Computability on Subsets of Locally Compact Spaces

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Abstract. In this paper we investigate aspects of effectivity and computability on closed and compact subsets of locally compact spaces. We use the framework of the representation approach, TTE, where continuity and computability on finite and infinite sequences of symbols are defined canonically and transferred to abstract sets by means of notations and representations. This work is a generalization of the concepts introduced in [4] and [22] for the Euclidean case and in [3] for metric spaces. Whenever reasonable, we transfer a representation of the set of closed or compact subsets to locally compact spaces and discuss its properties and their relations to each other.

1 Introduction

Computable Analysis connects Computability/Computational Complexity with Analysis/Numerical Computation by combining concepts of approximation and of computation. During the last 70 years various mutually non-equivalent models of real number computation have been proposed ([19], Chap. 9 in [22]). Among these models the representation approach (Type-2 Theory of Effectivity, TTE) proposed by Grzegorczyk and Lacombe [7,14] seems to be particularly realistic, flexible and expressive. So far the study of computability on sets of points, sets (open, closed, compact) and continuous functions has developed mainly bottom-up, i.e., from the real numbers to Euclidean space and metric spaces [26,4,24,22,27,3,28]. But often generalizations to more general spaces are needed (locally compact Hausdorff spaces [5], non-metrizable spaces [25], second countable T_0-spaces [17,8]).

In this article we investigate computability on locally compact spaces with the following motivation:

– Computability on metric spaces has been widely and deeply studied. However, the concept of a metric space is not powerful enough to capture all the interesting phenomena in computable analysis. Many results in classical topology, that hold for more general spaces such as Hausdorff spaces or locally compact spaces, can be tied with effectivity.

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– Locally compact spaces inherit some nice properties of metric spaces. Roughly speaking, a locally compact space with a countable base is metrizable. This demonstrates that locally compact spaces are "quite close" to metric spaces.
– Furthermore general topological spaces, especially locally compact spaces, have practical applications. For example in [5] Collins used locally compact spaces to study computability of reachable sets for nonlinear dynamic and control systems.

For these reasons it is necessary, reachable and meaningful to study computability on locally compact spaces.

In [4] and [22] and in [3] several representations are introduced for subsets of the Euclidean space and of a metric space respectively. We don't consider those, which are defined by means of the metric distance function. We prove that the properties $\delta_{union} \equiv \delta_{Sierpinski} \equiv \delta_{dom} \equiv \delta^{>} \equiv \delta_{fiber}$ and $\delta^{<} \leq \delta_{range}$ for closed sets and $\delta_{min-cover} \leq \delta_{cover} \equiv \delta_{\mathcal{K}}^{>}$ for compact sets, shown in [3] for metric spaces, hold true for locally compact spaces as well. A crucial point for the fact, that these results can be transfered to locally compact spaces, are two additional axioms that are required in the metric case for some of these reductions,

1. the existence of nice closed balls,
2. the effective covering property.

Both properties hold true for computably locally compact spaces. The first follows directly from the definition of computably locally compact spaces, as the closure of each base element is compact. The second property utilizes the completeness of the cover representation: a name lists all names of all finite basic covers of a compact set (shown in Lemma 1). Furthermore we introduce a new representation κ^{net} where a compact set is denoted by a decreasing sequence of finite basic covers whose intersection equals to the compact set. We show that κ^{net} is equivalent to the cover representation.

This article is organized as follows: In Section 2, we sketch some basic notions on TTE and provide some fundamental definitions and properties of representations of points and sets in computable T_0-spaces. In Section 3, we introduce and characterize computably locally compact spaces and computably Hausdorff spaces. In Section 4 we define and compare various representations of closed subsets and compact subsets of computably locally compact, computably Hausdorff spaces. The conclusion is drawn in the last Section. Since this is an extended abstract proofs of the main theorems are given in the Appendix.

2 Preliminaries

This Section consists of two parts. In Section 2.1, we sketch the concept of TTE. In Section 2.2, we introduce computable T_0-spaces and the underlying representations of points and sets.

2.1 Type-2 Theory of Effectivity (TTE)

In this article we use the framework of TTE (Type-2 theory of effectivity) [22] to explore several aspects of computability in locally compact spaces. The Type-2 theory of effectivity defines computability on Σ^* and Σ^ω via Type-2 machines and transfers a computability concept to "abstract" sets by means of naming systems.

We assume that Σ is a fixed finite alphabet containing the symbols 0 and 1 and consider computable functions on finite and infinite sequences of symbols Σ^* and Σ^ω, respectively, which can be defined, for example, by Type-2 machines, i.e., Turing machines reading from and writing on finite or infinite tapes. A Type-2 machine may have several one-way read-only input tapes, several two-way work tapes and a unique one-way write-only output tape. It permits infinite input or output, and has a finiteness property, that is, each group of prefixes of the inputs determines a unique prefix of the output. A partial function from X to Y is denoted by $f :\subseteq X \to Y$. For $\alpha_i \in \{*, \omega\}$, a function $f :\subseteq \Sigma^{\alpha_1} \times \ldots \times \Sigma^{\alpha_k} \to \Sigma^{\alpha_0}$ is called computable if $f = f_M$ for some Type-2 machine M.

The "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1 a_2 \ldots a_k) := 110a_1 0a_2 0 \ldots a_k 011$ encodes words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly. We consider standard functions for finite or countable tupling on Σ^* and Σ^ω denoted by $\langle \cdot \rangle$ and projections of the inverse π_i for $i \in \mathbb{N}$. By "$\lhd$" we denote the subword relation. A sequence $p \in \Sigma^* \cup \Sigma^\omega$ is called a list of M if $M = \{u \mid \iota(u) \lhd p\}$.

