Towards the Design of Parametric Model Predictive Controllers for Non-linear Constrained Systems

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Summary. The benefits of parametric programming for the design of optimal controllers for constrained systems are widely acknowledged, especially for the case of linear systems. In this work we attempt to exploit these benefits and further extend the theoretical contributions to multi-parametric Model Predictive Control (mp-MPC) for non-linear systems with state and input constraints. The aim is to provide an insight and understanding of multi-parametric control and its benefits for non-linear systems and outline key issues for ongoing research work.

1 Introduction

The explicit, multi-parametric MPC (mp-MPC) has been extensively investigated for linear systems. Major results have been presented for the discrete-time case ([2]) and recently for the linear, continuous-time systems case ([20]). The key advantage of mp-MPC is that the on-line optimization, typically involved in MPC, can be performed off-line to produce an explicit mapping of the optimal control actions and the objective function in the space of the current states. The on-line implementation of the controller is then reduced to a simple function evaluation ([17]).

Although mp-MPC has received much attention for the linear systems case, there is relatively little progress for the non-linear systems case. Most of the research on MPC has focused in on-line implicit MPC methods that usually rely, for the case of continuous-time systems, on numerical dynamic optimization techniques ([5, 6, 13]) or for the case of discrete-time systems, on on-line Non-linear Optimization ([15, 18]). A first attempt towards the design of approximate, linear mp-MPC controllers for the non-linear MPC problem, is presented in [11] and [12]. The first work proposes a method for a local mp-QP approximation to the continuous-time, mp-NLP control problem. The second work focuses on the MPC problem for non-linear systems with linear state and input constraints, and a quadratic cost.

This work aims to provide an insight and understanding of multi-parametric control and outline the benefits of non-linear mp-MPC. Two approaches are

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presented here. The first approach presents a method for obtaining a Piecewise Affine (PWA) approximation to the solution of the non-linear MPC for discrete-time systems, by exploiting the fact that the non-linear MPC problem for non-linear systems with non-linear state and input constraints, and non-linear objective, is a non-linear optimization problem where the input is the optimisation variable and the initial state is the optimisation parameter. This approach is based on recent developments on parametric programming techniques ([7]).

The second approach deals with the explicit solution of the non-linear MPC, for certain classes of non-linear, continuous-time models for which there exists an analytical solution to the dynamic systems arising from the first order optimality conditions. The optimisation is then solved off-line, based on a recently developed multi-parametric, dynamic optimisation algorithm ([20]) - the control law is then derived as an explicit, non-linear function of the states. In both approaches the implementation of the controller is simply reduced to a sequence of function evaluations, instead of solving the on-line, non-linear optimal control problem, which is usually the typical procedure of non-linear MPC. The two procedures are then applied, in the end of this paper, on the classical, constrained Brachistochrone problem to illustrate the key features of the new developments.

2 Piecewise Affine Approximation to the Discrete - Time Non-linear MPC

The main multi-parametric programming problem that is frequently encountered in various engineering applications, including non-linear MPC, is the following

$$z(\theta) = \min_{x} f(x) \tag{1a}$$

$$\text{s.t. } g(x) \leq b + F\theta \tag{1b}$$

$$x \in \mathcal{X} \tag{1c}$$

$$\theta \in \Theta \tag{1d}$$

where x is a vector of continuous variables, f a scalar, continuously differentiable function of x, g a vector of continuously differentiable functions of x, b a constant vector, F are constant matrices of appropriate dimensions, θ a vector of parameters and $\mathcal{X}$ and Θ are compact subsets of the x and θ-space respectively. A representative example of this problem is the discrete-time constrained linear quadratic regulator problem ([2]), where x is the sequence of control inputs over a finite time horizon, $f(x)$ is a strictly convex quadratic function of x, $g(x)$ is a linear function of x, θ is the initial state and $\mathcal{X}$ and Θ are convex, polyhedral sets. Although, solving (1) has been proved to be a difficult task, an algorithm was presented recently in [7, 8] which can obtain a linear, PWA approximation to $z(\theta)$ with a prescribed accuracy. The value function $z(\theta)$ as well as the optimization variable $x(\theta)$ are linear, PWA function of θ. Given a value of θ then $z(\theta)$ and $x(\theta)$ can be obtained by simple function evaluations.

