Chapter 6

GPS Observation Equations and Equivalence Properties

In this chapter, first the general mathematical model of GPS observation and its linearisation are discussed. All partial derivatives of the observational function are given in detail. These are necessary for forming GPS observation equations. Then, linear transformation and covariance propagation are outlined. In the data combinations section, all meaningful and useful data combinations are discussed, such as ionosphere-free, geometry-free, code-phase combinations, ionospheric residuals, as well as differential Doppler and Doppler integration. In the section of data differentiation, single, double and triple differences as well as their related observation equations and weight propagation are discussed. The parameters in the equations are greatly reduced through difference forming; however, the covariance derivations are tedious. In the last two sections, the equivalent properties between the uncombined and combining as well as undifferenced and differencing algorithms are discussed. A unified GPS data processing method is proposed in detail. The method is selectively equivalent to the zero-, single-, double-, triple-, and user-defined differential methods.

6.1 General Mathematical Models of GPS Observations

Recalling the discussions in Chap. 4, the GPS code pseudorange, carrier phase and Doppler observables are formulated as (cf. Eqs. 4.7, 4.18, 4.23):

$$R_i^k(t_\mathrm{r},t_\mathrm{e}) = \rho_i^k(t_\mathrm{r},t_\mathrm{e}) - (\delta t_\mathrm{r} - \delta t_\mathrm{k})c + \delta_\mathrm{ion} + \delta_\mathrm{trop} + \delta_\mathrm{tide} + \delta_\mathrm{rel} + \varepsilon_\mathrm{c}\,, \tag{6.1}$$

$$\lambda \Phi_i^k(t_\mathrm{r},t_\mathrm{e}) = \rho_i^k(t_\mathrm{r},t_\mathrm{e}) - (\delta t_\mathrm{r} - \delta t_\mathrm{k})c + \lambda N_i^k - \delta_\mathrm{ion} + \delta_\mathrm{trop} + \delta_\mathrm{tide} + \delta_\mathrm{rel} + \varepsilon_\mathrm{p} \quad \text{and} \tag{6.2}$$

$$D = \frac{\mathrm{d}\rho_i^k(t_\mathrm{r},t_\mathrm{e})}{\lambda \mathrm{d}t} - f\frac{\mathrm{d}(\delta t_\mathrm{r} - \delta t_\mathrm{k})}{\mathrm{d}t} + \delta_{\mathrm{rel}\_f} + \varepsilon_\mathrm{d}\,. \tag{6.3}$$

Where ionospheric effects can be approximated as (cf. Sect. 5.1.2, Eq. 5.26)

$$\delta_\mathrm{ion} = \frac{A_1}{f^2} + \frac{A_2}{f^3}\,,$$

and R is the observed pseudorange, Φ is the observed phase, D is Doppler measurement, t_e denotes the GPS signal emission time of the satellite k, t_r denotes the GPS signal reception time of the receiver i, c denotes the speed of light, subscript i and superscript k denote the receiver and satellite, and δt_r and δt_k denote the clock errors of the receiver and satellite at the time t_r and t_e, respectively. The terms δ_{ion}, δ_{trop}, δ_{tide}, and δ_{rel} denote the ionospheric, tropospheric, tidal and relativistic effects, respectively. Tidal effects include Earth tide and ocean loading tide effects. The multipath effect has been discussed in Sect. 5.6 and is omitted here. ε_c, ε_p and ε_d are the remaining errors, respectively. f is the frequency, wavelength is denoted by λ, A_1 and A_2 are ionospheric parameters, N_i^k is the ambiguity related to receiver i and satellite k, δ_{rel_f} is the frequency correction of the relativistic effects, the ρ_i^k is the geometric distance, and (cf. Eq. 4.6)

$$\rho_i^k(t_r, t_e) = \rho_i^k(t_r) + \frac{d\rho_i^k(t_r)}{dt}\Delta t \ , \tag{6.4}$$

where Δt denotes the signal transmitting time and $\Delta t = t_r - t_e$. $d\rho_i^k(t_r)/dt$ denotes the time derivation of the radial distance between the satellite and receiver at the time t_r. All terms in Eqs. 6.1 and 6.2 have units of length (meters).

Considering Eq. 6.4 in the ECEF coordinate system, the geometric distance is a function of station state vector $(x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)$ (denoted by X_i) and satellite state vector $(x_k, y_k, z_k, \dot{x}_k, \dot{y}_k, \dot{z}_k)$ (denoted by X_k). GPS observation Eqs. 6.1, 6.2 and 6.3 can then be generally presented as

$$O = F(X_i, \ X_k, \ \delta t_i, \ \delta t_k, \ \delta_{ion}, \ \delta_{trop}, \ \delta_{tide}, \ \delta_{rel}, \ N_i^k, \ \delta_{rel\_f}) \ , \tag{6.5}$$

where O denotes observation and F denotes implicit function. In other words, the GPS observable is a function of state vectors of the station and satellite, and numbers of physical effects as well as ambiguity parameters. In principle, through GPS observations, the desired parameters of the function in Eq. 6.5 can be solved for. This is why nowadays GPS has been widely used for positioning and navigation (to determine the state vector of the station), for orbit determination (to determine the state vector of satellite), for timing (to synchronise clocks), for meteorological applications (i.e. troposphere profiling), and for ionospheric occultation (i.e. ionosphere sounding). In turn, the satellite orbit is a function of the gravitational field of the Earth and numbers of disturbing effects such as solar radiation pressure and atmospheric drags. GPS is now also used for gravity field mapping, as well as solar and Earth system study.

It is obvious that Eq. 6.5 is a non-linear one. The straightforward mathematical method to solve problem 6.5 is to search for the optimal solution by using some effective search algorithms. The so-called ambiguity function (AF, see Sect. 8.5 and Sect. 12.2) method is one of the examples. Generally speaking, solving a non-linear problem is much more complicated than first linearising the problem and then solving the linearised problem.

It is notable that the satellite state vector and the station state vector shall be represented in the same coordinate system; otherwise coordinate transformation discussed in Chap. 2 shall be made. Because the rotations are "distance keeping" transformations, the distances computed in two different coordinate systems must be the same. However, because of the Earth's rotation, the velocities expressed in the ECI and ECEF coordinate systems are not the same. Generally, the station coordinates and ionospheric effects as well as tropospheric effects are given and presented in the ECEF system. A satellite state vector may be given in both the ECSF system and the ECEF system. This depends on the need of the concerned applications.

6.2
Linearisation of the Observational Model

The non-linear multivariable function F in Eq. 6.5 can be further generalised as

$$O = F(Y) = F(y_1, y_2, \ldots, y_n) \ , \tag{6.6}$$

where variable vector Y has n elements. The linearisation is accomplished by expanding the function in a Taylor series to the first order (linear term) as

$$O = F(Y^0) + \left.\frac{\partial F(Y)}{\partial Y}\right|_{Y^0} \cdot \mathrm{d}Y + \varepsilon(\mathrm{d}Y) \ , \tag{6.7}$$

where

$$\frac{\partial F(Y)}{\partial Y} = \begin{pmatrix} \frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} & \cdots & \frac{\partial F}{\partial y_n} \end{pmatrix}, \quad \text{and} \quad \mathrm{d}Y = \left(Y - Y^0\right) = \begin{pmatrix} \mathrm{d}y_1 \\ \mathrm{d}y_2 \\ \vdots \\ \mathrm{d}y_n \end{pmatrix} ;$$

the symbol $|_{Y^0}$ means that the partial derivative $\partial F(Y)/\partial Y$ takes the value of $Y = Y^0$ and ε is the truncating error, which is a function of the second order partial derivative and $\mathrm{d}Y$. Y^0 is called the initial value vector. Equation 6.7 turns out then to be

$$O - C = \begin{pmatrix} \frac{\partial F}{\partial y_1} & \frac{\partial F}{\partial y_2} & \cdots & \frac{\partial F}{\partial y_n} \end{pmatrix}_{Y^0} \cdot \begin{pmatrix} \mathrm{d}y_1 \\ \mathrm{d}y_2 \\ \vdots \\ \mathrm{d}y_n \end{pmatrix} + \varepsilon \ , \tag{6.8}$$

where $F(Y^0)$ is denoted by C (or say, the computed value). So GPS observation Eq. 6.6 is linearised as a linear equation (Eq. 6.8). Denoting the observational error and truncating error as v and $O - C$ as l, partial derivative $(\partial F/\partial y_j)|_{Y^0} = a_j$, then Eq. 6.8 can be written as

$$l_i = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y_1 \\ \mathrm{d}y_2 \\ \vdots \\ \mathrm{d}y_n \end{pmatrix} + v_i \ , \quad (i = 1, 2, \ldots, m) \ , \tag{6.9}$$

where l is also often called "observable" in adjustment or $O - C$ (observed minus computed), and j and i are indices of unknowns and the observations. Equation 6.9 is a linear error equation. A set of GPS observables then forms a linear error equation system:

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}y_1 \\ \mathrm{d}y_2 \\ \vdots \\ \mathrm{d}y_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} ,$$

or in matrix form (denotes $\mathrm{d}Y$ by X)

$$L = AX + V , \tag{6.10}$$

where m is the observable number. A number of adjustment and filtering methods (cf. Chap. 7) can be applied for solving the GPS problem 6.10. The solved parameter vector is X (or $\mathrm{d}Y$). The original unknown vector Y can be obtained by adding $\mathrm{d}Y$ to Y^0. V is the residual vector. Statistically, V shall be assumed to be a random vector, and is normally distributed with zero expectation and variance var(V). To characterise the different qualities and correlation situations of the observables, a so-called weight matrix P is introduced to Eq. 6.10. Supposing all observations are linearly independent or un-correlated, the covariance of observable vector L is

$$Q_{LL} = \mathrm{cov}(L) = \sigma^2 E \tag{6.11}$$

or

$$P = Q_{LL}^{-1} = \frac{1}{\sigma^2} E , \tag{6.12}$$

where E is an identity matrix of dimension $m \times m$, superscript $^{-1}$ is an inversion operator, and cov(L) is covariance of L.

Generally, only if the solved unknown vector $\mathrm{d}Y$ is small enough, the linearisation process can be considered done well. Therefore, the initial vector Y^0 has to be carefully given. In case the initial vector is not well-known or not well-given, the linearisation process has to be iterated. In other words, the initial vector that is not well-known has to be modified by the solved vector $\mathrm{d}Y$, and the linearisation process has to be made again until $\mathrm{d}Y$ converges. If $X = 0$, then $L = V$; therefore, the "observable" vector L is also called a residual vector sometimes. If the initial vector Y^0 is well-known or well-given, then the residual vector V can also be used as a criterion to judge the "goodness or badness" of the original observable vector. This property is used in robust Kalman filtering to adjust the weight of the observable (cf. Chap. 7).

6.3 Partial Derivatives of Observational Function

Partial Derivatives of Geometric Path Distance with Respect to the State Vector (x_i, y_i, z_i, $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$) of the GPS Receiver

The signal transmitting path is described by (cf. Eqs. 4.3 and 4.6 in Chap. 4)

$$\rho_i^k(t_\mathrm{r}, t_\mathrm{e}) = \sqrt{(x_k(t_\mathrm{e}) - x_i)^2 + (y_k(t_\mathrm{e}) - y_i)^2 + (z_k(t_\mathrm{e}) - z_i)^2} \quad , \quad \text{and} \tag{6.13}$$

$$\rho_i^k(t_r,t_e) \approx \rho_i^k(t_r,t_r) + \frac{d\rho_i^k(t_r,t_r)}{dt}\Delta t \quad , \tag{6.14}$$

where index k denotes the satellite, and the satellite coordinates are related to the signal emission time t_e, i denotes the station, and the station coordinates are related to the signal reception time t_r, $\Delta t = t_e - t_r$. Then one has

$$\frac{d\rho_i^k(t_r,t_r)}{dt} = \frac{1}{\rho_i^k(t_r,t_r)} \times \left((x_k - x_i)(\dot{x}_k - \dot{x}_i) \quad (y_k - y_i)(\dot{y}_k - \dot{y}_i) \quad (z_k - z_i)(\dot{z}_k - \dot{z}_i)\right), \tag{6.15}$$

where the satellite state vector is related to the time t_r, and

$$\frac{\partial\rho_i^k(t_r,t_e)}{\partial(x_i,y_i,z_i)} = \frac{-1}{\rho_i^k(t_r,t_e)}\left(x_k - x_i \quad y_k - y_i \quad z_k - z_i\right) , \tag{6.16}$$

$$\frac{\partial\rho_i^k(t_r,t_e)}{\partial(\dot{x}_i,\dot{y}_i,\dot{z}_i)} = \frac{-\Delta t}{\rho_i^k(t_r,t_r)}\left(x_k - x_i \quad y_k - y_i \quad z_k - z_i\right) . \tag{6.17}$$

Partial Derivatives of Geometric Path Distance with Respect to the State Vector ($x_k, y_k, z_k, \dot{x}_k, \dot{y}_k, \dot{z}_k$) of the GPS Satellite

Similar to above, one has

$$\frac{\partial\rho_i^k(t_r,t_e)}{\partial(x_k,y_k,z_k)} = \frac{1}{\rho_i^k(t_r,t_e)}\left(x_k - x_i \quad y_k - y_i \quad z_k - z_i\right) , \tag{6.18}$$

$$\frac{\partial\rho_i^k(t_r,t_e)}{\partial(\dot{x}_k,\dot{y}_k,\dot{z}_k)} = \frac{\Delta t}{\rho_i^k(t_r,t_r)}\left(x_k - x_i \quad y_k - y_i \quad z_k - z_i\right) . \tag{6.19}$$

Partial Derivatives of the Doppler Observable with Respect to the Velocity Vector of the Station

The time differentiation of the geometric signal path distance can be derived as

$$\frac{d\rho_i^k(t_r,t_e)}{dt} = \frac{1}{\rho_i^k(t_r,t_e)} \quad ; \tag{6.20}$$
$$((x_k(t_e)-x_i)(\dot{x}_k(t_e)-\dot{x}_i)+(y_k(t_e)-y_i)(\dot{y}_k(t_e)-\dot{y}_i)+(z_k(t_e)-z_i)(\dot{z}_k(t_e)-\dot{z}_i))$$

then one has

$$\frac{\partial(d\rho_i^k(t_r,t_e)/dt)}{\partial(\dot{x}_i,\dot{y}_i,\dot{z}_i)} = \frac{-1}{\rho_i^k(t_r,t_e)}\left(x_k(t_e) - x_i \quad y_k(t_e) - y_i \quad z_k(t_e) - z_i\right) . \tag{6.21}$$

Partial Derivatives of Clock Errors with Respect to the Clock Parameters

If the clock errors are modelled by Eq. 5.163 (cf. Sect. 5.5)

$$\delta t_i = b_i + d_i t + e_i t^2\,, \quad \delta t_k = b_k + d_k t + e_k t^2\,, \tag{6.22}$$

where *i* and *k* are the indices of the clock error parameters of the receiver and satellite, then one has

$$\frac{\partial \delta t_i}{\partial (b_i, d_i, e_i)} = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \quad \text{and}$$

$$\frac{\partial \delta t_k}{\partial (b_k, d_k, e_k)} = \begin{pmatrix} 1 & t & t^2 \end{pmatrix}. \tag{6.23}$$

If the clock errors are modelled by Eq. 5.164 (cf. Sect. 5.5)

$$\delta t_i = b_i\,, \quad \delta t_k = b_k\,, \tag{6.24}$$

then

$$\frac{\partial \delta t_i}{\partial b_i} = 1\,, \quad \frac{\partial \delta t_k}{\partial b_k} = 1\,. \tag{6.25}$$

The above derivatives are valid for both the code and phase observable equations. For the Doppler observable, denote (cf. Eq. 6.3)

$$\delta_{\text{clock}} = f\,\frac{\mathrm{d}(\delta t_i - \delta t_k)}{\mathrm{d}t}\,, \tag{6.26}$$

then for the clock error model of Eq. 6.22 one has

$$\frac{\partial \delta_{\text{clock}}}{\partial (d_i, e_i)} = \begin{pmatrix} 1 & 2t \end{pmatrix} f \quad \text{and} \quad \frac{\partial \delta_{\text{clock}}}{\partial (d_k, e_k)} = -\begin{pmatrix} 1 & 2t \end{pmatrix} f\,. \tag{6.27}$$

Partial Derivatives of Tropospheric Effects with Respect to the Tropospheric Parameters