We use the concept of multi-functions. A *multi-valued partial function*, or *multi-function* for short, from A to B is a triple $f = (A, B, \mathrm{R}_f)$ such that $\mathrm{R}_f \subseteq A \times B$ (the *graph* of f). Usually we will denote a multi-function f from A to B by $f :\subseteq A \rightrightarrows B$. For $X \subseteq A$ let $f[X] := \{b \in B \mid (\exists a \in X)(a, b) \in \mathrm{R}_f\}$ and for $a \in A$ define $f(a) := f[\{a\}]$. Notice that f is well-defined by the values $f(a) \subseteq B$ for all $a \in A$. We define $\mathrm{dom}(f) := \{a \in A \mid f(a) \neq \emptyset\}$. In the applications we have in mind, for a multi-function $f :\subseteq A \rightrightarrows B$, $f(a)$ is interpreted as the set of all results which are "acceptable" on input $a \in A$. Any concrete computation will produce on input $a \in \mathrm{dom}(f)$ some element $b \in f(a)$, but usually there is no method to select a specific one. In accordance with this interpretation the "functional" composition $g \circ f :\subseteq A \rightrightarrows D$ of $f :\subseteq A \rightrightarrows B$ and $g :\subseteq C \rightrightarrows D$ is defined by $\mathrm{dom}(g \circ f) := \{a \in A \mid a \in \mathrm{dom}(f) \text{ and } f(a) \subseteq \mathrm{dom}(g)\}$ and $g \circ f(a) := g[f(a)]$ (in contrast to "non-deterministic" or "relational" composition gf defined by $g\,f(a) := g[f(a)]$ for *all* $a \in A$).

Notations $\nu :\subseteq \Sigma^* \to M$ and representations $\delta :\subseteq \Sigma^\omega \to M$ are used for introducing relative continuity and computability on "abstract" sets M. For a representation $\delta :\subseteq \Sigma^\omega \to M$, if $\delta(p) = x$ then the point $x \in M$ can be identified by the "name" $p \in \Sigma^\omega$. For representations $\delta :\subseteq \Sigma^\omega \to M$ and $\delta' :\subseteq \Sigma^\omega \to M'$ define $[\delta, \delta'] :\subseteq \Sigma^\omega \to M \times M'$ by $[\delta, \delta']\langle p, p' \rangle := \delta(p) \times \delta(p')$ and $[\delta]^\omega :\subseteq \Sigma^\omega \to M^\omega$ by $[\delta]^\omega \langle p_0, p_1, p_2, \ldots \rangle := \delta(p_0) \times \delta(p_1) \times \delta(p_2) \times \ldots$.

For a naming systems $\gamma :\subseteq Y_i \to M_i$, $Y_i = \Sigma^*$ or Σ^ω, the set X is called γ-open (γ-clopen, γ-r.e. , γ-decidable), iff $\gamma^{-1}[X]$ is open (clopen, r.e. open, decidable) in $\mathrm{dom}(\gamma)$.

For naming systems $\gamma_i :\subseteq Y_i \to M_i$ $(i = 0,\ldots,k)$, a function $h :\subseteq Y_1 \times \ldots \times Y_k \to Y_0$ is a $(\gamma_1,\ldots,\gamma_k,\gamma_0)$-realization of $f :\subseteq M_1 \times \ldots \times M_k \rightrightarrows M_0$, if $\gamma_0 \circ h(p_1,\ldots,p_k) \in f(\gamma_1(p_1),\ldots,\gamma_k(p_k))$ whenever $f(\gamma_1(p_1),\ldots,\gamma_k(p_k))$ exists. The function h is called a strong realization of f, if $h(p_1,\ldots,p_k) = \uparrow$ for all $\langle p_1,\ldots,p_k\rangle \in \mathrm{dom}([\gamma_1,\ldots,\gamma_k])$ with $[\gamma_1,\ldots,\gamma_k](p_1,\ldots,p_k) \notin \mathrm{dom}(f)$. The multi-function f is $(\gamma_1,\ldots,\gamma_k,\gamma_0)$-continuous (-computable), if it has a continuous (computable) $(\gamma_1,\ldots,\gamma_k,\gamma_0)$-realization.

For naming systems $\gamma :\subseteq Y \to M$ and $\gamma' :\subseteq Y' \to M'$ $(Y, Y' \in \{\Sigma^*, \Sigma^\omega\})$, let $\gamma \le_t \gamma'$ (t-reducible) and $\gamma \le \gamma'$ (reducible), iff there is some continuous or computable function $f :\subseteq Y \to Y'$ such that $\gamma(y) = \gamma' f(y)$ for all $y \in \mathrm{dom}(\gamma)$, respectively. Define t-equivalence and equivalence as follows: $\gamma \equiv_t \gamma' \iff (\gamma \le_t \gamma'$ and $\gamma' \le_t \gamma)$ and $\gamma \equiv \gamma' \iff (\gamma \le \gamma'$ and $\gamma' \le \gamma)$, respectively.

Two representations induce the same continuity or computability, iff they are t-equivalent or equivalent, respectively. If multi-functions on represented sets have realizations, then their composition is realized by the composition of the realizations. In particular, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on represented sets are closed under flowchart programming with indirect addressing [23]. This result allows convenient informal construction of new computable multi-functions on multi-represented sets from given ones.

Let $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$ be some standard notation of the natural numbers, ρ the standard representation of $\mathbb{R}$. A $\rho^<$-name represents a real number by lower rational bounds. $\rho^<(p) = x$, if p is a list of all rational numbers $a < x$ and η^{ab} a standard representation of F^{ab}, the partial continuous functions $f :\subseteq \sigma^a \to \sigma^b$ with open or G_δ domain, if $b = *$ or $b = \omega$ respectively, with properties $\mathrm{utm}(\eta^{ab})$ and $\mathrm{smn}(\eta^{ab})$.