The mathematical framework for the discrete-time nonlinear MPC can be shortly summarised as the following constrained non-linear programming (NLP) problem ([15, 18])

$$z^o(x_t) = \min_U J(U, x_t) \tag{2a}$$

$$\text{s.t } h(U, x_t) \le b \tag{2b}$$

$$U_L \le U \le U_U \ , \ x_L \le x \le x_U \tag{2c}$$

where x_t is the state at the current time instant, $U = \{u_t, u_{t+1}, \ldots, u_{t+N-1}\}$ is the sequence of control inputs over the prediction horizon N, $J(U, x_t)$ is a scalar objective function, $h(U, x_t)$ is a vector of non-linear functions, U_L, U_U are lower and upper bounds for U and x_L and x_U are lower and upper bounds for x. The functions $J(U, x_t)$ and $h(U, x_t)$ are generally non-linear, although the analysis that follows can be applied for the linear case as well, and may include any terminal cost function and terminal constraints respectively to ensure stability ([15]).

Transforming the NLP (2) to (1) can been done in two steps. First, if $J(U, x_t)$ is only a function of U then simply replace (2a) by simply $J(U)$ and the objective function of (2) is the same with (1). Otherwise, introduce a new scalar $\epsilon \in \mathbb{R}$ and transform (2) into the following NLP

$$\bar{z}(x_t) = \min_U \epsilon \tag{3a}$$

$$\text{s.t. } J(U, x_t) \le \epsilon \ , \ h(U, x_t) \le b \tag{3b}$$

$$U_L \le U \le U_U \ , \ x_L \le x \le x_U \tag{3c}$$

or simply to

$$\bar{z}(x_t) = \min_U \epsilon \tag{4a}$$

$$\text{s.t. } \bar{h}(U, x_t) \le \bar{b} \ , \ U_L \le U \le U_U \ , \ x_L \le x \le x_U \tag{4b}$$

where $\bar{h}(U, x_t) = [J(U, x_t) \ h^T(U, x_t)]^T$ and $\bar{b} = [\epsilon \ b^T]^T$.

A simple but conservative way to solve the above problem is by linearising the inequalities in (4) and solving off-line the linearized problem. More specifically, choose an initial x_t^* and solve (4) to acquire U^*. Then linearize the inequalities in (4) over x_t^*, U^* to obtain the following approximating, mp-LP problem over x_t and U

$$\breve{z}(x_t) = \min_U \epsilon \tag{5}$$

$$\bar{h}(U^*, x_t^*) + \frac{\partial \bar{h}(U^*, x_t^*)}{\partial U}(U - U^*) \le \bar{b} - \frac{\partial \bar{h}(U^*, x_t^*)}{\partial x_t}(x_t - x_t^*) \tag{6}$$

$$U_L \le U \le U_U \ , \ x_L \le x \le x_U \tag{7}$$

which now of form (1), where x is U and θ is x_t. The solution to the mp-LP (5) is a linear, PWA function of x_t, $\breve{z}(x_t)$ ([8]). The control sequence $U(x_t)$ is also a linear PWA function of x_t and hence the first control input $u_t(x_t)$ of the control sequence, is a linear PWA function of x_t. The solutions $\breve{z}(x_t)$ and $u_t(x_t)$ are only valid in a critical region $\mathcal{CR}$ of x_t which is defined as the feasible region of x_t associated with an optimal basis ([7, 8]). In the next step choose x_t outside the region of $\mathcal{CR}$ and repeat the procedure until the space of interest is covered.

A different procedure for transforming the NLP (4) into (1) is obtained if one considers that a nonlinear function $\bar{h}_i(U, x_t)$ consists of the addition, subtraction, multiplication and division of five simple non-linear functions of $U_i, x_{t,i}$ ([7, 9, 16, 21]): a) linear $f_L(U_i, x_{t,i})$, b) bilinear $f_B(U_i, x_{t,i})$, c) fractional $f_F(U_i, x_{t,i})$, d) exponential $f_{exp}(U_i, x_{t,i})$ and e) univariate concave $f_{uc}(U_i, x_{t,i})$ functions of $U_i, x_{t,i}$. If $f_L(U_i, x_{t,i})$, $f_B(U_i, x_{t,i})$, $f_F(U_i, x_{t,i})$, $f_{exp}(U_i, x_{t,i})$ and $f_{uc}(U_i, x_{t,i})$ are simply functions of U_i then they are simply retained without further transforming them. If, however, they are functions of both $U_i, x_{t,i}$ then a new variable is assigned for each of the non-linear functions and a convex approximating function can be obtained which is linear with respect to $x_{t,i}$. For example consider the non-linear inequality

$$\sin(U_i) + \frac{1}{U_i x_{t,j} + 1} \leq 0 \tag{8}$$

The term $\sin U_i$ is preserved without further manipulation as it is a non-linear function of U_i. Set $w = U_i x_{t,j} + 1$. This equality contains a bilinear term of $U_i x_{t,j}$. A convex approximation can then be obtain for this equality by employing the McCormick ([7, 9, 16]) over- and underestimators for bilinear functions

