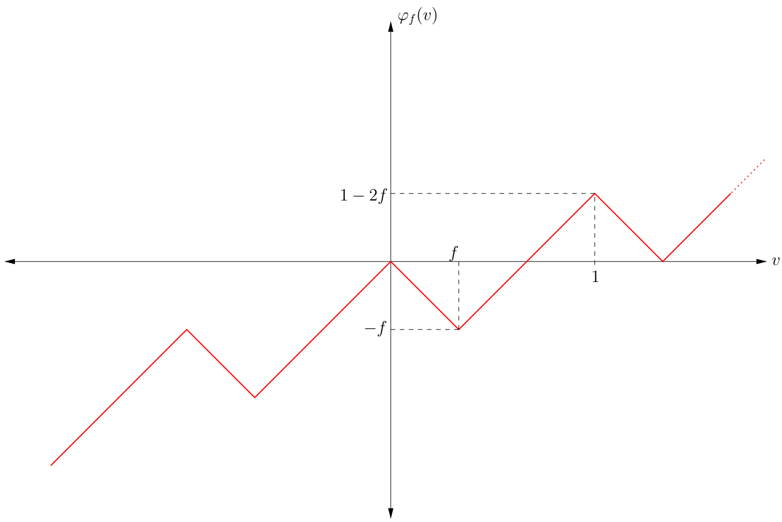

Fig. 3. Conic mixed-integer rounding function.

Next we show that mixed-integer rounding (MIR) inequalities [21, 20] for linear mixed-integer programming can be obtained as conic MIR inequalities. Consider a linear mixed-integer set

$$ax - y \leq b,\ x \geq 0,\ y \geq 0,\ x \in \mathbb{Z}^n,\ y \in \mathbb{R} \tag{17}$$

and the corresponding valid MIR inequality

$$\sum_j \left(\lfloor a_j \rfloor + \frac{(f_j - f)^+}{1-f} \right) x_j - \frac{1}{1-f} y \leq \lfloor b \rfloor, \tag{18}$$

where $f_j := a_j - \lfloor a_j \rfloor$ for $j = 1, \ldots, n$ and $f := b - \lfloor b \rfloor$.

Proposition 4. *Every MIR inequality is a conic MIR inequality.*

Proof. We first rewrite inequalities $ax - y \leq b$ and $y \geq 0$, in the conic form

$$-ax + 2y + b \geq |ax - b|$$

and then split the terms involving integer variables x on the right hand side into their integral and fractional parts as

$$-ax+2y+b \geq \left| \left(\sum_{f_j \leq f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j \right) + \sum_{f_j \leq f} f_j x_j - \sum_{f_j > f} (1 - f_j) x_j - b \right| .$$

Then, since $z = \sum_{f_j \leq f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j$ is integer and $y^+ = \sum_{f_j \leq f} f_j x_j \in \mathbb{R}_+$ and $y^- = \sum_{f_j > f} (1 - f_j) x_j \in \mathbb{R}_+$, we write the simple conic MIR inequality (8)

$$\begin{aligned} -ax + 2y + b + \sum_{f_j \leq f} f_j x_j + \sum_{f_j > f} (1 - f_j) x_j \\ \geq (1 - 2f) \left(\sum_{f_j \leq f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j - \lfloor b \rfloor \right) + f \ . \end{aligned}$$

After rearranging this inequality as

$$2y+2(1-f)\lfloor b \rfloor \geq \sum_{f_j \leq f} ((1-2f)\lfloor a_j \rfloor - f_j + a_j) x_j + \sum_{f_j > f} ((1-2f)\lceil a_j \rceil - (1-f_j) + a_j) x_j$$

and dividing it by $2(1 - f)$ we obtain the MIR inequality (18). □

Example 2. In this example we illustrate that conic mixed-integer rounding cuts can be used to generate valid inequalities that are difficult to obtain by Chvátal-Gomory (CG) integer rounding in the case of pure integer programming. It is well-known that CG rank of the polytope given by inequalities

$$-kx_1 + x_2 \leq 1, \ kx_1 + x_2 \leq k + 1, \ x_1 \leq 1, \ x_1, x_2 \geq 0$$

for a positive integer k equals exactly k [20]. Below we show that the non-trivial facet $x_2 \leq 1$ of the convex hull of integer points can be obtained by a single application of the conic MIR cut.

