Conditions for MPC Based Stabilization of Sampled-Data Nonlinear Systems Via Discrete-Time Approximations

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Summary. This paper is devoted to the stabilization problem of nonlinear continuous-time systems with piecewise constant control functions. The controller is to be computed by the receding horizon control method based on discrete-time approximate models. Multi-rate - multistep control is considered and both measurement and computational delays are allowed. It is shown that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant. The conditions are formulated in terms of the original continuous-time models and the design parameters so that they should be verifiable in advance.

1 Introduction

One of the most popular methods to design stabilizing controllers for nonlinear systems is the receding horizon control, also known as model predictive control. In receding control, a finite horizon optimal control problem is repeatedly solved and the input applied to the system is based on the obtained optimal open-loop control. As a result of substantial efforts of many researchers, several theoretically well-established versions of this method have been proposed in the past one and a half decade both for continuous- and discrete-time models; see e.g. [24], [4], [6], [21] for surveys and the references therein.

In continuous-time setting a great deal of the investigations is devoted to the idealized situation, when the optimization procedure is solved at all time instants, and the initial value of optimal control is applied to the plant (to mention just a few examples, see [23], [13], [14], [1], [3]). This turns out almost always to be an intractable task in practice. A more realistic assumption is that the optimization problem is solved only at disjoint time instants and the resulting optimal control function is implemented in between, which leads to a sampled-data nonlinear model predictive control scheme (see e.g. [2], [18],

⋆ The financial support from the Hungarian National Science Foundation for Scientific Research, grant no. T037491 is gratefully acknowledged.

R. Findeisen et al. (Eds.): Assessment and Future Directions, LNCIS 358, pp. 35–48, 2007.
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[8]). Being the optimal control in general merely measurable, the troubles of the implementation of such a function are not negligible. The effect of the "sampling and zero-order hold" is considered in [17] assuming the existence of a global control Lyapunov function (CLF), and in [22], where the stabilizing property of a piecewise constant NMPC computed from - and applied to the continuous-time model is investigated without taking into account any approximation in the plant model.

In several technical respects the situation is simpler, if the model of the plant is given in discrete-time. However, such models frequently derived from some continuous-time models as "good" approximations. For this reason, it is important to know conditions which guarantee that the same family of controllers that stabilizes the approximate discrete-time model of the plant also practically stabilizes the exact model of the plant. Sufficient conditions for a controller having these properties are presented in [26] and [25]. As it is emphasized by the title of the latter paper, these results provide a framework for controller design relying on the approximate discrete-time models, but they do not explain how to find controllers that satisfy the given conditions. Within this framework some optimization-based methods are studied in [11]: the design is carried out either via an infinite horizon optimization problem or via an optimization problem over a finite horizon with varying length. To relax the computational burden of these approaches, one can apply a suitable version of the receding horizon control method. Some sets of conditions are formulated and stability results are proved in [15] and [5] for sampled-data receding horizon control method without and with delays based on approximate discrete-time models. In this work we shall investigate the stability property of the equilibrium under a different set of assumptions that are verifyable in advance.

In receding horizon control method, a Bolza-type optimal control problem is solved, in which the design parameters are the horizon length $0 \leq t_1 \leq \infty$, the stage cost l, the terminal cost g (which are usually assumed to be at least non-negative valued) and the terminal constraint set $\mathcal{X}_f$. It is well-known that, if no further requirements for these parameters are stated, then one can show even linear examples, where the resulting closed-loop system is unstable. On the other hand, if that minimal requirement is satisfied that the origin is a locally asymptotically stable equilibrium for the closed-loop system, then one expect to have the largest domain of attraction possible, the least computational efforts possible for finding the controller, and certain robustness, as well. The domain of attraction can probably be increased by increasing the time-horizon, but this involves the increase of the necessary computational efforts, too. Under the terminal constraint one can expect a relatively large domain of attraction with relatively short time horizon. This is the reason, why this constraint is frequently applied in receding horizon. However, if a terminal constraint is included in the optimization problem, then the corresponding value function will not have suitable regularity in general, which ensures an expected robustness. In [9] several examples are presented which show the realization of this phenomenon. Therefore, if stabilization is aimed via an approximate model, the terminal constraint may not be considered explicitly. Several results show that an appropriate choice of the terminal cost g may also enforce stability: in fact, if g is a strict control Lyapunov function within one of its level sets, then the receding horizon controller makes the origin to be asymptotically stable with respect to the closed-loop system with a domain of attraction containing the above mentioned level set of g (the terminal constraint set is implicit e.g. in [14], [18]). This domain of attraction can be enlarged up to an arbitrary compact set, which is asymptotically controllable to the origin, by a suitable - finite - choice of the horizon length. For a substantial class of systems well-established methods exist for the construction of a suitable terminal cost (see e.g. [3], [1], [29]). Sometimes it may be difficult – if not impossible – to derive an appropriate terminal cost. Lately, it has been proven by [19] and [10] that the required stability can be enforced merely by a sufficiently large time horizon, having obvious advantages, but at the cost of a – possibly substantial – enlargement of the computational burden.

Here we consider in details the case when the terminal cost is a control Lyapunov function, and we shall make some remarks on the case of general terminal cost.