$$-w + x_{t,j}^L U_i \leq U_i^L x_{t,j}^L - U_i^L x_{t,j} \ , \ -w + x_{t,j}^U U_i \leq U_i^U x_{t,j}^U - U_i^U x_{t,j} \tag{9a}$$

$$w - x_{t,j}^U U_i \leq -U_i^L x_{t,j}^U + U_i^L x_{t,j} \ , \ w - x_{t,j}^L U_i \leq -U_i^U x_{t,j}^L + U_i^U x_{t,j} \tag{9b}$$

Moreover, (8) can be re-written as $\sin U_i + 1/w \leq 0$. It can be easily noticed that the above inequality and (9) have the same form with the inequalities in (1). Convex approximations to non-linear functions have been extensively investigated in [7, 9, 16, 21]. Since it is difficult to fully present the theory of convex approximations in this paper due to lack of space, the interested reader can look in the relevant literature and the references within, cited here in [7, 9, 16, 21].

Following the above procedure, one can transform the NLP (4) to the mp-NLP problem (1) as following

$$\hat{z}(x_t) = \min_{U,W} \epsilon \tag{10a}$$

$$\text{s.t. } \hat{h}(U, W) \leq \hat{b} + \hat{F} x_t \tag{10b}$$

where W is the vector of all new variables w which were introduced to replace the non-linear terms $f_L(U_i, x_{t,i})$, $f_B(U_i, x_{t,i})$, $f_F(U_i, x_{t,i})$, $f_{exp}(U_i, x_{t,i})$ and $f_{uc}(U_i, x_{t,i})$. The algorithm in [7, 8] can then be used to solve the above problem and obtain a linear, PWA approximation to the non-linear MPC problem for u_t. The control input u_t as well as the value function $\hat{z}(x_t)$ are both PWA function of x_t hence a feedback control policy is obtained.

The main disadvantage of the above method is that both problems (5) and (10) only provide an approximation for the optimal solution of (2). This could result to violation of the constraints of (2), although the constraints in both (5) and (10) are satisfied, thus resulting into state and input constraints violation for the system. However, as far as the authors are aware of, there is currently no alternative multi-parametric MPC method which can guarantee constraint satisfaction for non-linear, discrete-time systems, since most methods rely on the approximation of the initial non-linear programming problem (2). An alternative method, for obtaining the optimal solution and guarantee constraint satisfaction is to address the problem in continuous-time and not in discrete-time. This will be shown in the next section.

3 Multi-parametric Non-linear Optimal Control Law for Continuous - Time Dynamic Systems

It is a common practise to deal with the problem of non-linear MPC in discrete time by transforming the continuous-time optimal control problem involved into a discrete-time one. The interest of the relevant research has long being focused on solving the discrete-time non-linear MPC problem. However, the continuous-time case remain of great importance since in practise most of the systems of interest are continuous-time. In this section a novel approach is presented that derives off-line the optimal control law in a continuous-time optimal control problem with state and input constraints. More specifically consider the following continuous-time, optimal control problem

$$\hat{\phi} = \min_{x(t),u(t)} \phi(x_{t_f,t_f}) \tag{11a}$$

$$\text{s.t. } \dot{x} = f(x(t), u(t), t) \tag{11b}$$

$$\psi^g(x_{t_f}) \leq 0 \tag{11c}$$

$$g(x(t), u(t)) \leq 0 \tag{11d}$$

$$x(t_0) = x_0 \tag{11e}$$

$$t_0 \leq t \leq t_f \tag{11f}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ are the systems states, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ are the control variables, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$ are the path constraints and $\psi^g : \mathbb{R}^n \to \mathbb{R}^{Q_g}$ is the terminal constraint. The objective function $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous, differentiable, non-linear function of $x(t_f)$ at the final time t_f.

The objective is to obtain the solution of problem (11) i.e. the optimal value of the performance index $\hat{\phi}$ and the optimal profiles of the control inputs $u(t)$, as explicit function of the initial states x_0. Hence, by treating x_0 as a parameter, the optimal control problem (11) is recast as a multi-parametric Dynamic Optimization (mp-DO) problem where $\hat{\phi}$ is the value function, $u(t)$ the optimal control profiles and x_0 is the parameter of the problem. Problem (11) has been thoroughly studied for the case of the continuous-time, linear quadratic optimal control problem ([20]), however this is the first time this problem is treated for non-linear systems. Our purpose here is to extend the results of [20] for the continuous-time, non-linear optimal control problem described in (11).

