Chapter 8

Cycle Slip Detection and Ambiguity Resolution

In phase measurement there is an ambiguity problem. If the signal happens with a loss of lock, the phase measurement has to be initiated again. This phenomenon is called cycle slips, i.e., the cycle counting has a new beginning because of an interruption of signal. The consequence of the cycle slips is that the adjacent carrier phase observable jumps by an integer number of cycles, and in the related observation model the ambiguity parameter should be a new one. Correct cycle slip detection becomes a guarantee for a correct ambiguity parameterisation. After the discussion of cycle slip detection, emphasis given to the integer ambiguity resolution problem includes the criteria of the integer ambiguity search. The historical ambiguity function method is also outlined and discussed.

8.1 Cycle Slip Detection

Recalling the discussions made in Sect. 6.5, several methods of cycle slip detection can be summarised as follows.

1. Phase-Code Comparison

Using the first equation of 6.88

$$\Delta_t R_j = \lambda_j \Delta_t \Phi_j - \lambda_j \Delta_t N_j + \varepsilon \,, \tag{8.1}$$

cycle slips of the phase observable in working frequency j can be detected. Δ_t, R_j, Φ_j, N_j, λ_j, ε, and j are the time difference operator, code range, phase, ambiguity, wavelength, residual, and index of the frequency, respectively. In the case of no cycle slips, the time difference of the ambiguity should be zero, i.e. $\Delta_t N_j = 0$. Because the noise level of the code range is much higher than that of the phase, this method can only be used for big cycle slip detection.

2. Phase-Phase Ionospheric Residual

Using Eq. 6.80

$$\lambda_1 \Delta_t \Phi_1(t_j) - \lambda_2 \Delta_t \Phi_2(t_j) = \lambda_1 \Delta_t N_1 - \lambda_2 \Delta_t N_2 - \Delta_t \Delta\delta_{\mathrm{ion}}(t_j) + \Delta_t \Delta\varepsilon_{\mathrm{p}} \,, \tag{8.2}$$

cycle slips of the two phase observables in frequency 1 and 2 can be detected. $\Delta_t \Delta\delta_{\mathrm{ion}}(t_j)$ is the so-called ionospheric residual. Generally speaking, the computed ionospheric residual of the two adjacent epochs should be very small. Any unusual change of the ionospheric residual may indicate cycle slips in one or two phases. However, two special cycle slips, ΔN_1 and ΔN_2, can lead also to a very small combination of $\lambda_1 \Delta_t N_1 - \lambda_2 \Delta_t N_2$. Examples of such combinations can be found, e.g., in (Hofmann-Wellenhof et al. 1997). Therefore, a big ionospheric residual indicates the cycle slips, whereas a small ionospheric residual does not guarantee that there are no cycle slips. Another shortcoming of this method is that the ionospheric residual itself provides no possibility to check in which phase the cycle slips happen.

3. Doppler Integration

Using Eq. 6.87

$$\Delta_t N_j = \Delta_t \Phi_j - \int_{t_{j-1}}^{t_j} D_j \mathrm{d}t + \varepsilon\,, \quad j = 1, 2, 5\,, \tag{8.3}$$

cycle slips of the phase observable in working frequency j can be detected. D_j is the Doppler observable of frequency j. Recalling the discussions made in Chap. 4, the phase is measured by keeping track of the partial phase and accumulating the integer count. If there is any loss of lock of the signal during this time, the integer accumulation will be wrong, i.e., cycle slip happens. Therefore, an external instantaneous Doppler integration is a good choice for cycle slip detection. The integration can be made first by fitting the Doppler data with a polynomial of suitable order, and then integrating that within the desired time interval. Polynomial fitting and numerical integration methods can be found in Sect. 11.5.2 and 3.4.

4. Differential Phases (of Time)

Using the first equation of 6.86

$$\lambda_j \Delta_t \Phi_j = \Delta_t \rho - \Delta_t (\delta t_\mathrm{r} - \delta t_\mathrm{e}) c + \lambda_j \Delta_t N_j + \varepsilon_\mathrm{p}\,, \quad j = 1,2, \tag{8.4}$$

cycle slips can be detected. Except for the ambiguity term, all other terms on the right side are of low variation ones. Any cycle slips will lead to a sudden jump of the time difference of the phases. The differenced data may be fitted with polynomials, and the polynomials can be used for interpolating or extrapolating the data at the checking epoch; the computed and differenced data then can be compared to decide if there are any cycle slips.

8.2 Method of Dealing with Cycle Slips

As soon as the cycle slips have been detected, there are two ways to deal with them. One is to repair the cycle slips, the other is to set a new ambiguity unknown parameter in the GPS observation equations. To repair the cycle slips, the cycle slips have to be known exactly. Any incorrect reparation will affect all observations later. Setting a new unknown ambiguity parameter after a cycle slip is a more secure method. It seems that in this way there will be more unknowns in the observation equations. However, there exists a condition between the former ambiguity parameter $N(1)$ and the new one $N(2)$, i.e.,

$$N(1,i,j,k) = N(2,i,j,k) + I(i,j,k)\,, \tag{8.5}$$

where I is an integer constant and i, j and k are indices of the receiver, satellite, and observing frequency, respectively. For any solution of $N(1)$ and $N(2)$ with good qualities, the integer constant should be able to be easily distinguished. If $I = 0$, then no cycle slips have really happened.

If instrumental biases have not been modelled, the biases may destroy the integer property of the original ambiguity parameters. However, in such a case, the double differenced ambiguities are still integers.

8.3
A General Criterion of Integer Ambiguity Search

An integer ambiguity search method based on conditional adjustment theory is proposed in this section. By taking the coordinate and ambiguity residuals into account, a general criterion for ambiguity searching is derived. The search can be carried out in both ambiguity and coordinate domains. The optimality and uniqueness properties of the general criterion are also discussed. A numerical explanation of the general criterion is outlined. An equivalent criterion of the general criterion is derived based on a diagonalised normal equation. It shows that the commonly used least squares ambiguity search (LSAS) criterion is just one of the terms of the equivalent general criterion. Numerical examples are given to illustrate the two components of the equivalent criterion.

