Working with the LR Degrees*

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Abstract. We say that $A \leq_{LR} B$ if every B-random number is A-random. Intuitively this means that if oracle A can identify some patterns on some real γ, oracle B can also find patterns on γ. In other words, B is at least as good as A for this purpose. We propose a methodology for studying the LR degrees and present a number of recent results of ours, including sketches of their proofs.

1 Introduction

The present paper is partly a short version of a longer draft [1] with the full proofs of the results presented here, but it also contains additional very recent material which does not appear in [1]. One of the goals of this work is to present a uniform approach to studying the LR degrees, both globally and locally. So far the known results about this degree structure have mostly been scattered and in papers dealing with a wider range of themes in algorithmic randomness (see for example [11]). An exception is Simpson's recent paper [16] which deals with themes like almost everywhere domination which are very closely related to the LR degrees.

Also, a number of results in this area have been proved via a mix of frameworks like martingales, prefix-free complexity and Martin-Löf tests, with more than one framework sometimes appearing in the same proof (see [11,12]). In contrast, we present proofs of new and old results using only the Martin-Löf approach, i.e. Σ^0_1 classes and (in the relativised case) c.e. operators. We work in the Cantor space 2^ω with the usual topology generated by the basic open intervals $[\sigma] = \{\beta \mid \beta \in 2^\omega \ \wedge \ \sigma \subseteq \beta\}$ (where σ is a finite binary string and $\sigma \subseteq \beta$ denotes that σ is a prefix of β) and the Lebesgue measure generated by $\mu([\sigma]) = 2^{-|\sigma|}$.

We systematically confuse sets of finite strings U with the class of reals which extend some string in U. Thus

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– we write $\mu(U)$ for the measure of the corresponding class of reals
– all subset relations $U \subset V$ where U, V are sets of strings actually refer to the corresponding classes of reals
– Boolean operations on sets of strings actually refer to the same operations on the corresponding classes of reals.

In section 2 we review the basic definition of an oracle Martin-Löf test and make the simple observation that there exists a universal test with certain nice properties that will later be useful. It is worth noting that not all universal (even unrelativised) Martin-Löf tests have the same properties and for some arguments it is convenient to assume that we hold a special test. This is not new—see for example [9] where a property of the test derived by its construction (and not the general definition) is used to show that random sets are effectively immune. In section 3 we give the definition of $\leq_{LR}$ and the induced degree structure and mention some known properties. In section 4 we show that there is a continuum of reals which are LR-reducible to the halting problem and then extend this argument to show that the same is true of any α which is not GL_2. We also give a method for diagonalization in the LR degrees.

In section 5 we study the structure of the computably enumerable LR degrees. The main goal here is to show how techniques from the theory of the c.e. Turing degrees can be transferred to the c.e. LR degrees. We deal with two fundamental techniques: Sacks coding and Sacks restraints. First we show that if A has intermediate c.e. Turing degree then the lower cone of c.e. LR degrees below it properly extends the corresponding cone of c.e. Turing degrees. The second example demonstrates the use of Sacks restraints in the LR context and is a splitting theorem for the LR degrees: every c.e. set can be split into two c.e. sets of incomparable LR degree. Also some results are given concerning further connections between the LR and the Turing degrees.

Most of the proofs are either omitted or given as sketches. The exceptions are the proofs of theorems 7 and 9 which are given in full. For the full proofs we refer the reader to the draft [1] which is available online. Theorems 9 and 14 do not appear in [1].

2 Oracle Martin-Löf Tests

An oracle Martin-Löf test (U_e) is a uniform sequence of oracle machines which output finite binary strings such that if U_e^β denotes the range of the e-th machine with oracle $\beta \in 2^\omega$ then for all $\beta \in 2^\omega$ and $e \in \mathbb{N}$ we have that $\mu(U_e^\beta) < 2^{-(e+1)}$ and $U_e^\beta \supseteq U_{e+1}^\beta$. A real α is called β-random if for every oracle Martin-Löf test (U_e) we have $\alpha \notin \cap_e U_e^\beta$. A universal oracle Martin-Löf test is an oracle Martin-Löf test (U_e) such that for every $\alpha, \beta \in 2^\omega$, α is β-random iff $\alpha \notin \cap_e U_e^\beta$. The following theorem concerns oracle-enumerations of random sets.