Let define the order of a path constraint before we proceed

Definition 1. *The constraint $g_i(x, u)$ is said to be of order $\hat{l} \geq 1$ with respect to the dynamics, if*

$$\frac{\partial g_i(x,u)^j}{\partial u_k} = 0, \; j = 1, 2, \ldots, \hat{l} - 1, \; k = 1, \ldots, m$$

$$\frac{\partial g_i(x,u)^{\hat{l}}}{\partial u_k} \neq 0, \; \textit{for at least one } k, \; k = 1, \ldots, m$$

where the index j denotes time derivatives. The constraint $g_i(x, u)$ is said to be of zero-th order if

$$\frac{\partial g_i(x,u)}{\partial u_k} \neq 0, \; \textit{for at least one } k, \; k = 1, \ldots, m$$

The Karush-Kuhn-Tucker conditions for the optimal control problem (11) derived from the Euler-Lagrange equations and for $\hat{l} \geq 1$ are given as ([1, 4, 14])

ORDINARY DIFFERENTIAL EQUATION (ODE)

$$\dot{x} = f(x(t), u(t), t), \; t_0 \leq t \leq t_f \tag{12}$$

BOUNDARY CONDITIONS FOR THE ADJOINTS

$$x(t_0) = x_0 \tag{13}$$

$$\lambda(t_f) = \left(\frac{\partial \phi(x_{t_f}, t_f)}{\partial x(t_f)}\right)^T + \left(\frac{\partial \psi^g(x(t_f))}{\partial x(t_f)}\right)^T \cdot \nu \tag{14}$$

COMPLEMENTARITY CONDITIONS

$$0 = \nu_j \cdot \psi_j^g(x(t_f)) \tag{15}$$

$$\nu_j \geq 0, \; j = 1, \ldots, Q_g \tag{16}$$

ADJOINT DIFFERENTIAL SYSTEM

$$\mu_i(t) \geq 0, \; g_i(x(t), u(t)) \cdot \mu_i(t) = 0, \; i = 1, \ldots, q \tag{17}$$

$$\dot{\lambda}(t) = -\left(\frac{\partial f(x(t), u(t), t)}{\partial x(t)}\right)^T \cdot \lambda(t) - \sum_{i=1}^{q} \left(\frac{\partial g_i^{\hat{l}_i}(x(t), u(t))}{\partial x(t)}\right)^T \cdot \mu_i(t) \tag{18}$$

$$0 = \left(\frac{\partial f(x(t), u(t), t)}{\partial u(t)} \right)^T \cdot \lambda(t) + \sum_{i=1}^{q} \left(\frac{\partial g_i^{\hat{l}_i}(x(t), u(t))}{\partial u(t)} \right)^T \cdot \mu_i(t) \tag{19}$$

$$t_0 \le t \le t_f \tag{20}$$

$$\text{Assume: } t_{n_{kt}+n_{kx}+1} = t_f, \text{ and Define:} \tag{21}$$

$$t_{kt} \equiv \text{Entry point if } \mu_j(t_{kt}^-) = 0,\ \mu_j(t_{kt}^+) \ge 0,\ k = 1, 2, \ldots, n_{kt} \tag{22}$$

$$t_{kx} \equiv \text{Exit point if } \mu_j(t_{kx}^+) = 0,\ \mu_j(t_{kx}^-) \ge 0,\ k = 1, 2, \ldots, n_{kx} \tag{23}$$

$$\text{For at least one } j = 1, 2, \ldots, q \tag{24}$$

JUNCTION CONDITIONS (ENTRY POINT)

$$0 = g_i^j(x(t_{kt}), u(t_{kt})),\ j = 0, \ldots, \hat{l}_i - 1 \tag{25}$$

$$0 = g_i^{\hat{l}_i}(x(t_{kt}^+), u(t_{kt}^+)),\ k = 1, 2, \ldots n_{kt},\ i = 1, \ldots, q \tag{26}$$

