Techniques for Uniting Lyapunov-Based and Model Predictive Control*

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Summary. This paper presents a review of recent contributions that unite predictive control approaches with Lyapunov-based control approaches at the implementation level (Hybrid predictive control) and at the design level (Lyapunov-based predictive control) in a way that allows for an explicit characterization of the set of initial conditions starting from where closed–loop stability is guaranteed in the presence of constraints.

1 Introduction

Virtually all operation of chemical processes is subject to constraints on their manipulated inputs and state variables. Input constraints arise as a manifestation of the physical limitations inherent in the capacity of control actuators (e.g., bounds on the magnitude of valve opening), and are enforced at all times (hard constraints). State constraints, on the other hand, arise either due to the necessity to keep the state variables within acceptable ranges to avoid, for example, runaway reactions (in which case they need to be enforced at all times, and treated as hard constraints) or due to the desire to maintain them within desirable bounds dictated by performance considerations (in which case they may be relaxed, and treated as soft constraints). Constraints automatically impose limitations on our ability to steer the dynamics of the closed-loop system at will, and can cause severe deterioration in the nominal closed-loop performance and may even lead to closed-loop instability if not explicitly taken into account at the stage of controller design.

Currently, model predictive control (MPC), also known as receding horizon control (RHC), is one of the few control methods for handling state and input

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constraints within an optimal control setting and has been the subject of numerous research studies that have investigated the stability properties of MPC. In the literature, several nonlinear model predictive control (NMPC) schemes have been developed (e.g., see [2, 4, 17, 19, 20, 27, 30]) that focus on the issues of stability, constraint satisfaction, uncertainty and performance optimization for nonlinear systems. One of the key challenges that impact on the practical implementation of NMPC is the inherent difficulty of characterizing, *a priori* (i.e., before controller implementation or testing for feasibility), the set of initial conditions starting from where a given NMPC controller is guaranteed to stabilize the closed–loop system. Specifically, the stability guarantee in various MPC formulations (with or without stability conditions, and with or without robustness considerations) is contingent upon the assumption of initial feasibility, and the set of initial conditions starting from where feasibility and stability is guaranteed is not explicitly characterized. For finite–horizon MPC, an adequate characterization of the stability region requires an explicit characterization of the complex interplay between several factors, such as the initial condition, the size of the constraints and uncertainty, the horizon length, the penalty weights, etc. Use of conservatively large horizon lengths to address stability only increases the size and complexity of the optimization problem and could make it intractable.

The desire to implement control approaches that allow for an explicit characterization of their stability properties has motivated significant work on the design of stabilizing control laws using Lyapunov techniques that provide explicitly–defined regions of attraction for the closed–loop system; the reader may refer to [15] for a survey of results in this area, for a more recent review, see [5]. In [6, 7, 8], a class of Lyapunov–based bounded robust nonlinear controllers, inspired by the results on bounded control originally presented in [18], was developed. While these Lyapunov–based controllers have well–characterized stability and constraint–handling properties, they cannot, in general, be designed to be optimal with respect to a pre–specified, arbitrary cost function.

From the above discussion, it is clear that both MPC and Lyapunov–based analytic control approaches possess, by design, their own, distinct stability and optimality properties. Motivated by these considerations, this paper presents a review of recent contributions [9, 10, 22, 23, 24, 25] that unite predictive control approaches with Lyapunov-based control approaches at the implementation level (Hybrid predictive control) and at the design level (Lyapunov-based predictive control) in a way that allows for an explicit characterization of the set of initial conditions starting from where closed–loop stability is guaranteed in the presence of constraints.

2 Preliminaries

We focus on the problem of nonlinear systems with input constraints of the form:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{1}$$

$$\|u\| \leq u_{max} \tag{2}$$

where $x = [x_1 \cdots x_n]' \in \mathbb{R}^n$ denotes the vector of state variables, $u = [u_1 \cdots u_m]'$ is the vector of manipulated inputs, $u_{max} \geq 0$ denotes the bound on the manipulated inputs, $f(\cdot)$ is a sufficiently smooth $n \times 1$ nonlinear vector function, and $g(\cdot)$ is a sufficiently smooth $n \times m$ nonlinear matrix functions. Without loss of generality, it is assumed that the origin is the equilibrium point of the unforced system (i.e. $f(0) = 0$). Throughout the paper, the notation $\|\cdot\|$ will be used to denote the standard Euclidean norm of a vector, while the notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x'Qx$ for all $x \in \mathbb{R}^n$, where Q is a positive–definite symmetric matrix and x' denotes the transpose of x. In order to provide the necessary background for our results in sections 3 and 4, we will briefly review in the remainder of this section the design procedure for, and the stability properties of, both the bounded and model predictive controllers, which constitute the basic components of our controllers. We focus on the state feedback control problem where measurements of $x(t)$ are assumed to be available for all t.