2 Stabilization Results with CLF Terminal Cost

2.1 The Models and the Method

Consider the nonlinear control system described by

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $\mathcal{X}$ is the state space, U is the control constraint set, $f\colon \mathbb{R}^n \times U \to \mathbb{R}^n$, with $f(0,0) = 0$, U is closed and $0 \in \mathcal{X}$, $0 \in U$. We shall assume that f is continuous and Lipschitz continuous with respect to x in any compact set. Let $\Gamma \subset \mathcal{X}$ be a given compact set containing the origin and consisting of all initial states to be taken into account.

Consider an auxiliary function $l : \mathbb{R}^n \times U \to \mathbb{R}_+$ with analogous regularity properties as f satisfying the condition $l(0,0) = 0$, and consider the augmented system (1) with

$$\dot{\chi}(t) = l(x(t), u(t)), \quad \chi(0) = 0. \tag{2}$$

For convenience we introduce the notation $\mathcal{Y}_\rho = \mathcal{Y} \cap \mathcal{B}_\rho$, where $\mathcal{B}_\rho$ denotes the ball around the origin with the radius ρ.

The system is to be controlled digitally using piecewise constant control functions $u(t) = u(iT) =: u_i$, if $t \in [iT, (i+1)T)$, $i \in \mathbb{N}$, where $T > 0$ is the control sampling period. We assume that for any $\overline{x} \in \mathcal{X}_{\Delta'}$ and $\overline{u} \in \mathcal{U}_{\Delta''}$, equation (1)–(2) with $u(t) \equiv \overline{u}$, ($t \in [0,T]$) and initial condition $x(0) = \overline{x}$, $\chi(0) = 0$) has a unique solution on $[0,T]$ denoted by $(\phi^E(.,\overline{x},\overline{u}), \varphi^E(.,\overline{x},\overline{u}))$. Then, the augmented *exact discrete-time model* of the system (1)–(2) can be defined as

$$x^E_{i+1} = F^E_T(x^E_i, u_i), \qquad x^E_0 = x_0, \tag{3}$$

$$\chi^E_{i+1} = \chi^E_i + l^E_T\left(x^E_i, u_i\right), \qquad \chi^E_0 = 0, \tag{4}$$

where $F^E_T(x,u) := \phi^E(T; x, u)$, and $l^E_T(x,u) = \varphi^E(T, x, u)$.

We note that, ϕ^E and φ^E are not known in most cases, therefore, the controller design can be carried out by means of an approximate discrete-time model

$$x_{k+1}^A = F_{T,h}^A\left(x_k^A, u_k\right), \qquad x_0^A = x_0, \tag{5}$$

$$\chi_{k+1}^A = \chi_k^A + l_{T,h}^A\left(x_k^A, u_k\right), \qquad \chi_0^A = 0, \tag{6}$$

where $F_{T,h}^A(x,u)$ and $l_{T,h}^A(x,u)$ are typically derived by multiple application of some numerical approximation formula with (possibly variable) step sizes bounded by the parameter h. Given $\mathbf{u} = \{u_0, u_1, \ldots\}$ and initial conditions $x_0^E = x'$ and $x_0^A = x''$, the trajectories of the discrete-time systems (3) and (5)–(6) are denoted, by $\phi_k^E(x', \mathbf{u})$ and $\phi_k^A(x'', \mathbf{u})$, $\varphi_k^A(x'', \mathbf{u})$, respectively.

Concerning the parameters T and h, in principle two cases are possible: $T = h$, and T can be adjusted arbitrarily; $T \neq h, T$ is fixed and h can be chosen arbitrarily small. Having less number of parameters, the first case seems to be simpler, but in practice there exists a lower bound to the smallest achievable T. Since the second case has much more practical relevance, here we shall discuss it in details, and we shall only point out the differences arising in case $T = h$, when appropriate. In what follows, we assume that $T > 0$ is given.

In this paper we address the problem of state feedback stabilization of (3) under the assumption that state measurements can be performed at the time instants jT^m, $j = 0, 1, \ldots$:

$$y_j := x^E(jT^m), \qquad j = 0, 1, \ldots.$$

The result of the measurement y_j becomes available for the computation of the controller at $jT^m + \tau_1$, where $\tau_1 \geq 0$, while the computation requires $\tau_2 \geq 0$ length of time i.e. the (re)computed controller is available at $T_j^* := jT^m + \tau_1 + \tau_2$, $j = 0, 1, \ldots$. We assume that $\tau_1 = \ell_1 T$, $\tau_2 = \ell_2 T$ and $T^m = \ell T$ for some integers $\ell_1 \geq 0$, $\ell_2 \geq 0$ and $\ell \geq \ell_1 + \ell_2 =: \overline{\ell}$.

If $\ell = 1$, $\ell_1 = \ell_2 = 0$, then we can speak about a single rate, one-step receding horizon controller without delays, if $\ell > 1$, then we have multi-rate, multistep controller with or without delays depending on values of ℓ_1 and ℓ_2. Papers [2] and [7] consider the problem of computational delay in connection with the receding horizon control for exact continuous-time models, while [28] develops results analogous to that of [25] for the case of multi-rate sampling with measurement delays.