8.3.1
Introduction

It is well-known that the ambiguity resolution is a key problem that has to be solved in GPS precise positioning. Some well-derived ambiguity fixing and searching algorithms have been published during the last ten years. There are four types of methods that are categorized. The first type includes Remondi's static initialisation approach (cf., e.g., Remondi 1984; Wang et al. 1988; Hofmann-Wellenhof et al. 1997), which requires a static survey time to solve the ambiguity unknowns even after a complete loss of lock. Normally, the results are good enough to take a round up ambiguity fixing. The second type includes the so-called phase-code combined methods (cf., e.g., Goad and Remondi 1984; Han and Rizos 1997; Sjoeberg 1999); the phase and code have to be used in the derivation as if they have the same precision, and in the case of anti-spoofing (AS), the C/A code has to be used. A search process is still needed in this case. The third type is the so-called ambiguity function method (Remondi 1984; Han and Rizos 1997); its search domain is a geometric one. The fourth type includes approaches; their search domain is only in domain of ambiguity, including some optimal algorithms to reduce the search area and to accelerate the search process (cf., e.g., Euler and Landau 1992; Teunissen 1995; Cannon et al. 1997; Han and Rizos 1997). Because of the statistic character of validation criteria, sometimes no valid result is obtained at the end of the search processes. Gehlich and Lelgemann (1997) separated the ambiguities from the other parameters; this is similar to the equivalent method (cf. Sect. 6.7).

The effort to develop KSGsoft (**K**inematic/**S**tatic **GPS** **Soft**ware) at the GeoForschungs-Zentrum (GFZ) in Potsdam began at the beginning of 1994 due to the requirement of kinematic GPS positioning in aerogravimetry applications (Xu et al. 1998). An optimal ambiguity resolution method is needed in order to implement it into the software; however, selecting the published algorithms has turned out to be a difficult task. This has led to the independent development of this so-called integer ambiguity search method. It turns out to be a very promising algorithm. Using this general criterion, an optimal solution vector can be searched for and found out. The searched result is the optimal one under the least squares principle and integer ambiguity property.

In the following sections, a brief summary of the conditional adjustment is given for the convenience of discussion. Then the ambiguity searches in the ambiguity domain, and both ambiguity and coordinate domains are discussed. Properties of the general criterion are discussed. An equivalent criterion of the general criterion is derived. Numerical examples, conclusions and comments are given.

8.3.2 Summary of Conditional Least Squares Adjustment

The principle of least squares adjustment with condition equations can be summarised as below (for details cf. Sect. 7.4; Gotthardt 1978; Cui et al. 1982):

1. The linearised observation equation system can be represented by

$$V = L - AX\,, \quad P \tag{8.6}$$

   where L is the observation vector of dimension m, A is the coefficient matrix of dimension $m \times n$, X is the unknown vector of dimension n, V is the residual vector of dimension m, n and m are numbers of unknowns and observations, and P is the symmetric and quadratic weight matrix of dimension $m \times m$.
2. The condition equation system can be written as

$$CX - W = 0\,, \tag{8.7}$$

   where C is the coefficient matrix of dimension $r \times n$, W is the constant vector of dimension r, and r is the number of conditions.
3. The least squares criterion for solving the observation equations with condition equations is well-known as

$$V^T PV = \min\,, \tag{8.8}$$

   where V^T is the transpose of the related vector V.
4. The solution of the conditional problem in Eqs. 8.6 and 8.7 under the least squares principle of Eq. 8.8 is then

$$\begin{aligned} X_c &= (A^T PA)^{-1}(A^T PL) - (A^T PA)^{-1} C^T K \\ &= (A^T PA)^{-1}(A^T PL - C^T K) \end{aligned} \tag{8.9}$$

and

$$K = (CQC^T)^{-1}(CQW_1 - W)\,, \tag{8.10}$$

where A^T and C^T are the transpose matrices of A and C, superscript $^{-1}$ is an inversion operator, $Q = (A^T PA)^{-1}$, K is a gain vector (of dimension r), index c is used to denote the variables related to the conditional solution, and $W_1 = A^T PL$.

5. The precisions of the solutions are then

$$p[i] = s_d\sqrt{Q_c[i][i]}\,, \tag{8.11}$$

where i is the element index of a vector or a matrix, s_d is the standard deviation (or sigma) of unit weight, $p[i]$ is the i^{th} element of the precision vector, $Q_c[i][i]$ is the i^{th} diagonal element of the quadratic matrix Q_c, and

$$Q_c = Q - QC^T Q_2 CQ\,, \tag{8.12}$$

$$Q_2 = (CQC^T)^{-1}\,, \tag{8.13}$$

$$s_d = \sqrt{\frac{(V^T PV)_c}{m-n+r}}\,, \quad \text{if} \quad (m > n-r)\,. \tag{8.14}$$

6. For recursive convenience, $(V^T PV)_c$ can be calculated by using

$$(V^T PV)_c = L^T PL - (A^T PL)^T X_c - W^T K\,. \tag{8.15}$$

Above are the complete formulas of conditional least squares adjustment. The application of such an algorithm for the purpose of integer ambiguity search will be further discussed in later sections.

8.3.3 Float Solution

GPS observation equation can be represented with Eq. 8.6. Considering the case without condition (Eq. 8.7), i.e., $C = 0$ and $W = 0$, the least squares solution of Eq. 8.6 is

$$X_0 = Q(A^T PL) = QW_1\,, \tag{8.16}$$

and

$$(V^T PV)_0 = L^T PL - (A^T PL)^T X_0\,, \tag{8.17}$$

$$s_d = \sqrt{\frac{(V^T PV)_0}{m-n}}\,, \quad \text{if} \quad (m > n) \quad \text{and} \tag{8.18}$$

$$p[i] = s_d\sqrt{Q[i][i]}\,, \tag{8.19}$$

where index 0 is used for convenience to denote the variables related to the least squares solution without conditions. X_0 is the complete unknown vector including coordinates and ambiguities and is called a float solution later on. Solution X_0 is the optimal one under the least squares principle. However, because of the observation and model errors as well as method limitations, float solution X_0 may not be exactly the right one, e.g., the ambiguity parameters are real numbers and do not fit to the integer property. Therefore, one sometimes needs to search for a solution, say X, which not only fulfils some special conditions, but also meanwhile keeps the deviation of the solution as small as possible (minimum). This can be represented by

$$V_x^T P V_x = \min , \tag{8.20}$$

or equivalently by a symmetric quadratic form of (cf. also Eq. 8.35 derived later)

$$(X_0 - X)^T Q^{-1}(X_0 - X) = \min . \tag{8.21}$$

In Eq. 8.20, V_x is the residual vector in the case of solution X. For simplification, let:

$$X = \begin{pmatrix} Y \\ N \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad W_1 = A^T PL = \begin{pmatrix} W_{11} \\ W_{12} \end{pmatrix},$$

$$M = A^T PA = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M = Q^{-1} , \tag{8.22}$$

where Y is the coordinate vector, N is the ambiguity vector (generally, a real vector). The float solution is denoted by

$$X_0 = \begin{pmatrix} Y_0 \\ N_0 \end{pmatrix} = \begin{pmatrix} Q_{11}W_{11} + Q_{12}W_{12} \\ Q_{21}W_{11} + Q_{22}W_{12} \end{pmatrix},$$

where X_0 is the solution of Eq. 8.6 without Condition 8.7.