2.2 Representations of Points and Sets in Computable T_0-Spaces

In this Section we introduce computable T_0-spaces together with some fundamental representations of points and sets.

A topological space $\mathbf{X} = (X, \tau)$ is a T_0-space, if for all $x, y \in X$ such that $x \ne y$, there is an open set $O \in \tau$ such that $x \in O$ iff $y \notin O$. In a T_0-space, every point can be identified by the set of its neighborhoods $O \in \tau$. $\mathbf{X}$ is called second-countable, if it has a countable base [6].

In the following we consider only second countable T_0-spaces. For introducing concepts of effectivity we assume that some notation ν of a base β with recursive domain is given.

Definition 1 (computable T_0-space)
A computable T_0-space *is a tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that* (X, τ) *is a second countable T_0-space and* $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ *with recursive domain,* $U \ne \emptyset$ *for* $U \in \beta$ *and* $\mathbf{X}$ *has* computable intersection*: there is a computable function* $h :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ *such that for all* $u, v \in \mathrm{dom}(\nu)$,

$$\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid w \in \mathrm{dom}(\nu) \ \ \textit{and} \ \ \iota(w) \lhd h(u,v)\}\,. \tag{1}$$

Call two computable T_0-spaces $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ recursively related, if and only if there are computable functions $g, g' : \subseteq \Sigma^* \to \Sigma^\omega$ such that

$$\nu_1(u) = \bigcup_{\iota(w) \lhd g(u)} \nu_2(w) \quad \text{and} \quad \nu_2(v) = \bigcup_{\iota(w) \lhd g'(v)} \nu_1(w)\,. \tag{2}$$

We are interested in computability concepts which are "robust", that is, which do not change if a space is replaced by a recursively related one.

In the following let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space. Now we introduce the standard representation of X.

Definition 2 (standard representation δ of X). *Define the standard representation $\delta : \subseteq \Sigma^\omega \to X$ as follows: $\delta(p) = x$ iff*

- $u \in \mathrm{dom}(\nu)$ *if* $\iota(u) \lhd p$
- $\{u \in \mathrm{dom}(\nu) \mid x \in \nu(u)\} = \{u \mid \iota(u) \lhd p\}$.

A δ-name p of an element $x \in X$ is a list of all words u such that $x \in \nu(u)$. The definition of δ corresponds to the definition of $\delta'_{\mathbf{S}}$ in Lemma 3.2.3 of [22], in particular, δ is admissible with final topology τ (Sec. 3.2 in [22]).

Definition 3 (union representation of open and closed sets)

1. *Define the union representation $\theta^{un} : \subseteq \Sigma^\omega \to \tau$ by*

$$\mathrm{dom}(\theta^{un}) := \{q \in \Sigma^\omega | u \in \mathrm{dom}(\nu) \textit{ if } \iota(u) \lhd q\} \quad \textit{and} \quad \theta^{un}(p) := \bigcup_{\iota(u) \lhd p} \nu(u)\,.$$

2. *Define the union representation $\psi^{un} : \subseteq \Sigma^\omega \to \tau^c$ by $\psi^{un}(p) := X \setminus \theta^{un}(p)$.*

Thus, $\theta^{un}(p)$ is the union of all $\nu(u)$ such that u is listed by p. The union representation of the closed sets is defined by the union representation of their complements.

For technical reasons we define a notation $\nu^* : \subseteq \Sigma^* \to \{M \subseteq \beta \mid M \text{ is finite}\}$ of all finite sets of base elements by $\mathrm{dom}(\nu^*) := \{w \in \Sigma^* \mid u \in \mathrm{dom}(\nu) \text{ if } \iota(u) \lhd w\}$ and

$$\nu^*(w) := \{\nu(u) \mid \iota(u) \lhd w\}$$

and a notation $\theta^* : \subseteq \Sigma^* \to \tau^{fin}$ of all open sets that can be written as the union of finitely many base elements by

$$\theta^*(w) := \bigcup \nu^*(w).$$

The representations δ and θ^{un} are not only very natural, but they can be characterized up to equivalence as maximal elements among representations for which the element relation is open or r.e., respectively. Furthermore the following properties hold.

Lemma 1. *For computable T_0-spaces,*

1. *"$O \neq \emptyset$" is θ^{un}-r.e.,*
2. *countable union on τ is $([\theta^{un}]^\omega, \theta^{un})$-computable,*
3. *intersection is $(\theta^{un}, \theta^{un}, \theta^{un})$-computable,*
4. *finite intersection is (ν^*, θ^{un})-computable.*

Proof. Omitted.

Equivalently to the computability of finite intersection on the base is the existence of an r.e. set $I \subseteq \Sigma^* \times \mathrm{dom}(\nu)$, such that for all $w \in \Sigma^*$ and $u \in \mathrm{dom}(\nu)$

$$\bigcap_{\iota(v) \lhd w} \nu(v) = \bigcup_{(w,u) \in I} \nu(u). \tag{3}$$

Furthermore the set $P := \{w \in \Sigma^* \mid (\exists u \in \mathrm{dom}(\nu))\,(w,u) \in I\}$ of all finite prefixes of δ-names is r.e..

A topological space is a T_2-space (also called Hausdorff space), if for all $x, y \in X$ such that $x \neq y$, there are disjoint open sets $O, O' \in \tau$ such that $x \in O$ and $y \in O'$. A subset $K \subseteq X$ of a Hausdorff space (X, τ) is compact, if every open cover of K by elements of the base has a finite subcover. Let

$$\mathcal{K}(\mathbf{X}) := \{K \subseteq X \mid K \text{ compact}\}$$

denote the set of all compact subsets of a Hausdorff space (X, τ). We write $\mathcal{K}$ instead of $\mathcal{K}(\mathbf{X})$, if there is no need to specify the space or if it's obvious which space we refer to.