A "new" controller computed according to the measurement $y_j = x^E(jT^m)$ will only be available from T_j^*, thus in the time interval $[jT^m, T_j^*)$ the "old" controller has to be applied. Since the corresponding exact trajectory is unknown, an approximation ζ_j^A to the exact state $x^E\left(T_j^*\right)$ can only be used, which can be defined as follows. Assume that a control sequence $\left\{u_0\left(\zeta_{j-1}^A\right), \ldots, u_{\ell-1}\left(\zeta_{j-1}^A\right)\right\}$ has been defined for $j \geq 1$. Let $\mathbf{v}^p\left(\zeta_{j-1}^A\right) = \left\{u_{\ell-\overline{\ell}}\left(\zeta_{j-1}^A\right), \ldots, u_{\ell-1}\left(\zeta_{j-1}^A\right)\right\}$ and define ζ_j^A by

$$\zeta_j^A = \mathcal{F}_{\overline{\ell}}^A\left(y_j, \mathbf{v}^p\left(\zeta_{j-1}^A\right)\right), \quad \zeta_0^A = \phi_{\overline{\ell}}^A(x, \mathbf{u}^c), \tag{7}$$

where $\mathcal{F}_{\overline{\ell}}^{A}\left(y,\{u_0,\ldots,u_{\overline{\ell}-1}\}\right) = F_{T,h}^{A}\left(\ldots F_{T,h}^{A}\left(F_{T,h}^{A}\left(y,u_0\right),u_1\right)\ldots,u_{\overline{\ell}-1}\right)$, and $\mathbf{u}^c$ is some precomputed controller (independent of state measurements). Let $\mathbf{v}^{(j)} = \left\{u_0^{(j)},\ldots,u_{\ell-1}^{(j)}\right\}$ be computed for ζ_j^A and let the ℓ-step exact discrete-time model be described by

$$\xi_{j+1}^{E} = \mathcal{F}_{\ell}^{E}(\xi_j^E,\mathbf{v}^{(j)}), \qquad \xi_0^E = \phi_{\overline{\ell}}^{E}(x,\mathbf{u}^c), \tag{8}$$

where $\mathcal{F}_{\ell}^{E}(\xi_j^E,\mathbf{v}) = \phi_{\ell}^{E}(\xi_j^E,\mathbf{v})$. In this way the right hand side of (8) depends on $y_j = x^E(jT^m)$ so that (7-8) represents an unconventional feedback system.

Our aim is to define a measurement based algorithm for solving the following problem: for given T, T^m, τ_1 and τ_2 find a control strategy

$$\mathbf{v}_{\ell,h}\colon \widetilde{\Gamma} \to \underbrace{U\times U\times\ldots\times U}_{\ell\text{ times}}$$

$\mathbf{v}_{\ell,h}(x) = \{u_0(x),\ldots,u_{\ell-1}(x)\}$, using the approximate model (5), (7) which stabilizes the origin for the exact system (3) in an appropriate sense, where $\widetilde{\Gamma}$ is a suitable set containing at least Γ.

Remark 1. If $T = T^m$, $\ell = 1$ and $\ell_1 = \ell_2 = 0$, then the single-rate delay-free case is recovered, therefore it is sufficient to discuss the general case in details.

In order to find a suitable controller $\mathbf{v}$, we shall apply a multistep version of the receding horizon method. To do so, we shall consider the following cost function.

Let $0 < N \in \mathbb{N}$ be given. Let (5) be subject to the cost function

$$J_{T,h}(N,x,\mathbf{u}) = \sum_{k=0}^{N-1} l_{T,h}^{A}(x_k^A,u_k) + g(x_N^A),$$

where $\mathbf{u} = \{u_0,u_1,\ldots,u_{N-1}\}$, $x_k^A = \phi_k^A(x,\mathbf{u})$, $k = 0,1,\ldots,N$ denote the solution of (5), $l_{T,h}^A$ is defined as in (6) and g is a given function.

Consider the optimization problem

$$P_{T,h}^{A}(N,x)\colon\ \min\left\{J_{T,h}(N,x,\mathbf{u}) : u_k \in U\right\}.$$

If this optimization problem has a solution denoted by $\mathbf{u}^*(x) = \{u_0^*(x),\ldots, u_{N-1}^*(x)\}$, then the first ℓ elements of $\mathbf{u}^*$ are applied at the state x i.e.

$$\mathbf{v}_{\ell,h}(x) = \left\{u_0^*(x),\ldots,u_{\ell-1}^*(x)\right\}.$$

In what follows we shall use the notation $V_N^A(x) = J_{T,h}(N,x,\mathbf{u}^*(x))$.

2.2 Assumptions and Basic Properties

To ensure the existence and the stabilizing property of the proposed controller, several assumptions are needed.

We might formulate this assumptions in part with respect to the approximate discrete-time model as it was done e.g. in [5] and [15]. However, it turns out that in several cases the verification of some conditions is much more tractable for the original model than the approximate one. For this reason, we formulate the assumptions with respect to the exact model (and to the applied numerical approximation method). For the design parameters l and g, we shall make the following assumption.

Assumption 1. (i) $g : \mathbb{R}^n \to \mathbb{R}$ is continuous, positive definite, radially unbounded and Lipschitz continuos in any compact set.

(ii) l is continuous with respect to x and u and Lipschitz continuous with respect to x in any compact set.