8.3.4 Integer Ambiguity Search in Ambiguity Domain

To use the conditional adjustment algorithm for integer ambiguity searching in the ambiguity domain, the condition shall be selected as $N = W$; here W of course is an integer vector. Generally, letting $C = (0, E)$, then Condition 8.7 turns out to be:

$$N = W . \tag{8.23}$$

Using the definitions of C and Q, one has

$$CQ = \begin{pmatrix} Q_{21} & Q_{22} \end{pmatrix} \quad \text{and}$$

$$CQC^T = Q_{22} .$$

The gain K_N can be computed by using Eq. 8.10:

$$K_N = Q_{22}^{-1}(CQW_1 - W) = Q_{22}^{-1}(N_0 - W)\,. \tag{8.24}$$

So under Condition 8.23, the conditional least squares solution in Eq. 8.9 can be written as

$$X_c = \begin{pmatrix} Y_c \\ N_c \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} W_{11} \\ W_{12} - K_N \end{pmatrix} = \begin{pmatrix} Y_0 \\ N_0 \end{pmatrix} - \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} K_N\,. \tag{8.25}$$

Simplifying Eq. 8.25, one gets:

$$Y_c = Y_0 - Q_{12}K_N \tag{8.26}$$

and

$$N_c = N_0 - Q_{22}K_N = N_0 - Q_{22}Q_{22}^{-1}(N_0 - W) = W\,. \tag{8.27}$$

The precision computing formulas under Condition 8.23 can be derived as below:

$$Q_c = Q - QC^T Q_{22}^{-1} CQ = \begin{pmatrix} Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \tag{8.28}$$

$$\begin{aligned}(V^T PV)_c &= L^T PL - (A^T PL)^T X_c - W^T K_N \\ &= L^T PL - (A^T PL)^T X_0 + (A^T PL)^T \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} K_N - W^T K_N \\ &= (V^T PV)_0 + \begin{pmatrix} W_1^T & W_2^T \end{pmatrix} \begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix} K_N - W^T K_N \\ &= (V^T PV)_0 + (N_0 - W)^T K_N \\ &= (V^T PV)_0 + (N_0 - W)^T Q_{22}^{-1}(N_0 - W)\end{aligned}\quad , \tag{8.29}$$

where $(V^TPV)_0$ is the value obtained without Condition 8.23. The second term on the right side of the last line in Eq. 8.29 is the often-used least squares ambiguity search (LSAS) criterion for an integer ambiguity search in the ambiguity domain, which can be expressed as

$$\delta(\mathrm{d}N) = (N_0 - N)^T Q_{22}^{-1}(N_0 - N)\,. \tag{8.30}$$

It indicates that any ambiguity fixing will cause an enlargement of the standard deviation. However, one may also notice that here only the enlargement of the standard deviation caused by ambiguity parameter changing has been considered. Furthermore, the Condition 8.23 does not really exist. Ambiguities are integers, however, they are unknowns. The formula to compute the accuracy vector of the ambiguity does not exist too, because the ambiguity condition is considered exactly known in conditional adjustment.

8.3.5 Integer Ambiguity Search in Coordinate and Ambiguity Domains

In order to see the enlargement of the standard deviation caused by the fixed solution, the condition shall be selected as $X = W$; here, W consists of two sub-vectors (coordinate and ambiguity parameter related sub-vectors). And only the ambiguity parameter related sub-vector is an integer one. Letting $C = E$, Condition 8.7 is then:

$$X = W\ . \tag{8.31}$$

One has

$$CQ = CQC^T = Q\ .$$

Denote $X_0 = QW_1$; here X_0 is the solution of Eq. 8.6 without Condition 8.31. The gain K can be computed by using Eq. 8.10:

$$K = Q^{-1}(CQW_1 - W) = Q^{-1}(X_0 - W)\ . \tag{8.32}$$

So under Condition 8.31, the conditional least squares solution in Eq. 8.9 can be written as

$$X_c = X_0 - QK = X_0 - QQ^{-1}(X_0 - W) = W\ . \tag{8.33}$$

Precision computing formulas under Condition 8.31 can be derived as below:

$$\begin{aligned} Q_c &= 0\ , \\ (V^TPV)_c &= L^TPL - (A^TPL)^T X_c - W^TK \\ &= L^TPL - (A^TPL)^T X_0 + (A^TPL)^T(X_0 - X_c) - W^TK \\ &= (V^TPV)_0 + W_1^TQK - W^TK \\ &= (V^TPV)_0 + (X_0 - W)^T K \\ &= (V^TPV)_0 + (X_0 - W)^T Q^{-1}(X_0 - W) \end{aligned}\ , \tag{8.34}$$

where $(V^TPV)_0$ is the value obtained without Condition 8.31.

Condition 8.31 will force the observation Eq. 8.6 to take the condition W as the solution and will take the zero value as the precision of the conditional solution (i.e., the precision is undefined). The reason for this is that the condition is considered exactly known in conditional adjustment. The second term on the right side of Eq. 8.34 is denoted as

$$\delta = (X_0 - X)^T Q^{-1}(X_0 - X)\ . \tag{8.35}$$

This term in Eq. 8.34 indicates that any solution vector X, which is different from the float solution vector X_0, will enlarge the weighted squares residuals. It is well-known that the float solution is the optimal solution under the least squares principle. Therefore, statistically, the optimal solution X shall be that X which takes the minimum value of δ in Eq. 8.35. Mathematically speaking, Eq. 8.35 is the "distance" between vector X and X_0 in the solution space (of dimension n). If one considers $n = 3$ and Q^{-1} to be a diagonal matrix, then δ is the geometric distance of point X and X_0 in a cubic space. So Eq. 8.35 can be used as a general criterion to express the nearness of the two vectors. By using criterion of Eq. 8.35, one may search for solution X in the area being searched so that the value of δ reaches the minimum. Under such a criterion, the deviation of the result vector X related to the float vector X_0 is homogenously considered.

