A Computationally Efficient Scheduled Model Predictive Control Algorithm for Control of a Class of Constrained Nonlinear Systems

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Summary. We present an overview of our results on stabilizing scheduled output feedback Model Predictive Control (MPC) algorithm for constrained nonlinear systems based on our previous publications [19, 20]. Scheduled MPC provides an important alternative to conventional nonlinear MPC formulations and this paper addresses the issues involved in its implementation and analysis, within the context of the NMPC05 workshop. The basic formulation involves the design of a set of local output feedback predictive controllers with their estimated regions of stability covering the desired operating region, and implement them as a single scheduled output feedback MPC which on-line switches between the set of local controllers and achieves nonlinear transitions with guaranteed stability. This algorithm provides a general framework for scheduled output feedback MPC design.

1 Introduction

Most practical control systems with large operating regions must deal with nonlinearity and constraints under output feedback control. Nonlinear Model Predictive Control (NMPC) is a powerful design technique that can stabilize processes in the presence of nonlinearities and constraints. Comprehensive reviews of state feedback NMPC algorithms can be found in [15]. In state feedback NMPC, full state information can be measured and is available as initial condition for predicting the future system behavior. In many applications, however, the system state can not be fully measured, and only output information is directly available for feedback. An output feedback NMPC algorithm can be formulated by combining the state feedback NMPC with a suitable state observer, e.g., moving horizon observer (MHE)[6] [14], extended Kalman filter [16], etc. A good overview of the observer based output feedback NMPC algorithms is provided in [4]

Besides developing efficient techniques such as multiple shooting for solving NLP [6] and parallel programming for control of nonlinear PDE systems [10], researchers have proposed various methods to simplify NMPC on-line computation. In [18], it was proposed that instead of the global optimal solution, an

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improved feasible solution obtained at each sampling time is enough to ensure stability. In [11], a stabilizing NMPC algorithm was developed with a few control moves and an auxiliary controller implemented over the finite control horizon. In [7], stability is guaranteed through the use of an a priori control Lyapunov function (CLF) as a terminal cost without imposing terminal state constraints. In [1], nonlinear systems were approximated by linear time varying (LTV) models, and the optimal control problem was formulated as a min-max convex optimization. In [9], nonlinear systems were approximated as linear parameter varying (LPV) models, and a scheduling quasi-min-max MPC was developed with the current linear model known exactly and updated at each sampling time. A hybrid control scheme was proposed in [3] for nonlinear systems under state feedback. This control scheme embeds the implementation of MPC within the stability regions of the bounded controllers and employs these controllers as fall-back in the event that MPC is unable to achieve closed-loop stability [2, 3, 13].

For a control system with a large operating region, it is desirable for the controller to achieve satisfactory performance of the closed-loop system around all setpoints while allowing smooth transfer between them. Pseudolinearization was used in the quasi-infinite horizon NMPC formulation to obtain a closed form expression for the controller parameters as a function of the setpoint [5]. A novel gain scheduling approach was introduced in [12], in which a set of off-line local controllers are designed with their regions of stability overlapping each other, and supervisory scheduling of the local controllers can move the state through the intersections of the regions of stability of different controllers to the desired operating point with guaranteed stability.

In [20], we developed a scheduled output feedback MPC for nonlinear constrained systems, based on the scheduling ideas of [19] and [12]. The basic ideas are (1) locally represent the nonlinear system around an equilibrium point as a linear time varying (LTV) model and develop a local predictive controller with an estimate of its region of stability; (2) expand the region of stability about the desired operating point by piecing together the estimated regions of stability of a set of local predictive controllers; (3) schedule the local predictive controllers based on the local region of stability that contains the system state. The key to estalishing stability of local predictive controllers and stability of scheduling of local predictive controllers is to design an exponentially stable state feedback controller and require the state observer to deliver bounded observer error to ensure asymptotic stability of the output feedback controller. In order to faciliate a finite dimensional formulation for enforcement of exponential stability, we can either represent the local nonlinearity as a LTV model and parameterize the infinite control horizon in terms of a linear feedback law, or we can use the nonlinear model and only enforce the constraint over a finite control horizon with a terminal constraint and a terminal cost. While [20] only used the former formulation, the current paper generalizes a framework for design of scheduled output feedback MPC, which covers both finite dimensional formulations.

2 Local Output Feedback MPC for Constrained Nonlinear Systems

2.1 State Feedback and No Disturbances

Consider a discrete-time nonlinear dynamical system described by

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

where $x(k) \in X \subseteq \mathbb{R}^n,\ u(k) \in U \subseteq \mathbb{R}^m$ are the system state and control input, respectively, X and U are compact sets. Assume $f(x,u) = \left[\, f_1(x,u) \ \cdots \ f_n(x,u)\,\right]^T$ are continuous differentiable in x and u.

