Robustness and Robust Design of MPC for Nonlinear Discrete-Time Systems

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1 Introduction

In view of the widespread success of Model Predictive Control (MPC), in recent years attention has been paid to its robustness characteristics, either by examining the robustness properties inherent to stabilizing MPC algorithms, or by developing new MPC methods with enhanced robustness properties.

By restricting attention to nonlinear systems, this paper presents in a unified framework some of the robustness results available for nonlinear MPC. Specifically, the first part of the paper is concerned with the introduction of the main definitions and of the general results used in the sequel as well as with the description of a "prototype" nominal MPC algorithm with stability. Then, the considered class of model uncertainties and disturbances are defined.

In the second part of the paper, the inherent robustness properties of MPC algorithms designed on the nominal model are reviewed under the main assumption that the problem is unconstrained and feasibility is always guaranteed. The results reported rely on the decreasing property of the optimal cost function [5], [6], [42], [25]. Further robustness characteristics can be derived by showing that unconstrained MPC is inversely optimal, and as such has gain and phase margins [34].

The last part of the paper is devoted to present the approaches followed so far in the design of MPC algorithms with robustness properties for uncertain systems. A first method consists in minimizing a nominal performance index while imposing the fulfillment of constraints for each admissible disturbance, see [24]. This calls for the inclusion in the problem formulation of tighter state, control and terminal constraints and leads to very conservative solutions or even to unfeasible problems. With a significant increase of the computational burden, an alternative approach consists in solving a min-max optimization problem. Specifically, in an open-loop formulation the performance index is minimized with respect to the control sequence and maximized with respect to the disturbance sequence over the prediction horizon. However, this solution is still unsatisfactory, since the minimization with respect to a single control profile does not solve the feasibility problem. This drawback can be avoided as in [3], where the

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MPC control law is applied to an already robust stable system. Alternatively, the intrinsic feedback nature of every Receding Horizon (RH) implementation of MPC can be exploited by performing optimization with respect to closed-loop strategies, as discussed in [8], [29], [30], [33] where robust algorithms have been proposed for systems with perturbations vanishing at the origin.

2 Notations and Basic Definitions

We use Z_+ to denote the set of all nonnegative integers. Euclidean norm is denoted simply as $|\cdot|$. For any function $\phi : Z_+ \to R^n$, $\|\phi\| = \sup\{|\phi(k)| : k \in Z_+\} \leq \infty$. B_r is the closed ball of radius r, i.e. $B_r = \{x \in R^n | \, |x| \leq r\}$.

A continuous function $\alpha(\cdot) : R_+ \to R_+$ is a $\mathcal{K}$ function if $\alpha(0) = 0$, $\alpha(s) > 0$ for all $s > 0$ and it is strictly increasing. A continuous function $\beta : R_+ \times Z_+ \to R_+$ is a $\mathcal{KL}$ function if $\beta(s,t)$ is a $\mathcal{K}$ function in s for any $t \geq 0$ and for each $s > 0$ $\beta(s,\cdot)$ is decreasing and $\beta(s,t) \to 0$ as $t \to \infty$. $\mathcal{M}_\Omega$ is the set of signals in some subset Ω.

Definition 1 (Stability). *[22], [23] Given the discrete-time dynamic system*

$$x(k+1) = f(x(k)), k \geq t, \ x(t) = \bar{x} \tag{1}$$

with $f(0) = 0$ and a set $\Xi \subseteq R^n$ with the origin as an interior point:

1. *the origin is an asymptotically stable equilibrium in Ξ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall \bar{x} \in \Xi$ with $|\bar{x}| \leq \delta$, $|x(k)| < \varepsilon$, $k \geq t$, and $\lim_{k\to\infty} |x(k)| \to 0$;*
2. *the origin is a locally exponentially stable equilibrium point if there exist positive constants δ, α and $\rho < 1$ such that for any $\bar{x} \in B_\delta$, $|x(k)| < \alpha |\bar{x}| \rho^{k-t}$, $k \geq t$;*
3. *the origin is an exponentially stable equilibrium point in Ξ if there exist positive constants α and $\rho < 1$ such that for any $\bar{x} \in \Xi$, $|x(k)| < \alpha |\bar{x}| \rho^{k-t}$, $k \geq t$.*

Definition 2 (Lyapunov function). *[23] A function $V(\cdot)$ is called a Lyapunov function for system (1) if there exist two sets Ξ_1 and Ξ_2 with $\Xi_1 \subseteq \Xi_2$ and $\mathcal{K}$ functions α_1, α_2, and α_3 such that*

$$\begin{aligned} V(x) &\geq \alpha_1(|x|), \forall x \in \Xi_2 \\ V(x) &\leq \alpha_2(|x|), \forall x \in \Xi_1 \\ \Delta V(x) = V(f(x)) - V(x) &\leq -\alpha_3(|x|), \forall x \in \Xi_2 \end{aligned} \tag{2}$$