Furthermore, Condition 8.31 is considered exactly known in conditional adjustment. However, in integer ambiguity searching, we just know the ambiguities are integers, but their values are indeed not known, or say, they are known with uncertainty (precision) within an area around the float solution. So the best solution shall be searched for. For computing the precision of the searched X, the formulas of least squares adjustment shall be further used, and meanwhile the enlarged residuals shall be taken into account by

$$p[i] = s_\mathrm{d}\sqrt{Q[\mathrm{i}][\mathrm{i}]}\,,$$

$$s_\mathrm{d} = \sqrt{\frac{(V^T PV)_\mathrm{c}}{m-n}}\,, \quad \text{if} \quad (m > n) \quad \text{and}$$

$$(V^T PV)_\mathrm{c} = (V^T PV)_0 + \delta\,. \tag{8.36}$$

In other words, the original Q matrix and $(V^TPV)_0$ of the least squares problem in Eq. 8.6 are further used. The δ has the function of enlarging the standard deviation. The precision computing formulas have nothing to do with the conditions. Searching for a minimum δ leads to a minimum of standard deviation s_d and therefore the best precision values.

Equation 8.35 is called the general criterion of an integer ambiguity search, which may be used for searching for the optimal solution in the ambiguity domain, or both coordinate and ambiguity domains. In most cases, the search will be started from the ambiguity domain. An integer vector N can be selected in the searching area, then the related coordinate vector Y can be computed using the consistent relation of Y and N (cf. Eqs. 8.26 and 8.24). The optimal solution searched shall be that X which leads Eq. 8.35 to a minimum value.

In the case of searching in the ambiguity domain, X consists of the selected sub-vector of N_c in Eq. 8.27 and the computed coordinate sub-vector Y_c in Eq. 8.26, i.e.,

$$W = \begin{pmatrix} Y_\mathrm{c} \\ N_\mathrm{c} \end{pmatrix}\,. \tag{8.37}$$

8.3.6
Properties of the General Criterion

1. Equivalence of the Two Searching Scenarios

It should be emphasised that the same searching criterion of Eq. 8.35 and the same formulas of precision estimation in Eq. 8.36 are used in the two integer ambiguity search scenarios. And the same normal equation of 8.6 is used to compute the Y_c using the selected N_c if necessary. The two searching processes indeed deal with the same problem, just as different ways of searching are used.

Suppose by searching in the ambiguity domain, the vector $X = (Y_c \quad N_c)^T$ is found so that δ reaches the minimum, where N_c is the selected integer sub-vector and Y_c is the computed one. And in the case of searching in both coordinate and ambiguity domains, a candidate vector $X = (Y \quad N)^T$ is selected so that δ reaches the minimum, where N is the selected integer sub-vector and Y is the selected coordinate vector. Because of the optimality and uniqueness properties of the vector X in Eq. 8.35 (please refer to 2, which is discussed next), here the selected $(Y \quad N)^T$ must be equal to $(Y_c \quad N_c)^T$. So the theoretical equivalency of the two searching processes is confirmed.

2. Optimality and Uniqueness Properties

The float solution X_0 is the optimal and unique solution of Eq. 8.6 under the principle of least squares. A minimum of δ in Eq. 8.35 will lead to a minimum of $(V^TPV)_c$ in Eq. 8.36. Therefore using criterion of Eq. 8.35 analogously, the searched vector X is the optimal solution of Eq. 8.6 under the least squares principle and integer ambiguity properties. The uniqueness property is obvious. If X_1 and X_2 are such that $\delta(X_1) = \delta(X_2) = \min$ or $\delta(X_1) - \delta(X_2) = 0$, then by using Eq. 8.35, one may assume that X_1 must be equal to X_2.

3. Geometric Explanation of the General Criterion

Geometrically, $\delta = (X_0 - X)^T(Q)^{-1}(X_0 - X)$ is the "distance" between the vector X and float vector X_0. The distance contributed to enlarge the standard deviation s_d (cf. Eq. 8.36). Ambiguity searching is then the search for the solution vector, which owns the integer ambiguity property and has the minimum distance to the float solution vector.

8.3.7
An Equivalent Ambiguity Search Criterion and its Properties

Suppose undifferenced GPS observation equation and related LS normal equation are

$$V = L - \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad P \tag{8.38}$$

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \tag{8.39}$$

where

$$\begin{pmatrix} A_1^T P A_1 & A_1^T P A_2 \\ A_2^T P A_1 & A_2^T P A_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = M, \quad M^{-1} = Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \tag{8.40}$$

$$W_1 = A_1^T PL \quad \text{and} \quad W_2 = A_2^T PL .$$

Where all symbols have the same meanings as that of Eqs. 7.117 and 7.118. Equation 8.39 can be diagonalised as (cf. Sect. 7.6.1)

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{8.41}$$

where

$$\begin{aligned} &Q_{11} = M_1^{-1}, \qquad\qquad Q_{22} = M_2^{-1} \\ &Q_{12} = -M_{11}^{-1}(M_{12}Q_{22}), \quad Q_{21} = -M_{22}^{-1}(M_{21}Q_{11}) \end{aligned}. \tag{8.42}$$

The related equivalent observation equation of the diagonal normal Eq. 8.41 can be written (cf. Sect. 7.6.1)

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} L \\ L \end{pmatrix} - \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \tag{8.43}$$

where all symbols have the same meanings as that of Eqs. 7.140 and 7.142.

Suppose GPS observation equation is Eq. 8.38 and the related least squares normal equation is Eq. 8.39, where $X_2 = N$ (N is the ambiguity sub-vector) and $X_1 = Y$ (Y is the other unknown sub-vector). The general criterion is (cf. Eq. 8.35)

$$\delta(\mathrm{d}X) = (X_0 - X)^T Q^{-1} (X_0 - X), \tag{8.44}$$

where $X = (Y \;\; N)^T$, $X_0 = (Y_0 \;\; N_0)^T$, $\mathrm{d}X = X_0 - X$ and index 0 denotes the float solution. The search process in the ambiguity domain is a process to find out a solution X (which includes N in the searching area and the computed Y) so that the value of $\delta(\mathrm{d}X)$ reaches the minimum. The optimality property of this criterion is obvious.

For the equivalent observation Eq. 8.43, the related least squares normal equation is Eq. 8.41. The related equivalent general criterion is then (putting the diagonal cofactor of Eq. 8.41 into Eq. 8.44 and taking Eqs. 8.40 and 8.42 into account)

$$\begin{aligned} \delta_1(\mathrm{d}X) &= (Y_0 - Y)^T Q_{11}^{-1} (Y_0 - Y) + (N_0 - N)^T Q_{22}^{-1} (N_0 - N) \\ &= \delta(\mathrm{d}Y) + \delta(\mathrm{d}N) \end{aligned} \tag{8.45}$$

where index 1 is used to distinguish criterion of Eq. 8.45 from Eq. 8.44. The observation equations 8.38 and 8.43 are equivalent, and the related normal Eqs. 8.39 and 8.41 are also equivalent. Therefore, the Criterion 8.45 is called an equivalent criterion of the general Criterion 8.44.