In the following we generalize the representation κ_c of the compact subsets of the Euclidean space [22] and δ_{cover} of the compact subsets of a computable metric space [3] defined by listing all finite basic subcovers from a countable base.

Definition 4 (cover representation κ^c of compact sets). *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space and let (X, τ) be a Hausdorff space. Define a representation $\kappa^c :\subseteq \Sigma^\omega \to \mathcal{K}$ as follows: $K = \kappa^c(p)$ iff*

- *$w \in \mathrm{dom}(\theta^*)$ if $\iota(w) \lhd p$,*
- *$\{w \in \Sigma^* \mid \iota(w) \lhd p\} = \{w \in \Sigma^* \mid K \subseteq \theta^*(w)\}$.*

Roughly speaking, p is a name of K, if it is a list of all (!) names of all finite basic subcovers of K with base elements.

For technical reasons we define a representation of all finite sets of compact sets $\kappa^* :\subseteq \Sigma^\omega \to \{M \subseteq \mathcal{K} \mid M \text{ is finite }\}$ by

$$\kappa^*(p) = \{K_1, \ldots, K_k\} :\iff \begin{array}{l} p = 1^k 0 \langle p_1, \ldots, p_k \rangle \text{ and} \\ \kappa^c(p_i) = K_i \text{ for all } i \in \{1, \ldots, k\}. \end{array}$$

Lemma 2. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space and let (X, τ) be a Hausdorff space, then*

1. *"$K \subseteq O$" is (κ^c, θ^{un})-r.e.,*
2. *finite union on $\mathcal{K}$ is (κ^*, κ^c)-computable,*
3. *countable intersection on $\mathcal{K}$ is $([\kappa]^\omega, \kappa^c)$-computable.*

Proof. Omitted.

Every closed subset of a compact set is compact. The next Lemma gives an effective version.

Lemma 3. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable T_0-space and let (X, τ) be a Hausdorff space. The mapping $\mathrm{F} : \tau^c \times \mathcal{K} \to \mathcal{K}$ defined by*

$$\mathrm{F}(A, K) := A \cap K$$

is $(\psi^{un}, \kappa^c, \kappa^c)$-computable.

Proof. Omitted.

3 Effectivity in Locally Compact Hausdorff Spaces

In this Section we introduce an effective version of the Hausdorff property and an effective version of locally compactness.

Definition 5 (computably Hausdorff). *A computable T_0-space $\mathbf{X} = (X, \tau, \beta, \nu)$ is called* computably Hausdorff *if there exists an r.e. set $H \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ such that*

$$(\forall (u, v) \in H) \quad \nu(u) \cap \nu(v) = \emptyset, \tag{4}$$

$$(\forall x, y \in X \text{ with } x \neq y)\, (\exists (u, v) \in H)\; x \in \nu(u) \wedge y \in \nu(v). \tag{5}$$

Lemma 4. *For computable T_0-spaces,*

$$\mathbf{X} \text{ computably Hausdorff } \iff \{(x, y) \in X \times X \mid x \neq y\} \text{ is } (\delta, \delta) - \text{r.e.}.$$

Proof. Omitted.

Lemma 4 implies the robustness of the computably Hausdorff property, as δ is robust.

Lemma 5. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computably Hausdorff space.*

1. *The mapping $\mathrm{F}_1 :\subseteq X \times X \rightrightarrows \beta \times \beta$ defined by $\mathrm{dom}(F_1) = \{(x, y) \mid x \neq y\}$ and*

$$(U, V) \in \mathrm{F}_1(x, y) :\iff x \in U \text{ and } y \in V \text{ and } U \cap V = \emptyset$$

is $(\delta, \delta, \nu, \nu)$-computable.

2. *The mapping* $\mathrm{F}_2 :\subseteq X \times \mathcal{K} \rightrightarrows \beta \times \tau$ *defined by* $\mathrm{dom}(F_2) = \{(x, K) \mid x \notin K\}$ *and*

$$(U, O) \in \mathrm{F}_2(x, K) :\Longleftrightarrow x \in U \quad and \quad K \subseteq O \quad and \quad U \cap O = \emptyset$$

is $(\delta, \kappa^c, \nu, \theta^*)$*-computable.*

3. *The mapping* $\mathrm{F}_3 :\subseteq \mathcal{K} \times \mathcal{K} \rightrightarrows \tau \times \tau$ *defined by* $\mathrm{dom}(F_3) = \{(K, K') \mid K \cap K' = \emptyset\}$ *and*

$$(O, O') \in \mathrm{F}_3(K, K') :\Longleftrightarrow K \subseteq O \quad and \quad K' \subseteq O' \quad and \quad O \cap O' = \emptyset$$

is $(\kappa^c, \kappa^c, \theta^*, \theta^*)$*-computable.*

Proof. Omitted.

Every compact subspace of a Hausdorff space is closed. The next theorem is an effective version.

Theorem 1. *For computably Hausdorff spaces,* $\kappa^c \leq \psi^{un}$.

Proof. See Appendix.

A topological space (X, τ) is called locally compact, if for every point $x \in X$, there exists a neighborhood O of x such that the closure $\bar{O}$ is compact. Next we introduce an effective version of locally compactness by means of the representation κ^c of the compact subsets of a Hausdorff space.