If the tropospheric effects can be modelled by (cf. Sect. 5.2)

$$\text{I:} \quad \delta_{\text{trop}} = f_{\text{p}}\mathrm{d}\rho \quad \text{and}$$

$$\text{II:} \quad \delta_{\text{trop}} = \frac{f_{\text{z}}\mathrm{d}\rho}{F} + \frac{f_{\text{a}}\mathrm{d}\rho}{F_{\text{c}}}\,, \tag{6.28}$$

where $\mathrm{d}\rho$ is the tropospheric effect computed by using the standard tropospheric model, f_{p}, f_{z}, f_{a} are parameters of the tropospheric delay in path, zenith, azimuth directions, and F and F_c are the mapping and co-mapping functions discussed in Sect. 5.2. The derivatives with respect to the parameters f_p, f_z, f_a are then

$$\text{I:}\quad \frac{\partial \delta_{\text{trop}}}{\partial f_p} = \mathrm{d}\rho \quad \text{and}$$

$$\text{II:}\quad \frac{\partial \delta_{\text{trop}}}{\partial (f_z, f_a)} = \begin{pmatrix} \dfrac{\mathrm{d}\rho}{F} & \dfrac{\mathrm{d}\rho}{F_c} \end{pmatrix}. \tag{6.29}$$

Furthermore, if the tropospheric parameters are defined as a step function or first order polynomial (cf. Sect. 5.2) by

$$\text{I:}\quad f_p = f_z = f_j \quad \text{if} \quad t_{j-1} < t \le t_j, \quad j = 1, 2, \ldots, n \quad \text{and}$$

$$\text{II:}\quad f_p = f_z = f_{j-1} + (f_j - f_{j-1})\frac{t - t_{j-1}}{\Delta t} \quad \text{if} \quad t_{j-1} < t \le t_j, \quad j = 1, 2, \ldots, n+1, \tag{6.30}$$

where $\Delta t = (t_n - t_0) / n$, t_0 and t_n are the beginning and the ending times of the GPS survey, and Δt is usually selected by two to four hours. Then one has

$$\text{I:}\quad \frac{\partial f_p}{\partial f_j} = \frac{\partial f_z}{\partial f_j} = 1 \quad \text{and}$$

$$\text{II:}\quad \frac{\partial f_p}{\partial (f_{j-1}, f_j)} = \frac{\partial f_z}{\partial (f_{j-1}, f_j)} = \begin{pmatrix} 1 + \dfrac{-t + t_{j-1}}{\Delta t} & \dfrac{t - t_{j-1}}{\Delta t} \end{pmatrix}. \tag{6.31}$$

The azimuth dependency may be assumed to be (cf. Eq. 5.121)

$$f_a = g_1 \cos a + g_2 \sin a, \tag{6.32}$$

where a is the azimuth, and g_1 and g_2 are called azimuth-dependent parameters. Then one gets

$$\frac{\partial f_a}{\partial (g_1, g_2)} = \begin{pmatrix} \cos a & \sin a \end{pmatrix}. \tag{6.33}$$

If parameters g_1 and g_2 are also defined as step functions or first order polynomials like Eq. 6.30, the partial derivatives can be obtained in a similar manner to Eq. 6.31.

Partial Derivatives of the Phase Observable with Respect to the Ambiguity Parameters

Depending on which scale one prefers, there is

$$\frac{\partial \lambda N}{\partial \lambda N} = 1 \quad \text{or} \quad \frac{\partial \lambda N}{\partial N} = \lambda. \tag{6.34}$$

Partial Derivatives of Tidal Effects with Respect to the Tidal Parameters

If the Earth tide model in Eqs. 5.147 and 5.149 are used, then the tidal effects can be generally written as

$$\delta_{\text{earth-tide}} = s_1 h_2 + s_2 l_2 + s_3 h_3 \ , \tag{6.35}$$

where s_1, s_2 and s_3 are the coefficient functions, which are given in Sect. 5.4.2 in detail, and h_2, h_3 and l_2 are the love numbers and Shida number, respectively. Then one has

$$\frac{\partial \delta_{\text{earth-tide}}}{\partial (h_2, l_2, h_3)} = \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} \ . \tag{6.36}$$

Ocean loading tide effects can be modelled as

$$\delta_{\text{loading-tide}} = f_{\text{load}} \begin{pmatrix} \mathrm{d}x_{\text{load}} & \mathrm{d}y_{\text{load}} & \mathrm{d}z_{\text{load}} \end{pmatrix} \ , \tag{6.37}$$

where f_{load} is the factor of the computed ocean loading effect vector $(\mathrm{d}x_{\text{load}} \ \ \mathrm{d}y_{\text{load}} \ \ \mathrm{d}z_{\text{load}})$. Then one has

$$\frac{\partial \delta_{\text{loading-tide}}}{\partial f_{\text{load}}} = \begin{pmatrix} \mathrm{d}x_{\text{load}} & \mathrm{d}y_{\text{load}} & \mathrm{d}z_{\text{load}} \end{pmatrix} \ . \tag{6.38}$$

6.4 Linear Transformation and Covariance Propagation

For any linear equation system

$$L = AX \quad \text{or} \tag{6.39}$$

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ,$$

a linear transformation can be defined as a multiplying operation of matrix T to Eq. 6.39, i.e.,

$$TL = TAX \quad \text{or} \tag{6.40}$$

$$\begin{pmatrix} t_{11} & t_{12} & \dots & t_{1m} \\ t_{21} & t_{22} & \dots & t_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ t_{k1} & t_{k2} & \dots & t_{km} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1m} \\ t_{21} & t_{22} & \dots & t_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ t_{k1} & t_{k2} & \dots & t_{km} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} ,$$

where T is called the linear transformation matrix and has a dimension of $k \times$ m. An inverse transformation of T is denoted by T^{-1}. An invertible linear transformation does not change the property (and solutions) of the original linear equations. This may be verified by multiplying T^{-1} to Eq. 6.40. A non-invertible linear transformation is called a rank deficient (or not full rank) transformation.

The covariance matrix of L is denoted by cov(L) or Q_{LL} (cf. Sect. 6.2); then the covariance of the transformed L (i.e., TL) can be obtained by covariance propagation theorem by (cf., e.g., Koch 1988)

$$\mathrm{cov}(TL) = T\,\mathrm{cov}(L)T^T = TQ_{LL}T^T \ , \tag{6.41}$$

where superscript T denotes the transpose of the transformation matrix.

If transformation matrix T is a vector (i.e., $k = 1$) and L is an inhomogeneous and independent observable vector (i.e., covariance matrix Q_{LL} is a diagonal matrix with elements of σ_j^2, where σ_j^2 is the variance (σ_j is called standard deviation) of the observable l_j), then Eqs. 6.40 and 6.41 can be written as

$$\begin{pmatrix} t_1 & t_2 & \dots & t_m \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{pmatrix} = \begin{pmatrix} t_1 & t_2 & \dots & t_m \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and}$$

$$\mathrm{cov}(TL) = \begin{pmatrix} t_1 & t_2 & \dots & t_m \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_m^2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix} . \tag{6.42}$$

Denoting cov(TL) as σ_{TL}^2, one gets

$$\sigma_{TL}^2 = t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + \ldots + t_m^2\sigma_m^2 = \sum_{j=1}^{m} t_j^2\sigma_j^2 \ . \tag{6.43}$$

Equation 6.43 is called the error propagation theorem.

6.5 Data Combinations

Data combinations are methods of combining GPS data measured with the same receiver at the same station. Usually, observables are the code pseudoranges, carrier phases and Doppler at working frequencies such as C/A code, P_1 and P_2 code, L1 phase Φ_1 and L2 phase Φ_2, and Doppler D_1 and D_2. In the future, there will also be P_5 code, L5 phase Φ_5 and Doppler D_5. According to the observation equations of the observables, a suitable combination can be advantageous for understanding and solving GPS problems.

For convenience, code, phase and Doppler observables are simplified and rewritten as (cf. Eqs. 6.1–6.3)

$$R_j = \rho - (\delta t_\mathrm{r} - \delta t_\mathrm{k})c + \delta_\mathrm{ion}(j) + \delta_\mathrm{trop} + \delta_\mathrm{tide} + \delta_\mathrm{rel} + \varepsilon_\mathrm{c}\,, \tag{6.44}$$

$$\lambda_j \Phi_j = \rho - (\delta t_\mathrm{r} - \delta t_\mathrm{k})c + \lambda_j N_j - \delta_\mathrm{ion}(j) + \delta_\mathrm{trop} + \delta_\mathrm{tide} + \delta_\mathrm{rel} + \varepsilon_\mathrm{p}\,, \tag{6.45}$$

$$D_j = \frac{\mathrm{d}\rho}{\lambda_j \mathrm{d}t} - f_j \frac{\mathrm{d}(\delta t_\mathrm{r} - \delta t_\mathrm{k})}{\mathrm{d}t} + \varepsilon_\mathrm{d} \quad \text{and} \tag{6.46}$$

$$\delta_\mathrm{ion}(j) = \frac{A_1}{f_j^2} + \frac{A_2}{f_j^3}\ . \tag{6.47}$$

Where j is the index of frequency f, the means of the other symbols are the same as the notes of Eqs. 6.1–6.3. Equation 6.47 is an approximation for code.

A general code-code combination can be formed by $n_1R_1 + n_2R_2 + n_5R_5$, where n_1, n_2 and n_5 are arbitrary constants. However, in order to make such a combination that still has the sense of a code survey, a standardised combination has to be formed by

$$R = \frac{n_1R_1 + n_2R_2 + n_5R_5}{n_1 + n_2 + n_5}\ . \tag{6.48}$$

The newly-formed code R can then be interpreted as a weight-averaged code survey of R_1, R_2 and R_5. The mathematical model of the observable Eq. 6.44 is generally still valid for R. Denoting the standard deviation of code observable R_i as σ_{ci} $(i = 1, 2, 5)$, the newly-formed code observation R has the standard deviation of

$$\sigma_\mathrm{c}^2 = \frac{1}{(n_1 + n_2 + n_5)^2}(n_1^2\sigma_\mathrm{c1}^2 + n_2^2\sigma_\mathrm{c2}^2 + n_5^2\sigma_\mathrm{c5}^2)\ .$$

Because of

$$\left|\frac{n_1 + n_2 + \ldots + n_m}{m}\right| \le \sqrt{\frac{n_1^2 + n_2^2 + \ldots + n_m^2}{m}}\ ,$$

(cf., e.g., Wang et al. 1979; Bronstein and Semendjajew 1987), one has the property of

$$(n_1 + n_2 + \ldots + n_m)^2 \le m(n_1^2 + n_2^2 + \ldots + n_m^2)\ ,$$

where m is the maximum index. Therefore in our case, one has

$$\sigma_c^2 \ge m \cdot \min\left\{\sigma_\mathrm{c1}^2, \sigma_\mathrm{c2}^2, \sigma_\mathrm{c5}^2\right\}\ , \quad m = 2 \text{ or } 3$$

for combinations of two or three code observables.

A general phase-phase linear combination can be formed by

$$\Phi = n_1\Phi_1 + n_2\Phi_2 + n_5\Phi_5\,, \tag{6.49}$$

where the combined signal has the frequency and wavelength

$$f = n_1 f_1 + n_2 f_2 + n_5 f_5 \quad \text{and} \quad \lambda = \frac{c}{f}\,. \tag{6.50}$$

$\lambda\Phi$ means the measured distance (with ambiguity!) and can be presented alternatively as

$$\lambda\Phi = \frac{1}{f^2}\left(n_1 f_1 \lambda_1 \Phi_1 + n_2 f_2 \lambda_2 \Phi_2 + n_5 f_5 \lambda_5 \Phi_5\right). \tag{6.51}$$

Mathematical model of Eq. 6.45 is generally still valid for the newly-formed $\lambda\Phi$. Denoting the standard deviation of phase observable $\lambda_i \Phi_i$ as σ_i ($i = 1, 2, 5$), the newly-formed observation has a variance of

$$\sigma^2 = \frac{1}{f^2}(n_1^2 f_1^2 \sigma_1^2 + n_2^2 f_2^2 \sigma_2^2 + n_5^2 f_5^2 \sigma_5^2) \quad \text{and} \tag{6.52}$$

$$\sigma^2 \geq m \cdot \min\left\{\sigma_1^2, \sigma_2^2, \sigma_5^2\right\},$$

with $m = 2$ or 3 for combinations of two or three phases.

That is, the data combination will degrade the quality of the original data.

Linear combinations $\Phi_W = \Phi_1 - \Phi_2$ and $\Phi_X = 2\Phi_1 - \Phi_2$ are called wide-lane and x-lane combinations with wavelengths of about 86.2 cm and 15.5 cm. They reduce the first order ionospheric effects on frequency f_2 to 40% and 20%, respectively. $\Phi_N = \Phi_1 + \Phi_2$ is called a narrow-lane combination.

6.5.1 Ionosphere-Free Combinations

Due to Eqs. 6.44–6.47, phase-phase and code-code ionosphere-free combinations can be formed by (cf. Sect. 5.1)

$$\lambda\Phi = \frac{f_1^2 \lambda_1 \Phi_1 - f_2^2 \lambda_2 \Phi_2}{f_1^2 - f_2^2} = \lambda(f_1 \Phi_1 - f_2 \Phi_2) \quad \text{and} \tag{6.53}$$

$$R = \frac{f_1^2 R_1 - f_2^2 R_2}{f_1^2 - f_2^2}\,. \tag{6.54}$$

The related observation equations can be formed from Eqs. 6.44 and 6.45 as

$$R = \rho - (\delta t_r - \delta t_k)c + \delta_{trop} + \delta_{tide} + \delta_{rel} + \varepsilon_{cc} \quad \text{and} \tag{6.55}$$

$$\lambda\Phi = \rho - (\delta t_r - \delta t_k)c + \lambda N + \delta_{trop} + \delta_{tide} + \delta_{rel} + \varepsilon_{pc}\,, \tag{6.56}$$

where

$$N = f_1 N_1 - f_2 N_2\,, \quad \lambda = \frac{c}{f_1^2 - f_2^2}\,, \tag{6.57}$$

ε_{cc} and ε_{pc} denote the residuals after the combination of code and phase, respectively.

The advantages of such ionosphere-free combinations are that the ionospheric effects have disappeared from the observation Eqs. 6.55 and 6.56 and the other terms of the equations have remained the same. However, the combined ambiguity is not an integer anymore, and the combined observables have higher standard deviations. Equations 6.55 and 6.56 are indeed first order ionosphere-free combinations.

Second order ionosphere-free combinations can be formed by (see Sect. 5.1.2 for details)

$$\lambda\Phi = C_1\lambda_1\Phi_1 + C_2\lambda_2\Phi_2 + C_5\lambda_5\Phi_5 \quad \text{and} \tag{6.58}$$

$$R = C_1R_1 + C_2R_2 + C_5R_5 \,, \tag{6.59}$$

where

$$C_1 = \frac{f_1^3(f_5 - f_2)}{C_4}, \quad C_2 = \frac{-f_2^3(f_5 - f_1)}{C_4},$$

$$C_5 = \frac{f_5^3(f_2 - f_1)}{C_4}, \quad C_4 = f_1^3(f_5 - f_2) - f_2^3(f_5 - f_1) + f_5^3(f_2 - f_1),$$

$$\lambda = \frac{c}{C_4}, \quad N = C_4(C_1N_1 + C_2N_2 + C_5N_5)\,.$$

The related observation equations are the same as Eqs. 6.55 and 6.56, with λ and N given above.