Furthermore, Y and N shall be consistent to each other because they are presented in the same normal Eqs. 8.39 and 8.41. Using condition $W = N$ and notation of Eq. 8.42, one has from Eqs. 8.26 and 8.24

$$Y_0 - Y = Q_{12} Q_{22}^{-1} (N_0 - N). \tag{8.46}$$

Putting Eq. 8.46 into Eq. 8.45, one has

$$\delta_1(\mathrm{d}X) = (N_0 - N)^T [Q_{22}^{-1}(E + Q_{21} Q_{11}^{-1} Q_{12} Q_{22}^{-1})](N_0 - N). \tag{8.47}$$

It is notable that the second term $\delta(\mathrm{d}N)$ of the equivalent criterion Eq. 8.45 is exactly the same as the commonly used least squares ambiguity search (LSAS) criterion of Eq. 8.30 (cf., e.g., Teunissen 1995; Leick 1995; Hofmann-Wellenhof et al. 1997; Euler and Landau

1992; Han and Rizos 1997). Through Eq. 8.47 one may clearly see the differences between the criteria of Eqs. 8.30 and 8.45. When the results searched using Eq. 8.30 are different from that of using Eq. 8.45, the results from the search using Eq. 8.30 shall be only sub-optimal ones due to the optimality and uniqueness property of Eq. 8.45. The first term on the right side of Eq. 8.45 signifies an enlarging of the residuals due to the coordinate change caused by ambiguity fixing (cf. Sect. 8.3.3). The second term on the right side of Eq. 8.45 signifies an enlarging of the residuals due to the ambiguity change caused by ambiguity fixing (cf. Sect. 8.3.4). Equation 8.45 takes both effects into account.

1. Optimality and Uniqueness Properties of the Equivalent Criterion

The float solution X_0 is the optimal and unique solution of Eq. 7.117 under the least squares principle. Criterion Eq. 8.45 is equivalent to criterion Eq. 8.44. A X leads to the minimum of $\delta_1(\mathrm{d}X)$ in Eq. 8.45, which will lead to the minimum of $\delta(\mathrm{d}X)$ in Eq. 8.44 and consequentially the minimum of $(V^TPV)_c$ in Eq. 8.36; therefore using criterion of Eq. 8.45, analogously, the searched vector X is the optimal solution of Eq. 8.38 under the least squares principle and integer ambiguity properties. The uniqueness property is obvious. If one has X_1 and X_2 so that $\delta_1(\mathrm{d}X_1) = \delta_2(\mathrm{d}X_2) = \text{min.}$, or $\delta_1(\mathrm{d}X_1) - \delta_1(\mathrm{d}X_2) = 0$, then by using Eq. 8.45, one may assume that X_1 must be equal to X_2.

It is notable that Eqs. 8.44 and 8.45 are equivalent for use in searching; however, they are neither the same nor equal. For computing the precision, δ in Eq. 8.36 has to be computed using Eq. 8.44.

8.3.8
Numerical Examples of the Equivalent Criterion

Several numerical examples are given here to illustrate the behaviour of the two terms of the criterion. The first and second terms on the right-hand side of Eq. 8.45 are denoted as $\delta(\mathrm{d}Y)$ and $\delta(\mathrm{d}N)$, respectively. $\delta_1(\mathrm{d}X) = \delta(\mathrm{d}Y) + \delta(\mathrm{d}N)$ is the equivalent criterion of the general criterion and is denoted as $\delta(\text{total})$. The term $\delta(\mathrm{d}N)$ is the LSAS criterion. Of course, the search is made in the ambiguity domain. The search area is determined by the precision vector of the float solution. All possible candidates are tested one by one, and the related $\delta_1(\mathrm{d}X)$ are compared with each other to find out the minimum.

In the first example, precise orbits and dual frequency GPS data of 15 April 1999 at station Brst (N 48.3805°, E 355.5034°) and Hers (N 50.8673°, E 0.3363°) are used. The session length is 4 hours. The total search candidate number is 1 020. Results of the two delta components are illustrated as 2-D graphics with the 1st axis of search number and the 2nd axis of delta in Fig. 8.1. The red and blue lines represent $\delta(\mathrm{d}Y)$ and $\delta(\mathrm{d}N)$, respectively. $\delta(\mathrm{d}Y)$ reaches the minimum at the search No. 237, and $\delta(\mathrm{d}N)$ at 769. $\delta(\text{total})$ is plotted in Fig. 8.2, and it shows that the general criterion reaches the minimum at the search No. 493. For more detail, a part of the results are listed in Table 8.1.

$\delta(\mathrm{d}N)$ reaches the second minimum at search No. 771. This example shows that the minimum of $\delta(\mathrm{d}N)$ may not lead to the minimum of total delta, because the related $\delta(\mathrm{d}Y)$ is large. If the delta ratio criterion is used in this case, the LSAS method will reject the found minimum and explain that no significant ambiguity fixing can be made. However, because of the uniqueness principle of the general criterion, the search reaches the total minimum uniquely.