Definition 1. *Given a set U, a point $x_0 \in X$ is an equilibrium point of the system (1) if a control $u_0 \in int(U)$ exists such that $x_0 = f(x_0, u_0)$. We call a connected set of equilibrium points an equilibrium surface.*

Suppose (x^{eq}, u^{eq}) is a point on the equilibrium surface. Within a neighborhood around $(x^{\mathrm{eq}}, u^{\mathrm{eq}})$, i.e., $\Pi_x = \{x \in \mathbb{R}^n |\ |x_r - x_r^{\mathrm{eq}}| \le \delta x_r,\ r = 1, ..., n\} \subseteq X$, and $\Pi_u = \{u \in \mathbb{R}^m |\ |u_r - u_r^{\mathrm{eq}}| \le \delta u_r,\ r = 1, ..., m\} \subseteq U$, let $\bar{x} = x - x^{\mathrm{eq}}$ and $\bar{u} = u - u^{\mathrm{eq}}$. The objective is to minimize the infinite horizon quadratic objective function

$$\min_{\bar{u}(k+i|k)} \ J_\infty(k)$$

subject to

$$|\bar{u}_r(k+i|k)| \le \delta u_{r,\max},\ \ i \ge 0,\ r = 1, 2, ..., m \tag{2}$$

$$|\bar{x}_r(k+i|k)| \le \delta x_{r,\max},\ \ i \ge 0,\ r = 1, 2, ..., n \tag{3}$$

where $J_\infty(k) = \sum_{i=0}^{\infty}[\bar{x}(k+i|k)^T Q \bar{x}(k+i|k) + \bar{u}(k+i|k)^T R\, \bar{u}(k+i|k)]$ with $Q > 0,\ \ R > 0$. To derive an upper bound on $J_\infty(k)$, define a quadratic function $V(\bar{x}) = \bar{x}^T Q(k)^{-1} \bar{x},\ Q(k) > 0$. Suppose $V(x)$ satisfies the following exponential stability constraint

$$V(\bar{x}(k+i+1|k) \le \alpha^2 V(\bar{x}(k+i|k)),\ V(\bar{x}(k|k)) \le 1,\ \alpha < \mathbf{1} \tag{4}$$

There exists a $\gamma(k) > 0$ such that

$$\begin{aligned} V(\bar{x}(k+i+1|k)) - V(\bar{x}(k+i|k)) \le -\tfrac{1}{\gamma(k)} \big[&\bar{x}(k+i|k)^T \mathcal{Q} \bar{x}(k+i|k) \\ &+ \bar{u}(k+i|k)^T \mathcal{R}\, \bar{u}(k+i|k) \big] \end{aligned} \tag{5}$$

Summing (5) from $i = 0$ to $i = \infty$ and requiring $\bar{x}(\infty|k) = 0$ or $V(\bar{x}(\infty|k)) = 0$, it follows that $J_\infty(k) \le \gamma(k) V(\bar{x}(k|k)) \le \gamma(k)$. Therefore, the optimization is formulated as

$$\underset{\gamma(k), Q(k), \bar{u}(k+i|k), i \ge 0}{min} \gamma(k) \tag{6}$$

subject to (2)-(5).

Algorithm 1 (Exponentially stable MPC). Given the controller design parameter $0 < \alpha < 1$. At each sampling time k, apply $u(k) = \bar{u}(k) + u^{\mathrm{eq}}$ where $\bar{u}(k)$ is obtained from $\underset{\gamma(k),Q(k),\bar{u}(k+i|k),i\geq 0}{min} \gamma(k)$ subject to (2), (4), (5) and (8), where R is obtained offline from the maximization (7) subject to (2)-(5).

Assume that at each sampling time k, a state feedback law $\bar{u}(k) = F(\bar{x}(k))$ is used. Then an ellipsoidal feasible region of the optimization (6) can be defined as $\mathcal{S} = \left\{\bar{x} \in \mathbb{R}^n \;\middle|\; \bar{x}R^{-1}\bar{x} \leq 1\right\}$, where R is the optimal solution Q of the following maximization

$$\max_{\gamma,Q,F(\bullet)} \log\det Q \tag{7}$$

subject to (2)-(5). Then $J_\infty(k)$ is bounded by $\gamma_R\bar{x}(k)R^{-1}\bar{x}(k)$, where γ_R is the solution of γ in (7).

Replacing the state constraint (3) by $\bar{x}(k+i|k) \in \mathcal{S},\ i \geq 0$, or, equivalently

$$R - Q > 0 \tag{8}$$

which confines the current state and all future predicted states inside $\mathcal{S}$, we develop an exponentially stable MPC algorithm with an estimated region of stability.

Remark 1. Enforcement of the exponential stability constraint (4) involves an infinite control horizon. In order to faciliate a finite dimensional formulation, we can either represent the local nonlinearity as a LTV model and parameterize the infinite control horizon in terms of a linear feedback law (see [20]), or we can use the nonlinear model and only enforce the constraint over a finite control horizon with a terminal constraint and a terminal cost. In fact, the estimated region of stability $\mathcal{S} = \left\{\bar{x} \in \mathbb{R}^n \;\middle|\; \bar{x}R^{-1}\bar{x} \leq 1\right\}$ and the cost upper bound $\gamma_R\bar{x}^TR^{-1}\bar{x}$ can serve as the terminal constraint and the terminal cost, respectively. A significant difference between this paper and [20] is that this paper provides a generalized framework, which covers both of the above two finite dimensional formulations.

Theorem 1. *Consider the nonlinear system (1). Suppose* $(x^{\,eq}, u^{eq})$ *is locally stabilizable, then there exist a neighborhood* (Π_x, Π_u) *around* (x^{eq}, u^{eq}) *and a controller design parameter* $0 < \alpha < 1$ *such that Algorithm 1 exponentially stabilizes the closed-loop system with an estimated region of stability* $\mathcal{S} = \left\{\bar{x} \in \mathbb{R}^n \;\middle|\; \bar{x}^TR^{-1}\bar{x} \leq 1\right\}$.