6.5.2 Geometry-Free Combinations

Due to Eqs. 6.44–6.46, code-code, phase-phase and phase-code geometry-free combinations can be formed by

$$R_1 - R_2 = \delta_{\text{ion}}(1) - \delta_{\text{ion}}(2) + \Delta\varepsilon_{\text{c}} = \frac{A_1}{f_1^2} - \frac{A_1}{f_2^2} + \Delta\varepsilon_{\text{c}} \,, \tag{6.60}$$

$$\lambda_1\Phi_1 - \lambda_2\Phi_2 = \lambda_1N_1 - \lambda_2N_2 - \frac{A_1}{f_1^2} + \frac{A_1}{f_2^2} + \Delta\varepsilon_{\text{p}} \,, \tag{6.61}$$

$$\lambda_1D_1 - \lambda_2D_2 = \Delta\varepsilon_{\text{d}} \,, \tag{6.62}$$

$$\lambda_j\Phi_j - R_j = \lambda_jN_j - 2\delta_{\text{ion}}(j) + \Delta\varepsilon_{\text{pc}} \quad \text{and} \quad j{=}1{,}2{,}5 \,, \tag{6.63}$$

where

$$\Delta\delta_{\text{ion}} = \delta_{\text{ion}}(1) - \delta_{\text{ion}}(2) = \frac{A_1}{f_1^2} - \frac{A_1}{f_2^2} \,. \tag{6.64}$$

For an ionospheric model of the second order, one has approximately

$$\Delta\delta_{\text{ion}} = \delta_{\text{ion}}(1) - \delta_{\text{ion}}(2) = \frac{A_1}{f_1^2} - \frac{A_1}{f_2^2} + \frac{A_2}{f_1^3} - \frac{A_2}{f_2^3} \ .$$

The geometry-free code-code and phase-phase combinations cancel out all other terms in the observation equations except the ionospheric term and the ambiguity parameters. Recalling the discussions of Sect. 5.1, δ_{ion} is the ionospheric path delay and can be considered a mapping of the zenith delay δ_{ion}^z or $\delta_{\text{ion}} = \delta_{\text{ion}}^z F$, where F is the mapping function (cf. Sect. 5.1). So one has

$$\delta_{\text{ion}}(1) = \frac{A_1^z}{f_1^2} F = \frac{A_1}{f_1^2} \ , \tag{6.65}$$

where A_1 and A_1^z have the physical meaning of total electronic contents at the signal path direction and the zenith direction, respectively. A_1^z is then independent from the zenith angle of the satellite. If the variability of the electronic contents at the zenith direction is stable enough, A_1^z can be modelled by a step function or a first order polynomial with a reasonably short time interval Δt by

$$A_1^z = g_j \quad \text{if} \quad t_{j-1} < t \le t_j \ , \quad j = 1, 2, \ldots, n+1 \tag{6.66}$$

or

$$A_1^z = g_{j-1} + (g_j - g_{j-1}) \frac{t - t_{j-1}}{\Delta t} \quad \text{if} \quad t_{j-1} < t \le t_j, \quad j = 1, 2, \ldots, n+1 \ , \tag{6.67}$$

where $\Delta t = (t_n - t_0) / n$, and t_0 and t_n are the beginning and ending time of the GPS survey. Δt can be, e.g., selected by 30 minutes. g_j is the coefficient of the polynomial.

Geometry-free combinations of Eqs. 6.60, 6.61 and 6.63 (only for $j = 1$) can be considered a linear transformation of the original observable vector $L = (R_1 \ \ R_2 \ \ \lambda_1\Phi_1 \ \ \lambda_2\Phi_2)^T$ by

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & g \\ \lambda_1 & -\lambda_2 & -g \\ \lambda_1 & 0 & d \end{pmatrix} \cdot \begin{pmatrix} N_1 \\ N_2 \\ A_1 \end{pmatrix} + \begin{pmatrix} \Delta\varepsilon_{\text{c}} \\ \Delta\varepsilon_{\text{p}} \\ \Delta\varepsilon_{\text{pc}} \end{pmatrix}, \tag{6.68}$$

where Eq. 6.65 is used and

$$g = \left(\frac{1}{f_1^2} - \frac{1}{f_2^2} \right), \quad d = -\frac{2}{f_1^2} \quad \text{and} \quad T = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

Equation 6.68 is called an ambiguity-ionospheric equation. For any viewed GPS satellite, Eq. 6.68 is solvable. If the variance vector of the observable vector is

$$\begin{pmatrix}\sigma_c^2 & \sigma_c^2 & \sigma_p^2 & \sigma_p^2\end{pmatrix}^T ,$$

then the covariance matrix of the original observable vector is (cf. Sect. 6.2)

$$Q_{LL} = \begin{pmatrix} \sigma_c^2 & 0 & 0 & 0 \\ 0 & \sigma_c^2 & 0 & 0 \\ 0 & 0 & \sigma_p^2 & 0 \\ 0 & 0 & 0 & \sigma_p^2 \end{pmatrix} ,$$

and the covariance matrix of the transformed observable vector (left side of Eq. 6.68) is (cf. Sect. 6.4)

$$\text{cov}(TL) = TQ_{LL}T^T = \begin{pmatrix} 2\sigma_c^2 & 0 & -\sigma_c^2 \\ 0 & 2\sigma_p^2 & \sigma_p^2 \\ -\sigma_c^2 & \sigma_p^2 & \sigma_c^2 + \sigma_p^2 \end{pmatrix} ,$$

$$P = (\text{cov}(TL))^{-1} = \frac{1}{2}\begin{pmatrix} h + \sigma_c^{-2} & -h & 2h \\ -h & h + \sigma_p^{-2} & -2h \\ 2h & -2h & 4h \end{pmatrix}, \quad h = \frac{1}{\sigma_c^2 + \sigma_p^2} . \tag{6.69}$$

Taking all the data measured at a station into account, the ambiguity and the ionospheric parameters (as a step function of the polynomial) can be solved by using Eq. 6.68 with the weight of Eq. 6.69. Taking the data station by station into account, all ambiguity and ionospheric parameters can be determined. The different weights of the code and phase measurements are considered exactly here. Due to the physical property of the ionosphere, all solved ionospheric parameters shall have the same sign. Even though the observation Eq. 6.68 is already a linear equation system, an initialisation is still helpful to avoid numbers from ambiguities that are too big. The broadcasting ionospheric model can be used for initialisation of the related ionospheric parameters.

A geometry-free combination of Eq. 6.62 can be used as a quality check of the Doppler data.

6.5.3 Standard Phase-Code Combination

Traditionally, phase and code combinations are used to compute the wide-lane ambiguity (cf. Sjoeberg 1999; Hofmann-Wellenhof et al. 1997). The formulas can be derived as follows. Dividing λ_j into Eq. 6.63 and forming the difference for $j = 1$ and $j = 2$, one gets

$$\Phi_w - \frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2} = N_w - \frac{2A_1}{c}\left(\frac{1}{f_1} - \frac{1}{f_2}\right) , \tag{6.70}$$

where $\Phi_W = \Phi_1 - \Phi_2$, $N_W = N_1 - N_2$, and they are called wide-lane observable and ambiguity; c is the velocity of light and A_1 is the ionospheric parameter. The error term is omitted here. Equation 6.60 can be rewritten as (by omitting the error term)

$$A_1 = (R_1 - R_2)\frac{f_1^2 f_2^2}{f_2^2 - f_1^2} \ , \tag{6.71}$$

and then one gets

$$\frac{A_1}{c}\left(\frac{1}{f_1} - \frac{1}{f_2}\right) = \left(\frac{R_1}{\lambda_1 f_1} - \frac{R_2}{\lambda_2 f_2}\right)\frac{f_1 f_2}{f_2 + f_1} = \frac{R_1}{\lambda_1}\frac{f_2}{(f_1 + f_2)} - \frac{R_2}{\lambda_2}\frac{f_1}{(f_1 + f_2)} \ . \tag{6.72}$$

Substituting Eq. 6.72 into 6.70 yields

$$N_w = \Phi_w - \frac{f_1 - f_2}{f_1 + f_2}\left(\frac{R_1}{\lambda_1} + \frac{R_2}{\lambda_2}\right) \ . \tag{6.73}$$

Equation 6.73 is the most popular formula for computing wide-lane ambiguities using phase and code observables. The un-differenced ambiguity N_1 can be derived as follows. Setting $\Phi_2 = \Phi_1 - \Phi_W$, $N_2 = N_1 - N_W$ into Eq. 6.61 and omitting the error term, one has

$$\lambda_1 N_1 - \lambda_2 (N_1 - N_w) = \frac{A_1}{f_1^2} - \frac{A_1}{f_2^2} + \lambda_1 \Phi_1 - \lambda_2 (\Phi_1 - \Phi_w) \ ,$$

$$N_1 = \Phi_1 - (\Phi_w - N_w)\frac{f_1}{f_w} + \frac{A_1}{c}\frac{f_1 + f_2}{f_1 f_2} \quad \text{or}$$

$$N_1 = \Phi_1 - (\Phi_w - N_w)\frac{f_1}{f_w} - \frac{R_1}{\lambda_1}\frac{f_2}{f_w} + \frac{R_2}{\lambda_2}\frac{f_1}{f_w} \ , \tag{6.74}$$

where $f_w = f_1 - f_2$ is the wide-lane frequency.

Compared with the adjustment method derived in Sect. 6.5.2, it is obvious that the quality differences of the phase and code data are not considered by using Eqs. 6.73 and 6.74 for determining the ambiguity parameters. Therefore, the method proposed in Sect. 6.5.2 is suggested for use.

6.5.4 Ionospheric Residuals

Considering the GPS observables as a time series, the geometry-free combinations of Eqs. 6.60–6.64 can be rewritten as

$$R_1(t_j) - R_2(t_j) = \Delta\delta_{\text{ion}}(t_j) + \Delta\varepsilon_c \ , \tag{6.75}$$

$$\lambda_1 \Phi_1(t_j) - \lambda_2 \Phi_2(t_j) = \lambda_1 N_1 - \lambda_2 N_2 - \Delta\delta_{\text{ion}}(t_j) + \Delta\varepsilon_p \ \text{ and} \tag{6.76}$$

$$\lambda_i \Phi_i(t_j) - R_i(t_j) = \lambda_i N_i - 2\delta_{\text{ion}}(i, t_j) + \Delta\varepsilon_{pc} \ , \quad i=1,2,5 \ , \tag{6.77}$$

where

$$\Delta\delta_{\text{ion}}(t_j)=\delta_{\text{ion}}(1,t_j)-\delta_{\text{ion}}(2,t_j)=\frac{A_1(t_j)}{f_1^2}-\frac{A_1(t_j)}{f_2^2}\ ,\ \ j=1,2,\ldots,m\ . \tag{6.78}$$

The differences of the above observable combinations at the two succeeded epochs t_j and t_{j-1} can be formed:

$$\Delta_t R_1(t_j)-\Delta_t R_2(t_j)=\Delta_t\Delta\delta_{\text{ion}}(t_j)+\Delta_t\Delta\varepsilon_{\text{c}}\ , \tag{6.79}$$

$$\lambda_1\Delta_t\Phi_1(t_j)-\lambda_2\Delta_t\Phi_2(t_j)=\lambda_1\Delta_t N_1-\lambda_2\Delta_t N_2-\Delta_t\Delta\delta_{\text{ion}}(t_j)+\Delta_t\Delta\varepsilon_{\text{p}}\ \ \text{and} \tag{6.80}$$

$$\lambda_i\Delta_t\Phi_i(t_j)-\Delta_t R_i(t_j)=\lambda_i\Delta_t N_i-2\Delta_t\delta_{\text{ion}}(i,t_j)+\Delta_t\Delta\varepsilon_{\text{pc}}\ ,\ \ i=1,2,5\ , \tag{6.81}$$

where Δ_t is a time difference operator, for any time function $G(t)$, $\Delta_t G(t_j) = G(t_j) - G(t_{j-1})$ is valid.

Because the time differences of the ionospheric effects $\Delta_t\delta_{\text{ion}}$ and $\Delta_t\Delta\delta_{\text{ion}}$ are generally very small, they are called ionospheric residuals. In the case of no cycle slips, i.e., ambiguities N_1 and N_2 are constant, ΔN_1 and ΔN_2 equal zero. Equations 6.79–6.81 are called ionospheric residual combinations. The first combination of Eq. 6.79 can be used for a consistency check of two code measurements. Equations 6.80 and 6.81 can be used for a cycle slip check. Equation 6.81 is a phase-code combination, due to the lower accuracy of the code measurements; it can be used only to check for big cycle slips. Equation 6.80 is a phase-phase combination, and therefore it has higher sensibility related to the cycle slips. However, two special cycle slips ΔN_1 and ΔN_2 can lead to a very small combination of $\delta_1\Delta_t N_1 - \delta_2\Delta_t N_2$. Examples of the combinations can be found, e.g., in (Hofmann-Wellenhof et al. 1997). That is, even the ionospheric residual of Eq. 6.80 is very small; it may not guarantee that there are no cycle slips.

6.5.5 Differential Doppler and Doppler Integration

Differential Doppler

The numerical differentiation of the original observables given in Eqs. 6.44 and 6.45 at the two succeeded epochs t_j and t_{j-1} can be formed as

$$\frac{\Delta_t R_j}{\lambda_j\Delta t}=\frac{\Delta_t\rho}{\lambda_j\Delta t}-f_j\frac{\Delta_t(\delta t_{\text{r}}-\delta t_{\text{k}})}{\Delta t}+\frac{\Delta_t\varepsilon_c}{\lambda_j\Delta t}\ ,\ \ j=1,2\ ,\ \ \text{and} \tag{6.82}$$

$$\frac{\Delta_t\Phi_j}{\Delta t}=\frac{\Delta_t\rho}{\lambda_j\Delta t}-f_j\frac{\Delta_t(\delta t_{\text{r}}-\delta t_{\text{k}})}{\Delta t}+\frac{\Delta_t\varepsilon_p}{\lambda_j\Delta t}\ ,\ \ j=1,2\ , \tag{6.83}$$

where $\Delta_t/\Delta t$ is a numerical differentiation operator and $\Delta t = t_j - t_{j-1}$.

The left-hand side of Eq. 6.83 is called differential Doppler. Ionospheric residuals are negligible and omitted here. The third terms of Eqs. 6.82 and 6.83 on the right-hand side are small residual errors. For convenience of comparison, the Doppler observable model of Eq. 6.46 is copied below:

$$D_j = \frac{d\rho}{\lambda_j dt} - f_j \frac{d(\delta t_r - \delta t_k)}{dt} + \varepsilon_d . \qquad (6.84)$$

It is obvious that Eqs. 6.83 and 6.84 are nearly the same. The only difference is that in Doppler Eq. 6.84 the observed Doppler is an instantaneous one and its model is presented by theoretical differentiation, whereas the term on the left-hand side of Eq. 6.83 is the numerically differenced Doppler (formed by phases) and its model is presented by numerical differentiation. Doppler measurement measures the instantaneous motion of the GPS antenna, whereas differential Doppler describes a kind of average velocity of the antenna during the two succeeded epochs. The velocity solution of Eq. 6.83 (denoted by $(\dot{x}\ \ \dot{y}\ \ \dot{z})^T$) can be used to predict the future kinematic position by

$$\begin{pmatrix} x_{j+1} \\ y_{j+1} \\ z_{j+1} \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix} + \begin{pmatrix} \dot{x}_j \\ \dot{y}_j \\ \dot{z}_j \end{pmatrix} \cdot \Delta t . \qquad (6.85)$$

In other words, differential Doppler can be used as the system equation of a Kalman filter for kinematic positioning. The Kalman filter will be discussed in the next chapter. A Kalman filter using differential Doppler will be discussed in Sect. 9.8.

Doppler Integration

Integrating the instantaneous Doppler Eq. 6.84, one has

$$\lambda_j \int_{t_{j-1}}^{t_j} D_j dt = \Delta_t \rho - \Delta_t (\delta t_r - \delta t_k) c + \varepsilon_d .$$

Using the operator Δ_t to the un-differenced phase Eq. 6.45 and code Eq. 6.44, one gets

$$\lambda_j \Delta_t \Phi_j = \Delta_t \rho - \Delta_t (\delta t_r - \delta t_k) c + \lambda_j \Delta_t N_j + \varepsilon_p \quad \text{and}$$

$$\Delta_t R_j = \Delta_t \rho - \Delta_t (\delta t_r - \delta t_k) c + \varepsilon_c , \qquad (6.86)$$

where the same symbols are used for the error terms (later too). Differencing the first equation of Eq. 6.86 with the integrated Doppler leads to

$$\lambda_j \Delta_t N_j = \lambda_j \Delta_t \Phi_j - \lambda_j \int_{t_{j-1}}^{t_j} D_j dt + \varepsilon_1$$

or

$$\Delta_t N_j = \Delta_t \Phi_j - \int_{t_{j-1}}^{t_j} D_j dt + \varepsilon_1 , \quad j = 1,2,5 . \qquad (6.87)$$

That is, the integrated Doppler can be used for cycle slip detection. Such a cycle slip detection method is very reasonable. Phase is measured by keeping track of the partial phase and accumulating the integer count. If any loss of lock of the signal happens during the time, the integer accumulating will be wrong, i.e., cycle slip happens. Therefore, an external instantaneous Doppler integration can be used as an alternative method of cycle slip detection. The integration can be made first by fitting the Doppler with a suitable order polynomial, and then integrating that within the time interval.