JUMP CONDITIONS (ENTRY POINT - EXIT POINT)

$$\lambda(t_{kt}^+) = \lambda(t_{kt}^-) + \sum_{i=1}^{q} \sum_{j=0}^{\hat{l}_i - 1} \left(\frac{\partial g_i^j(x(t_{kt}), u(t_{kt}))}{\partial x(t_{kt})} \right)^T \cdot \varphi_{j,i}(t_{kt}) \tag{27}$$

$$H(t_{kt'}^+) = H(t_{kt'}^-),\ k = 1, 2, \ldots, n_{kt} \tag{28}$$

$$\lambda(t_{kx}^+) = \lambda(t_{kx}^-) \tag{29}$$

$$H(t_{kx'}^+) = H(t_{kx'}^-),\ k = 1, 2, \ldots, n_{kx} \tag{30}$$

$$H(t) = \dot{x}(t)\lambda(t) + g(x(t), u(t))^T \cdot \mu(t) \tag{31}$$

$$t_{k(t,x)} = \{\min(t_{k(t,x)'}, t_f) \vee \max(t_{k(t,x)'}, t_0)\} \tag{32}$$

where $\lambda(t) \in \mathbb{R}^n$ is the vector of adjoint (co-state) variables, $\mu(t) \in \mathbb{R}^q$ is the vector of Lagrange multipliers associated with the path constraints, $\nu(t) \in \mathbb{R}^{Q_g}$ is the vector of Lagrange multipliers of the end-point constraints, $\varphi_i \in \mathbb{R}^{\hat{l}_i}$, $i = 1, \ldots, q$ are the Lagrange multipliers linked with the jump conditions and $H(t)$ is the Hamiltonian function of the system. The time points where the jump conditions apply are called *corners* or *switching points*. The time intervals $t \in [t_k, t_{k+1}]$, $k = 1, \ldots, (n_{kt} + n_{kx})$ between two consecutive corners are termed as *constrained* or *boundary arcs* if at least one constraint is active or *unconstrained arcs* otherwise, where n_{kt} is the maximum number of entry points that may exist in the problem and n_{kx} is the maximum number of exit points.

Remark 1. For a zeroth order constraint, equations (25),(26) are omitted, (27), (28) are written as $\lambda(t_{kt}^{+}) = \lambda(t_{kt}^{-})$ and $H(t_{kt}^{+}) = H(t_{kt}^{-})$ respectively and $\varphi = 0$.

The following assumption is necessary for the analysis that will follow.

Assumption 3.1. *There exist an analytical solution to the differential algebraic equation (DAE) system arising from* (12), (18) *and* (19) *with boundary conditions the equations in* (13), (14), (25), (26), (27) *and* (29).

If the above assumption holds then $x(t,t^k,x_0)$, $\lambda(t,t^k,x_0)$, $\mu(t,t^k,x_0)$, $u(t,t^k,x_0)$ and $\xi(t^k,x_0) = [x_f^T \ \lambda_0^T \ \mu^T(t_1) \ \ldots \ \mu^T(t_{n_{kt}}) \ \varphi^T(t_1) \ \ldots \ \varphi^T(t_{n_{kt}}) \ \nu^T]$ are explicit, non-linear functions of time t, the switching points $t^k = \{t_1 \ t_2 \ \ldots \ t_{n_{kt}} + t_{n_{kx}}\} \equiv \{t_{1t} \ t_{1x} \ t_{2t} \ \ldots \ t_{n_{kx}}\}$ and the initial condition x_0. This allows the derivation of the optimal profiles of the control inputs in terms of x_0 and the determination of the compact regions in the space of the initial conditions where these functions hold.

In order to obtain the optimal control profiles the following algorithm can be followed:

Algorithm 3.1

1: Define an initial region CR^{IG} in which problem (11) is going to be solved
2: Select a realization in the parameter space of x_0 and compute the optimal number of switching points and (constrained and/or unconstrained) arcs for these points by solving the DO problem (12)-(32).
3: Given the sequence of switching points and considering x_0 as a free parameter, solve analytically the DAE system arising from (12), (18) and (19) with boundary conditions the equations in (13), (14), (25), (26), (27) and (29).to obtain, first $\hat{\xi}(t^k,x_0)$ and then the differential states $\hat{\lambda(t,t^k},x_0)$, $\hat{x}(t,t^k,x_0)$, $\hat{\mu}(t,t^k,x_0)$ and finally the algebraic variables $\hat{u}(t,t^k,x_0)$.
4: Substitute the values of $\hat{\xi}(t^k,x_0)$, $\hat{\lambda(t,t^k},x_0)$, $\hat{x}(t,t^k,x_0)$, $\hat{\mu}(t,t^k,x_0)$ and $\hat{u}(t,t^k,x_0)$ in the equations (28), (30) and (32) and solve the new system of non-linear, algebraic equations to obtain t^k as an explicit function of the free parameter x_0 and call it $t^k(x_0)$.
5: Substitute $t^k(x_0)$ into the expression of $u(t,t^k(x_0),x_0)$ to obtain the optimal parametric control profile.
6: Compute the critical region CR where the optimal parametric control profile is valid.
7: If CR is not empty then select a new initial condition x_0 outside CR and go to Step 2 else stop