2.1 Model Predictive Control

We describe here a symbolic MPC formulation that incorporates most existing MPC formulations as special cases. This is not a new formulation of MPC; the general description is only intended for the purpose of highlighting the fact that the hybrid predictive control structure can incorporate any available MPC formulation. In MPC, the control action at time t is conventionally obtained by solving, on–line, a finite horizon optimal control problem. The generic form of the optimization problem can be described as:

$$\begin{aligned} & u(\cdot) = argmin\{J_s(x,t,u(\cdot))\| u(\cdot) \in S\} \\ & s.t. \quad \dot{x}(t) = f(x(t)) + g(x)u \\ & x(0) = x_0, \;\; x(t+T) \in \Omega_{MPC}(x,t) \end{aligned} \tag{3}$$

$$J_s(x,t,u(\cdot)) = \int_t^{t+T} (x'(s)Qx(s) + u'(s)Ru(s))ds + F(x(t+T)) \tag{4}$$

and $S = S(t,T)$ is the family of piecewise continuous functions, with period Δ, mapping $[t, t+T]$ into the set of admissible controls and T is the horizon length. A control $u(\cdot)$ in S is characterized by the sequence $\{u[k]\}$ where $u[k] := u(k\Delta)$ with $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. J_s is the performance index, R and Q are strictly positive definite, symmetric matrices and the function $F(x(t+T))$ represents a penalty on the states at the end of the horizon. The set $\Omega_{MPC}(x,t)$ could be a fixed set, or may represent inequality constraints (as in the case of MPC formulations that require some norm of the state, or a Lyapunov function value, to decrease at the end of the horizon). The stability guarantees in MPC formulations depend on the assumption of initial feasibility and obtaining an explicit characterization of the closed–loop stability region of the predictive controller remains a difficult task.

2.2 Bounded Lyapunov-Based Control

Consider the system of Eqs.1-2, for which a family of control Lyapunov functions (CLFs), $V_k(x)$, $k \in \mathcal{K} \equiv \{1, \cdots, p\}$ has been found. Using each control Lyapunov function, we construct, using the results in [18] (see also [6, 7]), the following continuous bounded control law

$$u_k(x) = -k_k(x)(L_gV_k)'(x) \equiv b_k(x) \tag{5}$$

$$k_k(x) = \frac{L_fV_k(x) + \sqrt{(L_fV_k(x))^2 + (u_{max}\|(L_gV_k)'(x)\|)^4}}{\|(L_gV_k)'(x)\|^2 \left[1 + \sqrt{1 + (u_{max}\|(L_gV_k)'(x)\|)^2}\right]} \tag{6}$$

$L_fV_k(x) = \dfrac{\partial V_k(x)}{\partial x} f(x)$, $L_gV_k(x) = [L_{g_1}V_k(x) \cdots L_{g_m}V_k(x)]'$ and $g_i(x)$ is the i-th column of the matrix $g(x)$. For the above controller, it can be shown, using standard Lyapunov arguments, that for all initial conditions within the state–space region described by the set

$$\Omega_k(u_{max}) = \{x \in \mathbb{R}^n : V_k(x) \leq c_k^{max}\} \tag{7}$$

where $c_k^{max} > 0$ is the largest number for which $\Phi_k(u_{max}) \supset \Omega_k(u_{max}) \backslash \{0\}$ where

$$\Phi_k(u_{max}) = \{x \in \mathbb{R}^n : L_fV_k(x) < u_{max}\|(L_gV_k)'(x)\|\} \tag{8}$$

then the controller continues to satisfy the constraints, and the time-derivative of the Lyapunov function is negative–definite for all times. The union of the invariant regions described by the set

$$\Omega(u_{max}) = \bigcup_{k=1}^{p} \Omega_k(u_{max}) \tag{9}$$

then provides an estimate of the stability region, starting from where the origin of the constrained closed–loop system, under the appropriate control law from the family of Eqs.5-6, is guaranteed to be asymptotically stable. Note that CLF-based stabilization of nonlinear systems has been studied extensively in the nonlinear control literature (e.g., see [1, 11, 18, 29]). The construction of constrained CLFs (i.e. CLFs that take the constraints into account) remains a difficult problem (especially for nonlinear systems) that is the subject of ongoing research. For several classes of nonlinear systems that arise commonly in the modeling of practical systems, systematic and computationally feasible methods are available for constructing unconstrained CLFs (CLFs for the unconstrained system) by exploiting the system structure. Examples include the use of quadratic functions for feedback linearizable systems and the use of back-stepping techniques to construct CLFs for systems in strict feedback form. Furthermore, we note here that the bounded control law of Eqs.5-6 will be used in the remainder of the paper only to illustrate the basic idea of the proposed techniques for uniting