Code Smoothing

Comparing the two formulas of Eq. 6.86, one has

$$\Delta_t R_j = \lambda_j \Delta_t \Phi_j - \lambda_j \Delta_t N_j + \varepsilon_2 \quad \text{or}$$

$$\Delta_t R_j = \lambda_j \Delta_t \Phi_j + \varepsilon_3 \ . \tag{6.88}$$

Equation 6.88 can be used for smoothing the code survey by phase if there are no cycle slips.

Differential Phases

The first formula of Eq. 6.86 is the numerical difference of the phases at the two succeeded epochs t_j and t_{j-1}

$$\lambda_j \Delta_t \Phi_j = \Delta_t \rho - \Delta_t (\delta t_\mathrm{r} - \delta t_k) c + \lambda_j \Delta_t N_j + \varepsilon_\mathrm{p} \, , \quad j = 1,2 \ .$$

All other terms on the right-hand side are of low variation ones except the ambiguity term. Any cycle slips will lead to a sudden jump of the time difference of the phases. Therefore, the time differenced phase can be used as an alternative method of cycle slip detection.

6.6 Data Differentiations

Data differentiations are methods of combining GPS data (of the same type) measured at different stations. For the convenience of later discussions, tidal effects and relativistic effects are considered corrected before forming the differences. The original code, phase and Doppler observables as well as their standardised combinations can be re-written as (cf. Eqs. 6.44–6.47)

$$R_i^k(j) = \rho_i^k - c\delta t_i + c\delta t_k + \delta_\mathrm{ion}(j) + \delta_\mathrm{trop} + \varepsilon_\mathrm{c} \ , \tag{6.89}$$

$$\lambda_j \Phi_i^k(j) = \rho_i^k - c\delta t_i + c\delta t_k + \lambda_j N_i^k(j) - \delta_\mathrm{ion}(j) + \delta_\mathrm{trop} + \varepsilon_\mathrm{p} \ , \tag{6.90}$$

$$\delta_\mathrm{ion}(j) = \frac{A_1}{f_j^2} + \frac{A_2}{f_j^3} \quad \text{and} \tag{6.91}$$

$$D_i^k(j) = \frac{\mathrm{d}\rho_i^k}{\lambda_j \mathrm{d}t} - f_j \frac{\mathrm{d}(\delta t_i - \delta t_k)}{\mathrm{d}t} + \varepsilon_\mathrm{d} \ , \tag{6.92}$$

where j ($j = 1, 2, 5$) is the index of frequency f, subscript i is the index of station number and superscript k is the id number of satellite.

6.6.1 Single Differences

Single difference (SD) is the difference formed by data observed at two stations on the same satellite as

$$\mathrm{SD}_{i1,i2}^k(O) = O_{i2}^k - O_{i1}^k \ , \tag{6.93}$$

where O is the original observable, and $i1$ and $i2$ are two id number of the stations. Supposing the original observables have the same variance of σ^2, then the single difference observable has a variance of $2\sigma^2$. Considering Eqs. 6.89–6.92, one has

$$\mathrm{SD}_{i1,i2}^k(R(j)) = \rho_{i2}^k - \rho_{i1}^k - c\delta t_{i2} + c\delta t_{i1} + \mathrm{d}\delta_\mathrm{ion}(j) + \mathrm{d}\delta_\mathrm{trop} + \mathrm{d}\varepsilon_\mathrm{c} \ , \tag{6.94}$$

$$\begin{aligned}\mathrm{SD}_{i1,i2}^k(\lambda_j \Phi(j)) = {} & \rho_{i2}^k - \rho_{i1}^k - c\delta t_{i2} + c\delta t_{i1} + \lambda_j N_{i2}^k(j) - \lambda_j N_{i1}^k(j) \\ & - \mathrm{d}\delta_\mathrm{ion}(j) + \mathrm{d}\delta_\mathrm{trop} + \mathrm{d}\varepsilon_\mathrm{p}\end{aligned} \quad \text{and} \tag{6.95}$$

$$\mathrm{SD}_{i1,i2}^k(D(j)) = \frac{\dot\rho_{i2}^k - \dot\rho_{i1}^k}{\lambda_j} - f_j \frac{\mathrm{d}(\delta t_{i2} - \delta t_{i1})}{\mathrm{d}t} + \mathrm{d}\varepsilon_\mathrm{d} \ , \tag{6.96}$$

where $\dot\rho$ is the time differentiation of ρ, and $\mathrm{d}\delta_\mathrm{ion}(j)$ and $\mathrm{d}\delta_\mathrm{trop}$ are the differenced ionospheric and tropospheric effects at the two stations related to the satellite k, respectively.

The most important property of single differences is that the satellite clock error terms in the model are eliminated. However, it should be emphasised that the satellite clock error, which implicitly affects the computation of satellite position, still has to be carefully considered. Ionospheric and tropospheric effects are reduced through difference forming, especially for those stations that are not very far away from each other. Because of the identical mathematical models of the station clock errors and ambiguities, not all clock and ambiguity parameters can be resolved in the single difference equations of Eqs. 6.94–6.96.

For the original observable vector of station $i1$ and $i2$,

$$O = \begin{pmatrix} O_{i1}^{k1} & O_{i1}^{k2} & O_{i1}^{k3} & O_{i2}^{k1} & O_{i2}^{k2} & O_{i2}^{k3} \end{pmatrix}^T, \quad \mathrm{cov}(O) = \sigma^2 E \ ,$$

the single differences

$$\mathrm{SD}(O) = \begin{pmatrix} O_{i1,i2}^{k1} & O_{i1,i2}^{k2} & O_{i1,i2}^{k3} \end{pmatrix}^T \ ,$$

can be formed by a linear transformation

$\mathrm{SD}(O) = C \cdot O$ and

$$C = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -E & E \end{pmatrix} . \tag{6.97}$$

Where common satellites $k1$, $k2$, $k3$ are observed, E is an identity matrix that has the size of the observed satellite number; in the above example the size is 3×3.

The covariance matrix of the single differences is then

$$\mathrm{cov}(\mathrm{SD}(O)) = C \cdot \mathrm{cov}(O) \cdot C^T = \sigma^2 C \cdot C^T = 2\sigma^2 E \ , \tag{6.98}$$

i.e., the weight matrix is

$$P = \frac{1}{2\sigma^2} E \ .$$

That is, the single differences are un-correlated observables in the case of a single baseline. C in Eq. 6.97 is a general form, so C is denoted by $C_s = (-E_{n\times n} \;\; E_{n\times n})$, and n is the number of commonly viewed satellites.

Single differences can be formed for any baselines as long as the two stations have common satellites in sight. However, the baselines should be a set of "independent" ones. The most-used methods are to form the radial baselines or traverse baselines. Supposing the stations' id vector is $(i1, i2, i3, \ldots, i(m-1), im)$ and the baseline between station $i1$ and $i2$ is denoted by $(i1, i2)$, then the radial baselines could be formed, e.g., by $(i1, i2)$, $(i1, i3)$, …, $(i1, im)$, and the traverse baselines could be formed, e.g., by $(i1, i2)$, $(i2, i3)$, …, $(i(m-1), im)$. Station $i1$ is called a reference station and is freely selectable. In some cases, a mixed radial and traverse baselines have to be formed such as, e.g., by $(i1, i2)$, $(i1, i3)$, $(i3, i4)$, …, $(i3, i(m-1))$, $(i3, im)$. Sometimes the baselines have to be formed by several groups, and therefore several references have to be selected. A method of forming an independent and optimal baseline network will be discussed Sects. 9.1 and 9.2.

In case three stations are used to measure the GPS data, the original observable vector of station $i1$, $i2$ and $i3$ is

$$O_i = \begin{pmatrix} O_i^{k1} & \ldots & O_i^{kn} \end{pmatrix}^T , \quad \mathrm{cov}(O_i) = \sigma^2 E_{n\times n} \ , \quad i = i1, i2, i3 \ ,$$

where n is the commonly observed satellite number. The single differences of the baseline (i, j) are

$$\mathrm{SD}_{i,j}(O) = \begin{pmatrix} O_{i,j}^{k1} & \ldots & O_{i,j}^{kn} \end{pmatrix}^T \ , \quad i,j = i1, i2, i3, \quad i \neq j \ .$$

If the baselines are formed in a radial way, i.e., baselines are formed as $(i1, i2)$ and $(i1, i3)$, then one has

$$\begin{pmatrix} \mathrm{SD}_{i1,i2}(O) \\ \mathrm{SD}_{i1,i3}(O) \end{pmatrix} = \begin{pmatrix} -E & E & 0 \\ -E & 0 & E \end{pmatrix} \begin{pmatrix} O_{i1} \\ O_{i2} \\ O_{i3} \end{pmatrix}$$

$$\mathrm{cov}(\mathrm{SD}) = \sigma^2 \begin{pmatrix} -E & E & 0 \\ -E & 0 & E \end{pmatrix} \begin{pmatrix} -E & -E \\ E & 0 \\ 0 & E \end{pmatrix} = \sigma^2 \begin{pmatrix} 2E & E \\ E & 2E \end{pmatrix} \quad \text{and} \tag{6.99}$$

$$P_s = [\mathrm{cov}(\mathrm{SD})]^{-1} = \frac{1}{3\sigma^2} \begin{pmatrix} 2E & -E \\ -E & 2E \end{pmatrix} .$$

If the baselines are formed in a traverse way, i.e., baselines are formed as ($i1$, $i2$) and ($i2$, $i3$), then one has

$$\begin{pmatrix} \mathrm{SD}_{i1,i2}(O) \\ \mathrm{SD}_{i2,i3}(O) \end{pmatrix} = \begin{pmatrix} -E & E & 0 \\ 0 & -E & E \end{pmatrix} \begin{pmatrix} O_{i1} \\ O_{i2} \\ O_{i3} \end{pmatrix} ,$$

$$\mathrm{cov}(\mathrm{SD}) = \sigma^2 \begin{pmatrix} -E & E & 0 \\ 0 & -E & E \end{pmatrix} \begin{pmatrix} -E & 0 \\ E & -E \\ 0 & E \end{pmatrix} = \sigma^2 \begin{pmatrix} 2E & -E \\ -E & 2E \end{pmatrix} \quad \text{and}$$

$$P_s = [\mathrm{cov}(\mathrm{SD})]^{-1} = \frac{1}{3\sigma^2} \begin{pmatrix} 2E & E \\ E & 2E \end{pmatrix} .$$

It is obvious that the single differences are correlated if the station numbers are more than two. And the correlation depends on the ways the baselines are formed. Therefore, a general covariance formula of the single differences of a network is not possible to be derived. Furthermore, the commonly viewed satellite number n could be different from baseline to baseline, so the formulation of the covariance matrix could be more complicated.

A baseline-wise processing of the GPS data of a network by using single differences is equivalent to an omission of the correlation between the baselines.

6.6.2 Double Differences

Double differences are formed between two single differences related to two observed satellites as

$$\mathrm{DD}_{i1,i2}^{k1,k2}(O) = \mathrm{SD}_{i1,i2}^{k2}(O) - \mathrm{SD}_{i1,i2}^{k1}(O) \tag{6.100}$$

or

$$\mathrm{DD}_{i1,i2}^{k1,k2}(O) = (O_{i2}^{k2} - O_{i1}^{k2}) - (O_{i2}^{k1} - O_{i1}^{k1}) , \tag{6.101}$$

where $k1$ and $k2$ are the two id numbers of the satellites. Supposing the original observables have the same variance of σ^2, then the double differenced observables have a variance of $4\sigma^2$. Considering Eqs. 6.89–6.92, one has

$$\mathrm{DD}_{i1,i2}^{k1,k2}(R(j)) = \rho_{i2}^{k2} - \rho_{i1}^{k2} - \rho_{i2}^{k1} + \rho_{i1}^{k1} + \mathrm{dd}\delta_{\mathrm{ion}}(j) + \mathrm{dd}\delta_{\mathrm{trop}} + \mathrm{dd}\varepsilon_{\mathrm{c}} \ , \tag{6.102}$$

$$\begin{aligned}\mathrm{DD}_{i1,i2}^{k1,k2}(\lambda_j \Phi(j)) = \rho_{i2}^{k2} - \rho_{i1}^{k2} - \rho_{i2}^{k1} + \rho_{i1}^{k1} + \lambda_j(N_{i2}^{k2}(j) - N_{i1}^{k2}(j) \\ - N_{i2}^{k1}(j) + N_{i1}^{k1}(j)) - \mathrm{dd}\delta_{\mathrm{ion}}(j) + \mathrm{dd}\delta_{\mathrm{trop}} + \mathrm{dd}\varepsilon_{\mathrm{p}}\end{aligned} \quad \text{and} \tag{6.103}$$

$$\mathrm{DD}_{i1,i2}^{k1,k2}(D(j)) = \frac{\dot\rho_{i2}^{k2} - \dot\rho_{i1}^{k2} - \dot\rho_{i2}^{k1} + \dot\rho_{i1}^{k1}}{\lambda_j} + \mathrm{dd}\varepsilon_{\mathrm{d}} \ , \tag{6.104}$$

where $\mathrm{dd}\delta_{\mathrm{ion}}(j)$ and $\mathrm{dd}\delta_{\mathrm{trop}}$ are the differenced ionospheric and tropospheric effects at the two stations related to the two satellites, respectively. For the ionosphere-free combined observables (denoted by $j = 4$ for distinguishing), the ionospheric error terms have vanished from above equations.

The most important property of the double differences is that the clock error terms in the equation (model) are completely eliminated. It should be emphasised that the clock error, which implicitly affects the computation of the position of the satellite, still has to be carefully considered. Ionospheric and tropospheric effects are reduced greatly through difference forming, especially for those stations that are not very far away from each other. Double differenced Doppler directly describes the geometry change. Double differenced ambiguities can be denoted by

$$N_{i1,i2}^{k1,k2}(j) = N_{i2}^{k2}(j) - N_{i1}^{k2}(j) - N_{i2}^{k1}(j) + N_{i1}^{k1}(j) \ . \tag{6.105}$$

The original ambiguities used in Eq. 6.103 are for convenience in case of reference satellite changing.

For the single difference observable vector

$$\mathrm{SD}(O) = \begin{pmatrix} O_{i1,i2}^{k1} & O_{i1,i2}^{k2} & O_{i1,i2}^{k3} \end{pmatrix}^T \quad \text{and} \quad \mathrm{cov}(\mathrm{SD}(O)) = 2\sigma^2 E \ , \tag{6.106}$$

the double differences

$$\mathrm{DD}(O) = \begin{pmatrix} O_{i1,i2}^{k1,k2} & O_{i1,i2}^{k1,k3} \end{pmatrix}^T \tag{6.107}$$

can be formed by a linear transformation

$$\mathrm{DD}(O) = C_{\mathrm{d}} \cdot \mathrm{SD}(O) \ , \tag{6.108}$$

$$C_{\mathrm{d}} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -I_m & E_{m\times m} \end{pmatrix} \quad (\text{here } m = 2) \ , \tag{6.109}$$

where E is an identity matrix of size $m \times m$, I is a 1 vector of size m (all elements of the vector are 1), m is the number of formed double differences, and $m = n - 1$. The covariance matrix of the double differences is then

$$\mathrm{cov}(\mathrm{DD}(O)) = C_{\mathrm{d}} \cdot \mathrm{cov}(\mathrm{SD}(O)) \cdot C_{\mathrm{d}}^T = 2\sigma^2 C_{\mathrm{d}} \cdot C_{\mathrm{d}}^T = 2\sigma^2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \ . \tag{6.110}$$

For single and double differences

$$\mathrm{SD}(O)=\begin{pmatrix}O_{i1,i2}^{k1} & O_{i1,i2}^{k2} & O_{i1,i2}^{k3} & O_{i1,i2}^{k4}\end{pmatrix}^T, \quad \mathrm{cov}(\mathrm{SD}(O))=2\sigma^2E \quad \text{and} \tag{6.111}$$

$$\mathrm{DD}(O)=\begin{pmatrix}O_{i1,i2}^{k1,k2} & O_{i1,i2}^{k1,k3} & O_{i1,i2}^{k1,k4}\end{pmatrix}^T, \tag{6.112}$$

the linear transformation matrix C_d and the covariance matrix can be obtained by

$$C_\mathrm{d}=\begin{pmatrix}-1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}-I & E\end{pmatrix} \quad \text{and} \tag{6.113}$$

$$\mathrm{cov}(\mathrm{DD}(O))=C_\mathrm{d}\cdot\mathrm{cov}(\mathrm{SD}(O))\cdot C_\mathrm{d}^T=2\sigma^2C_\mathrm{d}\cdot C_\mathrm{d}^T=2\sigma^2\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}. \tag{6.114}$$

For the general case of

$$\mathrm{SD}(O)=\begin{pmatrix}O_{i1,i2}^{k1} & O_{i1,i2}^{k2} & O_{i1,i2}^{k3} & \dots & O_{i1,i2}^{kn}\end{pmatrix}^T, \quad \mathrm{cov}(\mathrm{SD}(O))=2\sigma^2E, \quad \text{and}$$

$$\mathrm{DD}(O)=\begin{pmatrix}O_{i1,i2}^{k1,k2} & O_{i1,i2}^{k1,k3} & \dots & O_{i1,i2}^{k1,km}\end{pmatrix}^T, \tag{6.115}$$

it is obvious that the general transformation matrix C_d and the related covariance matrix can be represented as

$$C_\mathrm{d}=\begin{pmatrix}-I_m & E_{m\times m}\end{pmatrix} \quad \text{and} \tag{6.116}$$

$$\mathrm{cov}(\mathrm{DD}(O))=C_\mathrm{d}\,\mathrm{cov}(\mathrm{SD}(O))C_\mathrm{d}^T=2\sigma^2C_\mathrm{d}C_\mathrm{d}^T=2\sigma^2\left(I_{m\times m}+E_{m\times m}\right), \tag{6.117}$$

where $I_{\mathrm{m}\times\mathrm{m}}$ is an $m\times m$ matrix whose elements are all 1, and the weight matrix has the form of

$$P=[\mathrm{cov}(\mathrm{DD}(O))]^{-1}=\frac{1}{2\sigma^2n}\left(nE_{m\times m}-I_{m\times m}\right), \tag{6.118}$$

where $n = m + 1$. Equation 6.118 can be verified by an identity matrix test (i.e., $P\cdot\mathrm{cov}(\mathrm{DD}(O)) = E$).