Lemma 1. *[23]Let Ξ_2 be a positive invariant set for system (1) that contains a neighborhood Ξ_1 of the origin and let $V(\cdot)$ be an associated Lyapunov function. Then:*

1. *the origin is an asymptotically stable equilibrium in Ξ_2;*
2. *if $\alpha_1(|x|) = \alpha_1 |x|^p$, $\alpha_2(|x|) = \alpha_2 |x|^p$, $\alpha_3(|x|) = \alpha_3 |x|^p$, for some real positive α_1, α_2, α_3 and p, the origin is locally exponentially stable. Moreover, if the inequality (2) holds for $\Xi_1 = \Xi_2$, then the origin is exponentially stable in Ξ_2.*

Definition 3 (Output admissible set). *Consider the system*

$$x(k+1) = f(x(k), u(k)),\ k \geq t,\ x(t) = \bar{x} \tag{3}$$

where k is the discrete time index, $x(k) \in R^n$, $u(k) \in R^m$, *and* $f(0,0) = 0$. *The state and control variables are required to fulfill the following constraints*

$$x \in X, u \in U \tag{4}$$

where X and U are compact subsets of R^n *and* R^m, *respectively, both containing the origin as an interior point. Consider the control law*

$$u = \kappa(x). \tag{5}$$

Then, the term output admissible set [11], referred to the closed-loop system (3), (5) denotes a positively invariant set $\bar{X} \subseteq X$ *which is a domain of attraction of the origin and such that* $\bar{x} \in \bar{X}$ *implies* $\kappa(x(k)) \in U$, $k \geq t$.

3 Nominal Model Predictive Control

Given the system (3) and the state and control constraints (4), we assume that $f(\cdot,\cdot)$ is a C^1 function with Lipschitz constant L_f, $\forall x \in X$ and $\forall u \in U$.

To introduce the MPC algorithm, first let $u_{t_1,t_2} := [u(t_1)\ u(t_1+1) \ldots u(t_2)]$, $t_2 \geq t_1$, then define the following finite-horizon optimization problem.

Definition 4 (FHOCP). *Consider a stabilizing auxiliary control law* $\kappa_f(\cdot)$ *and an associated output admissible set* X_f. *Then, given the positive integer N, the stage cost* $l(\cdot,\cdot)$ *and the terminal penalty* $V_f(\cdot)$, *the Finite Horizon Optimal Control Problem (FHOCP) consists in minimizing, with respect to* $u_{t,t+N-1}$, *the performance index*

$$J(\bar{x}, u_{t,t+N-1}, N) = \sum_{k=t}^{t+N-1} l(x(k), u(k)) + V_f(x(t+N)) \tag{6}$$

subject to

(i) the state dynamics (3) with $x(t) = \bar{x}$;
(ii) the constraints (4), $k \in [t, t+N-1]$;
(iii) the terminal state constraint $x(t+N) \in X_f$.

It is now possible to define a "prototype" *Nonlinear Model Predictive Control* ($NMPC$) algorithm: at every time instant t, define $\bar{x} = x(t)$ and find the optimal control sequence $u^o_{t,t+N-1}$ by solving the $FHOCP$. Then, according to the Receding Horizon approach, define

$$\kappa^{MPC}(\bar{x}) = u^o_{t,t}(\bar{x}) \tag{7}$$

where $u^o_{t,t}(\bar{x})$ is the first column of $u^o_{t,t+N-1}$, and apply the control law

$$u = \kappa^{MPC}(x) \tag{8}$$

In order to guarantee the stability of the origin of the closed-loop system (3), (8), many different choices of the stabilizing control law $\kappa_f(\cdot)$, of the terminal set X_f and of the terminal cost function V_f have been proposed in the literature, see [37], [40], [2], [6], [28], [32], [7], [16], [14], [20]. Irrespective of the specific algorithm applied, a general result can be stated under the following assumptions which will always be considered in the sequel.

Assumption 3.1. *$l(x,u)$ is Lipschitz with Lipschitz constant L_l and is such that $\alpha_l(|x|) \le l(x,u) \le \beta_l(|(x,u)|)$ where α_l and β_l are $\mathcal{K}$ functions.*

Assumption 3.2. *Let $\kappa_f(\cdot)$, $V_f(\cdot)$, X_f be such that*

1. *$X_f \subseteq X$, X_f closed, $0 \in X_f$*
2. *$\kappa_f(x) \in U$, $\forall x \in X_f$*
3. *$\kappa_f(x)$ is Lipschitz in X_f with Lipschitz constant $L_{\kappa f}$*
4. *$f(x, \kappa_f(x)) \in X_f$, $\forall x \in X_f$*
5. *$\alpha_{V_f}(|x|) \le V_f(x) \le \beta_{V_f}(|x|)$, α_{V_f} and β_{V_f} $\mathcal{K}$ functions*
6. *$V_f(f(x,\kappa_f(x))) - V_f(x) \le -l(x, \kappa_f(x))$, $\forall x \in X_f$*
7. *V_f is Lipschitz in X_f with Lipschitz constant L_{Vf}*