Proof. The proof can be found in the Appendix. ■

2.2 State Feedback and Asymptotically Decaying Disturbances

Consider the nonlinear system (1) subject to the unknown additive asymptotically decaying disturbance $d(k)$, $x^p(k+1) = f(x^p(k), u(k)) + d(k)$, where we have made a distinction between the state of the perturbed system, $x^p(k)$, and the state of the unperturbed system, $x(k)$. In order for $x^p(k+1)$ to remain in the region of stability S , we develop a sufficient condition between the norm bound of $d(k)$ and the controller design parameter α. Let $\bar{x}^p(k) = x^p(k) - x^{\mathrm{eq}}$, $\bar{x}(k+1) = f(x^p(k), u(k)) - x^{\mathrm{eq}}$. Suppose $\bar{x}^p(k) \in \mathcal{S}$, (i.e., $\|\bar{x}^p(k)\|^2_{R^{-1}} \leq 1$), $\|\bar{x}^p(k+1)\|^2_{R^{-1}} = \|\bar{x}(k+1) + d(k)\|^2_{R^{-1}} = \|\bar{x}(k+1)\|^2_{R^{-1}} + 2\bar{x}(k+1)^T R^{-1} d(k) + \|d(k)\|^2_{R^{-1}}$, where $u(k)$ is computed by Algorithm 1. From (8) and (4), we know that $\|\bar{x}(k+1)\|^2_{R^{-1}} \leq \|\bar{x}(k+1)\|^2_{Q(k)^{-1}} \leq \alpha^2 \|\bar{x}^p(k)\|^2_{Q(k)^{-1}} \leq \alpha^2$. Therefore, invariance is guaranteed if $\|\bar{x}^p(k+1)\|^2_{R^{-1}} \leq \alpha^2 + 2\alpha \|d(k)\|_{R^{-1}} + \|d(k)\|^2_{R^{-1}} = (\alpha + \|d(k)\|_{R^{-1}})^2 \leq 1$. A sufficient condition for $x^p(k+1)$ to remain in the region of stability $\mathcal{S}$ is $\|d(k)\|_{R^{-1}} \leq 1 - \alpha$, which means that the disturbance should be bounded in a region $\mathcal{S}^d \triangleq \{d \in \mathbb{R}^n \mid d^T R^{-1} d \leq (1-\alpha)^2\}$. As $d(k)$ is asymptotically decaying, the closed-loop trajectory asymptotically converges to the equilibrium $(x^{\mathrm{eq}}, u^{\mathrm{eq}})$.

2.3 Output Feedback

Consider the nonlinear system (1) with a nonlinear output map

$$y(k) = h(x(k)) \in \mathbb{R}^q \tag{9}$$

where $h(x) = \left[\, h_1(x) \cdots h_q(x) \,\right]^T$ are continuous differentiable. For all $x, \hat{x} \in \Pi_x$ and $u \in \Pi_u$, consider a full order nonlinear observer with a constant observer gain L_p,

$$\hat{x}(k+1) = f(\hat{x}(k), u(k)) + L_p(h(x(k)) - h(\hat{x}(k))) \tag{10}$$

The error dynamic system is $e(k+1) = f(x(k), u(k)) - f(\hat{x}(k), u(k)) - L_p(h(x(k)) - h(\hat{x}(k)))$. Define a quadratic function $V_e(x) = e^T P e,\ P > 0$. Suppose for all time $k \geq 0$, $x(k), \hat{x}(k) \in \Pi_x$ and $u(k) \in \Pi_u$, and $V_e(e)$ satisfies the following exponential convergent constraint

$$V_e(e(k+i+1|k)) \leq \rho^2 V_e(e(k+i|k)) \tag{11}$$

In order to facilitate the establishment of the relation between $\|d\|_{R^{-1}}$ and $\|e\|_P$ in §2.4, we want to find a P as close to R^{-1} as possible. Therefore, we minimize γ such that

$$\gamma R^{-1} \geq P \geq R^{-1} \tag{12}$$

Algorithm 2. Consider the nonlinear system (1) and (9) within (Π_x, Π_u) around $(x^{\mathrm{eq}}, u^{\mathrm{eq}})$. Given the observer design parameter $0 < \rho < 1$, the constant observer gain L_p of the full order observer (10) is obtained from $\min\limits_{\gamma, P, L_p} \gamma$ subject to (11) and (12).

Theorem 2. *Consider the nonlinear system (1) and (9). Suppose (x^{eq}, u^{eq}) is locally observable, then there exist a neighborhood (Π_x, Π_u) around (x^{eq}, u^{eq}) and an observer design parameter $0 < \rho < 1$ such that the minimization in Algorithm 2 is feasible. Furthermore, if for all time $k \geq 0$, $x(k), \hat{x}(k) \in \Pi_x$ and $u(k) \in \Pi_u$, then the observer in Algorithm 2 is exponentially convergent.*

Algorithm 3 (Local output feedback MPC for constrained nonlinear systems). Consider the nonlinear system (1) and the output map (9) within the neighborhood (Π_x, Π_u) around $(x^{\rm eq}, u^{\rm eq})$. Given the controller and observer design parameters $0 < \alpha < 1$ and $0 < \rho < 1$. At sampling time $k > 0$, apply $u(k) = F(k; (\hat{x}(k) - x^{\rm eq})) + u^{\rm eq}$, where $\hat{x}(k)$ is solved by the observer in Algorithm 2 with the output measurement $y(k-1)$ and $F(k; \bullet)$ is solved by the state feedback MPC in Algorithm 1 based on $\bar{x}(k) = \hat{x}(k) - x^{\rm eq}$.