Writing constraints $-kx_1 + x_2 \leq 1$ and $kx_1 + x_2 \leq k + 1$ in conic form, we obtain

$$\left| kx_1 - \frac{k}{2} \right| \leq \frac{k}{2} + 1 - x_2 \ . \tag{19}$$

Dividing the conic constraint (19) by k and treating $1/2 + 1/k - x_2/k$ as a continuous variable, we obtain the conic MIR cut

$$\frac{1}{2} \leq \frac{1}{2} + \frac{1}{k} - \frac{x_2}{k}$$

which is equivalent to $x_2 \leq 1$.

Conic Aggregation

We can generate other cuts for the second order conic mixed integer set C by aggregating constraints (4) in conic form: for $\lambda, \mu \in \mathbb{R}^m_+$, we have $\lambda' t \geq \lambda'(Ax + Gy - b)$, and $\mu' t \geq \mu'(-Ax - Gy + b)$. Writing these two inequalities in conic form, we obtain

$$\begin{aligned}&\left(\frac{\lambda+\mu}{2}\right)' t + \left(\frac{\mu-\lambda}{2}\right)' (Ax+Gy) + \left(\frac{\lambda-\mu}{2}\right)' b \\ &\quad \geq \left| \left(\frac{\mu-\lambda}{2}\right)' t + \left(\frac{\lambda+\mu}{2}\right)' (Ax+Gy) - \left(\frac{\lambda+\mu}{2}\right)' b \right| .\end{aligned} \tag{20}$$

Then we can write the corresponding conic MIR inequalities for (20) by treating the left-hand-side of inequality (20) as a single continuous variable. Constraint (20) allows us to utilize multiple polyhedral conic constraints (4) simultaneously.

4 Preliminary Computational Results

In this section we report our preliminary computational results with the conic mixed-integer rounding inequalities. We tested the effectiveness of the cuts on SOCMIP instances with cones $\mathcal{Q}^2$, $\mathcal{Q}^{25}$, and $\mathcal{Q}^{50}$. The coefficients of A, G, and b were uniformly generated from the interval [0,3]. All experiments were performed on a 3.2 GHz Pentium 4 Linux workstation with 1GB main memory using CPLEX[1] (Version 10.1) second-order conic MIP solver. CPLEX uses a barrier algorithm to solve SOCPs at the nodes of a branch-and-bound algorithm.

Conic MIR cuts (15) were added only at the root node using a simple separation heuristic. We performed a simple version of conic aggregation (20) on pairs of constraints using only $0-1$ valued multipliers λ and μ, and checked for violation of conic MIR cut (15) for each integer variable x_j with fractional value for the continuous relaxation.

In Table 1 we report the size of the cone (m), number (n) of integer variables in the formulation, the number of cuts, the integrality gap (the percentage gap between the optimal solution and the continuous relaxation), the number of nodes explored in the search tree, and CPU time (in seconds) with and without adding the conic mixed-integer rounding cuts (15). Each row of the table represents the averages for five instances. We have used the default settings of CPLEX except that the primal heuristics were turned off. CPLEX added a small number of MIR cuts (18) to the formulations in a few instances.

We see in Table 1 the conic MIR cuts have been very effective in closing the integrality gap. Most of the instances had 0% gap at the root node after adding the cuts and were solved without branching. The remaining ones were solved within only a few nodes. These preliminary computational results are quite encouraging on the positive impact of conic MIR cuts on solving conic mixed-integer programs.

[1] CPLEX is a registered trademark of ILOG, Inc.

Table 1. Effectiveness of conic MIR cuts (15)

m	n	without cuts			with cuts			
		`% gap`	`nodes`	`time`	`cuts`	`% gap`	`nodes`	`time`
2	100	95.8	19	0	87	0.4	1	0
	200	90.8	29	0	192	0.6	1	0
	300	90.3	38	0	248	0.6	1	0
	400	85.2	62	0	322	0.0	0	0
	500	86.4	71	0	349	0.7	1	0
25	100	8.6	10	0	35	2.6	2	0
	200	41.2	80	2	101	4.5	12	1
	300	46.1	112	4	20	0.0	0	2
	400	68.3	5951	295	99	17.8	63	12
	500	74.6	505	24	116	3.4	6	3
50	100	24.5	7	1	42	0.0	0	1
	200	51.3	67	6	44	0.0	0	1
	300	52.6	105	13	51	3.2	3	2
	400	55.6	158	20	49	5.4	7	5
	500	66.9	233	43	62	1.3	2	3

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