Theorem 1. *Let $X^{MPC}(N)$ be the set of the states such that a feasible solution for the $FHOCP$ exists. Given an auxiliary control law κ_f, a terminal set X_f, a terminal penalty V_f and a cost $l(\cdot,\cdot)$ satisfying Assumptions 3.1, 3.2, the origin is an asymptotically stable equilibrium point for the closed-loop system formed by (3) and (8) with output admissible set $X^{MPC}(N)$ and $V(\bar{x}, N) := J(\bar{x}, u^o_{t,t+N-1}, N)$ is an associated Lyapunov function. Moreover if $\alpha_l(|x|) = \alpha_l |x|^p$, $\beta_{V_f}(|x|) = \beta_{V_f} |x|^p$, $p > 0$, then the origin is an exponentially stable equilibrium point in $X^{MPC}(N)$.*

Proof of Theorem 1. First note that

$$V(x,N) := J(x, u^o_{t,t+N-1}, N) \ge l(x, \kappa^{MPC}(x)) \ge \alpha_l(|x|) \tag{9}$$

Moreover, letting $u^o_{t,t+N-1}$ be the solution of the $FHOCP$ with horizon N at time t, in view of Assumption 3.2

$$\tilde{u}_{t,t+N} = [u^o_{t,t+N-1}, \kappa_f(x(t+N))]$$

is an admissible control sequence for the $FHOCP$ with horizon $N+1$ with

$$J(x, \tilde{u}_{t,t+N}, N+1) = V(x,N) - V_f(x(t+N)) + V_f(x(t+N+1))$$
$$+ l(x(t+N), \kappa_f(x(t+N))) \le V(x,N)$$

so that

$$V(x, N+1) \le V(x,N), \quad \forall x \in X^{MPC}(N) \tag{10}$$

with $V(x,0) = V_f(x)$, $\forall x \in X_f$. Then

$$V(x,N+1) \leq V(x,N) \leq V_f(x) \leq \beta_{V_f}(|x|), \quad \forall x \in X_f \tag{11}$$

Finally

$$\begin{aligned} V(x,N) &= l(x,\kappa^{MPC}(x)) + J(f(x,\kappa^{MPC}(x)), u^o_{t+1,t+N-1}, N-1) \\ &\geq l(x,\kappa^{MPC}(x)) + V(f(x,\kappa^{MPC}(x)), N) \\ &\geq \alpha_l(|x|) + V(f(x,\kappa^{MPC}(x)), N), \quad \forall x \in X^{MPC}(N) \end{aligned} \tag{12}$$

Then, in view (9), (11) and (12) $V(x,N)$ is a Lyapunov function and in view of Lemma 1 the asymptotic stability in $X^{MPC}(N)$ and the exponential stability in X_f are proven. In order to prove exponential stability in $X^{MPC}(N)$, let B_ρ be the largest ball such that $B_\rho \in X_f$ and $\bar{V}$ be a constant such that $V(x,N) \leq \bar{V}$ for all $x \in X^{MPC}(N)$. Now define

$$\bar{\alpha}_2 = \max\left(\frac{\bar{V}}{\rho^p}, \beta_{V_f}\right),$$

then it is easy to see [27] that

$$V(x,N) \leq \bar{\alpha}_2 |x|^p, \quad \forall x \in X^{MPC}(N) \tag{13}$$

4 Robustness Problem and Uncertainty Description

Let the uncertain system be described by

$$x(k+1) = f(x(k),u(k)) + g(x(k),u(k),w(k)), \; k \geq t, \; x(t) = \bar{x} \tag{14}$$

or equivalently

$$x(k+1) = \tilde{f}(x(k),u(k),w(k)), \; k \geq t, \; x(t) = \bar{x} \tag{15}$$

In (14), $f(x,u)$ is the nominal part of the system, $w \in \mathcal{M}_\mathcal{W}$ for some compact subset $\mathcal{W} \subseteq R^p$ is the disturbance and $g(\cdot,\cdot,\cdot)$ is the uncertain term assumed to be Lipschitz with respect to all its arguments with Lipschitz constant L_g.

The perturbation term $g(\cdot,\cdot,\cdot)$ allows one to describe modeling errors, aging, or uncertainties and disturbances typical of any realistic problem. Usually, only partial information on $g(\cdot,\cdot,\cdot)$ is available, such as an upper bound on its absolute value $|g(\cdot,\cdot,\cdot)|$.

For the robustness analysis the concept of Input to State Stability (ISS) is a powerful tool.