Proof. The proof can be found in the Appendix. ∎

Now we combine the state feedback MPC in Algorithm 1 with the observer in Algorithm 2 to form a local output feedback MPC for the constrained nonlinear system.

2.4 Stability Analysis of Output Feedback MPC

For the output feedback MPC in Algorithm 3 to be feasible and asymptotically stable, it is required that for all time $k \geq 0$, $x(k), \hat{x}(k) \in \Pi_x$. In this subsection, we study conditions on $x(0)$ and $\hat{x}(0)$ such that $x(k), \hat{x}(k) \in \mathcal{S}$ is satisfied for all times $k \geq 0$. Consider the closed-loop system with the output feedback MPC in Algorithm 3,

$$\begin{aligned} x(k+1) &= f(\hat{x}(k), u(k)) + d_1(k) \\ \hat{x}(k+1) &= f(\hat{x}(k), u(k)) + d_2(k) \end{aligned}$$

with $d_1(k) = f(x(k), u(k)) - f(\hat{x}(k), u(k))$ and $d_2(k) = L_p(h(x(k)) - h(\hat{x}(k)))$. At time k, $u(k)$ is obtained by using the state feedback MPC in Algorithm 1 based on $\bar{x}(k|k) = \hat{x}(k) - x^{\rm eq}$.

Since f is continuous differentiable, within (Π_x, Π_u) there exist $\beta_1, \beta_2 > 0$ such that $\|d_1(k)\|_{R^{-1}} \leq \beta_1 \|e(k)\|_P$ and $\|d_2(k)\|_{R^{-1}} \leq \beta_2 \|e(k)\|_P$. Suppose initially $x(0), \hat{x}(0) \in \mathcal{S}$ and $\|e(0)\|_P \leq \eta := \frac{1-\alpha}{\max\{\beta_1, \beta_2\}}$, then $\|d_1(0)\|_{R^{-1}} \leq 1 - \alpha$ and $\|d_2(0)\|_{R^{-1}} \leq 1 - \alpha$, which in turn lead to $x(1), \hat{x}(1) \in \mathcal{S}$ (see §2.2)and $\|e(1)\|_P \leq \eta$ (see §2.3). And so on. Since for all time $k \geq 0$, $x(k), \hat{x}(k) \in \mathcal{S}$, the state feedback MPC in Algorithm 1 is exponentially stable, the observer in Algorithm 2 is exponentially convergent, and the combination of both asymptotically stabilizes the closed-loop system.

Theorem 3. *Consider the nonlinear system (1) and (9). Suppose* (x^{eq}, u^{eq}) *is locally stabilizable and observable, then there exist a neighborhood* (Π_x, Π_u) *around* (x^{eq}, u^{eq}) *and controller and observer design parameters* $0 < \alpha < 1$ *and* $0 < \rho < 1$ *such that the output feedback MPC in Algorithm 3 asymptotically stabilizes the closed-loop system for any* $x(0), \hat{x}(0) \in \mathcal{S} = \left\{ x \in \mathbb{R}^n \,\middle|\, (x - x^{eq})^T R^{-1} (x - x^{eq}) \leq 1 \right\}$ *satisfying* $\|x(0) - \hat{x}(0)\|_P \leq \eta$.

Remark 2. In fact, $\|d_1(k)\|_{R^{-1}} \leq \beta_1 \|e(k)\|_P \leq 1 - \alpha$ defines two ellipsoidal regions, i.e., $\mathcal{S}^d \triangleq \{d_1 \in \mathbb{R}^n \,|\, d_1^T R^{-1} d_1 \leq (1-\alpha)^2\}$ and $\mathcal{S}^e \triangleq \{e \in \mathbb{R}^n \,|\, \beta_1^2 e^T P e \leq (1-\alpha)^2\} \subset \mathcal{S}^d$. The effect of the optimization in (12) on the observer performance is to find the maximum $\mathcal{S}^e$ within $\mathcal{S}^d$.

2.5 Observability Analysis of Output Feedback MPC

For the output feedback MPC in Algorithm 3, the state is not measured, but from the output of the system and the estimated state, we can observe the exponential decay of the norm bound of the state estimation error, and thus observe the real state incrementally.