The algorithm starts with the definition of the space CR^{IR} of initial conditions x_0, in which the mp-DO problem is going to be solved. In step 2 the switching points and the corresponding arcs and active constraints are obtained by solving the DO (12)-(32) for a fixed value of x_0. In step 3 the DAE system that consists of the system's dynamic model and the optimality conditions corresponding to the switching points and active constraints, derived in step 2, is solved symbolically to obtain the optimal profiles of $\hat{\xi}(t^k, x_0)$, $\lambda(t, \hat{t^k}, x_0)$, $\hat{x}(t, t^k, x_0)$, $\hat{\mu}(t, t^k, x_0)$ and $\hat{u}(t, t^k, x_0)$. The vector $t^k(x_0)$ is calculated in the step 4 by solving symbolically the non-linear, algebraic equalities of the Jump conditions (28), (30) and (32). In step 5 the optimal parametric control profile is obtained by substituting $t^k(x_0)$ into $\hat{u}(t, t^k, x_0)$. Finally the critical region in which the optimal control profile is valid, is calculated in step 6, following the procedure which will be described in the following. The algorithm then repeats the procedure until the whole initial region CR^{IR} is covered.

A critical region CR in which the optimal control profiles are valid, is the region of initial conditions x_0 where the active and inactive constraints, obtained in step 2 of algorithm 3.1, remain unaltered ([20]). Define the set of inactive constraints $\breve{g}$, the active constraints $\tilde{g}$ and $\tilde{\hat{\mu}} > 0$ the Lagrange multipliers associated with the active constraints $\tilde{g}$; obviously the Lagrange multipliers μ associated with the inactive constraints are 0. The critical region CR is then identified by the following set of inequalities

$$\begin{aligned} CR \triangleq \{x_0 \in \mathbb{R}^n \mid \breve{g}(\hat{x}(t, t^k(x_0), x_0), \hat{u}(t, t^k(x_0), x_0)) < 0\ ,\ \tilde{\hat{\mu}}(t, t^k(x_0), x_0) > 0 \\ \tilde{\nu}(t, t^k(x_0), x_0) > 0\} \end{aligned} \tag{33}$$

In order to characterize CR one has to obtain the boundaries of the set described by inequalities (33). These boundaries obviously are obtained when each of the linear inequalities in (33) is critically satisfied. This can be achieved by solving the following parametric programming problems, where time t is the variable and x_0 is the parameter.

- Take first the inactive constraints through the complete time horizon and derive the following parametric expressions:

$$\breve{G}_i(x_0) = \max_t \{\breve{g}_i(\hat{x}(t, t^k(x_0), x_0), \hat{u}(t, t^k(x_0), x_0)) | t \in [t_0, t_f]\},\ i = 1, \ldots, \breve{q} \tag{34}$$

  where $\breve{q}$ is the number of inactive constraints.
- Take the path constraints that have at least one constrained arc $[t_{i,\tilde{k}t}, t_{i,\tilde{k}x}]$ and obtain the following parametric expression

$$\tilde{G}_i(x_0) = \max_t \{\tilde{g}_i(\hat{x}(t, t^k(x_0), x_0), \hat{u}(t, t^k(x_0), x_0)) | t \in [t_0, t_f]\} \wedge \{t \not\in [t_{i,\tilde{k}t}, t_{i,\tilde{k}x}]]\} \tag{35}$$

$$k = 1, 2, \ldots, n_{i,\tilde{k}t},\ i = 1, 2, \ldots, \tilde{q}$$

  where $n_{i,\tilde{k}t}$ is the total number of entry points associated with the ith active constraint and $\tilde{q}$ is the number of active constraints.
- Finally, take the multipliers of the active constraints and obtain the following parametric expressions

$$\breve{\mu}(x_0) = \min_t \{\tilde{\hat{\mu}}(t, t^k(x_0), x_0) | t = t_{i,kt} = t_{i,kx}, k = 1, 2, \ldots, n_{i,kt}\},\ i = 1, 2, \ldots, \tilde{q} \tag{36}$$

One should notice that the multipliers assume their minimum value when the corresponding constraint is critically satisfied, hence, the path constraint reduces to a point constraint. This property is captured in the equality constraint $t = t_{i,kt} = t_{i,kx}$.