(iii) There exist such class-$\mathcal{K}_\infty$ functions $\varphi_1, \overline{\varphi}_1, \varphi_2$ and $\overline{\varphi}_2$ that

$$\varphi_1(\|x\|) + \overline{\varphi}_1(\|u\|) \leq l(x,u) \leq \varphi_2(\|x\|) + \overline{\varphi}_2(\|u\|), \tag{9}$$

holds for all $x \in \mathcal{X}$ and $u \in U$.

Remark 2. The lower bound in (9) can be substituted by different conditions: e.g. $\overline{\varphi}_1$ may be omitted, if U is compact. If the stage cost for the discrete-time optimization problem is directly given, other conditions ensuring the existence and uniform boundedness of the optimal control sequence can be imposed, as well (see e.g. [10], [15] and [20]). However, having a $\mathcal{K}_\infty$ lower estimation with respect to $\|x\|$ is important in the considerations of the present paper.

The applied numerical approximation scheme has to ensure the closeness of the exact and the approximate models in the following sense.

Assumption 2. For any given $\Delta' > 0$ and $\Delta'' > 0$ there exists a $h_0^* > 0$ such that

(i) $F_{T,h}^A(0,0) = 0$, $l_{T,h}^A(0,0) = 0$, $l_{T,h}^A(x,u) > 0$, $x \neq 0$, $F_{T,h}^A$ and $l_{T,h}^A$ are continuous in both variables uniformly in $h \in (0, h_0^*]$, and they preserve the Lischitz continuity of the exact models, uniformly in h;

(ii) there exists a $\gamma \in \mathcal{K}$ such that

$$\|F_T^E(x,u) - F_{T,h}^A(x,u)\| \leq T\gamma(h), \qquad \|l_T^E(x,u) - l_{T,h}^A(x,u)\| \leq T\gamma(h),$$

for all $x \in \mathcal{B}_{\Delta'}$, all $u \in U_{\Delta''}$, and $h \in (0, h_0^*]$.

Remark 3. We note that Assumption A2 depends on the numerical approximation method, and it can be proven for reasonable discretization formulas.

Definition 1. *System (3) is asymptotically controllable from a compact set Ω to the origin, if there exist a $\beta(.,.) \in \mathcal{KL}$ and a continuous, positive and non-decreasing function $\sigma(.)$ such that for all $x \in \Omega$ there exists a control sequence $\mathbf{u}(x)$, $u_k(x) \in U$, such that $\|u_k(x)\| \leq \sigma(\|x\|)$, and the corresponding solution ϕ^E of (3) satisfies the inequality*

$$\left\|\phi_k^E(x, \mathbf{u}(x))\right\| \leq \beta(\|x\|, kT), \qquad k \in \mathbb{N}.$$

The next assumption formulates, roughly speaking, a necessary condition for the existence of a stabilizing feedback.

Assumption 3. (i) The exact discrete-time system (3) is asymptotically controllable from a set Ω containing Γ to the origin.

(ii) There exists a $\Delta_0 > 0$, and a control sequence $\mathbf{u}^c = \left\{u_0^c, \dots, u_{\bar{l}-1}^c\right\}$ ($u_i^c \in U$) can be given so that $\Gamma \subset \Omega_{\Delta_0}$, $\phi_k^E(x, \mathbf{u}^c) \in \Omega_{\Delta_0}$, $\phi_k^A(x, \mathbf{u}^c) \in \Omega_{\Delta_0}$, $k = 0, 1, \dots, \overline{\ell}$ for all $x \in \Gamma$.

In what follows let $\Delta_1 = \beta(\Delta_0, 0)$ and $\Delta_2 = \sigma(\Delta_0)$, where β and σ are given in Definition 1.

Finally, the next assumption implies that the final state penalty has to be a local control Lyapunov function within the sampled data controllers.

Assumption 4. There exist a positive number η and a class-$\mathcal{K}$ function α_g such that for all $x \in \mathcal{G}_\eta = \{x \in \mathcal{X} : g(x) \leq \eta\}$ there is a $\kappa(x) \in U_{\Delta_2}$ such that for $u_0 = \kappa(x)$

$$g\left(F_T^E(x, u_0)\right) - g(x) + l_T^E(x, u_0) \leq -\alpha_g(\|x\|). \tag{10}$$

Remark 4. Sometimes it may be more convenient to verify the analogue of Assumption A4 for the approximate discrete-time system (c.f. [15]). This is the case e.g. if the model has a controllable linearization (c.f. [2], [22]). In other cases, as e.g. in [16], the present form is more advantageous.

Let us consider now the auxiliary problem of the minimization of the cost function

$$J_T^E(N, x, \mathbf{u}) = \sum_{k=0}^{N-1} l_T^E(x_k^E, u_k) + g(x_N^E),$$

subject to the exact system (3), and introduce the notation

$$V_N^E(x) = \inf \left\{J_T^E(N, x, \mathbf{u})\colon \mathbf{u} = \{u_0, \dots, u_{N-1}\},\ u_k \in U\right\}$$

Lemma 1. *If Assumptions A1, A3 and A4 hold true, then there exists a constant $V_{\max}^E$ independent of N, such that $V_N^E(x) \leq V_{\max}^E$, for all $x \in \Omega_{\Delta_0}$ and $N \in \mathbb{N}$.*

Proof. The proof is similar to that of the analogous statement in [15], therefore it is omitted here. ■