In the case of three stations, supposing n common satellites ($k1, k2, \dots, kn$) are viewed, then the single and double differences can be written as

$$\mathrm{SD}_{i,j}(O)=\begin{pmatrix}O_{i,j}^{k1} & O_{i,j}^{k2} & O_{i,j}^{k3} & \dots & O_{i,j}^{kn}\end{pmatrix}^T \quad \text{and}$$

$$\mathrm{DD}_{i,j}(O)=\begin{pmatrix}O_{i,j}^{k1,k2} & O_{i,j}^{k1,k3} & \dots & O_{i,j}^{k1,km}\end{pmatrix}^T \quad i,j=i1,i2,i3, \quad i\neq j. \tag{6.119}$$

Then one has the transformation and covariance

$$\begin{pmatrix} \mathrm{DD}_{i1,i2}(O) \\ \mathrm{DD}_{i1,i3}(O) \end{pmatrix} = \begin{pmatrix} C_\mathrm{d} & 0 \\ 0 & C_\mathrm{d} \end{pmatrix} \begin{pmatrix} \mathrm{SD}_{i1,i2}(O) \\ \mathrm{SD}_{i1,i3}(O) \end{pmatrix} \quad \text{and}$$

$$\mathrm{cov(DD)} = \begin{pmatrix} C_\mathrm{d} & 0 \\ 0 & C_\mathrm{d} \end{pmatrix} \mathrm{cov}(SD) \begin{pmatrix} C_\mathrm{d} & 0 \\ 0 & C_\mathrm{d} \end{pmatrix}^T = \sigma^2 \begin{pmatrix} 2E & -E \\ -E & 2E \end{pmatrix} (C_\mathrm{d} C_\mathrm{d}^T) .$$

Because of the dependency of the cov(SD) on the baselines forming, cov(DD) is also dependent on the baselines forming. A baseline-wise processing of a network GPS data using double differences is equivalent to an omission of the correlation between the baselines.

6.6.3 Triple Differences

Triple differences are formed between two double differences related to the same stations and satellites at the two adjacent epochs as

$$\mathrm{TD}_{i1,i2}^{k1,k2}(O(t1,t2)) = \mathrm{DD}_{i1,i2}^{k1,k2}(O(t2)) - \mathrm{DD}_{i1,i2}^{k1,k2}(O(t1))$$

or

$$\begin{aligned} \mathrm{TD}_{i1,i2}^{k1,k2}(O(t1,t2)) = {} & O_{i2}^{k2}(t2) - O_{i1}^{k2}(t2) - O_{i2}^{k1}(t2) + O_{i1}^{k1}(t2) \\ & - O_{i2}^{k2}(t1) + O_{i1}^{k2}(t1) + O_{i2}^{k1}(t1) - O_{i1}^{k1}(t1) \end{aligned} , \tag{6.120}$$

where $t1$ and $t2$ are two adjacent epochs. Supposing the original observables have the same variance of σ^2, then the triple differenced observables have a variance of $8\sigma^2$. Considering Eqs. 6.102–6.104, one has

$$\begin{aligned} \mathrm{TD}_{i1,i2}^{k1,k2}(R(j,t1,t2)) = {} & \rho_{i2}^{k2}(t2) - \rho_{i1}^{k2}(t2) - \rho_{i2}^{k1}(t2) + \rho_{i1}^{k1}(t2) - \rho_{i2}^{k2}(t1) \\ & + \rho_{i1}^{k2}(t1) + \rho_{i2}^{k1}(t1) - \rho_{i1}^{k1}(t1) + td\varepsilon_\mathrm{c} \end{aligned} , \tag{6.121}$$

$$\begin{aligned} \mathrm{TD}_{i1,i2}^{k1,k2}(\lambda_j \Phi(j,t1,t2)) = {} & \rho_{i2}^{k2}(t2) - \rho_{i1}^{k2}(t2) - \rho_{i2}^{k1}(t2) + \rho_{i1}^{k1}(t2) - \rho_{i2}^{k2}(t1) \\ & + \rho_{i1}^{k2}(t1) + \rho_{i2}^{k1}(t1) - \rho_{i1}^{k1}(t1) + \delta N + t\mathrm{d}\varepsilon_\mathrm{p} \end{aligned} \quad \text{and} \tag{6.122}$$

$$\begin{aligned} \mathrm{TD}_{i1,i2}^{k1,k2}(D(j,t1,t2)) = {} & \frac{\dot\rho_{i2}^{k2}(t2) - \dot\rho_{i1}^{k2}(t2) - \dot\rho_{i2}^{k1}(t2) + \dot\rho_{i1}^{k1}(t2)}{\lambda_j} \\ & - \frac{\dot\rho_{i2}^{k2}(t1) - \dot\rho_{i1}^{k2}(t1) - \dot\rho_{i2}^{k1}(t1) + \dot\rho_{i1}^{k1}(t1)}{\lambda_j} + t\mathrm{d}\varepsilon_\mathrm{d} \end{aligned} , \tag{6.123}$$

where

$$\delta N = \lambda_j (N_{i1,i2}^{k1,k2}(j,t2) - N_{i1,i2}^{k1,k2}(j,t1)) . \tag{6.124}$$

Ionospheric and tropospheric effects are eliminated. If there are no cycle slips during the time, the term of Eq. 6.124 is zero. Therefore, triple differences of Eq. 6.122 can also be used as a check for the cycle slips. Through triple difference forming, the systematic cycle slip turns out to be an effect like an outlier.

The most important property of the triple differences is that only the geometric changing is left in the models. Triple differences of Doppler describe the acceleration of the position.

For double differences

$$\mathrm{DD}(O(t)) = \begin{pmatrix} O_{i1,i2}^{k1,k2}(t) & O_{i1,i2}^{k1,k3}(t) & \ldots & O_{i1,i2}^{k1,km}(t) \end{pmatrix}^T , \tag{6.125}$$

one has

$$\mathrm{TD}(O(t1,t2)) = C_T \cdot \begin{pmatrix} \mathrm{DD}(O(t1)) \\ \mathrm{DD}(O(t2)) \end{pmatrix} , \tag{6.126}$$

where

$$C_T = \begin{pmatrix} -E_{m\times m} & E_{m\times m} \end{pmatrix} . \tag{6.127}$$

Then the related covariance matrix can be represented as

$$\begin{aligned} \mathrm{cov}(\mathrm{TD}(O(t1,t2))) &= C_T \cdot \mathrm{cov}(\mathrm{DD}(O)) \cdot C_T^T \\ &= C_T \cdot C_{d2}\,\mathrm{cov}(\mathrm{SD}(O)) \cdot C_{\mathrm{d2}}^T C_T^T = 2\sigma^2 C_T C_{\mathrm{d2}} C_{\mathrm{d2}}^T C_T^T \end{aligned} , \tag{6.128}$$

where C_{d2} is the double difference transformation matrix of two epochs. Because double differences are independent epoch wise, C_{d2} is a diagonal matrix of C_{d}, i.e.,

$$C_{\mathrm{d2}} = \begin{pmatrix} C_{\mathrm{d}} & 0 \\ 0 & C_{\mathrm{d}} \end{pmatrix} . \tag{6.129}$$

It is notable that the triple differences formed by epochs ($t1$, $t2$) are correlated to the differences formed by epochs ($t0$, $t1$) and ($t1$, $t2$). Such a correlation makes a sequential processing of the triple difference data very complicated. Sequentially using the above covariance formula indicates an omission of the correlation related to the previous epoch and the next epoch.

Taking the correlation between the baselines into account, an exact correlation description of the triple differences of a GPS network turns out to be very complicated.

6.7 Equivalence of the Uncombined and Combining Algorithms

Uncombined and combining algorithms are standard GPS data processing methods, which can often be found in the literature (cf., e.g., Leick 2004, Hofmann-Wellenhof et al. 2001). Different combinations own different properties and are beneficial for dealing with the data and solving the problem in different cases (Hugentobler et al. 2001, Kouba and Heroux 2001, Zumberge et al. 1997). The equivalence between the undifferenced and differencing algorithms were proved, and a unified equivalent data processing method was proposed by Xu (2002, cf. Sect. 6.8). The question of whether the uncombined and combining algorithms are also equivalent is an interesting topic and will be addressed here in detail (cf. Xu et al. 2006a).

6.7.1 Uncombined GPS Data Processing Algorithms

Original GPS Observation Equations

The original GPS code pseudorange and carrier phase measurements represented in Eqs. 6.44 and 6.45 (cf. Sect. 6.5) can be simplified as

$$R_j = C_\rho + \delta_{\text{ion}}(j) \ , \tag{6.130}$$

$$\lambda_j \Phi_j = C_\rho + \lambda_j N_j - \delta_{\text{ion}}(j) \ , j = 1,2 \tag{6.131}$$

where

$$C_\rho = \rho - (\delta t_{\text{r}} - \delta t_{\text{k}})\text{c} + \delta_{\text{trop}} + \delta_{\text{tide}} + \delta_{\text{rel}} + \varepsilon_i \ , \quad i = c, p \ , \tag{6.132}$$

$$\delta_{\text{ion}}(j) = \frac{A_1}{f_j^2} = \frac{A_1^z}{f_j^2} F = \frac{f_s^2 B_1}{f_j^2} = \frac{f_s^2 B_1^z}{f_j^2} F \ . \tag{6.133}$$

Where symbols have the same meanings as those of Eqs. 6.44–6.47. j is the index of the frequency f and wavelength λ. A_1 and A_1^z are the ionospheric parameters in the path and zenith directions; B_1 and B_1^z are scaled A_1 and A_1^z with f_s^2 for numerical reasons. c denotes the speed of light, index c denotes code. C_ρ is called geometry and N_j is the ambiguity. For simplicity, the residuals of the codes (and phases) are denoted with the same symbol ε_c (and ε_p) and have the same standard deviations of σ_c (and σ_p). Equations 6.130 and 6.131 can be written in a matrix form with weight matrix P as (Blewitt 1998)

$$\begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & f_s^2/f_1^2 & 1 \\ 0 & 0 & f_s^2/f_2^2 & 1 \\ 1 & 0 & -f_s^2/f_1^2 & 1 \\ 0 & 1 & -f_s^2/f_2^2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} , \ P = \begin{pmatrix} \sigma_c^2 & 0 & 0 & 0 \\ 0 & \sigma_c^2 & 0 & 0 \\ 0 & 0 & \sigma_p^2 & 0 \\ 0 & 0 & 0 & \sigma_p^2 \end{pmatrix}^{-1} . \tag{6.134}$$

Solutions of Uncombined Observation Equations

Equation 6.134 includes the observations of one satellite viewed by one receiver at one epoch. Alternatively, Eq. 6.134 can be considered a transformation between the observations and unknowns, and the transformation is a linear and invertible one. Denoting

$$a = \frac{f_1^2}{f_1^2 - f_2^2}, \quad b = \frac{-f_2^2}{f_1^2 - f_2^2}, \quad g = \frac{1}{f_1^2} - \frac{1}{f_2^2}, \quad q = g f_s^2 \ , \tag{6.135}$$

then one has relations of

$$1-a=b,\quad \frac{1}{f_1^2 g}=b,\quad \frac{1}{f_2^2 g}=-a \tag{6.136}$$

and

$$\begin{pmatrix} 0 & 0 & f_s^2/f_1^2 & 1 \\ 0 & 0 & f_s^2/f_2^2 & 1 \\ 1 & 0 & -f_s^2/f_1^2 & 1 \\ 0 & 1 & -f_s^2/f_2^2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1-2a & -2b & 1 & 0 \\ -2a & 2a-1 & 0 & 1 \\ 1/q & -1/q & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix} = T\ . \tag{6.137}$$

Where a and b are the coefficients of the ionosphere-free combinations of the observables of L1 and L2. The solution of Eq. 6.134 has a form of (by multiplying the transformation matrix T to Eq. 6.134)

$$\begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} = \begin{pmatrix} 1-2a & -2b & 1 & 0 \\ -2a & 2a-1 & 0 & 1 \\ 1/q & -1/q & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1 \Phi_1 \\ \lambda_2 \Phi_2 \end{pmatrix}. \tag{6.138}$$

The related covariance matrix of the above solution vector is then

$$Q=\mathrm{cov}\begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} = T \begin{pmatrix} \sigma_c^2 & 0 & 0 & 0 \\ 0 & \sigma_c^2 & 0 & 0 \\ 0 & 0 & \sigma_p^2 & 0 \\ 0 & 0 & 0 & \sigma_p^2 \end{pmatrix} T^T$$

$$= \begin{pmatrix} (1-2a)^2+4b^2+\frac{\sigma_p^2}{\sigma_c^2} & 4a^2-4ab-2a+2b & \frac{1-2a+2b}{q} & a-2a^2-2b^2 \\ 4a^2-4ab-2a+2b & 8a^2-4a+1+\frac{\sigma_p^2}{\sigma_c^2} & \frac{1-4a}{q} & -2a^2+2ab-b \\ \frac{1-2a+2b}{q} & \frac{1-4a}{q} & \frac{2}{q^2} & \frac{a-b}{q} \\ a-2a^2-2b^2 & -2a^2+2ab-b & \frac{a-b}{q} & a^2+b^2 \end{pmatrix} \sigma_c^2 \tag{6.139}$$

Equation 6.139 can be simplified by using the relation of $1-a=b$ and neglecting the terms of $(\sigma_p/\sigma_c)^2$ (because (σ_p/σ_c) is less than 0.01) as well as letting $f_s=f_1$ (so that $q=1/b$). Taking the relationships of ratios of the frequencies into account ($f_1=154f_0$ and $f_2=120f_0$, f_0 is the fundamental frequency), one has approximately

$$\operatorname{cov}\begin{pmatrix}\lambda_1 N_1\\ \lambda_2 N_2\\ B_1\\ C_\rho\end{pmatrix}=\begin{pmatrix}26.2971 & 33.4800 & 11.1028 & -15.1943\\ 33.4800 & 42.6629 & 14.1943 & -19.2857\\ 11.1028 & 14.1943 & 4.7786 & -6.3243\\ -15.1943 & -19.2857 & -6.3243 & 8.8700\end{pmatrix}\sigma_c^2\ . \tag{6.140}$$

The precisions of the solutions will be further discussed in Sect. 6.7.3. The parameterisation of the GPS observation models is an important issue and can be found in Chap. 9 or (Blewitt 1998; Xu 2004), if interested.