Definition 6 (computably locally compact space). *A computable* T_0*-space* $\mathbf{X}' = (X, \tau, \beta', \nu')$ *is called a* computably locally compact space *if* (X, τ) *is a Hausdorff space and there is some computable* T_0*-space* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that* $\mathrm{CLS} : \beta \to \mathcal{K}(\mathbf{X})$ *defined by* $\mathrm{CLS}(U) := \bar{U}$ *is* (ν, κ^c)*-computable and* $\mathbf{X}'$ *and* $\mathbf{X}$ *are recursively related.*

The definition of computably locally compactness ensures its robustness. In the following if $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computably locally compact space, we suppose CLS to be (ν, κ^c)-computable (without changing the base or its notation).

If $\mathbf{X}$ is a computably locally compact space, then it is locally compact since the closure of each base element is compact. Therefore $\mathbf{X}$ is Tychonoff, thus regular (and a Hausdorff space) and even metrizable since $\mathbf{X}$ is second countable ([6]).

Lemma 6. *For computably locally compact spaces,*

1. *"*$\bar{U} \subseteq O$*" is* (ν, θ^{un})*-r.e.,*
2. *"*$\bar{O} \subseteq O'$*" is* (θ^*, θ^{un})*-r.e..*

Proof. Omitted.

In [3] and [5] property 1 is defined as the "effective covering property" of a computable metric space. As this property holds true for each computably locally compact space, we do not need an additional axiom.

For every compact subspace K of a locally compact space X and every open set $V \subseteq X$ that contains K, there exists an open set $U \subseteq X$ such that $K \subseteq U \subseteq \bar{U} \subseteq V$ and $\bar{U}$ is compact. The next Lemma gives an effective version.

Lemma 7. *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be a computably locally compact space. The mapping* $\mathrm{F} :\subseteq \mathcal{K} \times \tau \rightrightarrows \tau$ *defined by* $\mathrm{dom}(F) = \{(K, O) \in \mathcal{K} \times \tau \mid K \subseteq O\}$ *and*

$$U \in \mathrm{F}(K, O) :\Longleftrightarrow K \subseteq U \subseteq \bar{U} \subseteq O$$

is $(\kappa^c, \theta^{un}, \theta^*)$*-computable.*

Proof. Omitted.

Computably regular spaces have been introduced in [17] and [9]. The following theorem gives an effective version of the classical hierarchy, every locally compact space is regular and every regular space is a Hausdorff space.

Theorem 2. *1. A computable* T_0*-space is computably regular, if it is computably locally compact and computably Hausdorff.*
2. A computable T_0*-space is computably Hausdorff, if it is computably regular.*

Proof. Omitted.

4 Computability on Subsets of Computably Locally Compact, Computably Hausdorff Spaces

4.1 Computability on Closed Subsets

In this Section we study several representations of the closed subsets of a computably locally compact, computably Hausdorff space.

Definition 7 (representations of closed sets). *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be a computably locally compact, computably Hausdorff space. Let* $\eta_p^{\omega *}$, $\eta_p^{\omega\omega}$ *be standard representation of the set of continuous functions of* $F :\subseteq \Sigma^\omega \to \Sigma^*$ *and* $F :\subseteq \Sigma^\omega \to \Sigma^\omega$*, respectively.*

1. *Define the domain representation* $\psi^{dom} :\subseteq \Sigma^\omega \to \tau_c$ *by*

$$\psi^{dom}(p) = A :\Longleftrightarrow \eta_p^{\omega *} \text{ is a strong } (\delta, \nu_{\mathbb{N}})\text{-realization of } f :\subseteq X \to \mathbb{N} \text{ such that } \mathrm{dom}(f) = A^c.$$

2. *Define the Sierpinski representation* $\psi^{sie} :\subseteq \Sigma^\omega \to \tau_c$ *by*

$$\psi^{sie}(p) = A :\Longleftrightarrow \eta_p^{\omega\omega} \text{ is a } (\delta, \rho^<)\text{-realization of } \mathrm{cf}_A : X \to \mathbb{R}.$$

   Here $\mathrm{cf}_A : X \to \mathbb{R}$ *means that*

$$x \mapsto \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise} \end{cases}$$

3. *Define the fiber representation* $\psi^{fiber} :\subseteq \Sigma^\omega \to \tau_c$ *by*

$$\psi^{fiber}(p) = A :\Longleftrightarrow \eta_p^{\omega\omega} \text{ is a } (\delta, \rho)\text{-realization of } f : X \to \mathbb{R} \text{ such that } f^{-1}\{0\} = A.$$

4. *Define the inner representation* $\psi^< : \subseteq\Sigma^\omega \to \tau_c$ *as follows:* $\psi^<(p) = A$ *iff*
   - $u \in \mathrm{dom}(\nu)$ *if* $\iota(u) \lhd p$,
   - $\{w \mid \iota(w) \lhd p\} = \{w \mid \nu(w) \cap A \neq \emptyset\}$.
5. *Define the enumeration representation* $\psi^{range} : \subseteq\Sigma^\omega \to \tau_c$ *by*

$$\begin{aligned}\psi^{range}(0^\omega) \quad &:= \emptyset,\\ \psi^{range}(0^k 1p) &:= cls \circ range([\nu_{\mathbb{N}} \to \delta]_{\mathbb{N}}(p)).\end{aligned}$$

6. *Define the outer representation* $\psi^> : \subseteq\Sigma^\omega \to \tau_c$ *as follows* $\psi^>(p) = A$ *iff*
   - $u \in \mathrm{dom}(\nu)$ *if* $\iota(u) \lhd p$,
   - $\{w \mid \iota(w) \lhd p\} = \{w \mid \overline{(\nu(w))} \cap A = \emptyset\}$.

All these representations of closed sets with the exception of $\psi^>$ are well-defined and robust even for computable T_0-spaces.