Consider the output feedback MPC in Algorithm 3 which can stabilize any $x(0), \hat{x}(0) \in \mathcal{S}$ satisfying $\|x(0) - \hat{x}(0)\|_P \leq \eta$. Let $T > 0$. At time $k - T \geq 0$, let $x(k-T), \hat{x}(k-T) \in \mathcal{S}$ satisfying $\|x(k-\mathcal{T}) - \hat{x}(k-\mathcal{T})\|_P \leq \eta$. During T steps, an input sequence $\{u(k-T), ..., u(k-1)\} \subset \Pi_u$ is obtained by the controller based on $\{\hat{x}(k-T), ..., \hat{x}(k-1)\} \subset \mathcal{S}$. We know that the state evolution starting from $x(k-T)$ driven by $\{u(k-T), ..., u(k-1)\}$ is inside $\mathcal{S} \subset \Pi_x$. Suppose that the state evolution $\tilde{x}(k+1) = f(\tilde{x}(k), u(k))$ starting from $\tilde{x}(k-T) = \hat{x}(k-T)$ driven by $\{u(k-T), ..., u(k-1)\}$ is also inside Π_x, then we can get

$$
\begin{aligned}
x(k+1) - \tilde{x}(k+1) &= f(x(k), u(k)) - f(\tilde{x}(k), u(k)) \\
y(k) - \tilde{y}(k) &= h(x(k)) - h(\tilde{x}(k))
\end{aligned}
$$

Let $V_{\mathcal{T}} := \sum_{j=k-\mathcal{T}}^{k-1} \|y(j) - \tilde{y}(j)\|^2$. Suppose $(x^{\text{eq}}, u^{\text{eq}})$ is locally observable, then there exist a neighborhood (Π_x, Π_u) around $(x^{\text{eq}}, u^{\text{eq}})$ and $T, \mu > 0$ such that $V_{\mathcal{T}} \geq \mu \|x(k-\mathcal{T}) - \hat{x}(k-\mathcal{T})\|_P^2$, or equivalently, $\|x(k) - \hat{x}(k)\|_P^2 \leq \frac{\rho^{\mathcal{T}} V_{\mathcal{T}}}{\mu}$.

Theorem 4. *Consider the nonlinear system (1) and (9). Suppose* (x^{eq}, u^{eq}) *is locally stabilizable and observable, then there exist a neighborhood* (Π_x, Π_u) *around* (x^{eq}, u^{eq}) *and controller and observer design parameters* $0 < \alpha < 1$ *and* $0 < \rho < 1$, *and* $T, \mu > 0$ *such that the output feedback MPC in Algorithm 3 is asymptotically stable for any* $x(0), \hat{x}(0) \in \mathcal{S} = \left\{x \in \mathbb{R}^n \,\middle|\, (x - x^{eq})^T R^{-1} (x - x^{eq}) \leq 1\right\}$ *satisfying* $\|x(0) - \hat{x}(0)\|_P \leq \eta$. *On-line, let* $x(0), \hat{x}(0) \in \mathcal{S}$ *and* $\|x(0) - \hat{x}(0)\|_P \leq \eta$. *Apply the output feedback controller. At time* $k \geq T$, *if the state evolution starting from* $\hat{x}(k-T)$ *driven by the input sequence* $\{u(k-\mathcal{T}), ..., u(k-1)\}$ *from the controller is inside* Π_x, *then* $\|x(k) - \hat{x}(k)\|_P^2 \leq \frac{\rho^{\mathcal{T}} V_{\mathcal{T}}}{\mu}$.

Proof. The proof can be found in the Appendix. ■

3 Scheduled Output Feedback MPC for Constrained Nonlinear Systems

Algorithm 4 (Design of scheduled output feedback MPC). For the nonlinear system (1) and the output map (9), given an equilibrium surface and a desired equilibrium point $(x^{(0)}, u^{(0)})$. Let $i := 0$.

1. Specify a neighborhood $\left(\Pi_x^{(i)},\ \Pi_u^{(i)}\right)$ around $(x^{(i)}, u^{(i)})$ satisfying $\Pi_x^{(i)} \subseteq X$ and $\Pi_u^{(i)} \subseteq U$.

2. Given $0 < \alpha^{(i)} < 1$ and $0 < \rho^{(i)} < 1$, design Controller $\#i$ (Algorithm 3) with its explicit region of stability

$$\mathcal{S}^{(i)} = \left\{ x \in \mathbb{R}^n \;\middle|\; \left(x - x^{(i)}\right)^T \left(R^{(i)}\right)^{-1} \left(x - x^{(i)}\right) \le 1 \right\}$$

Store $x^{(i)}$, $u^{(i)}$, $\left(R^{(i)}\right)^{-1}$, $P^{(i)}$, $\eta^{(i)}$, $T^{(i)}$ and $\mu^{(i)}$ in a lookup table;
3. Select $(x^{(i+1)}, u^{(i+1)})$ satisfying $x^{(i+1)} \in int\left(\mathcal{S}_\theta^{(i)}\right)$ with

$$\mathcal{S}_\theta^{(i)} = \left\{ x \in \mathbb{R}^n \;\middle|\; \left(x - x^{(i)}\right)^T \left(R^{(i)}\right)^{-1} \left(x - x^{(i)}\right) \le (\theta^{(i)})^2 < 1 \right\}$$

Let $i := i + 1$ and go to step 1, until the region $\cup_{i=0}^{M} \mathcal{S}^{(i)}$ with $M = \max i$ covers a desired portion of the equilibrium surface.