6.7.2 Combining Algorithms of GPS Data Processing

Ionosphere-free Combinations

Letting transformation matrix

$$T_1=\begin{pmatrix}1 & -1 & 0 & 0\\ a & b & 0 & 0\\ 0 & 0 & a & b\\ 1/2 & 0 & 1/2 & 0\end{pmatrix}, \tag{6.141}$$

and applying the transform to the Eq. 6.134, one has

$$T_1\begin{pmatrix}R_1\\ R_2\\ \lambda_1\Phi_1\\ \lambda_2\Phi_2\end{pmatrix}=\begin{pmatrix}0 & 0 & q & 0\\ 0 & 0 & 0 & 1\\ a & b & 0 & 1\\ 1/2 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1 N_1\\ \lambda_2 N_2\\ B_1\\ C_\rho\end{pmatrix}. \tag{6.142}$$

The ionosphere parameter in Eq. 6.142 is free in the last three equations, which are traditionally called ionosphere-free combinations. To solve the ionosphere-free equations or the whole Eq. 6.142 will lead to the same results. Equation 6.142 has a unique solution vector of

$$\begin{pmatrix}\lambda_1 N_1\\ \lambda_2 N_2\\ B_1\\ C_\rho\end{pmatrix}=\begin{pmatrix}0 & -2 & 0 & 2\\ 0 & (2a-1)/b & 1/b & -2a/b\\ 1/q & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}T_1\begin{pmatrix}R_1\\ R_2\\ \lambda_1\Phi_1\\ \lambda_2\Phi_2\end{pmatrix}, \tag{6.143}$$

or (noticing $(1 - a) = b$, cf., Eq. 6.136)

$$\begin{pmatrix}\lambda_1 N_1\\ \lambda_2 N_2\\ B_1\\ C_\rho\end{pmatrix}=\begin{pmatrix}1-2a & -2b & 1 & 0\\ -2a & 2a-1 & 0 & 1\\ 1/q & -1/q & 0 & 0\\ a & b & 0 & 0\end{pmatrix}\begin{pmatrix}R_1\\ R_2\\ \lambda_1\Phi_1\\ \lambda_2\Phi_2\end{pmatrix}. \tag{6.144}$$

Equations 6.144 and 6.138 are identical. Therefore the covariance matrix of the solution vector on the left side of Eq. 6.144 is the same as that given in Eq. 6.139. This shows that the uncombined algorithms and the ionosphere-free combinations are equivalent in this discussed case.

Geometry-free Combinations

Letting transformation matrix

$$T_2 = \begin{pmatrix} a & b & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \tag{6.145}$$

and applying the transformation to Eq. 6.134, one has

$$T_2 \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1 \Phi_1 \\ \lambda_2 \Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & q & 0 \\ 1 & -1 & -q & 0 \\ 1 & 0 & -2f_s^2 / f_1^2 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix}. \tag{6.146}$$

The geometric component in Eq. 6.146 is free in the last three equations, which are traditionally called geometry-free combinations. The geometry-free equations must be solved or Eq. 6.146 will lead to the same results. Equation 6.146 has a unique solution vector of

$$\begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} = \begin{pmatrix} 0 & 2/(f_1^2 g) & 0 & 1 \\ 0 & 2/(f_1^2 g) - 1 & -1 & 1 \\ 0 & 1/q & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} T_2 \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1 \Phi_1 \\ \lambda_2 \Phi_2 \end{pmatrix}, \tag{6.147}$$

or (noticing $1/(f_1{}^2 g) = b$, cf., Eq. 6.136)

$$\begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} = \begin{pmatrix} 2b-1 & -2b & 1 & 0 \\ 2b-2 & 1-2b & 0 & 1 \\ 1/q & -1/q & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1 \Phi_1 \\ \lambda_2 \Phi_2 \end{pmatrix}. \tag{6.148}$$

Taking the relations of Eq. 6.136 (i.e., $b = 1 - a$) into account, Eqs. 6.148 and 6.138 are identical. Therefore the covariance matrix of the solution vector on the left side of Eq. 6.148 is identical with Eq. 6.139. This shows that the uncombined algorithms and the geometry-free combinations are equivalent in this discussed case.

Ionosphere-free and Geometry-free Combinations

Letting transformation matrix

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \tag{6.149}$$

one then has

$$T_3T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & a & b \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ a & b & 0 & 0 \\ -a & -b & a & b \\ 1/2-a & -b & 1/2 & 0 \end{pmatrix}. \tag{6.150}$$

Applying the transformation 6.150 to Eq. 6.134 or applying the transformation 6.149 to Eq. 6.142 leads to the same results, and one has

$$T_3T_1 \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} \quad \text{or} \tag{6.151}$$

$$\begin{pmatrix} R_1 - R_2 \\ aR_1 + bR_2 \\ a\lambda_1\Phi_1 + b\lambda_2\Phi_2 - aR_1 - bR_2 \\ (\lambda_1\Phi_1 + R_1)/2 - aR_1 - bR_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix}. \tag{6.152}$$

The ionosphere and geometry are both free in the last two equations, which are called ionosphere-geometry-free combinations. Solving the ionosphere-free and geometry-free equations or directly solving Eq. 6.152 will lead to the same results. Eq. 6.152 has a unique solution vector of

$$\begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1/b & -2a/b \\ 1/q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} T_3T_1 \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix}, \tag{6.153}$$

or (noticing $(1-a)/b = 1$, cf., Eq. 6.136)

$$\begin{pmatrix} \lambda_1 N_1 \\ \lambda_2 N_2 \\ B_1 \\ C_\rho \end{pmatrix} = \begin{pmatrix} 1-2a & -2b & 1 & 0 \\ -2a & 2a-1 & 0 & 1 \\ 1/q & -1/q & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix}. \tag{6.154}$$

Equations 6.154 and 6.138 are identical. This shows that the uncombined algorithms and the ionosphere-geometry-free combinations are equivalent in this discussed case.

Diagonal Combinations

Letting transformation matrix

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.155}$$

one has

$$T_4T_3T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2a \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ a & b & 0 & 0 \\ -a & -b & a & b \\ 1/2-a & -b & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ a & b & 0 & 0 \\ -2ab & b(2a-1) & 0 & b \\ 1/2-a & -b & 1/2 & 0 \end{pmatrix}. \tag{6.156}$$

If applying the transformation 6.156 to Eq. 6.134 or applying the transformation 6.155 to Eq. 6.151, one has the same results of

$$T_4T_3T_1 \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1N_1 \\ \lambda_2N_2 \\ B_1 \\ C_\rho \end{pmatrix}. \tag{6.157}$$

In the above equation, the ionosphere and geometry as well as the ambiguities are diagonal to each other. Such combinations are called diagonal ones. The solution vector of Eq. 6.157 may be easily derived:

$$\begin{pmatrix} \lambda_1N_1 \\ \lambda_2N_2 \\ B_1 \\ C_\rho \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1/b & 0 \\ 1/q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} T_4T_3T_1 \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix} \tag{6.158}$$

or

$$\begin{pmatrix} \lambda_1N_1 \\ \lambda_2N_2 \\ B_1 \\ C_\rho \end{pmatrix} = \begin{pmatrix} 1-2a & -2b & 1 & 0 \\ -2a & 2a-1 & 0 & 1 \\ 1/q & -1/q & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1\Phi_1 \\ \lambda_2\Phi_2 \end{pmatrix}. \tag{6.159}$$

Equations 6.159 and 6.138 are identical. This shows that the uncombined algorithms and the diagonal combinations are equivalent in the discussed case.

General Combinations

For arbitrary combinations, as soon as the transformation matrix is an invertible one, the transformed equations are equivalent to the original ones based on algebra theory. Both the solution vector and the variance-covariance matrix are identical. That is, no matter what kinds of combinations are used, neither different solutions nor different precisions of the solutions will be obtained. The different combinations lead to an easier dealing of the related special problems.

Wide- and Narrow-lane Combinations

Denoting

$$T_5 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1/b & 0 \\ 1/q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{6.160}$$

and letting transformation matrix

$$T_6 = \begin{pmatrix} \dfrac{1}{\lambda_1} & \dfrac{-1}{\lambda_2} & 0 & 0 \\ \dfrac{1}{\lambda_1} & \dfrac{1}{\lambda_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.161}$$

one may form the wide and narrow lanes (Petovello 2006) directly by multiplying Eq. 6.161 to Eq. 6.158 to obtain the related wide- and narrow-lane ambiguities

$$\begin{pmatrix} N_1 - N_2 \\ N_1 + N_2 \\ B_1 \\ C_\rho \end{pmatrix} = T_6 T_5 T_4 T_3 T_1 \begin{pmatrix} R_1 \\ R_2 \\ \lambda_1 \Phi_1 \\ \lambda_2 \Phi_2 \end{pmatrix}. \tag{6.162}$$

Indeed, there is $T_5 T_4 T_3 T_1 = T$. Because of the unique property of the solutions of different combinations, any direct combinations of the solutions must be equivalent to each other. None of the combinations will lead to better solutions or better precisions of the solutions. From this rigorous theoretical aspect, the traditional wide-lane ambiguity fixing technique may lead to a more effective search, but not a better solution and precision of the ambiguity.

6.7.3
Secondary GPS Data Processing Algorithms

In the Case of More Satellites in View

Up to now, the discussions have been limited for the observations of one satellite viewed by one receiver at one epoch. The original observation equation is given in Eq. 6.134. The solution vector and its covariance matrix are given in Eqs. 6.138 and 6.139, respectively. The elements of the covariance matrix depend on the coefficients of Eq. 6.134, and the coefficients of the observation equation depend on the way of parameterisation. E.g., if instead of B_1, B_1^z is used, then Eq. 6.134 turns out to be

$$\begin{pmatrix} R_1(k) \\ R_2(k) \\ \lambda_1\Phi_1(k) \\ \lambda_2\Phi_2(k) \end{pmatrix} = \begin{pmatrix} 0 & 0 & F_k f_s^2/f_1^2 & 1 \\ 0 & 0 & F_k f_s^2/f_2^2 & 1 \\ 1 & 0 & -F_k f_s^2/f_1^2 & 1 \\ 0 & 1 & -F_k f_s^2/f_2^2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 N_1(k) \\ \lambda_2 N_2(k) \\ B_1^z \\ C_\rho(k) \end{pmatrix}, \tag{6.163}$$

where k is the index of the satellite. Ionospheric mapping function F_k is dependent on the zenith distance of the satellite k. The solution vector of Eq. 6.163 is then similar to that of Eq. 6.138:

$$\begin{pmatrix} \lambda_1 N_1(k) \\ \lambda_2 N_2(k) \\ B_1^z \\ C_\rho(k) \end{pmatrix} = \begin{pmatrix} 1-2a & -2b & 1 & 0 \\ -2a & 2a-1 & 0 & 1 \\ 1/q_k & -1/q_k & 0 & 0 \\ a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} R_1(k) \\ R_2(k) \\ \lambda_1\Phi_1(k) \\ \lambda_2\Phi_2(k) \end{pmatrix}, \quad Q(k)\,, \tag{6.164}$$

where $q_k = qF_k$ and $Q(k)$ is the covariance matrix, which can be similarly derived and given by adding the index k to q in Q of Eq. 6.139. The terms on the right-hand side can be considered secondary "observations" of the unknowns on the left-hand side. If K satellites are viewed, one has the observation equations of one receiver

$$\begin{pmatrix} \lambda_1 N_1(1) \\ \lambda_2 N_2(1) \\ B_1^z \\ C_\rho(1) \\ \vdots \\ \lambda_1 N_1(K) \\ \lambda_2 N_2(K) \\ B_1^z \\ C_\rho(K) \end{pmatrix} = \begin{pmatrix} 1-2a & -2b & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -2a & 2a-1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1/q_1 & -1/q_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-2a & -2b & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -2a & 2a-1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1/q_K & -1/q_K & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} R_1(1) \\ R_2(1) \\ \lambda_1\Phi_1(1) \\ \lambda_2\Phi_2(1) \\ \vdots \\ R_1(K) \\ R_2(K) \\ \lambda_1\Phi_1(K) \\ \lambda_2\Phi_2(K) \end{pmatrix}, \tag{6.165}$$

and variance matrix

$$Q_K = \begin{pmatrix} Q(1) & \ldots\ldots. & 0 \\ : & :::::: & : \\ 0 & \ldots\ldots. & Q(K) \end{pmatrix}. \tag{6.166}$$

Multiplying a transformation matrix

$$T(K) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots\ldots. & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots\ldots. & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/K & 0 & \ldots\ldots. & 0 & 0 & 1/K & 0 \\ 0 & 0 & 0 & 1 & \ldots\ldots. & 0 & 0 & 0 & 0 \\ : & : & : & : & :::::: & : & : & : & : \\ 0 & 0 & 0 & 0 & \ldots\ldots. & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots\ldots. & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots\ldots. & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.167}$$

to Eq. 6.165, one has the solutions of GPS observation equations of one station

$$\begin{pmatrix} \lambda_1 N_1(1) \\ \lambda_2 N_2(1) \\ B_1^z \\ C_\rho(1) \\ : \\ \lambda_1 N_1(K) \\ \lambda_2 N_2(K) \\ C_\rho(K) \end{pmatrix} = T(K) \begin{pmatrix} 1-2a & -2b & 1 & 0 & \ldots\ldots. & 0 & 0 & 0 & 0 \\ -2a & 2a-1 & 0 & 1 & \ldots\ldots. & 0 & 0 & 0 & 0 \\ 1/q_1 & -1/q_1 & 0 & 0 & \ldots\ldots. & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & \ldots\ldots. & 0 & 0 & 0 & 0 \\ : & : & : & : & :::::: & : & : & : & : \\ 0 & 0 & 0 & 0 & \ldots\ldots. & 1-2a & -2b & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots\ldots. & -2a & 2a-1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots\ldots. & 1/q_K & -1/q_K & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots\ldots. & a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} R_1(1) \\ R_2(1) \\ \lambda_1 \Phi_1(1) \\ \lambda_2 \Phi_2(1) \\ : \\ R_1(K) \\ R_2(K) \\ \lambda_1 \Phi_1(K) \\ \lambda_2 \Phi_2(K) \end{pmatrix}, \tag{6.168}$$

and the related

$$Q = T(K) Q_K (T(K))^T \tag{6.169}$$

where mapping function is used to combine the K ionosphere parameters into one. Similar discussions can be made for the cases of using more receivers. The original observation vector and the so-called secondary "observation" vector are

$$\begin{pmatrix} R_1(k) \\ R_2(k) \\ \lambda_1 \Phi_1(k) \\ \lambda_2 \Phi_2(k) \end{pmatrix}, \begin{pmatrix} \lambda_1 N_1(k) \\ \lambda_2 N_2(k) \\ B_1(k) \\ C_\rho(k) \end{pmatrix}. \tag{6.170}$$

Both vectors are equivalent as proved in Sect. 6.7.2 and they can be transformed uniquely from one to the other. Any further data processing can be considered processing based on the secondary "observations". The secondary "observations" own the equivalence property whether they are uncombined or combining ones. Therefore the equivalence property is valid for further data processing based on the secondary "observations".

GPS Data Processing Using Secondary "Observations"

A by-product of the above equivalence discussions is that the GPS data processing can be performed directly by using the so-called secondary observations. Besides the two ambiguity parameters (scaled with the wavelengths), the other two secondary observations are the electronic density in the observing path (scaled by square of f_1) and the geometry. The geometry includes the whole observation model except the ionosphere and ambiguity terms. For a time series of the secondary "observations", the electron density (or, for simplicity, "ionosphere") and the "geometry" are real time observations, whereas the "ambiguities" are constants in case no cycle-slip occurs (Langley 1998a, b). Sequential adjustment or filtering methods can be used to deal with the observation time series. It is notable that the secondary "observations" are correlated with each other (see the covariance matrix Eq. 6.139). However, the "ambiguities" are direct observations of the ambiguity parameters, and the "ionosphere" and "geometry" are modelled by Eqs. 6.132 and 6.133, respectively. The "ambiguity" observables are ionosphere-geometry-free. The "ionosphere" observable is geometry-free and ambiguity-free. The "geometry" observable is ionosphere-free. It is notable that some algorithms may be more effective; however, the results and the precisions of the solutions are equivalent no matter which algorithms are used. It should be emphasized that all the above discussions are based on the observation Model 6.134. The problem concerning the parameterisation of the GPS observation model will not affect the conclusions of the discussions and will be further discussed in Chap. 9.