Let us introduce the notations $V_{\max}^A = V_{\max}^E + 1$, $\Delta_2^* = \overline{\varphi}_1^{-1}\left(V_{\max}^A / T\right)$, and

$$\Gamma_{\max}(h_0) = \left\{x \in \mathcal{X}\colon V_N^A(x) \leq V_{\max}^A,\ h \in (0, h_0]\right\},$$

$$M_f(\Delta', \Delta'') = \max_{x \in \mathcal{X}_{\Delta'}} \max_{u \in U_{\Delta''}} \|f(x, u)\|.$$

Theorem 1. *Suppose that Assumptions A1–A4 are valid, and inequality* $s \geq 2TM_f(2s, \Delta_2^*)$ *holds true, if* $s \geq \Delta_0$. *Then there exist constants* N^*, r_0^*, Δ_1^* *and functions* $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ *so that for any fixed* $N \geq N^*$, $r_0 \in (0, r_0^*]$ *and* $\delta > 0$ *there exists a* $\overline{h} > 0$ *such that for all* $h \in (0, \overline{h}]$

$$\Gamma \subset \Omega_{\Delta_0} \subset \Gamma_{\max}(h) \subset \mathcal{B}_{\Delta_1^*}, \tag{11}$$

$$\sigma_1(\|x\|) \leq V_N^A(x) \leq \sigma_2(\|x\|), \tag{12}$$

$$\left\|\phi_k^A(x, \mathbf{u}^*(x))\right\| \leq \Delta_1^*, \quad \|u_k^*(x)\| \leq \Delta_2^*, \quad k = 0, 1, \ldots, N-1, \tag{13}$$

if $x \in \Gamma_{\max}(h) \backslash \mathcal{B}_{r_0}$, *and for all* $k = 1, \ldots, \ell$

$$V_N^A\left(\phi_k^A(x, \mathbf{u}^*(x))\right) - V_N^A(x) \leq -l_{T,h}^A(x, u_0^*(x)) + \delta, \tag{14}$$

where $\mathbf{u}^*(x)$ *denotes the optimal solution of* $P_{T,h}^A(N, x)$. *Moreover,* V_N^A *is locally Lipschitz continuous in* $\Gamma_{\max}(h)$ *uniformly in* $h \in (0, \overline{h}]$.

Proof. The proof is given in the Appendix. ∎

Remark 5. If the sampling parameter T and the discretization parameter h coincide and T can be arbitrary adjusted, then – besides some technical problems that can easily be handled – the main difficulty originates from the fact that the lower bound of $l_{T,h}^A$ is no longer independent of the adjustable parameter. Nevertheless, a uniform lower bound for V_N^A can be given for this case, as well (see [15]).

Remark 6. If $\mathcal{X}$ is bounded, then the condition $s \geq 2T\ M_f(2s, \Delta_2^*)$, if $s \geq \Delta_0$ in Theorem 1 is not needed, otherwise the set of possible initial states, the choice of T and the growth of f have to be fitted together.

2.3 Multistep Receding Horizon Control

In this section we outline an approach to the problem how the occurring measurement and computational delays can be taken into account in the stabilization of multi-rate sampled-data systems by receding horizon controller.

Suppose that a precomputed control sequence $\mathbf{u}^c$ satisfying Assumption A3 is given. Then the following Algorithm can be proposed.

Algorithm. Let $N \geq N^*$ be given, let $j = 0$, $T_{-1}^* = 0$ and let $\mathbf{u}^{(0)} = \mathbf{u}^{(p,0)} = \mathbf{u}^c = \{u_0^c, \ldots, u_{\ell-1}^c\}$. Measure the initial state $y(0) = x_0$.
Step j.

(i) Apply the controller $\mathbf{u}^{(j)}$ to the exact system over the time interval $[T_{j-1}^*, T_j^*]$.
(ii) Predict the state of the system at time T_j^* from $y(j)$ by the approximation let $\zeta_j^A = \phi_\ell^A(y(j), \mathbf{u}^{(p,j)})$.

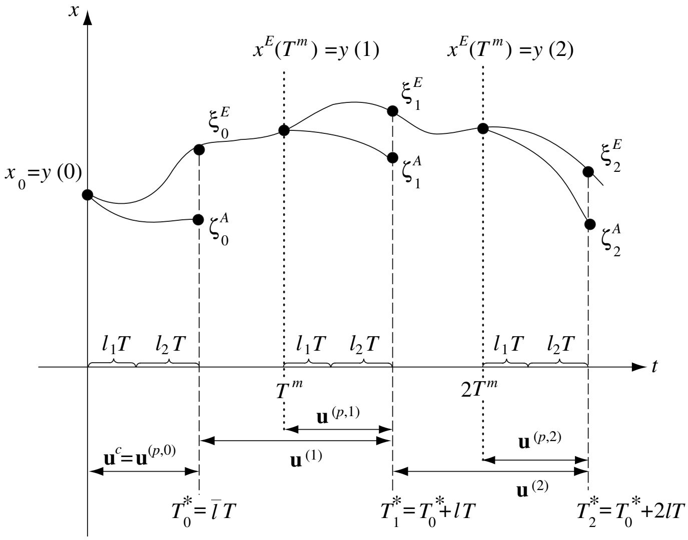

Fig. 1. Sketch to the Algorithm

(iii) Find the solution $\mathbf{u}^* = \{u_0^*, \ldots, u_{N-1}^*\}$ to the problem $P_{T,h}^A(N, \zeta_j^A)$, let $\mathbf{u}^{(j+1)} = \{u_0^*, \ldots, u_{\ell-1}^*\}$ and $\mathbf{u}^{(p,j+1)} = \{u_{\ell-\overline{\ell}}^*, \ldots, u_{\ell-1}^*\}$.