Lyapunov-based and predictive controllers. Our choice of using this particular design is motivated by its explicit structure and well-defined region of stability. However, our results are not restricted to this particular design and any other analytical bounded control law, with an explicit structure and well-defined region of stability, can be used.

3 Hybrid Predictive Control

By comparing the bounded controller and MPC designs presented in the previous section, some tradeoffs with respect to their stability and optimality properties are evident. The bounded controller, for example, possesses a well-defined region of admissible initial conditions that guarantee constrained closed–loop stability. However, its performance may not be optimal with respect to an arbitrary performance criterion. MPC, on the other hand, provides the desired optimality requirement, but poses implementation difficulties and lacks an explicit characterization of the stability region. In this section, we reconcile the two approaches by means of a switching scheme that provides a safety net for the implementation of MPC to nonlinear systems.

3.1 Formulation of the Switching Problem

Consider the constrained nonlinear system of Eqs.1-2, for which the bounded controllers of Eqs.5-6 and predictive controller of Eqs.3-4 have been designed. The control problem is formulated as the one of designing a set of switching laws that orchestrate the transition between MPC and the bounded controllers in a way that guarantees asymptotic stability of the origin of the closed–loop system starting from any initial condition in the set $\Omega(u_{max})$ defined in Eq.9, respects input constraints, and accommodates the optimality requirements whenever possible. For a precise statement of the problem, the system of Eq.1 is first cast as a switched system of the form

$$\dot{x} = f(x) + g(x)u_{i(t)};\ \|u_i\| \leq u_{max};\ i(t) \in \{1, 2\} \tag{10}$$

where $i : [0, \infty) \rightarrow \{1, 2\}$ is the switching signal, which is assumed to be a piecewise continuous (from the right) function of time, implying that only a finite number of switches, between the predictive and bounded controllers, is allowed on any finite interval of time. The index, $i(t)$, which takes values in the set $\{1, 2\}$, represents a discrete state that indexes the control input $u(\cdot)$, with the understanding that $i(t) = 1$ if and only if $u_i(x(t)) = M(x(t))$ and $i(t) = 2$ if and only if $u_i(x(t)) = b_k(x(t))$ for some $k \in \mathcal{K}$. Our goal is to construct a switching law $i(t) = \psi(x(t), t)$ that provides the set of switching times that ensure stabilizing transitions between the predictive and bounded controllers, in the event that the predictive controller is unable to enforce closed–loop stability. This in turn determines the time-course of the discrete state $i(t)$. While various switching schemes that focus on closed–loop stability and performance considerations to various degrees are possible [9, 10, 22, 24], we next present one example of a switching scheme (formalized in Theorem 1 below; for the proof, see [9]) that addresses the above problem while focusing on achieving closed–loop stability.

3.2 Controller Switching Logic

Theorem 1. *Consider the constrained nonlinear system of Eq.10, with any initial condition* $x(0) \equiv x_0 \in \Omega_k(u_{max})$*, for some* $k \in \mathcal{K} \equiv \{1, \cdots, p\}$*, where* Ω_k *was defined in Eq.7, under the model predictive controller of Eqs.3-4. Also let* $\bar{T} \geq 0$ *be the earliest time for which either the closed–loop state, under MPC, satisfies*

$$L_f V_k(x(\bar{T})) + L_g V_k(x(\bar{T}))M(x(\bar{T})) \geq 0 \tag{11}$$

or the MPC algorithm fails to prescribe any control move. Then, the switching rule given by

$$i(t) = \left\{ \begin{array}{l} 1, \;\; 0 \leq t < \bar{T} \\ 2, \;\; t \geq \bar{T} \end{array} \right\} \tag{12}$$

where $i(t) = 1 \Leftrightarrow u_i(x(t)) = M(x(t))$ *and* $i(t) = 2 \Leftrightarrow u_i(x(t)) = b_k(x(t))$*, guarantees that the origin of the switched closed–loop system is asymptotically stable.*