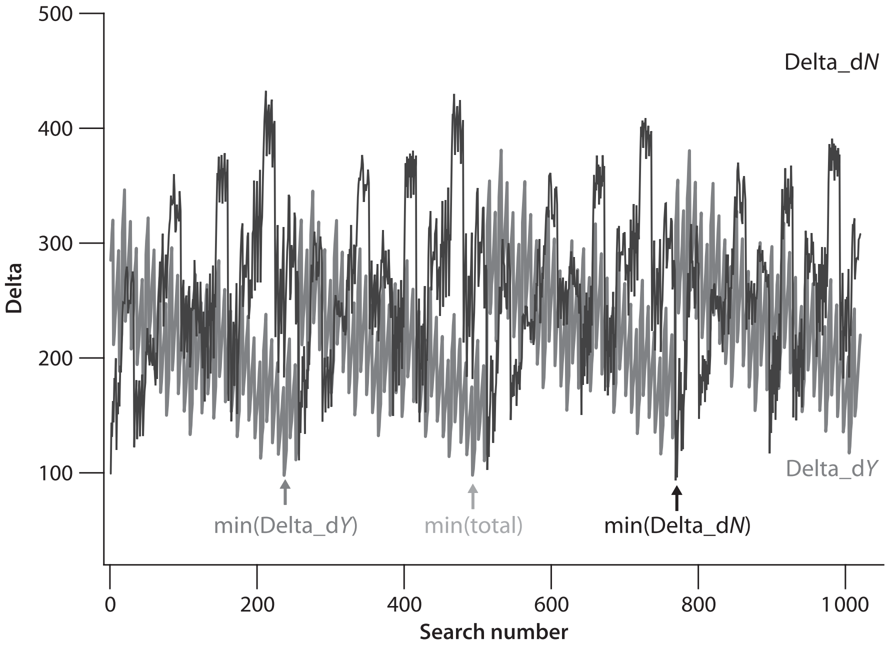

Fig. 8.1. Two components of the equivalent ambiguity search criterion

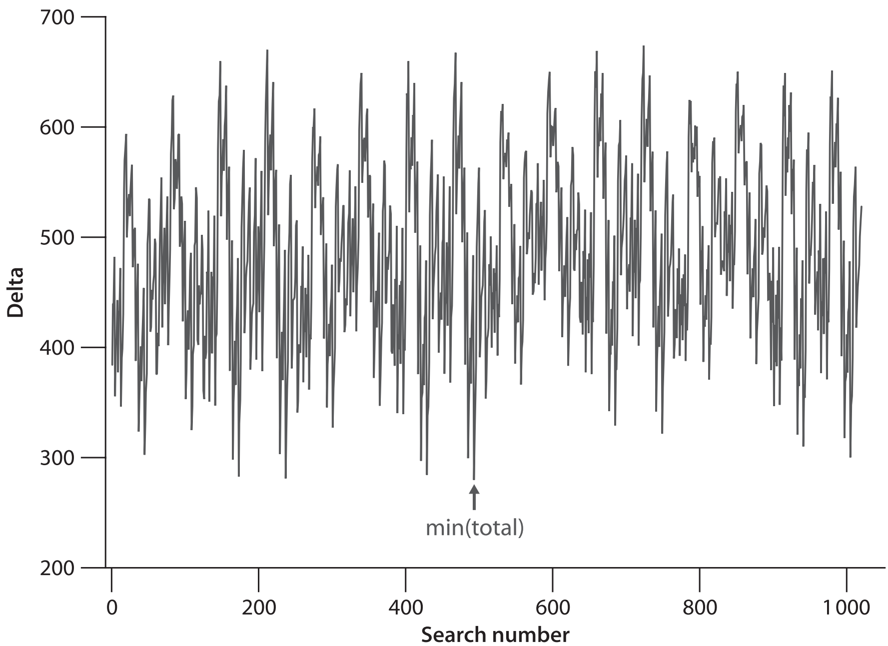

Fig. 8.2. Equivalent ambiguity search criterion

The second example is very similar to the first one. The delta values of the search process are plotted in Fig. 8.3, where $\delta(\mathrm{d}Y)$ is much smaller than $\delta(\mathrm{d}N)$. $\delta(\mathrm{d}N)$ reaches the minimum at the search No. 5 and $\delta(\mathrm{d}Y)$ at 171. δ(total) reaches the minimum at the search No. 129. The total 11 ambiguity parameters are fixed and listed in Table 8.2. Two ambiguity fixings have just one cycle difference at the 6^{th} ambiguity parameter. The related coordinate solutions after the ambiguity fixings are listed in Table 8.3. The coordinate differences at component x and z are about 5 mm. Even the results are very similar; however, two criteria do give different results.

In the third example, real GPS data of 3 October 1997 at station Faim (N 38.5295°, E 331.3711°) and Flor (N 39.4493°, E 328.8715°) are used. The delta values of the search process are listed in Table 8.4. Both $\delta(\mathrm{d}N)$ and δ(total) reach the minimum at the search No. 5. This indicates that the LSAS criterion may sometimes reach the same result as that of the equivalent criterion being used.

Table 8.1. Delta values of searching process

Search No.	$\delta(\mathrm{d}N)$	$\delta(\mathrm{d}Y)$	δ(total)
237	183.0937	97.8046	280.8984
493	181.7359	97.9494	279.6853
769	93.3593	315.2760	408.6353
771	96.0678	343.5736	439.6414

Table 8.2. Two kinds of ambiguity fixing due to two criteria

Ambiguity No.	1	2	3	4	5	6	7	8	9	10	11
LSAS fixing	0	0	1	0	0	0	–1	0	0	–1	–1
General fixing	0	0	1	0	0	–1	–1	0	0	–1	–1

Table 8.3. Ambiguity fixed coordinate solutions (in meters)

Coordinates	x	y	z
LSAS fixing	0.2140	–0.0449	0.1078
General fixing	0.2213	–0.0465	0.1127

Table 8.4. Deltas of the ambiguity search process

Search No.	$\delta(\mathrm{d}N)$	$\delta(\mathrm{d}Y)$	δ(total)
1	248.5681	129.0555	377.6236
2	702.6925	58.9271	761.6195
3	889.5496	107.9330	997.4825
4	452.1952	42.3226	494.5178
5	186.7937	112.3030	299.0967
6	739.0487	55.9744	795.0231
7	931.4125	89.9074	1021.3199
8	592.1887	38.0969	630.2856

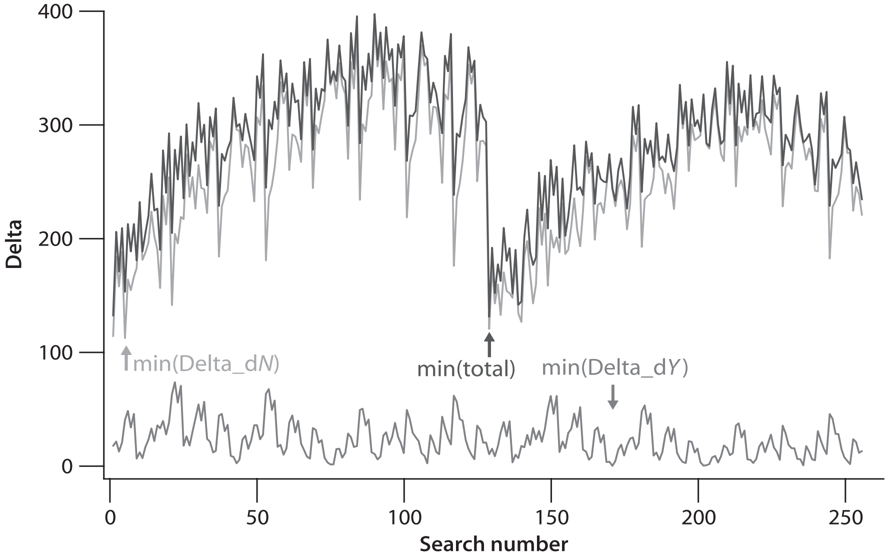

Fig. 8.3. Example of equivalent ambiguity search criterion

8.3.9
Conclusions and Comments

1. Conclusions

A general criterion and its equivalent criterion of integer ambiguity searching are proposed in this section. Using these two criteria, the searched result is optimal and unique under the least squares minimum principle and under the condition of integer ambiguities. The general criterion has a clear geometrical explanation. The theoretical relationship between the equivalent criterion and the commonly used least squares ambiguity search (LSAS) criterion is obvious. It shows that the LSAS criterion is just one of the terms of the equivalent criterion of the general criterion (this does not take into account the residual enlarging effect caused by coordinate change due to ambiguity fixing). Numerical examples show that a minimum $\delta(\mathrm{d}N)$ may have a relatively large $\delta(\mathrm{d}Y)$, and therefore a minimum $\delta(\mathrm{d}N)$ may not guarantee a minimum $\delta(\text{total})$. For an optimal search, the equivalent criterion or the general criterion shall be used.