Remark 3. In general, the scheduled MPC in Algorithm 4 requires a specified path on the equilibrium surface so as to extend the region of stability. An optimal path can be defined by the steady state optimization, which provides a set of operating conditions with optimal economic costs. For the same path on the equilibrium surface, the larger the number of controllers designed, the more overlap between the estimated regions of stability of two adjacent controllers, and the better the transition performance, because control switches can happen without moving the state trajectory close to the intermediate equilibrium points. Yet a larger number of controllers leads to a larger storage space for the lookup table and a longer time to do the search. So there is a trade-off between achievement of good transition performance and computational efficiency.

On-line, we implement the resulting family of local output feedback predictive controllers as a single controller whose parameters are changed if certain switching criteria are satisfied. We call such a controller scheme a scheduled output feedback MPC. For the case that $x(0), \hat{x}(0) \in \mathcal{S}^{(0)}$ satisfying $\|x(0) - \hat{x}(0)\|_{P^{(0)}} \le \eta^{(0)}$, according to Theorem 3, Controller $\#0$ asymptotically converges the closed-loop system to the desired equilibrium $\left(x^{(0)}, u^{(0)}\right)$. Similarly, for the case that $x(0), \hat{x}(0) \in \mathcal{S}^{(i)}$, $i \ne 0$ satisfying $\|x(0) - \hat{x}(0)\|_{P^{(i)}} \le \eta^{(i)}$, Controller $\#i$ asymptotically converges the closed-loop system to the equilibrium $\left(x^{(i)}, u^{(i)}\right)$. Because $x^{(i)} \in int\left(\mathcal{S}_\theta^{(i-1)}\right)$, both $x(k)$ and $\hat{x}(k)$ will enter $\mathcal{S}_\theta^{(i-1)}$ in finite time. At time k, in order to switch from Controller $\#i$ to $\#(i-1)$, we need to make sure that the initial conditions for stability of Controller $\#(i-1)$ are satisfied, i.e., $x(k), \hat{x}(k) \in \mathcal{S}^{(i-1)}$ and $\|x(k) - \hat{x}(k)\|_{P^{(i-1)}} \le \eta^{(i-1)}$.

Suppose $\hat{x}(k) \in \mathcal{S}_\theta^{(i-1)}$. We know that

$$\begin{aligned}
&\left\| x(k) - x^{(i-1)} \right\|_{\left(R^{(i-1)}\right)^{-1}} \\
&\le \|x(k) - \hat{x}(k)\|_{\left(R^{(i-1)}\right)^{-1}} + \left\| \hat{x}(k) - x^{(i-1)} \right\|_{\left(R^{(i-1)}\right)^{-1}} \\
&\le \|x(k) - \hat{x}(k)\|_{P^{(i-1)}} + \left\| \hat{x}(k) - x^{(i-1)} \right\|_{\left(R^{(i-1)}\right)^{-1}}
\end{aligned}$$

and $\|x(k) - \hat{x}(k)\|^2_{P^{(i-1)}} \leq \zeta^{i\to(i-1)} \|x(k) - \hat{x}(k)\|^2_{P^{(i)}}$ with $\zeta^{i\to(i-1)}$ solved by the following minimization

$$\min_{\zeta^{i\to(i-1)}} \zeta^{i\to(i-1)} \tag{13}$$

subject to $\zeta^{i\to(i-1)} > 0$ and $\zeta^{i\to(i-1)} P^{(i)} - P^{(i-1)} \geq 0$. Hence $x(k) \in \mathcal{S}^{(i-1)}$ and $\|x(k) - \hat{x}(k)\|_{P^{(i-1)}} \leq \eta^{(i-1)}$ are satisfied, if

$$\|x(k) - \hat{x}(k)\|^2_{P^{(i)}} \leq \frac{1}{\zeta^{i\to(i-1)}} \min\left(\left(1-\theta^{(i-1)}\right)^2, \left(\eta^{(i-1)}\right)^2\right) \tag{14}$$

From Theorem 4, we know that if Controller # i has been implemented for at least $T^{(i)}$ time steps, and if the state evolution starting from $\hat{x}(k - T^{(i)})$ driven by the input from the Controller $\#i$ is inside $\Pi_x^{(i)}$, then $\|x(k) - \hat{x}(k)\|^2_{P^{(i)}} \leq \frac{\left(\rho^{(i)}\right)^{\mathcal{T}^{(i)}} V_{\mathcal{T}}^{(i)}}{\mu^{(i)}}$. By imposing an upper bound $\delta^{i\to(i-1)}$ on $V_{\mathcal{T}}^{(i)}$, we can upper bound the state estimation error at current time k. Let

$$\delta^{i\to(i-1)} = \frac{\mu^{(i)}}{\zeta^{i\to(i-1)} \left(\rho^{(i)}\right)^{\mathcal{T}^{(i)}}} \min\left(\left(1-\theta^{(i-1)}\right)^2, \left(\eta^{(i-1)}\right)^2\right) \tag{15}$$

the satisfaction of (14) is guaranteed. Furthermore, because the observer is exponentially converging, for any finite $\delta^{i\to(i-1)}$, there exists a finite time such that $V_{\mathcal{T}}^{(i)} \leq \delta^{i\to(i-1)}$ is satisfied.