Remark 1. Theorem 1 describes a stability-based switching strategy for control of nonlinear systems with input constraints. The main components of this strategy include the predictive controller, a family of bounded nonlinear controllers, with their estimated regions of constrained stability, and a high–level supervisor that orchestrates the switching between the controllers. A schematic representation of the hybrid control structure is shown in Figure 1. The implementation procedure of this hybrid control strategy is outlined below:

- Given the system model of Eq.1, the constraints on the input and the family of CLFs, design the bounded controllers using Eqs.5-6. Given the performance objective, set up the MPC optimization problem.
- Compute the stability region estimate for each of the bounded controllers, $\Omega_k(u_{max})$, using Eqs.7-8, for $k = 1, \ldots, p$, and $\Omega(u_{max}) = \bigcup_{k=1}^{p} \Omega_k(u_{max})$.
- Initialize the closed–loop system under MPC, at any initial condition, x_0 within Ω, and identify a CLF, $V_k(x)$, for which the initial condition is within the corresponding stability region estimate, Ω_k.
- Monitor the temporal evolution of the closed–loop trajectory (by checking Eq.11 at each time) until the earliest time that either Eq.11 holds or the MPC algorithm prescribes no solution, $\bar{T}$.
- If such a $\bar{T}$ exists, discontinue MPC implementation, switch to the k-th bounded controller (whose stability region contains x_0) and implement it for all future times.

Remark 2. The main idea behind Theorem 1, and behind the hybrid predictive controller (including the designs that address issues of unavailability of

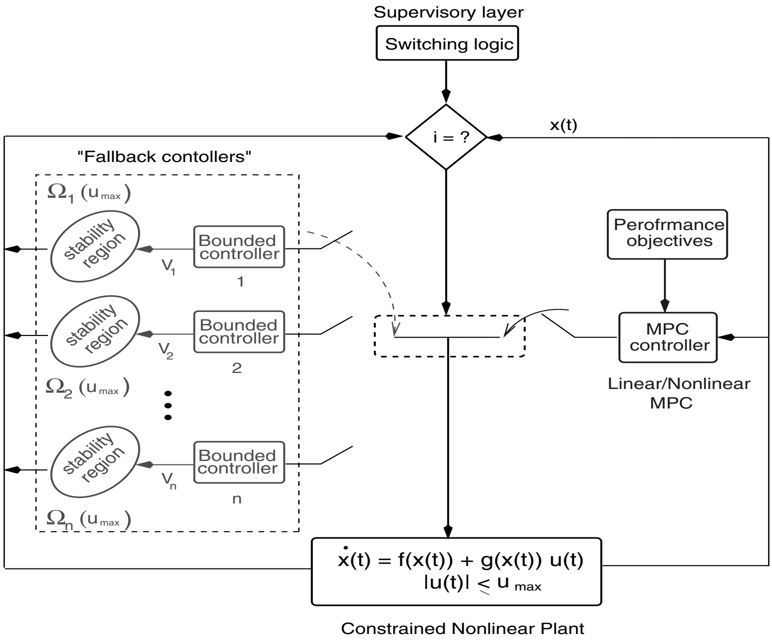

Fig. 1. Schematic representation of the hybrid control structure merging MPC and a family of fall–back bounded controllers with their stability regions

measurements and uncertainty) is as follows: first design a Lyapunov-based controller that allows for an explicit characterization of the set of initial conditions starting from where closed–loop stability is guaranteed in the presence of constraints. For an initial condition within the stability region of the Lyapunov-based controller, the predictive controller is implemented in the closed–loop system, while the supervisor monitors the evolution of the states of the closed–loop system. The supervisor checks switching rules designed to detect instability like behavior under the predictive controller and to guard against the possibility that the closed–loop trajectory under the predictive controller escapes out of the region where the Lyapunov-based controller provides the stability guarantees. In the theorem above, increase in the value of the Lyapunov-function (that is used in the design of the Lyapunov-based controller and in characterizing the stability region) is considered both as instability like behavior and to safeguard against the closed–loop state trajectory escaping the stability region (since the stability region is defined by a level set of the Lyapunov-function). The switching rule therefore dictates switching to the fall-back Lyapunov-based controller in the event of an increase in the value of the Lyapunov function.