2. Comments

The float solution is the optimal solution of the GPS problem under the least squares minimum principle. Using the equivalent general criterion, the searched solution is the optimal solution under the least squares minimum principle and under the condition of integer ambiguities. However, the ambiguity-searching criterion is just a statistic criterion. Statistic correctness does not guarantee correctness in all applications. Ambiguity fixing only makes sense when the GPS observables are good enough and the data processing models are accurate enough.

8.4 Ambiguity Function

It is well-known that in GPS precise positioning, ambiguity resolution is one of the key problems that has to be solved. Some well-derived ambiguity fixing and searching algorithms have been published in the past. One of these methods is the ambiguity function (AF) method, which can be found in many standard publications (Remondi 1984; Wang et al. 1988; Han and Rizos 1995; Hofmann-Wellenhof et al. 1997).

The principle of the ambiguity function method is to use the single-differenced phase observation

$$\Phi_j(t_k) = \frac{1}{\lambda}\rho_j(t_k) + N_j - \gamma(t_k), \tag{8.48}$$

to form an exponential complex function

$$\mathrm{e}^{i2\pi[\Phi_j(t_k)-\rho_j(t_k)/\lambda]} = \mathrm{e}^{i2\pi[N_j-\gamma(t_k)]} \quad \text{or} \tag{8.49}$$

$$\mathrm{e}^{i2\pi[\Phi_j(t_k)-\rho_j(t_k)/\lambda]} = \mathrm{e}^{-i2\pi\gamma(t_k)}, \tag{8.50}$$

where Φ is the phase observable, ρ is the geometric distance of the signal transmitting path, λ is the wavelength, index j denotes the observed satellite, t_k is the k^{th} observational time, N is ambiguity, γ is the model of the receiver clock errors, and i is the imaginary unit. All terms in Eq. 8.48 have the units of cycles and are single-differenced terms. Property

$$\mathrm{e}^{i2\pi N_j} = 1$$

is used in order to get Eq. 8.50.

Making a summation over all satellites and then taking the modulus operation, one has

$$\left|\sum_{j=1}^{n_j} \mathrm{e}^{i2\pi[\Phi_j(t_k)-\rho_j(t_k)/\lambda]}\right| = n_j(k), \tag{8.51}$$

where property

$$\left|\mathrm{e}^{-i2\pi\gamma(t_k)}\right| = 1$$

is used, n_j is the satellite number and $n_j(k)$ is the observed satellite number at epoch k.

Making a summation of Eq. 8.51 over all the observed time epochs, one has

$$\sum_{k=1}^{n_k}\left|\sum_{j=1}^{n_j} \mathrm{e}^{i2\pi[\Phi_j(t_k)-\rho_j(t_k)/\lambda]}\right| = \sum_{k=1}^{n_k} n_j(k), \tag{8.52}$$

where n_k is the total epochs number. The left side of Eq. 8.52 is called the ambiguity function, where unknowns are the coordinates of the remote station. The values of the ambiguity function have to be computed for all candidates of coordinates, and the optimum solution is found if the function reaches the maximum, i.e.,

$$\sum_{k=1}^{n_k} \left| \sum_{j=1}^{n_j} \mathrm{e}^{i2\pi[\Phi_j(t_k)-\rho_j(t_k)/\lambda]} \right| \Rightarrow \text{maximum} \,. \tag{8.53}$$

The search area can be determined by the standard deviations (σ) of the initial coordinates (e.g., a cube with side lengths of 3σ or a sphere with a radius of 3σ). The AF method is indeed an ambiguity free method. The ambiguity can be computed using the optimal coordinate solution of Eq. 8.53.

Further discussion on the AF method is given in the next sub-section.

8.4.1
Maximum Property of Ambiguity Function

The ambiguity function is discussed in Sect. 8.4. Here a numerical study of the maximum property of the ambiguity function (AF) is given. It seems that the maximum value of the AF trends to be reached at the boundary of any given search area. Numerical examples are given to illustrate the conclusion. However, a theoretical proof has still not been found up to now; even the author tried to find one, but failed.

Numerical Examples

Several numerical examples are given here to illustrate the behaviours of the ambiguity function criterion. The GPS data of the EU AGMASCO project (cf., e.g., Xu et al. 1997) are used. Data are combined with the data of IGS network and solved for precise coordinates as references. The station Faim (N 38.5295°, E 331.3711°) is used as the reference and Flor (N 39.4493°, E 328.8715°) is used as the remote station. The baseline length is about 240 km. The data length is about four hours of 3 October, 1997. KSGsoft (Xu et al. 1998) is used for computing a static solution of the coordinates of Flor. The differences of the KSGsoft solution and IGS solution are (0.26, 1.93, 1.37) cm in the global Cartesian coordinate system. Related standard deviations of the KSGsoft solution are (0.04, 0.04, 0.02) cm. The differences are caused partly by the different data lengths. This assures a good standard for the software being used.

The search step is selected as 1 mm. Tropospheric and ionospheric effects are corrected. In the first example, three hours of data are used. The search area is a 3-D cube with side lengths of ±(0.7, 0.7, 0.4) cm in (x, y, z). Results show that the AF maximum is reached at point (–0.7, 0.7, 0.4) cm, which is on the boundary of the area being searched.

A search process (with a search area of ±7 mm and one hour of data) is illustrated in 2-D graphics with the 1$^{\text{st}}$ axis containing search numbers and the 2$^{\text{nd}}$ axis containing AF values in Fig. 8.4. The graphic looks like a 3-D AF projection of the cubic searching area (the picture could be quite different in other examples). Figure 8.4 clearly shows the boundary maximum effect of the AF criterion. Expanding the searched area (and, of course, its boundary), the maximum is reached on the new boundary (of the new cubic surface).

Alternatively, the search may be made on a spherical surface with an expanding radius. The results of such an example are illustrated in Fig. 8.5, where only radii of 1, 2, …, 10 mm are given. As the radius expands, the AF maximum becomes greater and is always reached over the spherical surface with the maximum radius.