Definition 5 (Input-to-state stability). *The system*

$$x(k+1) = f(x(k),w(k)), k \geq t, \; x(t) = \bar{x} \tag{16}$$

with $w \in \mathcal{M}_\mathcal{W}$ is said to be ISS in Ξ if there exists a $\mathcal{KL}$ function β, and a $\mathcal{K}$ function γ such that

$$|x(k)| \leq \beta(|\bar{x}|, k) + \gamma(\|w\|), \; \forall k \geq t, \forall \bar{x} \in \Xi$$

Definition 6 (ISS-Lyapunov function). *A function $V(\cdot)$ is called an ISS-Lyapunov function for system (16) if there exist a set Ξ, $\mathcal{K}$ functions α_1, α_2, α_3, and σ such that*

$$
\begin{aligned}
V(x) &\geq \alpha_1(|x|), \forall x \in \Xi \\
V(x) &\leq \alpha_2(|x|), \forall x \in \Xi \\
\Delta V(x,w) = V(f(x,w)) - V(x) &< -\alpha_3(|x|) + \sigma(|w|), \forall x \in \Xi, \forall w \in \mathcal{M}_{\mathcal{W}}
\end{aligned}
\tag{17}
$$

Note that if the condition on ΔV is fulfilled with $\sigma(\cdot) = 0$, then the origin is asymptotically stable for any considered disturbance w.

Lemma 2. *[21] Let Ξ be a positive invariant set for system (16) that contains the origin and let $V(\cdot)$ be a ISS-Lyapunov function for system (16), then the system (16) is ISS in Ξ.*

5 Inherent Robustness of Nominal *MPC*

In this section, the robustness properties of nominal MPC algorithms are reviewed under the fundamental assumption that the presence of uncertainties and disturbances do not cause any loss of feasibility. This holds true when the problem formulation does not include state and control constraints and when any terminal constraint used to guarantee nominal stability can be satisfied also in perturbed conditions.

5.1 Inverse Optimality

It is well known that the control law solving an unconstrained optimal Infinite Horizon (IH) problem guarantees robustness properties both in the continuous and in the discrete time cases, see [12], [43], [10], [1]. Hence, the same robustness characteristics can be proven for MPC regulators provided that they can be viewed as the solution of a suitable IH problem. For continuous time systems, this has been proven in [34], while in the discrete time case, from the optimality principle we have

$$V(x,N) = \bar{l}(x(k), \kappa^{MPC}(x(k))) + V(f(x, \kappa^{MPC}(x)), N)$$

with

$$
\begin{aligned}
\bar{l}(x(k), \kappa^{MPC}(x(k))) : &= l(x(k), \kappa^{MPC}(x(k))) - V(f(x, \kappa^{MPC}(x)), N) \\
&+ V(f(x, \kappa^{MPC}(x)), N-1)
\end{aligned}
$$

Then $\kappa^{MPC}(x(k))$ is the solution of the Hamilton-Jacobi-Bellman equation for the IH optimal control problem with stage cost $\bar{l}(x,u)$. In view of Assumption 3.2 and (10) it follows that

$$\bar{l}(x(k), \kappa^{MPC}(x(k))) > l(x(k), \kappa^{MPC}(x(k)))$$

so that the stage cost is well defined and robustness of IH is guaranteed. Specifically, under suitable regularity assumptions on V, in [5] it has been shown that MPC regulators provide robustness with respect to gain perturbations due to actuator nonlinearities and additive perturbations describing unmodeled dynamics. Further results on gain perturbations can be achieved as shown in [1].

5.2 Robustness Provided by the *ISS* Property

The robustness analysis provided by ISS, see also [21], can be summarized by the following result.

Theorem 2. *Under Assumptions 3.1 and 3.2, if $V(x,N)$ is Lipschitz with Lipschitz constant L_V, the closed-loop system (14), (8) is ISS in $X^{MPC}(N)$ for any perturbation $g(x,u,w)$ such that $|g(x,u,0)| < \frac{\rho}{L_V}\alpha_l(|x|)$ where $0 < \rho < 1$ is an arbitrary real number.*

Proof. Note that (9) and (13) still hold. Moreover

$$\begin{aligned} &V(\tilde{f}(x,\kappa^{MPC}(x),w),N) - V(f(x,\kappa^{MPC}(x),N) \\ &\leq L_V \left|g(x,\kappa^{MPC}(x),w)\right| \leq L_V \left|g(x,\kappa^{MPC}(x),0)\right| + L_V L_g |w| \\ &\leq \rho\alpha_l(|x|) + L_V L_g |w| \end{aligned}$$

Hence

$$\begin{aligned} &V(\tilde{f}(x,\kappa^{MPC}(x),w),N) \\ &\leq V(x,N) - (1-\rho)\alpha_l(|x|) + L_V L_g |w| \end{aligned}$$

Remark 1. If $w = 0$ the result is equivalent to the one on robust stability reported in [6]. On the contrary if $w \neq 0$ then ISS guarantees that the system evolves towards a compact set which size depends on the bound on w. A way to estimate this size is given in [31]. Further results on the robustness with bounded and exponentially decaying disturbances are reported in [42], [25].