Algorithm 5. Off-line, construct $M+1$ local predictive controllers by Algorithm 4. On-line, given $x(0), \hat{x}(0) \in \mathcal{S}^{(i)}$ satisfying $\|x(0) - \hat{x}(0)\|_{P^{(i)}} \leq \eta^{(i)}$ for some i. Apply Controller $\#i$. Let $T^{(i)}$ be the time period during which Controller $\#i$ is implemented. If for Controller $\#i > 0$, (1) $T^{(i)} \geq \mathcal{T}^{(i)}$, (2) $\hat{x}(k) \in \mathcal{S}_\theta^{(i-1)}$, and (3) the state evolution starting from $\hat{x}(k - \mathcal{T}^{(i)})$ driven by the input from Controller $\#i$ is inside $\Pi_x^{(i)}$, and $V_{\mathcal{T}}^{(i)} \leq \delta^{i\to(i-1)}$, then, at the next sampling time, switch from Controller $\#i$ to Controller $\#(i-1)$; Otherwise, continue to apply Controller $\#i$.

Theorem 5. *Consider the nonlinear system (1) and the output map (9). Suppose $x(0), \hat{x}(0) \in \mathcal{S}^{(i)}$ satisfying $\|x(0) - \hat{x}(0)\|_{P^{(i)}} \leq \eta^{(i)}$ for some i, the scheduled output feedback MPC in Algorithm 5 asymptotically stabilizes the closed-loop system to the desired equilibrium $(x^{(0)}, u^{(0)})$.*

4 Example

Consider a two-tank system

$$\begin{aligned} \rho S_1 \dot{h}_1 &= -\rho A_1 \sqrt{2gh_1} + u \\ \rho S_2 \dot{h}_2 &= \rho A_1 \sqrt{2gh_1} - \rho A_2 \sqrt{2gh_2} \\ y &= h_2 \end{aligned} \tag{16}$$

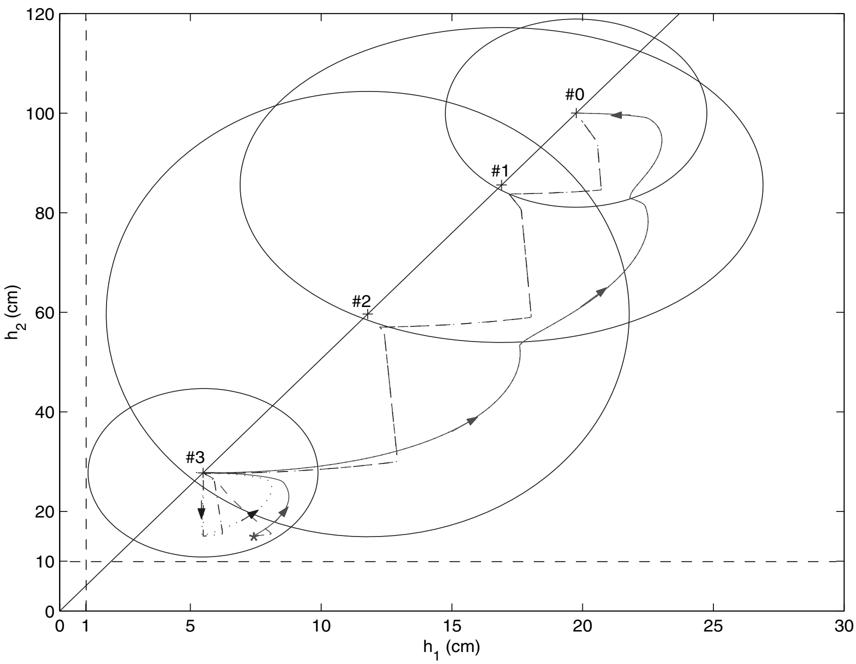

Fig. 1. Phase plots of the regulations from $x(0) = (7.5, 15)^T$ to the equilibrium $\left((19.753, 100)^T, 1.7710\right)$. First formulation: solid line - state; dotted line - estimated state. Second formulation: dashed line - state; dashed dotted line - estimated state.

where $\rho = 0.001\text{kg/cm}^3$, $g = 980\text{cm/s}^2$, $S_1 = 2500\text{cm}^2$, $A_1 = 9\text{cm}^2$, $S_2 = 1600\text{cm}^2$, $A_2 = 4\text{cm}^2$, $1\text{cm} \le h_1 \le 50\text{cm}$, $10\ \text{cm} \le h_2 \le 120\text{cm}$, and $0 \le u \le 2.5\text{kg/s}$. The sampling time is 0.5 sec. Let $Q = diag(0, 1)$, $R = 0.01$ and $\alpha = 0.998$ for all the controller designs, and $\rho = 0.99$ for all the observer designs. Let $\theta = 0.9$, $T = 10$ and $\delta = 10^{-5}$ for all switches.