Theorem 1. *For every $n \geq 1$ there exist sets which are n-random and which are properly n-c.e. in $\emptyset^{(n)}$.*

Given any oracle Martin-Löf test (U_e), each U_e can be thought of as a c.e. set of axioms $\langle \tau, \sigma \rangle$. If $\beta \in 2^\omega$ then $U_e^\beta = \{\sigma \mid \exists \tau (\tau \subset \beta \ \wedge \ \langle \tau, \sigma \rangle \in U_e)\}$ and for $\rho \in 2^{<\omega}$ we define $U_e^\rho = \{\sigma \mid \exists \tau (\tau \subseteq \rho \ \wedge \ \langle \tau, \sigma \rangle \in U_e)\}$. There is an analogy between oracle Martin-Löf tests as defined above and Lachlan functionals i.e. Turing functionals viewed as c.e. sets of axioms. This analogy will be exploited in a number of constructions below, especially in the constructions of c.e. LR degrees. The following lemma is easily proved and provides a universal oracle Martin-Löf test with properties which will later be useful.

Lemma 1. *There is an oracle Martin-Löf test* (U_e) *such that*

- *For every oracle Martin-Löf test* (V_e)*, uniformly on its c.e. index we can compute* $k \in \mathbb{N}$ *such that for every real* β *and all* e, $V_{e+k}^\beta \subseteq U_e^\beta$.
- *If* $\langle \tau_1, \sigma_1 \rangle, \langle \tau_2, \sigma_2 \rangle \in U_e$ *and* $\tau_1 \subseteq \tau_2$ *then* $\sigma_1 | \sigma_2$.
- *If* $\langle \tau, \sigma \rangle \in U_e$ *then* $|\tau| = |\sigma|$ *and* $\langle \tau, \sigma \rangle \in U_e[|\tau|] - U_e[|\tau| - 1]$.

From the properties of (U_e) as described in lemma 1 we get the following.

Corollary 1. *Let* (U_e) *be the universal oracle Martin-Löf test of lemma 1 and let* U *be any member of it. There is a computable function which, given any input* $\langle \tau, \tau' \rangle$ *such that* $\tau \subseteq \tau'$, *outputs the finite (clopen) set* $U^{\tau'} - U^{\tau}$.

Different Martin-Löf tests may have different properties and some are more useful than others. If we only require the first clause of theorem 1 we can achieve the stronger condition

> For every oracle Martin-Löf test (V_e), uniformly on its c.e. index we can compute $k \in \mathbb{N}$ such that $V_{e+k} \subseteq U_e$ (as sets of axioms) for all e. (1)

The following result demonstrates an application of property (1) of a universal Martin-Löf test (U_e).

Theorem 2. *If* U *is a member of an oracle Martin-Löf test satisfying property (1) and* $T \in \Sigma_1^0$, $\mu(T) < 1$ *then there are only finitely many* $\beta \in 2^\omega$ *such that* $U^\beta \subseteq T$. *Also, there are universal Martin-Löf tests which do not have this property.*

It is worth mentioning that there are tests which satisfy both the conditions of lemma 1 and the property of theorem 2. In fact, a standard construction of the oracle Martin-Löf test of theorem 2 gives a test with these properties. Note that if V is a member of an oracle Martin-Löf test and $T \in \Sigma_1^0$ then the class $\{\beta \mid V^\beta \subseteq T\}$ consists of the infinite paths through a $\mathbf{0}'$ computable tree. Since the paths through a $\mathbf{0}'$ computable tree with only finitely many infinite paths are Δ_2^0, by theorem 3 we get Nies' result [11] that all low for random sets are Δ_2^0.

3 *LR* Reducibility and Degrees

The LR reducibility was introduced in [13].

Definition 1. *[13] Let $A \leq_{LR} B$ if every B-random real is A-random. The induced degree structure is called the LR degrees.*

Intuitively this means that if oracle A can identify some patterns on some real γ, oracle B can also find patterns on γ. In other words, B is at least as good as A for this purpose. It is not hard to show (especially in view of theorem 3) that $\leq_{LR}$ is Σ^0_3 definable and this has been noticed by a number of authors. Being Σ^0_3 means that it has some things in common with $\leq_T$ (which is also Σ^0_3) and this can be seen more clearly in section 5 where techniques from the theory of c.e. Turing degrees are seen to be applicable in the c.e. LR degrees. For more examples of similar Σ^0_3 relations see [16]. We point out (after [13,16]) that a strict relativization of the notion of low for random [8] gives that A is low for random relative to B when $A \oplus B \leq_{LR} B$, which is different than $A \leq_{LR} B$. In particular, $\oplus$ does not define a least upper bound in the LR degrees and it is an open question as to whether any two degrees always have a least upper bound in this structure [13,16].