Remark 2. All the above results assume some regularity of the MPC control law and of the value function, see also [18], [19], [26]. It is well known that the MPC control law could be even discontinuous [38]. In [13], some examples of the loss of robustness have been presented. For a specific discussion on robustness of discontinuous MPC see [9].

6 Robust *MPC* Design with Restricted Constraints

The development of MPC algorithms robust with respect to persistent disturbances has received a great deal of attention both for linear systems, see e.g. [4], and in the nonlinear case. An approach to overcome the feasibility and stability problems consists in minimizing a nominal performance index while imposing the constraints fulfillment for any admissible disturbance. This implies the use of tighter and tighter state, control and terminal constraints, so leading to very conservative solutions or even to unfeasible problems. Algorithms with these characteristics have been described in [39] for continuous-time and in [24] for discrete time systems. The technique presented in [24] is now briefly summarized. To this aim, the following assumption must be introduced to allow for the analysis of the (worst-case) effects of the disturbance.

Assumption 6.1. *The uncertain term in (14) is bounded by* γ*, that is* $|g(\cdot,\cdot,\cdot)| \leq \gamma$ *for any* x *and* u *satisfying (4) and* $w \in \mathcal{M}_{\mathcal{W}}$.

In order to guarantee that at any future time instant in the prediction horizon the disturbance does not cause the state constraints violation, first introduce the following definition.

Definition 7 (Pontryagin difference). *Let* $A, B \subset R^n$*, be two sets, then the Pontryagin difference set is defined as* $A \sim B = \{x \in R^n | x + y \in A, \forall y \in B\}$.

Consider now the following sets $X_j = X \sim B^j_\gamma$ where B^j_γ is defined as

$$B^j_\gamma = \left\{ z \in R^n : |z| \leq \frac{L_f^j - 1}{L_f - 1}\gamma \right\}.$$

Definition 8 (NRFHOCP). *Consider a stabilizing auxiliary control law* $\kappa_f(\cdot)$ *and an associated output admissible set* X_f*. Then, given the positive integer* N*, the stage cost* $l(\cdot,\cdot)$ *and the terminal penalty* $V_f(\cdot)$*, the Nominal Robust Finite Horizon Optimal Control Problem (NRFHOCP) consists in minimizing, with respect to* $u_{t,t+N-1}$,

$$J(\bar{x}, u_{t,t+N-1}, N) = \sum_{k=t}^{t+N-1} l(x(k), u(k)) + V_f(x(t+N))$$

subject to:

(i) the state dynamics (3) with $x(t) = \bar{x}$;
(ii) the constraints $u(k) \in U$ *and* $x(k) \in X_{k-t+1}$*,* $k \in [t, t+N-1]$*, where* X_{k-t+1} *are given in Definition 7;*
(iii) the terminal state constraint $x(t+N) \in X_f$.

From the solution of the $NRFHOCP$, the Receding Horizon control law

$$u = \kappa^{MPC}(x) \tag{18}$$

is again obtained as in (7) and (8). Concerning the stability properties of the closed-loop system, the following hypothesis substitutes Assumption 3.2.

Assumption 6.2. *Let* $\kappa_f(\cdot)$, $V_f(\cdot)$, X_f *such that*

1. $\Phi_f := \{x \in R^n : V_f(x) \leq \alpha\} \subseteq X$, Φ_f *closed,* $0 \in \Phi_f$, α *positive constant*
2. $\kappa_f(x) \in U$, $\forall x \in \Phi_f$

3. $f(x, \kappa_f(x)) \in \Phi_f, \ \forall x \in \Phi_f$
4. $V_f(f(x, \kappa_f(x))) - V_f(x) \leq -l(x, \kappa_f(x)), \ \forall x \in \Phi_f$
5. $\alpha_{V_f}(|x|) \leq V_f(x) \leq \beta_{V_f}(|x|)$, α_{V_f}, β_{V_f} *are* $\mathcal{K}$ *functions*
6. $V_f(\cdot)$ *is Lipschitz in* Φ_f *with a Lipschitz constant* L_{Vf}
7. $X_f := \{x \in R^n : V_f(x) \leq \alpha_v\}$ *is such that for all* $x \in \Phi_f$, $f(x, \kappa_f(x)) \in X_f$, α_v *positive constant*

Then, the final theorem can be stated.

Theorem 3. *[24]Let* $X^{MPC}(N)$ *be the set of states of the system where there exists a solution of the* $NRFHOCP$*. Then the closed loop system (14), (18) is* ISS *in* $X^{MPC}(N)$ *if Assumption 6.1 is satisfied with*

$$\gamma \leq \frac{\alpha - \alpha_v}{L_{Vf} L_f^{N-1}}$$

The above robust synthesis method ensures the feasibility of the solution through a wise choice of the constrains (ii) and (iii) in the $NRFHOPC$ formulation. However, the solution can be extremely conservative or may not even exist, so that less stringent approaches are advisable.