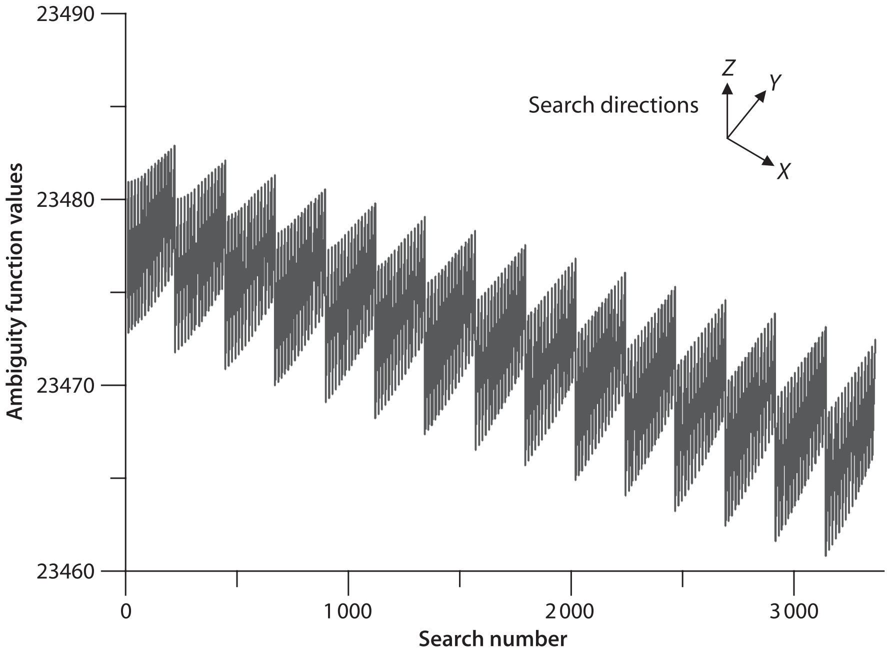

Fig. 8.4. 3-D coordinate search using ambiguity function

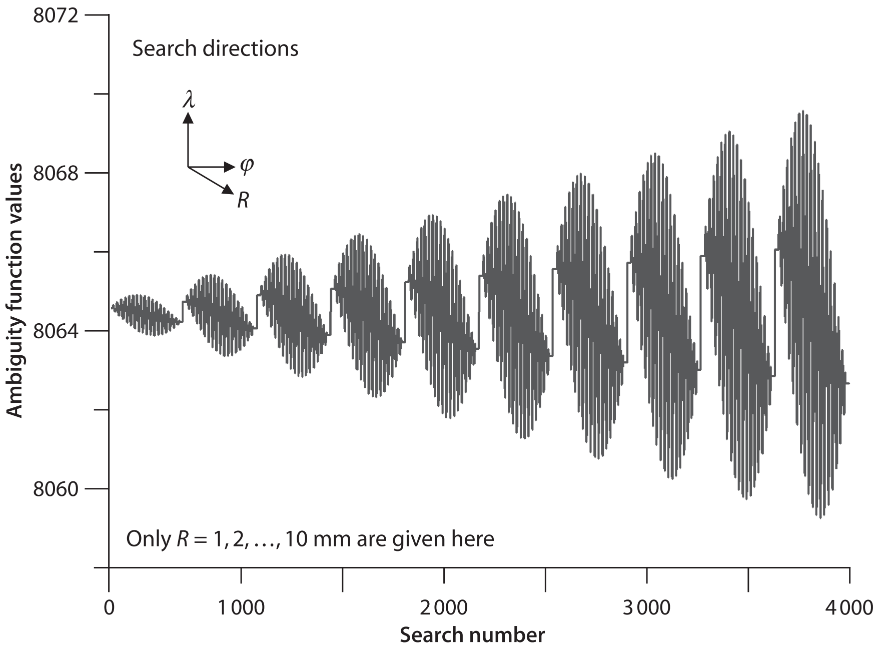

Fig. 8.5. Spherical coordinate search using ambiguity function

Theoretical Indications

The AF Eq. 8.53 is rewritten as

$$\sum_{k=1}^{n_k} G(t_k) \Rightarrow \max \,, \tag{8.54}$$

$$G(t_k) = |S_k|, \quad S_k = \sum_{j=1}^{n_j} \mathrm{e}^{i2\pi \cdot v_j(t_k)} \,, \tag{8.55}$$

$$v_j(t_k) = \Phi_j(t_k) - \rho_j(t_k)/\lambda, \quad Y \in \Omega \quad \text{and} \tag{8.56}$$

$$\mathrm{e}^{i2\pi \cdot v_j(t_k)} = \cos(2\pi v_j(t_k)) + i \sin(2\pi v_j(t_k)) \,, \tag{8.57}$$

where Y is the coordinate vector, Ω is to be the searched coordinate area and is a closed area (i.e., it includes the boundary Γ), $v_j(t_k)$ are the residuals of GPS observation equations (a continuous function of Y), S_k is a complex function of Y, and $G(t_k)$ is the modulus of S_k.

If the GPS data sampling intervals are sufficiently close and the numerical integration error is negligible (cf. Xu 1992), then one has

$$\frac{1}{n_k} \sum_{k=1}^{n_k} G(t_k) = \frac{1}{T} \int_{t_1}^{t_e} G(t) \cdot \mathrm{d}t \,, \tag{8.58}$$

where $T = t_e - t_1$, $t_e = t(n_k)$, and t_1 and t_e are the beginning and end time of the observations. According to the *middle value theorem* of the integration (cf., e.g., Bronstain and Semendjajew 1987; Wang et al. 1979) (such a theorem can be found in all integration related books), one has a time point ξ ($t_1 < \xi < t_e$) so that

$$\frac{1}{T} \int_{t_1}^{t_e} G(t) \cdot \mathrm{d}t = G(\xi) \,, \tag{8.59}$$

i.e., the AF can be represented by a unique $G(t)$ at time ξ (the constant factor is omitted here). Equation 8.54 turns out to be

$$G(\xi) \Rightarrow \max \,. \tag{8.60}$$

Because of the definition of AF, $G(\xi)$ is a modulus of a complex function.

In complex function analysis theory, there is a so-called maximum theorem (cf., e.g., Bronstain and Semendjajew 1987; Wang et al. 1979), i.e.:

Maximum Modulus Theorem: if complex function $f(z)$ is analytic within a limited area Z and is continuous over the closed Z, then modulus $|f(z)|$ reaches the maximum on the boundary Γ of Z.

However, such a theorem cannot be directly used for Eq. 8.60 because the theorem is valid only for the analytic complex function defined over a complex plane, whereas function $G(\xi)$ is a complicated three-dimensional complex function.

Maybe the interested reader will consider this in detail and find out a theoretical proof.