The representation $\psi^>$ is well-defined for computably locally compact spaces, as they are regular. For closed sets A and B, if

$$\{w \in \mathrm{dom}(\nu) \mid \overline{\nu(w)} \cap A = \emptyset\} = \{w \in \mathrm{dom}(\nu) \mid \overline{\nu(w)} \cap B = \emptyset\}$$

then

$$\{w \in \mathrm{dom}(\nu) \mid \overline{\nu(w)} \subseteq A^c\} = \{w \in \mathrm{dom}(\nu) \mid \overline{\nu(w)} \subseteq B^c\}.$$

Let $x \in A^c$ then there exists some $V \in \beta$ such that $x \in V \subseteq \bar{V} \subseteq A^c$ as X is regular. It follows $x \in V \subseteq B^c$, then $A^c \subseteq B^c$. By symmetry we can conclude that $A^c = B^c$, hence $A = B$.

However $\psi^>$ is not well-defined for Hausdorff spaces in general.

Example 1. Define a topological space $(\mathbb{R}, \tau)$ by $\tau := \{G \setminus E | G \in \tau_{\mathbb{R}}, E \subseteq \mathbb{Q}\}$, where $\tau_{\mathbb{R}}$ is the set of all open subsets of $\mathbb{R}$. The space $(\mathbb{R}, \tau)$ is a Hausdorff space, since for any two different $x_1, x_2 \in \mathbb{R}$, by the density of real numbers, there exists x between them, then we can find two open sets $(-\infty, x) \setminus E_1$, $(x, \infty) \setminus E_2$, which contains x_1, x_2 respectively, such that $((-\infty, x) \setminus E_1) \cap ((x, \infty) \setminus E_2) = \emptyset$.

Next we show that for any $G \in \tau_{\mathbb{R}}$, $G \setminus E_i$ have the same closure as G, $i \in \mathbb{N}$. In fact, we just need to prove $\bar{G} \subseteq \overline{G \setminus E}$, for any $E \subseteq \mathbb{Q}$. Suppose $x \in \bar{G}$ by the definition of closure, for any neighborhood $G' \setminus E'$ of x, $(G' \setminus E') \cap G \neq \emptyset$. Since G and G' are open sets of $\mathbb{R}$ with common topology, and by the density of irrational numbers we conclude that there exists some irrational number $y \in (G' \setminus E') \cap G$. As $E \subseteq \mathbb{Q}$, then $y \in (G' \setminus E') \cap (G \setminus E)$, that is, $(G' \setminus E') \cap (G \setminus E) \neq \emptyset$. Hence, $x \in \overline{G \setminus E}$, as required.

Given a closed subset $\mathbb{Q} \in \tau^c$, there is no open set $G \setminus E$ such that $G \cap \mathbb{Q} = \emptyset$, where G is the closure of $G \setminus E$.

Theorem 3. *For computably locally compact, computably Hausdorff spaces,*

$$\psi^{fiber} \equiv \psi^{dom} \equiv \psi^{sie} \equiv \psi^{un} \equiv \psi^>.$$

Proof. See Appendix.

Theorem 4. *For computably locally compact, computably Hausdorff spaces,*

$$\psi^{range} \leq \psi^{<}.$$

Proof. See Appendix.

In general $\psi^{<} \leq \psi^{range}$ does not hold. See [3] for a counterexample for computable metric spaces.

4.2 Computability on Compact Subsets

In this Section we study several representations of the set $\mathcal{K}$ of the compact subsets of a computably locally compact, computably Hausdorff space.

Definition 8 (representations of compact sets). *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computably locally compact, computably Hausdorff space.*

1. *Define the minimal cover representation $\kappa^{mc} :\subseteq \Sigma^\omega \to \mathcal{K}$ as follows: $K = \kappa^{mc}(p)$ iff*
   - $w \in \mathrm{dom}(\theta^*)$ *if* $\iota(w) \lhd p$,
   - $\{w \in \Sigma^* \mid \iota(w) \lhd p\} = \{w \in \Sigma^* \mid K \subseteq \theta^*(w)$ *and* $(\forall \iota(u) \lhd w)\, \nu(u) \cap K \neq \emptyset\}$.
2. *Define the union representation $\kappa^{un} :\subseteq \Sigma^\omega \to \mathcal{K}$ as follows: $\kappa^{un}\langle p, w\rangle = K$ iff*
   - $p \in \mathrm{dom}(\psi^{un})$ *and* $w \in \mathrm{dom}(\nu^*)$,
   - $K = \psi^{un}(p)$ *and* $K \subseteq \theta^*(w)$.
3. *Define the net representation $\kappa^{net} :\subseteq \Sigma^\omega \to \mathcal{K}$ as follows: $K = \kappa^{net}(p)$ iff*
   - $p = \langle w_1, w_2, \ldots\rangle$ *and* $w_i \in \mathrm{dom}(\nu^*)$ *for all* $i \in \mathbb{N}$,
   - $\overline{\theta^*(w_{i+1})} \subseteq \theta^*(w_i)$ *for all* $i \in \mathbb{N}$,
   - $K = \bigcap\limits_{i=1}^{\infty} \theta^*(w_i)$.

A κ^{mc}-name requires that each listed base element has nonempty intersection with K.

For any compact subset of a second countable locally compact space, there exists a "strictly" decreasing cover sequence converging to it. Since every locally compact space is a Hausdorff space, then the limit of a cover sequence is unique. Therefore, κ^{net} is well-defined. Comparing with the cover representation, the advantage of net representation is that it does not require all finite basic covers.

Theorem 5. *For computably locally compact, computably Hausdorff spaces,*

$$\kappa^{mc} \leq \kappa^{c} \equiv \kappa^{un} \equiv \kappa^{net}.$$

Proof. See Appendix.