7 Robust *MPC* Design with Min-Max Approaches

The design of MPC algorithms with robust stability has been first placed in an H_∞ setting in [44] for linear unconstrained systems. Since then, many papers have considered the linear constrained and unconstrained case, see for example [41]. For nonlinear continuous time systems, H_∞-MPC control algorithms have been proposed in [3], [30], [8], while discrete-time systems have been studied in [29], [15], [17], [33], [36]. In [29] the basic approach consists in solving a min-max problem where an H_∞-type cost function is maximized with respect to the admissible disturbance sequence, i.e. the "nature", and minimized with respect to future controls over the prediction horizon. The optimization can be solved either in open-loop or in closed-loop. The merits and drawbacks of these solutions are discussed in the sequel.

7.1 Open-Loop Min-Max *MPC*

Assume again that the perturbed system is given by

$$x(k+1) = \tilde{f}(x(k), u(k), w(k)), \ k \geq t, \ x(t) = \bar{x} \tag{19}$$

where now $\tilde{f}(\cdot, \cdot, \cdot)$ is a known Lipschitz function with Lipschitz constant $L_{\tilde{f}}$ and $\tilde{f}(0, 0, 0) = 0$. The state and control variables must satisfy the constraints (4), while the disturbance w is assumed to fulfill the following hypothesis.

Assumption 7.1. *The disturbance* w *is contained in a compact set* $\mathcal{W}$ *and there exists a* $\mathcal{K}$ *function* $\gamma(\cdot)$ *such that* $|w| \leq \gamma(|(x, u)|)$.

Letting $w_{t_1,t_2} := [w(t_1)\ w(t_1+1)\ \ldots\ w(t_2)]$, $t_2 \geq t_1$, the optimal min-max problem can now be stated.

Definition 9 (FHODG). *Consider a stabilizing auxiliary control law $\kappa_f(\cdot)$ and an associated output admissible set X_f. Then, given the positive integer N, the stage cost $l(\cdot,\cdot) - l_w(\cdot)$ and the terminal penalty $V_f(\cdot)$, the Finite Horizon Open-loop Differential Game (FHODG) problem consists in minimizing, with respect to $u_{t,t+N-1}$ and maximizing with respect to $w_{t,t+N-1}$ the cost function*

$$J(\bar{x}, u_{t,t+N-1}, w_{t,t+N-1}) = \sum_{k=t}^{t+N-1} \{l(x(k), u(k)) - l_w(w(k))\} + V_f(x(t+N))$$

subject to:

(i) the state dynamics (19) with $x(t) = \bar{x}$;
(ii) the constraints (4), $k \in [t, t+N-1]$;
(iii) the terminal state constraint $x(t+N) \in X_f$.

Once the $FHODG$ is solved and the optimal control sequence $u^o_{t,t+N-1}$ is available, according to the RH principle the feedback control law is again given by (7) and (8). To achieve robustness the idea could be to use a terminal set and a terminal penalty satisfying the following "robust" version of the sufficient conditions reported in Assumption 3.2.

Assumption 7.2. *Let $\kappa_f(\cdot)$, $V_f(\cdot)$, X_f such that*

1. *$X_f \subseteq X$, X_f closed, $0 \in X_f$*
2. *$\kappa_f(x) \in U$, $\forall x \in X_f$*
3. *$\tilde{f}(x, \kappa_f(x), w) \in X_f$, $\forall x \in X_f, \forall w \in \mathcal{W}$*
4. *$\alpha_{V_f}(|x|) \leq V_f(x) \leq \beta_{V_f}(|x|)$, α_{V_f} and β_{V_f} $\mathcal{K}$ functions*
5. *$V_f(\tilde{f}(x, \kappa_f(x), w)) - V_f(x) \leq -l(x,u) + l_w(w)$, $\forall x \in X_f, \forall w \in \mathcal{W}$*
6. *V_f is Lipschitz in X_f with Lipschitz constant L_{Vf}*

Along this line, one could argue again that the value function $V(x) = J(\bar{x}, u^o_{t,t+N-1}, w^o_{t,t+N-1})$ is a candidate to prove the stability of the closed-loop system. However, the following fundamental feasibility problem arises. Suppose that at time t an optimal (hence admissible) control sequence $u^o_{t,t+N-1}$ for the $FHODG$ is known. In other words, irrespective of the specific realization of w, this sequence steers the state x to X_f in N steps or less; hence, the abbreviated control sequence $u^o_{t+1,t+N-1}$ steers the state $x(t+1)$ to X_f at most in $N-1$ steps. Now, the major difficulty is to obtain a feasible control sequence $\tilde{u}_{t+1,t+N} := \left[u^o_{t+1,t+N-1}, v\right]$ required to complete the stability proof (see the proof of Theorem 1). In fact, Assumption 7.2 does not ensure the existence of a signal v with this property since the auxiliary control law $\kappa_f(x(t+N))$ can only provide a control value depending on $x(t+N)$, which in turn is a function of the particular realization of the disturbance w.