(iv) $j = j + 1$.

A schematic illustration of the Algorithm is sketched in Figure 1.

Theorem 2. *Suppose that the conditions of Theorem 1 hold true. Then there exists a $\beta \in \mathcal{KL}$, and for any $\overline{r} > 0$ there exists a $h^* > 0$ such that for any fixed $N \geq N^*$, $h \in (0, h^*]$ and $x_0 \in \Gamma$, the trajectory of the ℓ-step exact discrete-time system*

$$\xi_{k+1}^E = \mathcal{F}_\ell^E(\xi_k^E, \mathbf{v}_{\ell,h}(\zeta_k^A)), \qquad \xi_0^E = \phi_{\overline{\ell}}^E(x_0, \mathbf{u}^c) \tag{15}$$

with the ℓ-step receding horizon controller $\mathbf{v}_{\ell,h}$ obtained by the prediction

$$\zeta_{k+1}^A = \mathcal{F}_{\overline{\ell}}^A\left(y_{k+1}, \mathbf{v}^p\left(\zeta_k^A\right)\right), \qquad \zeta_0^A = \phi_{\overline{\ell}}^A(x_0, \mathbf{u}^c) \tag{16}$$

satisfies that $\xi_k^E \in \Gamma_{\max}(h)$ and

$$\left\|\xi_k^E\right\| \leq \max\left\{\beta\left(\left\|\xi_0^E\right\|, kT^m\right), \overline{r}\right\}$$

for all $k \geq 0$. Moreover, $\zeta_k^A \in \Gamma_{\max}(h)$, as well, and

$$\left\|\zeta_k^A\right\| \leq \max\left\{\beta\left(\left\|\zeta_0^A\right\|, kT^m\right) + \delta_1, \overline{r}\right\}$$

where δ_1 can be made arbitrarily small by suitable choice of h.

Proof. The proof is given in the Appendix. ■

Remark 7. In the proof of Theorem 2 one also obtains that $\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})$ converges to the ball $\mathcal{B}_{\overline{r}}$ as $j \to \infty$ for all k. Conclusions about the intersampling behavior can be made on the basis of [27].

Remark 8. We note that the statement of Theorem 2 is similar to the practical asymptotic stability of the closed-loop system (15)–(16) about the origin, but with respect to the initial state ξ_0^E, ζ_0^A. This is not true for the original initial state x_0, because – due to the initial phase – the ball $\mathcal{B}_{\overline{r}}$ is not invariant over the time interval $[0, \overline{\ell}T)$. In absence of measurement and computational delays, the theorem gives the practical asymptotic stability of the closed-loop system (15)–(16) about the origin in the usual sense.

3 Remarks on Other Choices of the Design Parameters

Recently, stability results have been proven for the case, when the terminal cost is not a CLF: see [19] for continuous-time and [10] discrete-time considerations. It is shown in both papers that stability can be achieved under some additional conditions, with arbitrary nonnegative terminal cost, if the time horizon is chosen to be sufficiently long. In respect of the subject of the present work the latter one plays crucial role. In fact, Theorem 1 of [10] provides a Lyapunov function having (almost) the properties which guarantee that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant. To this end, one has to ensure additionally the uniform Lipschitz-continuity of the Lyapunov function, presuming that assumptions of [10] are valid for the approximate discrete-time model. This assumptions can partly be transferred to the original continuous-time data similarly to the way of the previous section.

The main difficulty is connected with assumption SA4 of [10]. This assumption requires the existence of class-$\mathcal{K}_\infty$ upper bound of the value function independent of the horizon length. It is pointed out in [10] that such a bound exists if - roughly speaking - for the approximate discrete-time system the stage cost is exponentially controllable to zero with respect to an appropriate positive definite function. An appropriate choice of such a function is given in [10], if the discrete-time system is homogeneous. However, the derivation of the corresponding conditions for the original data of general systems requires further considerations. (We note that a certain version of the MPC approach for the sampled-data implementation of continuous-time stabilizing feedback laws is investigated in [12] under an assumption analogous to SA4 of [10].)

4 Conclusion

The stabilization problem of nonlinear continuous-time systems with piecewise constant control functions was investigated. The controller was computed by the receding horizon control method based on discrete-time approximate models.

Multi-rate - multistep control was considered and both measurement and computational delays were allowed. It was shown that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant. The conditions were formulated in terms of the original continuous-time models and the design parameters so that they could be verifiable in advance.