Remark 3. The presence of constraints limits the set of initial conditions starting from where closed–loop stability can be achieved (the so called null-controllable region, or X_{max}). While a given controller design typically provides stability from subsets of the null-controllable region, it is important to be able to estimate the set of initial conditions starting from where the controller can guarantee closed–loop stability. The difficulty in characterizing the set of initial conditions starting from where a given predictive controller is guaranteed to be stabilizing motivates the use of backup controllers within the hybrid predictive control structure that provide sufficiently non-conservative estimates of their stability region. The Lyapunov-based controller of Eqs.5-7 provides such an estimate of its stability region that compares well with the null controllable region (how well, is something that can only be determined on a case by case basis; see [21] for a comparison in the case of a linear system with constraints). Note also that estimating the stability region under the controller of Eqs.5-7 requires only algebraic computations and scales well with an increase in number of system states; see [12] for applications to a polyethylene reactor and [26] for an application in the context of fault-tolerant control.

Remark 4. In addition to constraints, other important factors that influence the stabilization problem are the lack of complete measurements of the process state variables and the presence of uncertainty. The problem of lack of availability of measurements is considered in [22] where the hybrid predictive output feedback controller design comprises of the state estimator, the Lyapunov-based and predictive controllers, together with the supervisor. In addition to the set of switching rules being different from the one under state feedback, an important characteristic of the hybrid predictive control strategy under output feedback is the inherent coupling, brought about by the lack of full state measurements, between the tasks of controller design, characterization of the stability region and supervisory switching logic design, on one hand, and the task of observer design, on the other. In [24] we consider the presence of uncertainty in the design of the individual controllers as well as the switching logic in a way that enhances the chances of the use of the predictive control algorithms while not sacrificing guaranteed closed–loop stability.

4 Lyapunov-Based Predictive Control

In this section, we review our recent results on the design of a Lyapunov-based predictive controller, where the design of the (Lyapunov-based) predictive controller uses a bounded controller, with its associated region of stability, only as an auxiliary controller. The Lyapunov-based MPC is shown to possess an explicitly characterized set of initial conditions, starting from where it is guaranteed to be feasible, and hence stabilizing, while enforcing both state and input constraints at all times.

4.1 System Description

Consider the problem of stabilization of continuous-time nonlinear systems with state and input constraints, with the following state-space description:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t);\, u \in U; \quad x \in X \tag{13}$$

where $x = [x_1 \cdots x_n]' \in \mathbb{R}^n$ denotes the vector of state variables, $u = [u^1 \cdots u^m]' \in \mathbb{R}^m$ denotes the vector of manipulated inputs, $U \subseteq \mathbb{R}^m, X \subseteq \mathbb{R}^n$ denote the constraints on the manipulated inputs and the state variables, respectively, $f(\cdot)$ is a sufficiently smooth $n \times 1$ nonlinear vector function, and $g(\cdot)$ is a sufficiently smooth $n \times m$ nonlinear matrix function. Without loss of generality, it is assumed that the origin is the equilibrium point of the unforced system (i.e., $f(0) = 0$).

4.2 Lyapunov-Based Predictive Control Design

Preparatory to the characterization of the stability properties of the Lyapunov-based predictive controller, we first state the stability properties of the bounded controller of Eqs.5–6 in the presence of both state and input constraints. For the controller of Eqs.5–6, one can show, using a standard Lyapunov argument, that whenever the closed–loop state, x, evolves within the region described by the set:

$$\Phi_{x,u} = \{x \in X : L_f^* V(x) \leq u^{max} \|(L_g V)'(x)\|\} \tag{14}$$

then the controller satisfies both the state and input constraints, and the time-derivative of the Lyapunov function is negative-definite. To compute an estimate of the stability region we construct a subset of $\Phi_{x,u}$ using a level set of V, i.e.,

$$\Omega_{x,u} = \{x \in \mathbb{R}^n : V(x) \leq c_{x,u}^{max}\} \tag{15}$$

where $c_{x,u}^{max} > 0$ is the largest number for which $\Omega_{x,u} \subseteq \Phi_{x,u}$. Furthermore, the bounded controller of Eqs.5-6 possesses a robustness property (with respect to measurement errors) that preserves closed–loop stability when the control action is implemented in a discrete (sample and hold) fashion with a sufficiently small hold time (Δ). Specifically, the control law ensures that, for all initial conditions in $\Omega_{x,u}$, the closed–loop state remains in $\Omega_{x,u}$ and eventually converges to some neighborhood of the origin (we will refer to this neighborhood as Ω^b) whose size depends on Δ. This property is exploited in the Lyapunov-based predictive controller design of Section 4.2 and is stated in Proposition 1 below (the proof can be found in [25]). For further results on the analysis and control of sampled-data nonlinear systems, the reader may refer to [13, 14, 28, 31].