One way to avoid this impasse is given in [3] where the MPC approach is applied to an already robust stable system, so that Assumption 7.2 is satisfied with $\kappa_f(\cdot) \equiv 0$. In this case a feasible control sequence is

$$\tilde{u}_{t+1,t+N} := \left[u^o_{t+1,t+N-1}, 0\right]$$

In order to obtain a system with a-priori robustness properties with respect to the considered class of disturbances, in [3] it has been suggested to pre-compensate the system under control by means of an inner feedback loop designed for example with the H_∞ approach.

7.2 Closed-Loop Min-Max *MPC*

The limitations of the open-loop min-max approach can be overcome by explicitly accounting for the intrinsic feedback nature of any RH implementation of MPC, see e.g. [41] for the linear case and [29] for nonlinear systems. In this approach, at any time instant the controller chooses the input u as a function of the current state x, so as to guarantee that the effect of the disturbance w is compensated for any choice made by the “nature”. Hence, instead of optimizing with respect to a control sequence, at any time t the controller has to choose a sequence of control laws $\kappa_{t,t+N-1} = [\kappa_0(x(t)) \; \kappa_1(x(t+1) \ldots \kappa_{N-1}(x(t+N-1)]$. Then, the following optimal min-max problem can be stated.

Definition 10 (FHCDG). *Consider a stabilizing auxiliary control law $\kappa_f(\cdot)$ and an associated output admissible set X_f . Then, given the positive integer N, the stage cost $l(\cdot,\cdot) - l_w(\cdot)$ and the terminal penalty $V_f(\cdot)$, the Finite Horizon Closed-loop Differential Game (FHCDG) problem consists in minimizing, with respect to $\kappa_{t,t+N-1}$ and maximizing with respect to $w_{t,t+N-1}$ the cost function*

$$J(\bar{x}, \kappa_{t,t+N-1}, w_{t,t+N-1}, N) = \sum_{k=t}^{t+N-1} \{l(x(k), u(k)) - l_w(w(k))\} + V_f(x(t+N))$$

subject to:

(i) the state dynamics (19) with $x(t) = \bar{x}$;
(ii) the constraints (4), $k \in [t, t+N-1]$;
(iii) the terminal state constraint $x(t+N) \in X_f$.

Finally, letting $\kappa^o_{t,t+N-1}$, $w^o_{t,t+N-1}$ the solution of the $FHCDG$ the feedback control law $u = \kappa^{MPC}(x)$ is obtained by setting

$$\kappa^{MPC}(x) = \kappa^o_0(x) \tag{20}$$

where $\kappa^o_0(x)$ is the first element of $\kappa^o_{t,t+N-1}$.

In order to derive the main stability and performance properties associated to the solution of $FHCDG$, the following assumption is introduced.

Assumption 7.3. $l_w(\cdot)$ *is such that* $\alpha_w(|w|) \le l_w(w) \le \beta_w(|w|)$ *where* α_w *and* β_w *are* $\mathcal{K}$ *functions.*

Then, the following result holds.

Theorem 4. *Let $X^{MPC}(N)$ be the set of states of the system where there exists a solution of the $FHCDG$ and $\kappa_{t,t+N-1}$ a vector of Lipschitz continuous control policies. Under Assumptions 7.1-7.3 the closed loop system Σ^{MPC} given by (19)-(20) is ISS with robust output admissible set $X^{MPC}(N)$, moreover if $\gamma(\cdot)$ is such that $\beta_w\left(\gamma\left(\left|x,\kappa^{MPC}(x)\right|\right)\right) - \alpha_l(|x|) < -\delta(|x|)$, where δ is a $\mathcal{K}$ function, the origin of the closed loop system Σ^{MPC} given by (19)-(20) is robustly asymptotically stable.*

Proof. First note that in view of Assumption 7.2, given $\tilde{w}_{t,t+N-1} = 0$, for every admissible $\kappa_{t,t+N-1}$

$$\begin{aligned} &J(\bar{x}, \kappa_{t,t+N-1}, 0, N) \\ = &\sum_{k=t}^{t+N-1} \{l(x(k), u(k))\} + V_f(x(t+N)) > 0, \forall x \in X^{MPC}(N)/\{0\} \end{aligned}$$

so that

$$\begin{aligned} V(x,N) \ : &= J(\bar{x}, \kappa^o_{t,t+N-1}, w^o_{t,t+N-1}, N) \geq \min_{\kappa_{t,t+N-1}} J(\bar{x}, \kappa_{t,t+N-1}, 0, N) \\ &> l(x, \kappa^{MPC}(x)) > \alpha_l(|x|), \quad \forall x \in X^{MPC}(N) \end{aligned} \tag{21}$$