Note that $\kappa^{c} \leq \kappa^{mc}$ is not true in general [22].

5 Conclusion and Future Work

In this article, we mainly generalize the representations of subsets of metric spaces to locally compact spaces and analyze which relations among these representations still hold. For this reason we define and characterize computably locally compactness and computably Hausdorff spaces.

The next step is to include representations of sets of functions and generalize this research to computable T_0-spaces. Moreover, we will apply these representations to examine the effectivity of certain theorems in general topology.

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6 Appendix: Proofs

Proof of Theorem 1:

Proof. Let $\kappa^c(p) = K$, I be the r.e. set for finite intersection defined in (3) and H the r.e. Hausdorff set. From p a sequence r can be computed such that $\iota(w) \lhd r$ iff

$$(\exists \iota(u) \lhd p \text{ listing } u_0 \cdots u_k)\,(\exists v \in \Sigma^* \text{ listing } v_0 \cdots v_k)$$
$$((u_0, v_0), \ldots, (u_k, v_k) \in H \quad \text{and} \quad (v, w) \in I).$$

Then $\nu(w) \subseteq K^c$ if $\iota(w) \lhd r$. Next we show, that $K^c \subseteq \bigcup_{\iota(w) \lhd r} \nu(w)$ holds. Let $y \in K^c$. For all $x \in K$ there is some $(u, v) \in H$ such that $x \in \nu(u)$ and $y \in \nu(v)$. The familiy of all these $\nu(u)$ covers K. As K is compact it has a finite subcover $\{\nu(u_i) \mid j \in J\}$ covering K. Let v list $\{v_i \mid j \in J\}$. As $y \in \bigcap_{j \in J} \nu(v_j) = \bigcap_{v_j \lhd v} \nu(v_j)$ there is some $w \in \Sigma^*$ such that

$$(v, w) \in I \quad \text{and} \quad y \in \nu(w)$$

Therefore r is a ψ^{un}-name of K.

Proof of Theorem 3:

Proof. $\psi^{fiber} \le \psi^{dom}$: (As in [4]:) Let A be a closed subset of X and $\psi^{fiber}(p) = A$, that is, $\eta_p^{\omega\omega}$ is a (δ, ρ)-realization of a function $f_p : X \to \mathbb{R}$ such that $f_p^{-1}\{0\} = A$. Define $g_p : \subseteq X \to \mathbb{N}$ as follows:

$$g_p(x) = \begin{cases} 1 & \text{if } f_p(x) \neq 0 \\ \uparrow & \text{otherwise} \end{cases}$$

for all $x \in X$. Then $\mathrm{dom}(g_p) = A^c$. As $\eta_p^{\omega\omega}$ is a realization of f_p, we have that, if $\rho\eta_p^{\omega\omega}(q) \neq 0$, $g_p\delta(q) = 1$; otherwise, $g_p\delta(q)$ diverges. By utm-theorem, there is a computable function $G : \subseteq\Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $g_p\delta(q) = \nu_{\mathbb{N}}G(p, q)$. And by smn-theorem for $\eta^{\omega*}$, there is a computable function $H : \Sigma^\omega \to \Sigma^\omega$ such that $g_p\delta(q) = \nu_{\mathbb{N}}\eta^{\omega*}_{H(p)}(q)$. Furthermore, for any $x \notin \mathrm{dom}(g_p)$, we have $f_p(x) = 0$, then $G(p, q)$ diverges. This shows that $\eta^{\omega*}_{H(p)}$ is a strong realization. Therefore, $H(p)$ is a ψ^{dom}-name of A, as required.

$\psi^{dom} \le \psi^{sie}$: (As in [3]) Let $\psi^{dom}(p) = A$, that is, $\eta^{\omega*}$ is a strong $(\delta, \nu_{\mathbb{N}})$-realization of function $f_p : \subseteq X \to \mathbb{N}$ such that $\mathrm{dom}(f_p) = A^c$. Let M be a Type-2 machine computing the universal function of $\eta^{\omega*}$. Define

$$H(p, q) = \begin{cases} 0^\omega & \text{if } M \text{ does not halt on input } (p, q) \\ 0^k1^\omega & \text{if } M \text{ halts on input } (p, q) \text{ after } k \text{ steps} \end{cases}$$

Then H is computable and by utm- and smn-theorem for $\eta^{\omega\omega}$ there exists a computable function F such that $\eta^{\omega\omega}_{F(p)}(q) = H(p, q)$. Now we have $\rho^<\eta^{\omega\omega}_{F(p)}(q) = \rho^< H(p, q) = \mathrm{cf}_A\delta(q)$, that is, $\eta^{\omega\omega}_{F(p)}$ is a $(\delta, \rho^<)$-realization of cf_A. Therefore, $F(p)$ is a ψ^{sie}-name of A, as required.

$\psi^{sie} \le \psi^{un}$: Let $\psi^{sie}(p) = A$ and M be a Type-2 machine computing the universal function of $\eta^{\omega\omega}$. There is a Type-2 machine that on input p computes a sequence r such that $\iota(u) \lhd r$ iff
$(\exists w \in \mathrm{dom}(\nu^*))$ (M on input (p, q) writes some $\langle a\rangle$ such that $\nu_{\mathbb{Q}}(a) > 0$ and M has at most read the prefix w of q) and $(w, u) \in I$, where I is the r.e. set for finite intersection defined in (3).

$\psi^{un} \le \psi^>$: Note that a ψ^{un}-name of A is just a θ^{un}-name of A^c. Since $\bar{U} \subseteq A^c$ is (ν, θ^{un})-r.e. by Lemma 6, we can now construct a Type-2 machine M, which on input a θ^{un}-name p outputs a list of u such that $\overline{\nu(u)} \subseteq A^c$, that is, $\overline{\nu(u)} \cap A = \emptyset$. Therefore, f_M translates ψ^{un} to $\psi^>$.