Consider the regulation from an initial state $h(0) = \begin{bmatrix} 7.5 \\ 15 \end{bmatrix}$ to the equilibrium $(h^{(0)}, u^{(0)}) = \left(\begin{bmatrix} 19.753 \\ 100 \end{bmatrix}, 1.7710 \right)$. Initial estimated state is $\hat{h}(0) = h^{(3)}$. Figure 1 shows four regions of stability defined by the optimization (7) for four equilibrium points. There are two controller formulations implemented within each stability region. The first formulation is to represent the local nonlinearity as a LTV model and parameterize the infinite control horizon in terms of a linear feedback law. Consider an equilibrium point $\left(\begin{bmatrix} h_1^{\text{eq}} \\ h_2^{\text{eq}} \end{bmatrix}, u^{\text{eq}} \right)$. The local nonlinearity within a neighborhood (Π_h, Π_u) is expressed as a polytopic

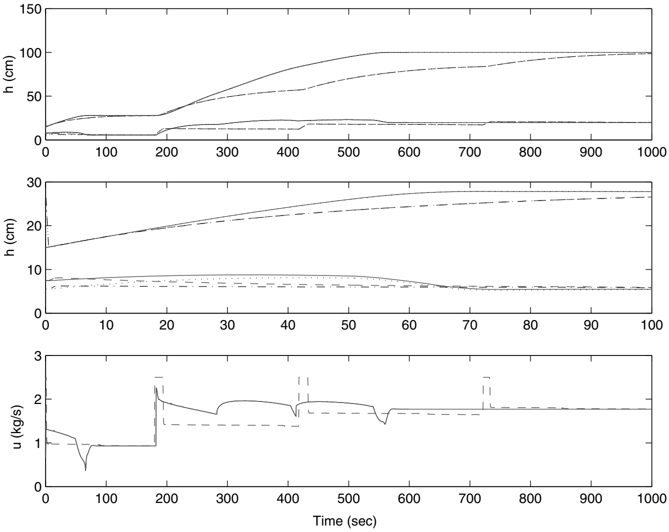

Fig. 2. Time responses of the regulations from $x(0) = (7.5, 15)^T$ to the equilibrium $\big((19.753, 100)^T, 1.7710\big)$. First formulation: solid line - state and input; dotted line - estimated state. Second formulation: dashed line - state and input; dashed dotted line - estimated state.

uncertainty Ω with four vertices $\{J(h_1^{\mathrm{eq}}+\delta h_1, h_2^{\mathrm{eq}}+\delta h_2)$, $J(h_1^{\mathrm{eq}}+\delta h_1, h_2^{\mathrm{eq}}-\delta h_2)$, $J(h_1^{\mathrm{eq}}-\delta h_1, h_2^{\mathrm{eq}}+\delta h_2)$, $J(h_1^{\mathrm{eq}}-\delta h_1, h_2^{\mathrm{eq}}-\delta h_2)\}$, where $J(h_1, h_2)$ is the Jacobian matrix at $(h_1^{\mathrm{eq}}, h_2^{\mathrm{eq}})^T$. The second formulation is to use the nonlinear model within a neighborhood (Π_h, Π_u) and only enforce the constraint over a finite control horizon of $N = 3$ with a terminal constraint and a terminal cost as specified in Remark 1. Figure 1 shows the transitions by using the two controller formulations. Figure 2 shows the time responses. Close-up views of the responses of the state and the estimated state are provided to show convergence of the observers.

The first formulation performs better than the second one, because control switches of the second formulation happen close to the intermediate equilibrium points. (For performance improvement see Remark 3). On a Gateway PC with Pentium III processor (1000MHz, Cache RAM 256KB and total memory 256MB) and using Matlab LMI toolbox for the first formulation and optimization toolbox for the second approach, the numerical complexity of the two controller formulations are 0.5 second and 0.15 second per step, respectively.

5 Conclusions

In this paper, we have proposed a stabilizing scheduled output feedback MPC formulation for constrained nonlinear systems with large operating regions. Since we were able to characterize explicitly an estimated region of stability of the designed local output feedback predictive controller, we could expand it by designing multiple predictive controllers, and on-line switch between the local controllers and achieve nonlinear transitions with guaranteed stability. This algorithm provides a general framework for the scheduled output feedback MPC design. Furthermore, we have shown that this scheduled MPC is easily implementable by applying it to a two tank process.

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Appendix

Proof. Proof of Theorem 1: Within a neighborhood (Π_x, Π_u) around $(x^{\rm eq}, u^{\rm eq})$, we locally represent the nonlinear system (1) by a LTV model $\bar{x}(k+1) = A(k)\bar{x}(k) + B(k)\,\bar{u}(k)$ with $\begin{bmatrix} A(k) \; B(k) \end{bmatrix} \in \Omega$. For all $x \in \Pi_x$ and $u \in \Pi_u$, the Jacobian matrix $\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}\right] \in \Omega$ with $\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$ and $\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$. It is straight forward to establish the closed-loop exponential stability within S based on the LTV model. Since the LTV model is a representation of a class of nonlinear systems including the given nonlinear system (1) within the neighborhood (Π_x, Π_u), the closed-loop nonlinear system is exponentially stable within S. ■

Proof. Proof of Theorem 2 and 4: Following the same procedure as in the proof for Theorem 1, we locally represent the nonlinear error dynamics as a LTV model $e(k+1) = (A(k) - L_p C(k))\, e(k)$ with $\left[\, A(k)^T \; C(k)^T \,\right]^T \in \Psi$. For all $x, \hat{x} \in \Pi_x$ and $u \in \Pi_u$, the Jacobian matrix $\left[\left(\frac{\partial f}{\partial x}\right)^T \; \left(\frac{\partial h}{\partial x}\right)^T \right]^T \in \Psi$ with $\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$ and $\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_q}{\partial x_1} & \cdots & \frac{\partial h_q}{\partial x_n} \end{bmatrix}$. It is straight forward to establish the exponential convergence of the observer and the norm bound of the state estimation error within S based on the LTV model and the nonlinear model. ■