In view of the Lipschitz assumption on $\kappa_{t,t+N-1}$ and Assumption 7.1, one can show that there exists a $\mathcal{K}$ function $\alpha_2(|x|)$ such that (17) is fulfilled for any $x \in X^{MPC}(N)$. Suppose now that $\kappa^o_{t,t+N-1}$ is the solution of the $FHCDG$ with horizon N and consider the following policy vector for the $FHCDG$ with horizon $N+1$

$$\tilde{\kappa}_{t,t+N} = \begin{cases} \kappa^o_{t,t+N-1} & t \leq k \leq t+N-1 \\ \kappa_f(x(t+N)) & k = t+N \end{cases}$$

Correspondingly

$$\begin{aligned} &J(\bar{x}, \tilde{\kappa}_{t,t+N}, w_{t,t+N}, N+1) \\ = &V_f(x(t+N+1)) - V_f(x(t+N)) \\ &+ l(x(t+N), u(t+N)) - l_w(w(t+N)) \\ &+ \sum_{k=t}^{t+N-1} \{l(x(k), u(k)) - l_w(w(k))\} + V_f(x(t+N)) \end{aligned}$$

so that in view of Assumption 7.2

$$\begin{aligned} &J(\bar{x}, \tilde{\kappa}_{t,t+N}, w_{t,t+N}, N+1) \\ \leq &\sum_{k=t}^{t+N-1} \{l(x(k), u(k)) - l_w(w(k))\} + V_f(x(t+N)) \end{aligned}$$

which implies

$$\begin{aligned}V(x,N+1) &\le \max_{w\in\mathcal{M}_{\mathcal{W}}} J(\bar{x},\tilde{\kappa}_{t,t+N-1},w_{t,t+N-1},N+1)\\ &\le \max_{w\in\mathcal{M}_{\mathcal{W}}} \sum_{k=t}^{t+N-1} \{l(x(k),u(k))-l_w(w(k))\}+V_f(x(t+N))\\ &= V(x,N) \end{aligned}\tag{22}$$

which holds $\forall x \in X^{MPC}(N), \forall w \in \mathcal{M}_{\mathcal{W}}$. Moreover

$$\begin{aligned}V(x,N) &= V(\tilde{f}(x,\kappa^{MPC}(x),w),N-1)\\ &\quad +l(x,\kappa^{MPC}(x))-l_w(w)\\ &\ge V(\tilde{f}(x,\kappa^{MPC}(x),w),N)+l(x,\kappa^{MPC}(x))-l_w(w)\end{aligned}$$

$\forall x \in X^{MPC}(N), \forall w \in \mathcal{M}_{\mathcal{W}}$ and

$$V(\tilde{f}(x,\kappa^{MPC}(x),w),N)-V(x,N) \le -l(x,\kappa^{MPC}(x))+l_w(w)$$

and the ISS is proven. Note also that in view of (22)

$$V(x,N) \le V(x,N-1) \le V(x,0) = V_f(x) \le \beta_{V_f}(|x|), \quad \forall x \in X_f \tag{23}$$

so that if $X_f = X^{MPC}(N)$ the Lipschitz assumption on $\kappa_{t,t+N-1}$ can be relaxed.

Finally, in view of Assumption 7.1 with $\gamma(\cdot)$ such that $\beta_w\left(\gamma\left(\left|x,\kappa^{MPC}(x)\right|\right)\right)-\alpha_l(|x|) < -\delta(|x|)$

$$\begin{aligned}V(\tilde{f}(x,\kappa^{MPC}(x),w),N)-V(x,N) &\le -\alpha_l(|x|)+\beta_w\left(\gamma\left(\left|x,\kappa^{MPC}(x)\right|\right)\right)\\ &\le -\delta(|x|), \forall x \in X^{MPC}(N), \forall w \in \mathcal{M}_{\mathcal{W}}\end{aligned}$$

and robust asymptotic stability is derived.

Remark 3. The major drawback of the closed-loop min-max approach is due to the need to perform optimization over an infinite dimensional space. However two comments are in order. First, one can resort to a finite dimensional parametrization of the control policies, see e.g. [35], [29], [8]. In this case, it is necessary that also the auxiliary control law shares the same structural properties. Second, similar results can be achieved using different prediction (N_p) and control (N_c) horizons, with $N_c \ll N_p$, see [29]. In this case, optimization has to be performed only with respect to N_c policies, while from the end of the control horizon onwards the auxiliary control law can be applied.

Remark 4. By means of the same kind of reasoning followed in the proof of Theorem 1 to derive an upper bound of V in $X^{MPC}(N)$, one can relax the hypothesis on Lipschitz continuity of $\kappa_{t,t+N-1}$ [27]. However, since in practice this sequence of control laws must be parametrized a priori, the continuity assumption in this case can be explicitly verified.

Remark 5. The computation of the auxiliary control law, of the terminal penalty and of the terminal inequality constraint satisfying Assumption 3.2, is not trivial at all. In this regard, a solution has been proposed for affine system in [29], where it is shown how to compute a non linear auxiliary control law based on the solution of a suitable H_∞ problem for the linearized system under control.

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