Theorem 3. *[7] For all $A, B \in 2^\omega$ the following are equivalent:*

- $A \leq_{LR} B$
- *For every $\Sigma^0_1(A)$ class T^A of measure < 1 there is a $\Sigma^0_1(B)$ class V^B such that $\mu(V^B) < 1$ and $T^A \subseteq V^B$.*
- *For some member U^A of a universal Martin-Löf test relative to A there is $V^B \in \Sigma^0_1(B)$ such that $\mu(V^B) < 1$ and $U^A \subseteq V^B$.*

The following result shows how two universal oracle Martin-Löf tests are related (or how 'similar' they are).

Theorem 4. *If (U_i) is an oracle Martin-Löf test, V is a member of a universal oracle Martin-Löf test and $\tau_0, \sigma_0 \in 2^{<\omega}$ such that $[\sigma_0] \not\subseteq \cap_{\gamma \supset \tau_0} V^\gamma$ then there exist $\tau, \sigma \in 2^{<\omega}$ and $m \in \mathbb{N}$ such that*

- $\tau \supset \tau_0$ *and* $\sigma \supset \sigma_0$
- *there is $\beta \supset \tau$ such that $[\sigma] \not\subseteq V^\beta$*
- *for all $\gamma \supset \tau$, $U^\gamma_m \cap [\sigma] \subseteq V^\gamma$.*

A natural question about reducibilities $\preceq$ on the reals is to determine the measure of upper and lower cones. For the Turing reducibility the lower cones are countable (hence they are null) and the non-trivial upper cones have measure 0 [14]. For $\leq_{LR}$ although lower cones are not always countable (see section 4) it is not difficult to show that they are null. Indeed, given A the A-random numbers have measure 1 and so it is enough to show that if β is A-random then $\beta \not\leq_{LR} A$. But this is obvious since β is not β-random.

Theorem 5. *For every A the set $\{\beta \mid \beta \leq_{LR} A\}$ has measure 0.*

For the upper cones it is tempting to think that a version of the majority vote technique which settled the question for $\leq_T$ (see [4] for an updated presentation of the argument) would work for $\leq_{LR}$ (especially if one thinks of randomness in terms of betting strategies). However Frank Stephan pointed out (in discussions with the first author) that the answer is most easily given by an application of van Lambalgen's theorem (a simple theorem with many applications) which asserts that $A \oplus B$ *is random iff* A *is random and* B *is* A*-random.*

Theorem 6 (Frank Stephan). *If* A *is random then it is* B*-random for almost all* $B \in 2^\omega$*. Also, any non-trivial upper cone in the* LR *degrees has measure* 0.

4 Global Structure

In computability theory we are used to structures in which every degree has only countably many predecessors. Below we show that the LR degrees do not have this property[1] and that, in fact, whenever α is not GL_2 the degree of α has an uncountable number of predecessors.

Lemma 2. *Let* U *be a member of an oracle Martin-Löf test,* $n \in \mathbb{N}$ *and* $\tau_0 \in 2^{<\omega}$*. Then there exists* $\tau_1 \supset \tau_0$ *such that for all* $\tau_2 \supset \tau_1$*,* $\mu(U^{\tau_2} - U^{\tau_1}) < 2^{-n}$*.*

In [13] (see [16] for a different proof and more detailed presentation) it was shown that the LR degrees are countable equivalence classes.

Theorem 7. *In the* LR *degrees the degree of* $\emptyset'$ *bounds* $2^{\aleph_0}$ *degrees.*

Proof. By cardinal arithmetic it is enough to show that the set $\mathcal{B} = \{\beta \mid \beta \leq_{LR} \emptyset'\}$ has cardinality $2^{\aleph_0}$. Let U be the second member of the universal oracle Martin-Löf test of lemma 1, so that by definition $\mu(U^\beta) < 2^{-2}$ for all $\beta \in 2^\omega$. It suffices to define a $\emptyset'$-computable perfect tree T (as a downward closed set of strings) such that $\mu(A) < \frac{1}{2}$ where $A = \cup_{\tau \in T} U^\tau$. Then $|[T]| = 2^{\aleph_0}$ (where $[T]$ is the set of infinite paths through T), and for all $\beta \in [T]$, $U^\beta \subseteq A$. Since A is $\emptyset'$-c.e., we have by theorem 3 that for all $\beta \in [T]$, $\beta \leq_{LR} \emptyset'$. We ask that $\mu(A) < \frac{1}{2}$ (rather than $\mu(A) < 1$) simply in order that figures used should be in line with what appears in the proof of theorem 8 in [1]. It remains to define such a tree T and verify the construction.