Proposition 1. *Consider the constrained system of Eq.1, under the bounded control law of Eqs.5–6 with $\rho > 0$ and let $\Omega_{x,u}$ be the stability region estimate under continuous implementation of the bounded controller. Let $u(t) = u(j\Delta)$ for all $j\Delta \leq t < (j+1)\Delta$ and $u(j\Delta) = b(x(j\Delta))$, $j = 0, \cdots, \infty$. Then, given any positive real number d, there exist positive real numbers Δ^*, δ' and ϵ^* such that if $\Delta \in (0, \Delta^*]$ and $x(0) := x_0 \in \Omega_{x,u}$, then $x(t) \in \Omega_{x,u} \subseteq X$ and $\limsup_{t\to\infty} \|x(t)\| \leq d$. Also, if $V(x_0) \leq \delta'$ then $V(x(\tau)) \leq \delta'$ $\forall\, \tau \in [0, \Delta)$ and if $\delta' < V(x_0) \leq c_{x,u}^{max}$, then $\dot{V}(x(\tau)) \leq -\epsilon^*$ $\forall\, \tau \in [0, \Delta)$.*

We present now a Lyapunov–based MPC formulation that guarantees feasibility of the optimization problem subject to hard constraints on the state and input, and hence constrained stabilization of the closed–loop system from an explicitly characterized set of initial conditions. For this MPC design, the control action at state x and time t is obtained by solving, on-line, a finite horizon optimal control problem of the form:

$$P(x,t) : \min\{J(x,t,u(\cdot))|u(\cdot) \in S,\ x \in X\} \tag{16}$$

$$s.t.\ \ \dot{x} = f(x) + g(x)u \tag{17}$$

$$\dot{V}(x(\tau)) \leq -\epsilon^* \ \ \forall\, \tau \in [t, t+\Delta)\ if\ \ V(x(t)) > \delta^{'} \tag{18}$$

$$V(x(\tau)) \leq \delta^{'}\ \forall\, \tau \in [t, t+\Delta)\ \ if\ \ V(x(t)) \leq \delta^{'} \tag{19}$$

where $S = S(t,T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into U and T is the horizon. Eq.17 is the nonlinear model describing the time evolution of the state x, V is the Lyapunov function used in the bounded controller design and $\delta^{'}$, ϵ^* are defined in Proposition 1. A control $u(\cdot)$ in S is characterized by the sequence $\{u[j]\}$ where $u[j] := u(j\Delta)$ and satisfies $u(t) = u[j]$ for all $t \in [j\Delta, (j+1)\Delta)$. The performance index is given by

$$J(x,t,u(\cdot)) = \int_{t}^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2\right] ds \tag{20}$$

where Q is a positive semi-definite symmetric matrix and R is a strictly positive definite symmetric matrix. $x^u(s;x,t)$ denotes the solution of Eq.1, due to control u, with initial state x at time t. The minimizing control $u^0(\cdot) \in S$ is then applied to the plant over the interval $[j\Delta, (j+1)\Delta)$ and the procedure is repeated indefinitely. Closed–loop stability and state and input constraint feasibility properties of the closed–loop system under the Lyapunov–based predictive controller are inherited from the bounded controller under discrete implementation and are formalized in Proposition 2 below (for a proof, please see [25]).

Proposition 2. *Consider the constrained system of Eq.1 under the MPC law of Eqs.16–20 with* $\Delta \leq \Delta^*$ *where* Δ^* *was defined in Proposition 1. Then, given any* $x_0 \in \Omega_{x,u}$*, where* $\Omega_{x,u}$ *was defined in Eq.15, the optimization problem of Eq.16-20 is feasible for all times,* $x(t) \in \Omega_{x,u} \subseteq X$ *for all* $t \geq 0$ *and* $\limsup_{t\to\infty} \|x(t)\| \leq d$.