$\psi^> \le \psi^{fiber}$: By Theorem 2 **X** is computably regular, if it is computably locally compact and a computably Hausdorff space. Then the multivalued Urysohn operator UR mappping every pair (A, B) of disjoint closed sets to all continuous functions $f : X \to [0; 1]$ such that $f[A] = 0$ and $f[B] = 1$ is $(\psi^>, \psi^>, [\delta \to \rho])$-computable ([9]).

Let $\psi^>(p) = A$ and let p be a list of $\{u_i \in \mathrm{dom}(\nu) \mid i \in \mathbb{N}\}$. There is a Type-2 machine M that on input p

- computes a $\psi^>$-name p_i of $\overline{\nu(u_i)}$ for all $i \in \mathbb{N}$ ($U \to \bar{U}$ is (ν, κ^c)-computable and $\kappa^c \le \psi^>$),

– computes a $[\delta \to \rho]$-name q_i of some $f_i \in \mathrm{UR}(\psi^{>}(p), \psi^{>}(p_i))$ for all $i \in \mathbb{N}$,
– computes a $[\delta \to \rho]$-name q of $f : X \to [0;1]$ defined by

$$f(x) := \sum_{i=0}^{\infty} 2^{-i} f_i(x).$$

Then $A = \psi^{fiber}(q)$.

Proof of Theorem 4:

Proof. Let $\psi^{range}(p) = A$. If $A = \emptyset$ then $p = 0^\omega$ and furthermore $\psi^{<}(0^\omega) = \emptyset$. If $A \neq \emptyset$ then

$$\begin{aligned} \nu(w) \cap A \neq \emptyset \iff & \text{ there exists some } x \in \mathrm{range}([\nu_{\mathbb{N}} \to \delta](p)) \text{ such that } x \in \nu(w) \\ \iff & \text{ there exists some } u \in \mathrm{dom}(\nu_{\mathbb{N}}) \text{ such that } w \text{ is listed by } \eta_p(u). \end{aligned}$$

The following Typ-2 machine M realizes the reduction: On input p the machine M copies all zeros on the input tape to the output tape until it reads a symbol $a \neq 0$. Then M writes $\iota(w)$ iff

$$(\exists u \in \mathrm{dom}(\nu_{\mathbb{N}}))\, \iota(w) \lhd u_\eta(p, u).$$

Proof of Theorem 5:

Proof. $\kappa^{mc} \leq \kappa^c$: Let $\kappa^{mc}(p) = K$. As $\mathrm{dom}(\nu^*)$ is r.e., there is a Type-2 machine that on input p computes a sequence $r \in \Sigma^\omega$ such that $\iota(w) \lhd r$ iff

$$(\exists \iota(w') \lhd p)(\exists w'' \in \mathrm{dom}(\nu^*))\, \{u \mid \iota(u) \lhd w\} = \{u \mid \iota(u) \lhd w'\} \cup \{u \mid \iota(u) \lhd w''\}.$$

$\kappa^c \leq \kappa^{un}$: By theorem 1 $\kappa^c \leq \psi^{un}$ for computably Hausdorff spaces. Furthermore if p is a κ^c-name of K then $K \subseteq \theta^*(w)$ for any $\iota(w) \lhd p$.

$\kappa^{un} \leq \kappa^c$: Let $\kappa^{un}\langle p, w\rangle = K$ where w lists $\{v_1, \ldots, v_k\}$. Then $K^c = \psi^{un}(p)$ and $K \subseteq B := \bigcup\{\overline{\nu(v_i)} \mid i = 1, \ldots k\}$ As $\mathbf{X}$ is computably locally compact, there is a Type-2 machine M that on input $(v_1, \ldots, v_k)$ computes a κ^*-name $\langle q_1, \ldots, q_k\rangle \in \Sigma^\omega$ such that $\kappa^c(q_i) = \overline{\nu(v_i)}$ for all $i \in \{1, \ldots, k\}$. By Lemma 2 from $\langle q_1, \ldots, q_k\rangle$ a sequence $q \in \Sigma^\omega$ can be computed with $\kappa^c(q) = B$. By Lemma 3 the mapping $(K, B) \to K \cap B$ is $(\psi^{un}, \kappa^c, \kappa^c)$-computable. As $K = K \cap B$ we have shown $\kappa^{un} \leq \kappa^c$.

$\kappa^{net} \leq \kappa^c$: Let $\kappa^{net}(p) = K$. There is a Type-2 machine that on input p computes a sequence r such that $\iota(w) \lhd r$ iff

$$(\exists \iota(w') \lhd p)\, \overline{\theta^*(w')} \subseteq \theta^*(w).$$

$\kappa^c \leq \kappa^{net}$: Let $\kappa^c(p) = K$, where p is a list of $\{w_i \in \mathrm{dom}(\theta^*) \mid i \in \mathbb{N}\}$ and F the $(\kappa^c, \theta^{un}, \theta^*)$-computable mapping from Lemma 7 with

$$U \in \mathrm{F}(K, O) :\iff K \subseteq U \subseteq \bar{U} \subseteq O.$$

There is a Type-2 machine that on input p

– starts with output $\iota(w_0)$.
– if $\iota(w'_0)\iota(w'_1)\ldots\iota(w'_i)$ has been written on the output tape, then M writes some $\iota(w'_{i+1})$ such that

$$\theta^*(w'_{i+1}) \in \mathrm{F}(\kappa^c(p), \theta^*(w'_i) \cap \theta^*(w_{i+1})).$$