In each of the above problems the critical time, where each of the inequalities $(\breve{g}_i(\hat{x}(t,t^k(x_0),x_0),\hat{u}(t,t^k(x_0),x_0)),\ \ \tilde{g}_i(\hat{x}(t,t^k(x_0),x_0),\hat{u}(t,t^k(x_0),x_0)),\ \ \tilde{\breve{\mu}}(t,t^k(x_0),x_0))$ is critically satisfied, is obtained as an explicit function of x_0 and then is replaced in the inequality to obtain a new inequality $(\breve{G}_i(x_0),\ \tilde{G}_i(x_0),\ \breve{\mu}(x_0))$ in terms of x_0. The critical region in which $\hat{u}(t,t^k(x_0),x_0)$ is valid, is given as follows

$$CR = \{\breve{G}(x_0) > 0\ ,\ \tilde{G}(x_0) > 0\ ,\ \breve{\mu}(x_0) > 0\ ,\ \tilde{\nu}(x_0) > 0\} \cap CR^{IG} \tag{37}$$

It is obvious the critical region CR is defined by a set of compact, non-linear inequalities. The boundaries of CR are represented by parametric non-linear expressions in terms of x_0. Moreover, (34), (35) and (36) imply that in every region calculated in Step 6 of the proposed algorithm, a different number and sequence of switching points and arcs holds.

Although, the optimal control profile $\hat{u}(t,t^k(x_0),x_0)$ constitutes an open-loop control policy, its implementation can be performed in a MPC fashion, thus resulting to a closed loop optimal control policy. More specifically, this is achieved by treating the current state $x(t^*) \equiv x_0$ as an initial state, where t^* is the time when the state value becomes available. The control action $\hat{u}(t,t^k(x(t^*)),x(t^*))$ is then applied for the time interval $[t^*, t^* + \Delta t]$, where Δt denotes the plants sampling time, and in the next time instant $t^* + \Delta t$ the state is updated and the procedure is repeated. Hence, this implementation results to the control law $u(x(t^*)) = \{\hat{u}(t,t^k(x(t^*)),x(t^*))|t^* \le t \le t^* + \Delta t\}$.

4 Example

We are going to illustrate the methods discussed above for the constrained Brachistochrone problem in which a beam slides on a frictionless wire between a point and a vertical plane $1m$ on the right of this point ([3]). The coordinates of the beam on every point on the wire satisfy the following system of differential equations

$$\dot{x} = (2gy)^{1/2} \cos\gamma \tag{38}$$

$$\dot{y} = (2gy)^{1/2} \sin\gamma \tag{39}$$

where x is the horizontal distance, y is the vertical distance (positive downwards), g is the acceleration due to gravity and γ is the angle the wire forms with the horizontal direction. The goal is to find the shape of the wire that will produce a minimum-time path between the two positions, while satisfying the inequality $y - 0.5x - 1 \le 0$. The above problem is of the form (11) where $\phi(x(t_f),t_f) = t_f$, (11b) is replaced by (38) and (39), $g(x(t),u(t)) = y - 0.5x - 1$

and $\psi^g(x(t_f)) = -x(t_f) + 1$. The last expression represents that at final time the beam should be positioned at a point where $x(t_f) \geq 1$. We also assume that $t_0 = 0$. Although, the problem has already been solved for a fixed initial point $x_0 = [0 \;\; 0]^T$ (as for example in [3]), here the optimal solution is derived for the first time as a function of the initial point coordinates.

The problem is first dealt in discrete-time as described in Section 2. The continuous-time system is turn into a discrete-time system assuming a sampling time Δt such that $[t_0, t_f]$ is divided in three equally spaced time intervals of Δt i.e. $[t_0, t_f] = 3\Delta t$. The discrete-time problem

$$\begin{aligned}
&\min_{\gamma_k} \Delta t \\
&x_{k+1} = x_k + (2gy_k)^{-1/2}\, cos\gamma_k\, \Delta t\ ,\ k = 0,1,2 \\
&y_{k+1} = y_k + (2gy_k)^{-1/2}\, sin\gamma_k\, \Delta t\ ,\ k = 0,1,2 \\
&y_k - 0.5x_k - 1 \leq 0\ ,\ k = 0,1,2,3 \\
&x_3 \geq 1
\end{aligned}$$

is then solved by transforming the above problem in (5) and solving the mp-LP problem to acquire the PWA solution. The continuous-time is solved next following Algorithm 3.1. The results for both the discrete-time case and the continuous-time case together with a simulation for $x_0 = [0 \quad 0]^T$ are shown in Figure 1 and 2. The straight line in both diagrams represents the boundary of the linear constraint $y - 0.5 - 1 \leq 0$. There are three control laws for the continuous-time case, depending in which region the initial state is. The control law in the unconstrained region (Figure 2.) is obtained by solving the following system of algebraic equalities with respect to c_1 and γ

$$\begin{aligned}
&0 = c_1^2 - \arccos(c_1\sqrt{y}) - xc_1^2 + c_1\sqrt{y}\sin\arccos c_1\sqrt{y} \\
&\gamma = -1/2(2g)^{1/2}c_1 t + \arccos c_1\sqrt{y}
\end{aligned}$$