Precision Analysis

If the sequential time series of the original observations are considered time independent as they traditionally have been, then the secondary "observations" and their precisions are also independent time series. From Eq. 6.140, the standard deviations of the L1 and L2 ambiguities are approximately $5.1281\sigma_c$ and $6.5317\sigma_c$. The standard deviation of ionosphere and geometry "observations" are about $2.1860\sigma_c$ and $2.9783\sigma_c$, respectively. That is, the precisions of the "observed" ambiguities are worse than that of the others at one epoch. If the standard deviation of the P code is about 1 decimetre (phase smoothed), then the precisions of the ambiguities determined by one epoch are worse than 0.5 meters. However, an average filter of m epoch data will raise the precisions by a factor of sqrt(m) (square root of m). After 100 or 10 000 epochs, the ambiguities are able to be determined with precisions of about 5 cm or 5 mm. "Ionosphere" data are observed with better precisions. However, due to the high dynamic of the electron movements, ionosphere effects may not be easily smoothed to raise the precision. The "Geometry" model is the most complicated one compared with the others, and discussions can be found from numerous publications for static, kinematic and dynamic applications (cf., e.g., ION proceedings, Chap. 10).

6.7.4 Summary

The equivalence properties between uncombined and combining algorithms are proved theoretically by algebraic linear transformations. The solution vector and the related covariance matrix are identical no matter which algorithms are used. Different combinations can lead to a more effective and easier dealing with the data. The so-called ionosphere-geometry-free and diagonal combinations are derived, which own better properties than that of the traditional combinations. A data processing algorithm using the uniquely transformed secondary "observations" is outlined and used to prove the equivalence. Because of the unique property of the solutions of different combinations, any direct combinations of the solutions must be equivalent to each other. None of the combinations will lead to better solutions or better precisions of the solutions than that of the others. From this aspect, the traditional wide-lane ambiguity fixing technique may lead to a more effective search of ambiguity, but it will not lead to a better solution and precision of the ambiguity.

6.8 Equivalence of Undifferenced and Differencing Algorithms

In Sect. 6.6 the single, double and triple differences as well as their related observation equations are discussed. The number of unknown parameters in the equations is greatly reduced through difference forming; however, the covariance derivations are tedious, especially for a GPS network.

In this section, a unified GPS data processing method based on equivalently eliminated equations is proposed and the equivalence between undifferenced and differencing algorithms is proved. The theoretic background of the method is given. By selecting the eliminated unknown vector as a vector of zero, a vector of satellite clock error, a vector of all clock error, a vector of clock and ambiguity parameters, or a vector of user-defined unknowns, the selectively eliminated equivalent observation equations can be formed, respectively. The equations are equivalent to the zero-, single-, double-, triple-, or user-defined differencing equations. The advantage of such a method is that the different GPS data processing methods are unified to a unique one, whereas the observational vector remains the original one and the weight matrix keeps the un-correlated diagonal form. In other words, by using this equivalent method, one may selectively reduce the unknown number; however, one does not have to deal with the complicated correlation problem. Several special cases of single-, double-, and triple-difference are discussed in detail to illustrate the theory. The reference-related parameters are dealt with using the a priori datum method.

6.8.1 Introduction

In GPS data processing practice, the commonly used methods are so-called zero-difference (un-differential), single-difference, double-difference and triple-difference methods (Bauer 1994; Hofmann-Wellenhof et al. 1997; King et al. 1987; Leick 1995; Remondi 1984; Seeber 1993; Strang and Borre 1997; Wang et al. 1988). It is well-known that the observation equations of the differencing methods can be obtained by carrying out a related linear transformation to the original equations. As soon as the weight matrix is similarly transformed according to the law of covariance propagation, all methods are equivalent, theoretically. A theoretical proof of the equivalence between the un-differential and differential methods can be found in Schaffrin and Grafarend (1986). A comparison of the advantages and disadvantages of the un-differential and differential methods can be found, e.g., in de Jong (1998). The advantage of the differential methods is that the unknown parameters are fewer so that the whole problem to be solved becomes smaller. The disadvantage of the differential methods is that there is a correlation problem, which appears in cases of multiple baselines of single-difference and all double difference as well as triple difference. The correlation problem is often complicated and not easy to be dealt with exactly (compared with the un-correlated problem). The advantages and disadvantages reach a balance. If one wants to deal with a reduced problem (cancellation of many unknowns), then one has to deal with the correlation problem. As an alternative, we use the equivalent observation equation approach to unify the un-differential and differential methods, while keeping all the advantages of the un-differential and differential methods.

In the next sections, the theoretical basis of the equivalently eliminated equations will be given based on the derivation of Zhou (1985). Several detailed cases are then discussed to illustrate the theory. The reference-related parameters are dealt with using the a priori datum method. A summary of the selectively eliminated equivalent GPS data processing method is outlined at the end.

6.8.2 Formation of Equivalent Observation Equations

For the convenience of later discussion, the method to form an equivalently eliminated equation system is outlined here. The theory is given in Sect. 7.6 in detail. In practice, sometimes only one group of unknowns is of interest; it is better to eliminate the other group of unknowns (called nuisance parameters), for example, because of their size. In this case, using the so-called equivalently eliminated observation equation system could be very beneficial (Wang et al. 1988; Xu and Qian 1986; Zhou 1985). The nuisance parameters can be eliminated directly from the observation equations instead of from the normal equations.

The linearised observation equation system can be represented using the matrix:

$$V = L - (A \quad B)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{and} \quad P, \tag{6.171}$$

where L is an observation vector of dimension n, A and B are coefficient matrices of dimension $n \times (s - r)$ and $n \times r$, X_1 and X_2 are unknown vectors of dimension $s - r$ and r, V is residual error, s is the total number of unknowns, and P is the weight matrix of dimension $n \times n$.

The related least squares normal equation can be formed then as:

$$(A \quad B)^T P (A \quad B) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = (A \quad B)^T PL \quad \text{or} \tag{6.172}$$

$$M_{11} X_1 + M_{12} X_2 = B_1 \quad \text{and} \tag{6.173}$$

$$M_{21} X_1 + M_{22} X_2 = B_2 \ , \tag{6.174}$$

where

$$B_1 = A^T PL \ , \quad B_2 = B^T PL \quad \text{and}$$

$$\begin{pmatrix} A^T PA & A^T PB \\ B^T PA & B^T PB \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} . \tag{6.175}$$

After eliminating the unknown vector X_1, the eliminated equivalent normal equation system is then

$$M_2 X_2 = R_2 \ , \tag{6.176}$$

where

$$M_2 = -M_{21} M_{11}^{-1} M_{12} + M_{22} = B^T PB - B^T PA\, M_{11}^{-1} A^T PB \quad \text{and} \tag{6.177}$$

$$R_2 = B_2 - M_{21} M_{11}^{-1} B_1 \ . \tag{6.178}$$

The related equivalent observation equation of Eq. 6.176 is then (cf. Sect. 7.6; Xu and Qian 1986; Zhou 1985)

$$U = L - (E - J) B X_2 \ , \quad P \ , \tag{6.179}$$

where

$$J = A M_{11}^{-1} A^T P \ . \tag{6.180}$$

E is an identity matrix of size n, L and P are the original observation vector and weight matrix, and U is the residual vector, which has the same property as V in Eq. 6.171. The advantage of using Eq. 6.179 is that the unknown vector X_1 has been eliminated; however, L vector and P matrix remain the same as the originals.

Similarly, the X_2 eliminated equivalent equation system is:

$$U_1 = L - (E - K) A X_1 \quad \text{and} \quad P \ , \tag{6.181}$$

where

$$K = B M_{22}^{-1} B^T P \ , \quad M_{22} = B^T PB \ ,$$

and U_1 is the residual vector (which has the same property as V).

We have separated the observation Eq. 6.171 into two equations, Eqs. 6.179 and 6.181; each equation contains only one of the unknown vectors. Each unknown vector can be solved independently and separately. Equations 6.179 and 6.181 are called equivalent observation equations of Eq. 6.171.

The equivalence property of Eqs. 6.171 and 6.179 is valid under three implicit assumptions. The first one is that the identical observation vector is used. The second is that the parameterisation of X_2 is identical. The third is that X_1 could be eliminated. Otherwise, the equivalence does not hold.

6.8.3 Equivalent Equations of Single Differences

In this section, the equivalent equations are formed to eliminate the satellite clock errors from the original zero-difference equations first, then the equivalency of the single differences (in two cases) related to the original zero-difference equations is proved.

Single differences cancel all the satellite clock errors out of the observation equations. This can also be achieved by forming equivalent equations where satellite clock errors are eliminated. Considering Eq. 6.171 the original observation equation and X_1 the vector of satellite clock errors, the equivalent equations of single differences can be formed as outlined in Sect. 6.8.2.

Suppose n common satellites ($k1, k2, \ldots, kn$) are observed at station $i1$ and $i2$. The original observation equation can then be written as

$$\begin{pmatrix} V_{i1} \\ V_{i2} \end{pmatrix} = \begin{pmatrix} L_{i1} \\ L_{i2} \end{pmatrix} - \begin{pmatrix} E & B_{i1} \\ E & B_{i2} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{and} \quad P = \frac{1}{\sigma^2} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \tag{6.182}$$

where X_1 is the vector of satellite clock errors and X_2 is the vector of other unknowns. For simplicity, clock errors are scaled by the speed of light c and directly used as unknowns; then the X_1-related coefficient matrix is an identity matrix, E.

Comparing Eq. 6.182 with Eq. 6.171, one has (cf. Sect. 6.8.2)

$$A = \begin{pmatrix} E \\ E \end{pmatrix}, \quad B = \begin{pmatrix} B_{i1} \\ B_{i2} \end{pmatrix}, \quad L = \begin{pmatrix} L_{i1} \\ L_{i2} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{i1} \\ V_{i2} \end{pmatrix},$$

and

$$M_{11} = \begin{pmatrix} E & E \end{pmatrix} \frac{1}{\sigma^2} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} E \\ E \end{pmatrix} = \frac{2}{\sigma^2} E,$$

$$J = \begin{pmatrix} E \\ E \end{pmatrix} \frac{\sigma^2}{2} E \begin{pmatrix} E & E \end{pmatrix} P = \frac{1}{2} \begin{pmatrix} E & E \\ E & E \end{pmatrix},$$

$$E_{2n \times 2n} - J = \frac{1}{2} \begin{pmatrix} E & -E \\ -E & E \end{pmatrix} \quad \text{and}$$

$$(E_{2n \times 2n} - J) B = \frac{1}{2} \begin{pmatrix} B_{i1} - B_{i2} \\ B_{i2} - B_{i1} \end{pmatrix}.$$

So the equivalently eliminated equation system of Eq. 6.182 is

$$\begin{pmatrix} U_{i1} \\ U_{i2} \end{pmatrix} = \begin{pmatrix} L_{i1} \\ L_{i2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} B_{i1} - B_{i2} \\ B_{i2} - B_{i1} \end{pmatrix} \cdot X_2 , \quad P = \frac{1}{\sigma^2}\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} , \tag{6.183}$$

where the satellite clock error vector X_1 is eliminated, and the observable vector and weight matrix are unchanged.

Denoting $B_s = B_{i2} - B_{i1}$, the least squares normal equation of Eq. 6.183 can then be formed as (cf. Chap. 7) (suppose Eq. 6.183 is solvable)

$$\frac{1}{2}\begin{pmatrix} -B_s^T & B_s^T \end{pmatrix} \cdot P \cdot \begin{pmatrix} -B_s \\ B_s \end{pmatrix} \cdot X_2 = \begin{pmatrix} -B_s^T & B_s^T \end{pmatrix} \cdot P \cdot \begin{pmatrix} L_{i1} \\ L_{i2} \end{pmatrix} \quad \text{or}$$

$$B_s^T B_s \cdot X_2 = B_s^T (L_{i2} - L_{i1}) \ . \tag{6.184}$$

Alternatively, a single difference equation can be obtained by multiplying Eq. 6.182 with a transformation matrix C_s

$$C_s = \begin{pmatrix} -E & E \end{pmatrix} ,$$

giving

$$C_s \cdot \begin{pmatrix} V_{i1} \\ V_{i2} \end{pmatrix} = C_s \cdot \begin{pmatrix} L_{i1} \\ L_{i2} \end{pmatrix} - C_s \cdot \begin{pmatrix} E & B_{i1} \\ E & B_{i2} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{or}$$

$$V_{i2} - V_{i1} = (L_{i2} - L_{i1}) - (B_{i2} - B_{i1}) X_2 \tag{6.185}$$

and

$$\text{cov}(\text{SD}(O)) = C_s \sigma^2 \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} C_s^T = 2\sigma^2 E \quad \text{and} \quad P_s = \frac{1}{2\sigma^2} E , \tag{6.186}$$

where P_s is the weight matrix of single differences, and cov(SD(O)) is the covariance of the single differences (SD) observational vector (O). Supposing Eq. 6.185 is solvable, the least squares normal equation system of Eq. 6.185 is then

$$(B_{i2} - B_{i1})^T (B_{i2} - B_{i1}) X_2 = (B_{i2} - B_{i1})^T (L_{i2} - L_{i1}) \ . \tag{6.187}$$

It is obvious that Eqs. 6.187 and 6.184 are identical. Therefore in the case of two stations, the single difference Eq. 6.185 is equivalent to the equivalently eliminated Eq. 6.183 and consequently equivalent to the original zero-difference equation.

Suppose n common satellites ($k1, k2, \ldots, kn$) are observed at station $i1$, $i2$ and $i3$. The original observation equation can then be written as

$$\begin{pmatrix} V_{i1} \\ V_{i2} \\ V_{i3} \end{pmatrix} = \begin{pmatrix} L_{i1} \\ L_{i2} \\ L_{i3} \end{pmatrix} - \begin{pmatrix} E & B_{i1} \\ E & B_{i2} \\ E & B_{i3} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{and} \quad P = \frac{1}{\sigma^2}\begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix} . \tag{6.188}$$

Comparing Eq. 6.188 with Eq. 6.171, one has (cf. Sect. 6.8.2)

$$A = \begin{pmatrix} E \\ E \\ E \end{pmatrix}, \quad B = \begin{pmatrix} B_{i1} \\ B_{i2} \\ B_{i3} \end{pmatrix}, \quad L = \begin{pmatrix} L_{i1} \\ L_{i2} \\ L_{i3} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_{i1} \\ V_{i2} \\ V_{i3} \end{pmatrix},$$

and

$$M_{11} = A^T PA = \frac{3}{\sigma^2} E \ ,$$

$$J = A\frac{\sigma^2}{3} EA^T P = \frac{1}{3}\begin{pmatrix} E & E & E \\ E & E & E \\ E & E & E \end{pmatrix},$$

$$E_{3n\times 3n} - J = \frac{1}{3}\begin{pmatrix} 2E & -E & -E \\ -E & 2E & -E \\ -E & -E & 2E \end{pmatrix} \quad \text{and}$$

$$(E_{3n\times 3n} - J)B = \frac{1}{3}\begin{pmatrix} 2B_{i1} - B_{i2} - B_{i3} \\ -B_{i1} + 2B_{i2} - B_{i3} \\ -B_{i1} - B_{i2} + 2B_{i3} \end{pmatrix} .$$

So the equivalently eliminated equation system of Eq. 6.188 is

$$\begin{pmatrix} U_{i1} \\ U_{i2} \\ U_{i3} \end{pmatrix} = \begin{pmatrix} L_{i1} \\ L_{i2} \\ L_{i3} \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 2B_{i1} - B_{i2} - B_{i3} \\ -B_{i1} + 2B_{i2} - B_{i3} \\ -B_{i1} - B_{i2} + 2B_{i3} \end{pmatrix} \cdot X_2 , \quad P = \frac{1}{\sigma^2}\begin{pmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix}, \tag{6.189}$$

and the related least squares normal equation can be formed as

$$\frac{1}{3}\begin{pmatrix} 2B_{i1} - B_{i2} - B_{i3} \\ -B_{i1} + 2B_{i2} - B_{i3} \\ -B_{i1} - B_{i2} + 2B_{i3} \end{pmatrix}^T \begin{pmatrix} 2B_{i1} - B_{i2} - B_{i3} \\ -B_{i1} + 2B_{i2} - B_{i3} \\ -B_{i1} - B_{i2} + 2B_{i3} \end{pmatrix} X_2 = \begin{pmatrix} 2B_{i1} - B_{i2} - B_{i3} \\ -B_{i1} + 2B_{i2} - B_{i3} \\ -B_{i1} - B_{i2} + 2B_{i3} \end{pmatrix}^T \begin{pmatrix} L_{i1} \\ L_{i2} \\ L_{i3} \end{pmatrix} . \tag{6.190}$$

Alternatively, for the Eq. system 6.188, single differences can be formed using transformation (cf. Sect. 6.6.1):

$$C_s = \begin{pmatrix} -E & E & 0 \\ 0 & -E & E \end{pmatrix}$$

and

$$P_s = [\text{cov(SD)}]^{-1} = \frac{1}{3\sigma^2}\begin{pmatrix} 2E & E \\ E & 2E \end{pmatrix} .$$

The correlation problem appears in the case of single differences of multiple baselines. The related observation equations and the least squares normal equation can be written as

$$\begin{pmatrix} V_{i2}-V_{i1} \\ V_{i3}-V_{i2} \end{pmatrix} = \begin{pmatrix} L_{i2}-L_{i1} \\ L_{i3}-L_{i2} \end{pmatrix} - \begin{pmatrix} B_{i2}-B_{i1} \\ B_{i3}-B_{i2} \end{pmatrix} X_2 \;, \quad P_{\mathrm{s}} \;\text{ and} \tag{6.191}$$

$$\begin{pmatrix} B_{i2}-B_{i1} \\ B_{i3}-B_{i2} \end{pmatrix}^T \begin{pmatrix} 2E & E \\ E & 2E \end{pmatrix} \begin{pmatrix} B_{i2}-B_{i1} \\ B_{i3}-B_{i2} \end{pmatrix} X_2 = \begin{pmatrix} B_{i2}-B_{i1} \\ B_{i3}-B_{i2} \end{pmatrix}^T \begin{pmatrix} 2E & E \\ E & 2E \end{pmatrix} \begin{pmatrix} L_{i2}-L_{i1} \\ L_{i3}-L_{i2} \end{pmatrix} . \tag{6.192}$$

Equations 6.190 and 6.192 are identical. This may be proved by expanding both equations and comparing the results. Again, this shows that the equivalently eliminated equations are equivalent to the single difference equations, however, without the need to deal with the correlation problem.