First find a string τ such that for any extension τ' of τ, $\mu(U^{\tau'} - U^\tau) < 2^{-4}$ and define $T(\emptyset) = \tau$. The existence of such a string is ensured by lemma 2. Note that $\mu(U^{T(\emptyset)}) < 2^{-2}$. Now for each of the one element extensions of $T(\emptyset)$, say τ_i, $i = 0, 1$ find some extension $\tau_i' \supseteq \tau_i$ such that for any $\tau' \supset \tau_i'$ we have $\mu(U^{\tau'} - U^{\tau_i'}) < 2^{-6}$. Define $T(0) = \tau_0'$, $T(1) = \tau_1'$ and note that $\mu((U^{T(0)} \cup U^{T(1)}) - U^{T(\emptyset)}) < 2 \cdot 2^{-4} = 2^{-3}$ by the previous step. Continue in the same way so that at the n-th stage, where we define $T(\sigma)$ for all σ with $|\sigma| = n$, we choose a value τ for $T(\sigma)$ such that for all $\tau' \supset \tau$ we have $\mu(U^{\tau'} - U^\tau) < 2^{-(2n+4)}$. Let

$$C_n = \{T(\sigma) \mid \sigma \in 2^{<\omega} \ \wedge \ |\sigma| \leq n\}$$

and note that $C_n \subseteq C_{n+1}$. Also let $A_n = \cup_{\tau \in C_n} U^\tau$ and note that $A_n \subseteq A_{n+1}$ and $A = \cup_n A_n$. By induction, for all n, $\mu(A_n) < \sum_{i=0}^n 2^i \cdot 2^{-(2i+2)} = \frac{1}{2}$. Note

[1] Joe Miller and Yu Liang have independently announced the existence of an LR degree with uncountably many predecessors.

that the factor 2^i in the above sum comes from the number of strings of level i in T (and where we say that τ is of level i in T if $\tau = T(\sigma)$ for σ of length i). It remains to show that we can run the construction of T computably in $\emptyset'$, but this follows immediately from corollary 1.

After we proved theorem 7 and since high degrees often resemble $\mathbf{0}'$, we considered showing that every high LR degree has uncountably many predecessors. Using a combination of highness techniques from [6,10,15] we succeeded in showing that if A is *generalized superhigh* (i.e. $A' \geq_{tt} (A \oplus \emptyset')'$) then A has uncountably many $\leq_{LR}$-predecessors. The following theorem is a stronger result showing that if A is merely $\overline{GL_2}$ (i.e. generalized non-low$_2$, $A'' >_T (A \oplus \emptyset')'$) then it has the same property. For other $\overline{GL_2}$ constructions we refer the reader to [10].

Theorem 8. *If α is $\overline{GL_2}$ then in the LR degrees the degree of α bounds $2^{\aleph_0}$ degrees.*

The basic idea behind the proof remains the same as in the proof of theorem 7 but now we need to define T using only an oracle for α (rather than an oracle for $\emptyset'$) and α-approximate a perfect tree $T^* \subseteq T$ during the course of the construction. By theorem 8 and a cardinality argument we obtain the following.

Corollary 2. *There are $A <_{LR} B$ such that for every $A_0 \equiv_{LR} A$, $B_0 \equiv_{LR} B$ we have $A_0 |_T B_0$. In fact for every $\overline{GL_2}$ set B there is A with the above property.*

Next, we provide a method for destroying LR reductions (a kind of diagonalization). As an illustration of this method we construct an antichain of LR degrees of cardinality $2^{\aleph_0}$.

Theorem 9. *There exists an antichain of cardinality $2^{\aleph_0}$ in the LR degrees.*

Proof. We wish to define a perfect tree T such that, for all distinct $A, B \in [T]$, $A \not\leq_{LR} B$. In order to do so, will make use of the following lemma which was originally proved by Kučera and which is frequently very useful in dealing with Π^0_1 classes of positive measure. For a very simple proof we refer the reader to [5].