Remark 5. Note that the predictive controller formulation of Eqs.16–20 requires that the value of the Lyapunov function decrease during the first step only. Practical stability of the closed–loop system is achieved since, due to the receding nature of controller implementation, only the first move of the set of calculated moves is implemented and the problem is re-solved at the next time step. If the optimization problem is initially feasible and continues to be feasible, then every control move that is implemented enforces a decay in the value of the Lyapunov function, leading to stability. Lyapunov-based predictive control approaches (see, for example, [16]) typically incorporate a similar Lyapunov function decay constraint, albeit requiring the constraint of Eq.18 to hold at the *end* of the prediction horizon as opposed to only the first time step. An input trajectory that only requires the value of the Lyapunov function value to decrease at the end of the horizon may involve the state trajectory leaving the level set (and, therefore, possibly out of the state constraint satisfaction region, violating the state constraints), and motivates using a constraint that requires the Lyapunov function to decrease during the first time step (this also facilitates the explicit characterization of the feasibility region).

Remark 6. For $0 < \Delta \leq \Delta^*$, the constraint of Eq.18, is guaranteed to be satisfied (the control action computed by the bounded controller design provides a feasible initial guess to the optimization problem). Note that the constraint requires the Lyapunov function value to decay, not at the *end* of the prediction horizon (as is customarily done in Lyapunov–based MPC approaches), but only during the first time step. Furthermore, since the state is initialized in $\Omega_{x,u}$, which is a level set of V, the closed–loop system evolves so as to stay within $\Omega_{x,u}$, thereby guaranteeing feasibility at future times. Since the level set $\Omega_{x,u}$ is completely contained in the set defining the state constraints, and the state trajectory under the predictive controller continues to evolve within this set, the state constraints are satisfied at all times.

Remark 7. In the event that measurements are not continuously available, but are available only at sampling times $\Delta_s > \Delta^*$, i.e., greater than what a given bounded control design can tolerate (and, therefore, greater than the maximum allowable discretization for the Lyapunov-based predictive controller), it is necessary to redesign the bounded controller to increase the robustness margin, and generate a revised estimate of the feasibility (and stability) region under the predictive controller. A larger value of Δ^* may be achieved by increasing the value of the parameter ρ in the design of the bounded controller. If the value of the sampling time is reasonable, an increase in the value of the parameter ρ, while leading to a shrinkage in the stability region estimate, can increase Δ^* to a value greater than Δ_s and preserve the desired feasibility and stability guarantees of the Lyapunov-based predictive controller.

4.3 Switched Systems with Scheduled Mode Transitions

In many chemical processes, the system is required to follow a prescribed switching schedule, where the switching times are prescribed via an operating schedule. This practical problem motivated the development of a predictive control framework for the constrained stabilization of switched nonlinear processes that transit between their modes of operation at prescribed switching times [23]. We consider the class of switched nonlinear systems represented by the following state-space description

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t)) + g_{\sigma(t)}(x(t))u_{\sigma(t)}(t) \\ u_{\sigma(t)} &\in \mathcal{U}_{\sigma};\ \sigma(t) \in \mathcal{K} := \{1, \cdots, p\} \end{aligned} \tag{21}$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of continuous-time state variables, $u_\sigma(t) = [u_\sigma^1(t) \cdots u_\sigma^m(t)]^T \in \mathcal{U}_\sigma \subset \mathbb{R}^m$ denotes the vector of constrained manipulated inputs taking values in a nonempty compact convex set $\mathcal{U}_\sigma := \{u_\sigma \in \mathbb{R}^m : \|u_\sigma\| \leq u_\sigma^{max}\}$, where $\|\cdot\|$ is the Euclidian norm, $u_\sigma^{max} > 0$ is the magnitude of the constraints, $\sigma : [0, \infty) \to \mathcal{K}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, i.e., $\sigma(t_k) = \lim_{t \to t_k^+} \sigma(t)$ for all k, implying that only a finite number of switches is allowed on any finite interval of time. p is the number of modes of the switched system, $\sigma(t)$, which takes different values in the finite index set $\mathcal{K}$, represents a discrete state that indexes the vector field $f(\cdot)$, the matrix $g(\cdot)$, and the control input $u(\cdot)$, which altogether determine $\dot{x}$.

Consider the nonlinear switched system of Eq.21, with a prescribed switching sequence (including the switching times) defined by $\mathcal{T}_{k,in} = \{t_{k_1^{in}}, t_{k_2^{in}}, \ldots\}$ and $\mathcal{T}_{k,out} = \{t_{k_1^{out}}, t_{k_2^{out}}, \ldots\}$. Also, assume that for each mode of the switched system, a Lyapunov–based predictive controller of the form of Eqs.16-20 has been designed and an estimate of the stability region generated. The control problem is formulated as the one of designing a Lyapunov-based predictive controller that guides the closed–loop system trajectory in a way that the schedule described by the switching times is followed and stability of the closed–loop system is achieved. The main idea (formalized in Theorem 2 below) is to design a Lyapunov–based predictive controller for each constituent mode in which the switched system operates, and incorporate constraints in the predictive controller design which upon satisfaction ensure that the prescribed transitions between the modes occur in a way that guarantees stability of the switched closed–loop system.