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Appendix

Proof. (Proof of Theorem 1) To obtain the properties of function V_N^A we shall subsequently introduce several notations. Let $\rho_1 > 0$ be such that $\mathcal{B}_{\rho_1} \subset \mathcal{G}_\eta$,

$$\tau(s) = \begin{cases} \min\left\{T,\ s/(2M_f(2\varDelta_0, \varDelta_2^*))\right\}, & \text{if } 0 \le s \le \varDelta_0, \\ T, & \text{if } \varDelta_0 < s, \end{cases} \tag{17}$$

$$\sigma_1(s) = \varphi_1\left(s/2\right)\tau\left(s\right)/2\ , \qquad \varDelta_1^* = \max\{\sigma_1^{-1}(V_{\max}^A),\ \varDelta_0\}, \tag{18}$$

$$\nu(s) = \max_{\|x\| \le s} g(x) + \alpha_g(s)\ , \qquad r_0^* = \min\{\nu^{-1}(\eta),\ 2TM_f(2\varDelta_0, \varDelta_2^*)\}, \tag{19}$$

$$\sigma_2(s) = \max\{\nu(s)\ ,\ \ \nu(\rho_1/2) + 2/\rho_1 V_{\max}^A\left(s - \rho_1/2\right)\}, \tag{20}$$

$$N^* = [(V_{\max}^A - \eta)\ /\ \varphi_1(\rho_1)] + 1. \tag{21}$$

First we observe that, under the conditions of the theorem, functions σ_1 and σ_2 defined by (17)–(18) and (19)–(20), respectively, as well as function $\overline{\varphi}_1$ belong to class-$\mathcal{K}_\infty$, therefore $\varDelta_1^*$ and $\varDelta_2^*$ are well-defined. Let $h_0^* > 0$ be given by

Assumption A2 with $\Delta' = \Delta_1^*$ and $\Delta'' = \Delta_2^*$. A straightforward computation shows that for any $x \in \mathcal{X}_{\Delta_1^*}$, $u \in U_{\Delta_2^*}$ we have $\left\|\phi^E(t,x,u)\right\| \leq 2\Delta_1^*$, if $t \in [0,T]$, and $l_T^E(x,u) \geq \tau(x)\varphi_1(\|x\|/2) + T\overline{\varphi}_1(\|u\|)$. Let $0 < h_1^* \leq h_0^*$ be such that $\tau(r_0)\varphi_1(r_0/2)/2 \geq T\gamma(h_1^*)$, where γ is defined by Assumption A2 (ii). Then for all $h \in (0, h_1^*]$

$$l_{T,h}^A(x,u) \geq \sigma_1(\|x\|) + T\overline{\varphi}_1(\|u\|),$$

if $x \in \mathcal{X}_{\Delta_1^*} \setminus B_{r_0}$ and $u \in U_{\Delta_2^*}$. Therefore for any $x \in \Omega_{\Delta_0}$ problem $P_{T,h}^A(N,x)$ has an optimal solution $\mathbf{u}^*(x)$, function V_N^A is continuous in its domain, $V_N^A(0) = 0$ and $V_N^A(x) > 0$ if $x \neq 0$. Being N fixed, from Assumption A2 and Lemma 1 it follows that there exists a $0 < h_2^* \leq h_0^*$ such that for all $h \in (0, h_2^*]$ estimation $V_N^A(x) \leq V_{\max}^A$ holds, which implies that $\Omega_{\Delta_0} \subset \Gamma_{\max}(h)$. Let $0 < h_3^* \leq h_0^*$ be so small that $(L_g+1)NT\gamma(h) \leq \alpha_g(r_0)$, if $h \in (0, h_3^*]$. Making use of Assumptions A2 and A4, one can show in a standard way that for any $x \in \mathcal{G}_\eta$

$$V_N^A(x) \leq g(x), \quad \text{if} \quad \|x\| \geq r_0, \quad \text{and} \quad V_N^A(x) \leq \nu(r_0) < \eta, \quad \text{if} \quad \|x\| < r_0. \tag{22}$$

Moreover, if $x \in \Omega_{\Delta_0}$ and for some $0 \leq j < N$, $\phi_j^A(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$, then $\phi_N^A(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$. As a consequence, we obtain that $\phi_N^A(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$ for any $x \in \Omega_{\Delta_0}$, if $N \geq N^*$ and $h \in (0, h']$, where h' is chosen as $h' = \min\{h_1^*,\ h_2^*,\ h_3^*\}$. Consider a $h \in (0, h']$. Being $l_{T,h}^A(\phi_k^A(x, \mathbf{u}^*(x)), u_k^*) \leq V_N^A(x)$, if $k = 0, 1, \ldots, N-1$, the lower estimation in (12), the inclusions in (11) and inequalities (13) follow immediately. Observing that $\sigma_2(\|x\|) \geq V_{\max}^A$, if $\|x\| \geq \rho_1$, and $\sigma_2(\|x\|) \geq g(x)$, the upper estimation in (12) is a consequence of (22). Let $k \in \{1, \ldots, \ell\}$. By repeated use of Assumption A4 together with A2 one can show that for any $h \in (0, h']$ and $x \in \Gamma_{\max}(h)$

$$V_N^A\left(\phi_k^A(x, \mathbf{u}^*(x))\right) - V_N^A(x) \leq -l_{T,h}^A(x, u_0^*(x)) + (L_g+1)kT\gamma(h).$$

Let $0 < h''$ be so small that $(L_g+1)\ell T\gamma(h'') \leq \delta$, then (14) holds true, if $h \in (0, \min\{h', h''\}]$. Finally, using Assumption A2 it can be shown by standard arguments that there exist an $h''' > 0$, $L_V > 0$, $\delta_V > 0$ such that for any $h \in (0, h''']$, $\left|V_N^A(x) - V_N^A(y)\right| \leq L_V \|x-y\|$ holds true for all $x, y \in \Gamma_{\max}(h)$ with $\|x-y\| \leq \delta_V$ (see the proof of Lemma 7 of [15]). Choosing $\overline{h} = \min\{h', h'', h'''\}$, all statements of the theorem are true. □