6.8.4 Equivalent Equations of Double Differences

Double differences cancel all the clock errors out of the observation equations. This can also be achieved by forming equivalent equations where all clock errors are eliminated. Considering Eq. 6.171 the original observation equation and X_1 the vector of all clock errors, the equivalent equation of double differences can be formed as outlined in Sect. 6.8.2.

In the case of two stations, supposing n common satellites ($k1, k2, \ldots, kn$) are observed at station $i1$ and $i2$, the equivalent single difference observation equation is then Eq. 6.183. Denoting $B_{\mathrm{s}1} = B_{i2} - B_{i1}$, the station clock error parameter as $\delta t_{i1} - \delta t_{i2}$ (cf. Eqs. 6.89–6.92), and assigning the coefficients of the first column to the station clock errors, i.e., $B_{\mathrm{s}1} = (I_{n\times 1}\ B_{\mathrm{s}})$, Eq. 6.183 turns out to be

$$\begin{pmatrix} U_{i1} \\ U_{i2} \end{pmatrix} = \begin{pmatrix} L_{i1} \\ L_{i2} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -I_{n\times 1} & -B_s \\ I_{n\times 1} & B_s \end{pmatrix} \begin{pmatrix} X_c \\ X_3 \end{pmatrix} \quad \text{and} \quad P = \frac{1}{\sigma^2} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} , \tag{6.193}$$

where X_{c} is the station clock error vector, X_3 is the other unknown vector, B_{s} is the X_3-related coefficient matrix, $I_{n\times 1}$ is a 1 matrix (where all elements are 1), and clock errors are scaled by the speed of light.

Comparing Eq. 6.193 with Eq. 6.171, one has (cf. Sect. 6.8.2)

$$A = \frac{1}{2} \begin{pmatrix} -I_{n\times 1} \\ I_{n\times 1} \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} -B_{\mathrm{s}} \\ B_{\mathrm{s}} \end{pmatrix}, \quad L = \begin{pmatrix} L_{i1} \\ L_{i2} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} U_{i1} \\ U_{i2} \end{pmatrix} ,$$

and

$$M_{11} = \frac{1}{4} \begin{pmatrix} -I_{n\times 1}^T & I_{n\times 1}^T \end{pmatrix} \frac{1}{\sigma^2} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} -I_{n\times 1} \\ I_{n\times 1} \end{pmatrix} = \frac{n}{2\sigma^2} ,$$

$$J = \begin{pmatrix} -I_{n\times 1} \\ I_{n\times 1} \end{pmatrix} \frac{\sigma^2}{2n} \begin{pmatrix} -I_{n\times 1}^T & I_{n\times 1}^T \end{pmatrix} \cdot P = \frac{1}{2n} \begin{pmatrix} I_{n\times n} & -I_{n\times n} \\ -I_{n\times n} & I_{n\times n} \end{pmatrix} \quad \text{and}$$

$$(E_{2n\times 2n}-J)\frac{1}{2}\begin{pmatrix}-B_s\\ B_s\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-E_{n\times n}+\frac{1}{n}I_{n\times n}\\ E_{n\times n}-\frac{1}{n}I_{n\times n}\end{pmatrix}B_s \ .$$

So the equivalently eliminated equation system of Eq. 6.193 is

$$\begin{pmatrix}U_{i1}\\ U_{i2}\end{pmatrix}=\begin{pmatrix}L_{i1}\\ L_{i2}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-E_{n\times n}+\frac{1}{n}I_{n\times n}\\ E_{n\times n}-\frac{1}{n}I_{n\times n}\end{pmatrix}B_s X_3 \quad \text{and} \quad P=\frac{1}{\sigma^2}\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}, \tag{6.194}$$

where the receiver clock error vector X_c is eliminated, observable vector and weight matrix are unchanged. The normal equation has a simple form of

$$B_s^T\left(E_{n\times n}-\frac{1}{n}I_{n\times n}\right)B_s X_3 = B_s^T\left(E_{n\times n}-\frac{1}{n}I_{n\times n}\right)(L_{i2}-L_{i1}) \ . \tag{6.195}$$

Alternatively, the traditional single difference observation Eqs. 6.185 and 6.186 can be rewritten as

$$V_{i2}-V_{i1}=(L_{i2}-L_{i1})-\begin{pmatrix}I_{n\times 1} & B_s\end{pmatrix}\begin{pmatrix}X_c\\ X_3\end{pmatrix} \quad \text{or}$$

$$\begin{pmatrix}V_{i2}^1-V_{i1}^1\\ V_{i2}^k-V_{i1}^k\end{pmatrix}=\begin{pmatrix}L_{i2}^1-L_{i1}^1\\ L_{i2}^k-L_{i1}^k\end{pmatrix}-\begin{pmatrix}1 & B_s^1\\ I_{m\times 1} & B_s^k\end{pmatrix}\begin{pmatrix}X_c\\ X_3\end{pmatrix} \tag{6.196}$$

and

$$\mathrm{cov}(\mathrm{SD}(O))=C_s\sigma^2\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}C_s^T=2\sigma^2E \quad \text{and} \quad P_s=\frac{1}{2\sigma^2}E \ ,$$

where $m = n - 1$, and the superscript 1 and k denote the first row and remaining rows of the matrices (or columns in case of vectors). The double difference transformation matrix and covariance are (cf. Sect. 6.6.2, Eqs. 6.116–6.118)

$$C_d=\begin{pmatrix}-I_{m\times 1} & E_{m\times m}\end{pmatrix} ,$$

$$\mathrm{cov}(\mathrm{DD}(O))=C_d\,\mathrm{cov}(\mathrm{SD}(O))C_d^T=2\sigma^2C_dC_d^T=2\sigma^2\left(I_{m\times m}+E_{m\times m}\right) \quad \text{and}$$

$$P_d=[\mathrm{cov}(\mathrm{DD}(O))]^{-1}=\frac{1}{2\sigma^2 n}\left(nE_{m\times m}-I_{m\times m}\right) .$$

The double difference observation equation and related normal equation are

$$C_d\begin{pmatrix}V_{i2}^1-V_{i1}^1\\ V_{i2}^k-V_{i1}^k\end{pmatrix}=C_d\begin{pmatrix}L_{i2}^1-L_{i1}^1\\ L_{i2}^k-L_{i1}^k\end{pmatrix}-C_d\begin{pmatrix}1 & B_s^1\\ I_{m\times 1} & B_s^k\end{pmatrix}\begin{pmatrix}X_c\\ X_3\end{pmatrix}$$

or

$$C_\mathrm{d}\begin{pmatrix} V_{i2}^1 - V_{i1}^1 \\ V_{i2}^k - V_{i1}^k \end{pmatrix} = C_\mathrm{d}\begin{pmatrix} L_{i2}^1 - L_{i1}^1 \\ L_{i2}^k - L_{i1}^k \end{pmatrix} - C_\mathrm{d}\begin{pmatrix} B_\mathrm{s}^1 \\ B_\mathrm{s}^k \end{pmatrix} X_3 \ ,$$

i.e.,

$$C_\mathrm{d}(V_{i2} - V_{i1}) = C_\mathrm{d}(L_{i2} - L_{i1}) - C_\mathrm{d} B_\mathrm{s} X_3 \tag{6.197}$$

and

$$B_\mathrm{s}^T C_\mathrm{d}^T P_\mathrm{d} C_\mathrm{d} B_\mathrm{s} X_3 = B_\mathrm{s}^T C_\mathrm{d}^T P_\mathrm{d} C_\mathrm{d} (L_{i2} - L_{i1}) \ , \tag{6.198}$$

where

$$C_\mathrm{d}^T P_\mathrm{d} C_\mathrm{d} = \frac{1}{2\sigma^2 n} \begin{pmatrix} -I_{m\times 1} & E_{m\times m} \end{pmatrix}^T \left(nE_{m\times m} - I_{m\times m} \right) \begin{pmatrix} -I_{m\times 1} & E_{m\times m} \end{pmatrix} \ , \tag{6.199}$$

$$\begin{pmatrix} -I_{m\times 1} & E_{m\times m} \end{pmatrix}^T \left(nE_{m\times m} - I_{m\times m} \right) = \begin{pmatrix} -I_{m\times 1} & nE_{m\times m} - I_{m\times m} \end{pmatrix}^T \quad \text{and} \tag{6.200}$$

$$\begin{pmatrix} -I_{m\times 1} & nE_{m\times m} - I_{m\times m} \end{pmatrix}^T \begin{pmatrix} -I_{m\times 1} & E_{m\times m} \end{pmatrix} = nE_{n\times n} - I_{n\times n} \ . \tag{6.201}$$

The above three equations can be proved readily. Substituting Eqs. 6.199–6.201 into Eq. 6.198, then Eq. 6.198 turns out to be the same as Eq. 6.195. So the equivalency between the double difference equation and the directly formed equivalent Eq. 6.193 is proved.

6.8.5 Equivalent Equations of Triple Differences

Triple differences cancel all the clock errors and ambiguities out of the observation equations. This can also be achieved by forming equivalent equations where all clock errors and ambiguities are eliminated. Considering Eq. 6.171 the original observation equation and X_1 the parameter vector of all clock errors and ambiguities, then the equivalent equations of triple differences can be formed as outlined in Sect. 6.8.2.

It is well-known that traditional triple differences are correlated between adjacent epochs and between baselines. In the case of sequential (epoch by epoch) data processing of triple differences, the correlation problem is difficult to be dealt with. However, using the equivalently eliminated equations, the weight matrix remains diagonal. The GPS observables remain the original ones.

6.8.6 Method of Dealing with the Reference Parameters

In differential GPS data processing, the reference-related parameters are usually considered known and are fixed (or not adjusted). This may be realised by the a priori datum method (for details cf. Sect. 7.8.2). Here we just outline the basic principle.

The equivalent observation Eq. system 6.179 can be rewritten as

$$U = L - \begin{pmatrix} D_1 & D_2 \end{pmatrix} \begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix} \quad \text{and} \quad P, \tag{6.202}$$

where

$$D = \begin{pmatrix} D_1 & D_2 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} X_{21} \\ X_{22} \end{pmatrix}.$$

Suppose there are a priori constraints of (cf. e.g. Zhou et al. 1997)

$$W = \bar{X}_{22} - X_{22} \quad \text{and} \quad P_2, \tag{6.203}$$

where $\bar{X}_{22}$ is the "directly observed" parameter sub-vector, P_2 is the weight matrix with respect to the parameter sub-vector X_{22}, and W is a residual vector, which has the same property as U. Usually, $\bar{X}_{22}$ is "observed" independently, so P_2 is a diagonal matrix. If X_{22} is a sub-vector of station coordinates, then the constraint of Eq. 6.203 is called a datum constraint. (This is also the reason why the name a priori datum is used). We consider here X_{22} a vector of reference-related parameters (such as clock errors and ambiguities of the reference satellite and reference station). Generally, the a priori weight matrix P_2 is given by covariance matrix Q_W and

$$P_2 = Q_W^{-1}. \tag{6.204}$$

In practice, the sub-vector $\bar{X}_{22}$ is usually a zero vector; this can be achieved through careful initialisation by forming observation Eq. 6.171.

The least squares normal equation of the a priori datum problem of Eqs. 6.202 and 6.203 can be formed (cf. Sect. 7.8.2). Compared with the normal equation of Eq. 6.202, the only difference between the two normal equations is that the a priori weight matrix P_2 has been added to the normal matrix. This indicates that the a priori datum problem can be dealt with simply by adding P_2 to the normal equation of observation Eq. 6.202.

If some diagonal components of the weight matrix P_2 is set to zero, then the related parameters (in X_{22}) are free parameters (or free datum) of the adjustment problem (without a priori constraints). Otherwise, parameters with a priori constraints are called a priori datum. Large weight indicates strong constraint and small weight indicates soft constraint. The strongest constraint is to keep the datum fixed. The reference-related datum (coordinates and clock errors as well as ambiguities) can be fixed by applying the strongest constraints to the related parameters, i.e., by adding the strongest constraints to the datum-related diagonal elements of the normal matrix.

6.8.7
Summary of the Unified Equivalent Algorithm

For any linearised zero-difference GPS observation Eq. system 6.171

$$V = L - \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \text{and} \quad P, \tag{6.205}$$

the X_1 eliminated equivalent GPS observation equation system is then Eq. 6.179:

$$U = L - (E - J)BX_2 \quad \text{and} \quad P\,, \tag{6.206}$$

where

$$J = AM_{11}^{-1}A^TP\,, \quad M_{11} = A^TPA\,,$$

E is an identity matrix, L is original observational vector, P is original weight matrix, and U is residual vector, which has the same property as V.

Similarly, the X_2 eliminated equivalent equation system is Eq. 6.181

$$U_1 = L - (E - K)AX_1 \quad \text{and} \quad P\,, \tag{6.207}$$

where

$$K = BM_{22}^{-1}B^TP\,, \quad M_{22} = B^TPB\,,$$

and U_1 is the residual vector (which has the same property as V).

Fixing the values of sub-vector X_{22} (of X_2) can be realised by adding the strongest constraints to the X_{22}-related diagonal elements of the normal matrix formed by Eq. 6.206. Alternatively, we may apply the strongest constraints directly to the normal equation formed by Eq. 6.205 first. In this way, the reference-related parameters (clock errors, ambiguities, coordinates, etc.) are fixed. And then we may form the equivalently eliminated observation Eq. 6.206. In this way, the relative and differential GPS data processing can be realised by using Eq. 6.206 after selecting the to be eliminated X_1.

The GPS data processing algorithm using Eq. 6.206 is then a selectively eliminated equivalent method. Selecting X_1 in Eq. 6.205 as a zero vector, then the algorithm is identical to the zero-difference method. Selecting X_1 in Eq. 6.205 as the satellite clock error vector, the vector of all clock errors, the clock error and ambiguity vector, and any user-defined vector, then the algorithm is equivalent to the single-difference method, double-difference method, triple-difference method, and user-defined eliminating method, respectively. The eliminated unknown X_1 can be solved separately if desired.

The advantages of this method are (compared with un-differential and differential methods):

- The un-differential and differential GPS data processing can be dealt with in an equivalent and unified way. The data processing scenarios can be selected by a switch and used in a combinative way;
- The eliminated parameters can be also solved separately with the same algorithm;
- The weight matrix remains the original diagonal one;
- The original observations are used; no differencing is required.

It is obvious that the described algorithm meanwhile has all the advantages of all un-differential and differential GPS data processing methods.