Theorem 2. *Consider the constrained nonlinear system of Eq.10, the control Lyapunov functions V_k, $k = 1, \cdots, p$, and the stability region estimates Ω_k, $k = 1, \cdots, p$ under continuous implementation of the bounded controller of Eqs.5-6 with fixed $\rho_k > 0$, $k = 1, \cdots, p$. Let $0 < T_{design} < \infty$ be a design parameter. Let t be such that $t_{k_r^{in}} \leq t < t_{k_r^{out}}$ and $t_{m_j^{in}} = t_{k_r^{out}}$ for some m, k. Consider the following optimization problem*

$$P(x,t) : \min\{J(x,t,u_k(\cdot)) | u_k(\cdot) \in S_k\} \tag{22}$$

$$J(x,t,u_k(\cdot)) = \int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u_k(s)\|_R^2\right] ds \tag{23}$$

where T is the prediction horizon given by $T = t_{k_r^{out}} - t$, if $t_{k_r^{out}} < \infty$ and $T = T_{design}$ if $t_{k_r^{out}} = \infty$, subject to the following constraints

$$\dot{x} = f_k(x) + g_k(x)u_k \tag{24}$$

$$\dot{V}_k(x(\tau)) \leq -\epsilon_k \;\; if \;\; V_k(x(t)) > \delta_k^{'}, \; \tau \in [t, t + \Delta_{k_r}) \tag{25}$$

$$V_k(x(\tau)) \leq \delta_k^{'} \;\; if \;\; V_k(x(t)) \leq \delta_k^{'},\; \tau \in [t, t+\Delta_{k_r}) \tag{26}$$

and if $t_{k_r^{out}} = t_{m_j^{in}} < \infty$

$$V_m(x(t_{m_j^{in}})) \leq \left\{ \begin{array}{cr} V_m(x(t_{m_{j-1}^{in}})) - \epsilon^* & , j > 1, V_m(x(t_{m_{j-1}^{in}})) > \delta_m^{'} \\ \delta_m^{'} & , j > 1, V_m(x(t_{m_{j-1}^{in}})) \leq \delta_m^{'} \\ c_m^{max} & , \qquad j = 1 \end{array} \right\} \tag{27}$$

where ϵ^* *is a positive real number. Then, given a positive real number* d^{max}*, there exist positive real numbers* Δ^* *and* $\delta_k^{'}$*,* $k = 1, \cdots, m$ *such that if the optimization problem of Eqs.22–27 is feasible at all times, the minimizing control is applied to the system over the interval* $[t, t + \Delta_{k_r}]$*, where* $\Delta_{k_r} \in (0, \Delta^*]$ *and* $t_{k_r^{out}} - t_{k_r^{in}} = l_{k_r}\Delta_{k_r}$ *for some integer* $l_{k_r} > 0$ *and the procedure is repeated, then,* $\limsup_{t\to\infty} \|x(t)\| \leq d^{max}$.

Remark 8. Note that the constraint of Eq.25 is guaranteed to be feasible between mode transitions, provided that the system is initialized within the stability region, and does not require the assumption of feasibility. This stability constraint ensures that the value of the Lyapunov function of the currently active mode keeps decreasing (recall that one of the criteria in the multiple Lyapunov-function stability analysis is that the individual modes of the switched system be stable). The constraint of Eq.25 expresses two transition requirements simultaneously: (1) the MLF constraints that requires that the value of the Lyapunov function be less than what it was the last time the system switched into that mode (required when the switching sequence is infinite, see [3] for details), and (2) the stability region constraint that requires that the state of the process reside within the stability region of the target mode at the time of the switch; since the stability regions of the modes are expressed as level sets of the Lyapunov functions, the MLF-constraint also expresses the stability region constraint. The understanding that it is a reasonably chosen switching schedule (that is, one that does not result in closed–loop instability), motivates assuming the feasibility of the transition constraints for all times. Note that the feasibility of the transition constraints can also be used to validate the switching schedule, and can be used to abort the switching schedule (i.e., to decide that the remaining switches should not be carried out) in the interest of preserving closed–loop stability.

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