Proof. (Proof of Theorem 2)

Let $\overline{r} > 0$ be arbitrary, let $d = \sigma_1(\sigma_2^{-1}(\sigma_1(\overline{r}))/2)$, let $r_0 = \sigma_2^{-1}(d)/2$ and let $\delta = \sigma_1(r_0)/2$. Let $\overline{h}$, δ_V, L_V be defined by Theorem 1 according to this r_0 and δ, and let $h' = \overline{h}$. The proof is based on the following claim:

Claim A. Let $k \in \{1, 2, \ldots, \ell\}$ be arbitrary and let d be defined above. If for $j \geq 1$ $\xi_{j-1}^E \in \Gamma_{\max}(h')$, $\zeta_{j-1}^A \in \Gamma_{\max}(h')$, and there exists a $\varepsilon_1 \in \mathcal{K}$ such that $\left\|\xi_{j-1}^E - \zeta_{j-1}^A\right\| \leq \varepsilon_1(h)$, if $0 < h \leq h'$, then there exist a $0 < h'' \leq h'$ such that for any $h \in (0, h'']$ inequality

$$\max\left\{V_N^A\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right), V_N^A(\xi_{j-1}^E)\right\} \geq d \tag{23}$$

implies that

$$V_N^A\left(\phi_k^E(\xi_{j-1}^E,\mathbf{u}^{(j)})\right) - V_N^A(\xi_{j-1}^E) \le -\sigma_1(\left\|\xi_{j-1}^E\right\|/2)/2,$$

where $\mathbf{u}^{(j)}$ is the optimal solution of problem $P_{T,h}^A(N,\zeta_{j-1}^A)$.

The proof of this claim can follow the same line as that of Theorem 2 in [26] by taking into account that with the given values of d and r_0, it makes no trouble that the estimations (12) are only valid outside of the ball $\mathcal{B}_{r_0}$. Thus we may return to the proof of the theorem. We observe first that the conditions of the claim for ξ_{j-1}^E and ζ_{j-1}^A, are valid if $j=1$ and ε_1 is chosen as $\varepsilon_1(h) = T\gamma(h)(e^{L_f\overline{\ell}T}-1)/(e^{L_fT}-1)$, where L_f is the Lipschitz constant of f. Assume that Claim A holds true for some $j\ge 1$. Let h'' be defined by this claim and consider a $h\in(0,h'']$. Suppose that $V_N^A(\xi_{j-1}^E)\ge d$. Then $\left\|\xi_{j-1}^E\right\|\ge\sigma_2^{-1}(d)=2r_0$, and

$$V_N^A\left(\phi_k^E(\xi_{j-1}^E,\mathbf{u}^{(j)})\right) - V_N^A(\xi_{j-1}^E) \le -\sigma_1(\left\|\xi_{j-1}^E\right\|/2)/2 \le -\sigma_1(r_0)/2,$$

hold true. Thus $\phi_k^E(\xi_{j-1}^E,\mathbf{u}^{(j)})\in\mathit{\Gamma}_{\max}(h)$, so that $\xi_j^E, y_j\in\mathit{\Gamma}_{\max}(h)$, as well, and

$$V_N^A\left(\xi_j^E\right) - V_N^A(\xi_{j-1}^E) \le -\sigma_1(r_0)/2. \tag{24}$$

Now we show that $\zeta_j^A\in\mathit{\Gamma}_{\max}(h)$. Let $0<h'''\le\min\{h',h''\}$ be so small that for any $h\in(0,h''']$ inequality $\varepsilon_1(h)\le\min\{\sigma_1(r_0)/L_V,\ 2\delta_V\}/2$ is satisfied. Then it can be shown that

$$\left\|\phi_k^E(y_j,\mathbf{u}^{(p,j)}) - \phi_k^A(y_j,\mathbf{u}^{(p,j)})\right\| \le \varepsilon_1(h).$$

and

$$V_N^A(\zeta_j^A) = V_N^A(\zeta_j^A) - V_N^A(\xi_j^E) + V_N^A(\xi_j^E) \le V_N^A(\xi_{j-1}^E) + L_V\varepsilon_1(h) - \sigma_1(r_0)/2 \le V_{\max}^A,$$

if $h\in(0,h''']$. Thus $\zeta_j^A\in\mathit{\Gamma}_{\max}(h)$, and the conditions of the claim hold also for $j+1$ as long as $V_N^A(\xi_{j-1}^E)\ge d$ holds. Therefore (24) implies that after finitely many steps $V_N^A(\xi_{j-1}^E)<d$ will occur. From the claim we get that $V_N^A\left(\phi_k^E(\xi_{j-1}^E,\mathbf{u}^{(j)})\right)<d$ must also be valid for $k=1,\ldots,\overline{\ell}$, thus for ξ_j^E, as well. Choosing $h^*=h'''$, one can show that the ball $\mathcal{B}_{\overline{r}}$ is positively invariant with respect to the exact and the approximate trajectories obtained during the application of the proposed Algorithm. The existence of a suitable function $\beta\in\mathcal{KL}$ can be constructed in the standard way. ■