In the constrained region the control law is obtained as following. First, the following system of equalities is solved

$$\begin{aligned}
&x(\tau'') - \frac{(2g)^{\frac{1}{2}}}{2c_1}\tau'' + \frac{1}{2c_1^2}\sin(2g)^{\frac{1}{2}}c_1(t_f - \tau'') = 1 - \frac{(2g)^{\frac{1}{2}}}{2c_1}t_f \\
&t_f = \frac{\arccos c_1\sqrt{y} + 0.5(2g)^{\frac{1}{2}}c_1\tau''}{0.5(2g)^{\frac{1}{2}}c_1}\ ,\ x(\tau'') = 0.1989g\tau''^2 + (2g^{\frac{1}{2}})0.896\sqrt{y_0}\tau'' + x_0 \\
&y(\tau'') = \left(0.222(2g)^{\frac{1}{2}}\tau'' + \sqrt{y_0}\right)^2\ ,\ 0.46 = 0.5(2g)^{\frac{1}{2}}c_1\,(t_f - \tau'')
\end{aligned}$$

which is a system of five equations with five unknowns $t_f, \tau'', c_1, x(\tau''), y(\tau'')$. Then, the control to be applied is given as

If $t \leq \tau''$ then $\gamma = 0.46 = \arctan(0.5)$ **Else If** $t \geq \tau''$ then $\gamma = 0.5(2g)^{\frac{1}{2}}c_1(t_f - t)$

As it can be observed from Fig. 1 the approximating, discrete-time, PWA solution is not the optimal one comparing to the optimal solution as it is given

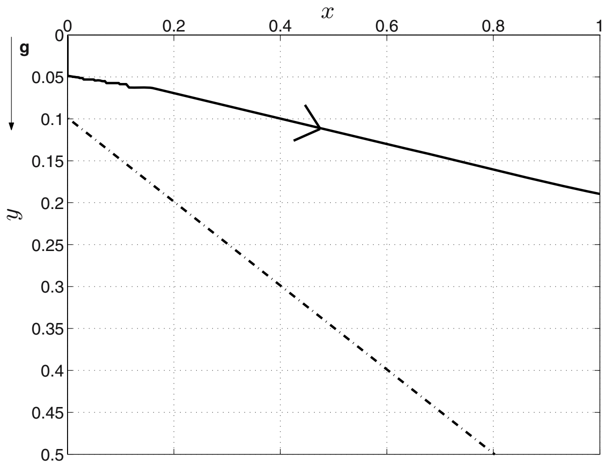

Fig. 1. Discrete-time Brachistochrone Problem

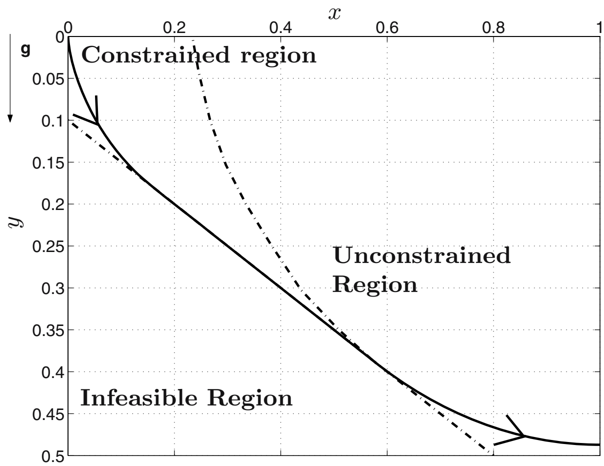

Fig. 2. Continuous-time Brachistochrone Problem

in [3], due to approximation error. On the other hand, the multi-parametric optimal control law illustrated in Fig. 2, is the optimal solution for each initial condition contained in the constrained and unconstrained regions.

5 Conclusions

In this paper the discrete-time MPC problem as well as the continuous-time optimal control problem were examined. A method was presented for obtaining a linear, PWA approximation to the discrete-time MPC problem where the objective and control are obtained as linear, PWA functions of the initial condition. Then, an algorithm was presented that solves the mp-DO problem arising from the non-linear, continuous-time, optimal control problem with state and input constraints, where the objective is a non-linear function of the state at the final time. It was shown that the optimal control profile is a non-linear function of the time and the state variables.

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