Lemma 3. *[9] Given any Π^0_1 class $\mathcal{P}$ of positive measure there exists a Π^0_1 class of positive measure $\mathcal{K}(\mathcal{P}) \subseteq \mathcal{P}$ such that the intersection of $\mathcal{K}(\mathcal{P})$ with any Π^0_1 class is either empty or of positive measure.*

Fix a member U of a universal oracle Martin-Löf test. Assume we are given an effective listing $\{V_e\}_{e\in\omega}$ of all c.e. operators V for which there exists $q \in \mathbb{Q}$ such that for all A, $\mu(V^A) < 1-q$. We say A is LR reducible to B via V_e if $U^A \subseteq V_e^B$. Clearly $A \leq_{LR} B$ iff $A \leq_{LR} B$ via some V_e. For each e we must ensure that for all distinct $A, B \in [T]$, A is not LR reducible to B via V_e. The following lemma provides a basic diagonalization technique for the $\leq_{LR}$ reducibility.

Lemma 4. *For any e and any $\tau_0, \tau_1, \mathcal{P}_0, \mathcal{P}_1$ such that $\tau_0, \tau_1 \in 2^{<\omega}$, $\mathcal{P}_0 \subseteq [\tau_0]$, $\mathcal{P}_1 \subseteq [\tau_1]$ and $\mathcal{P}_0$, $\mathcal{P}_1$ are Π^0_1 classes of positive measure there exist $\tau'_0, \tau'_1, \mathcal{P}'_0, \mathcal{P}'_1$ such that*

– $\tau_0' \supseteq \tau_0, \tau_1' \supseteq \tau_1$,
– $\mathcal{P}_0' \subseteq \mathcal{P}_0$, $\mathcal{P}_1' \subseteq \mathcal{P}_1$ *and* $\mathcal{P}_0', \mathcal{P}_1'$ *are* Π_1^0 *classes of positive measure,*
– $\mathcal{P}_0' \subseteq [\tau_0']$, $\mathcal{P}_1' \subseteq [\tau_1']$,
– *If* $A \in \mathcal{P}_0'$ *and* $B \in \mathcal{P}_1'$ *then* A *is not* LR *reducible to* B *via* V_e.

Proof. First we define $\mathcal{Q}_0 = \mathcal{K}(\mathcal{P}_0)$, where $\mathcal{K}$ is as defined in the statement of lemma 3. Now let A be any member of $\mathcal{Q}_0$ such that $A \not\leq_{LR} \emptyset$ (hence $\{\beta \mid A \leq_{LR} \beta\}$ is null). For any $\tau \subset A$ we have that $\mathcal{Q}_0 \cap [\tau]$ is of positive measure. We define $\mathcal{Q}_1$ to be the set of all $B \in \mathcal{P}_1$ such that A is not LR reducible to B via V_e. Since upper cones in the LR degrees are of measure 0, $\mathcal{Q}_1$ is of positive measure. We define for each σ, $\mathcal{Q}_{1,\sigma} = \{B : \ B \in \mathcal{P}_1 \text{ and } [\sigma] \not\subseteq V_e^B\}$. Since a countable union of sets of measure 0 is of measure 0 and $\mathcal{Q}_1 = \bigcup_{\sigma \in U^A} \mathcal{Q}_{1,\sigma}$ there exists $\sigma \in U^A$ such that $\mathcal{Q}_{1,\sigma}$ is of positive measure. Letting σ be such, we define $\tau_0' \supset \tau_0$ to be an initial segment of A such that $\sigma \in U^{\tau_0'}$. We define $\tau_1' = \tau_1$, $\mathcal{P}_0' = \mathcal{P}_0 \cap [\tau_0']$ and $\mathcal{P}_1' = \mathcal{Q}_{1,\sigma}$.

It is now clear how to use lemma 4 in order to define T. Suppose that at stage n we have already defined $T(\sigma)$ for all σ of length $\leq n$ and that for each leaf τ of T (as presently defined) we have specified some Π_1^0 class of positive measure $\mathcal{P}_\tau$ such that all strings in T extending τ must lie in $\mathcal{P}_\tau$. For each leaf τ we first choose two incompatible extensions τ_0, τ_1 such that for each $i \leq 1$, $\mathcal{P}_\tau \cap [\tau_i]$ is of positive measure. These are potential leaves of T for the next stage. Through successive applications of lemma 4 to all pairs of potential leaves we can then define $T(\sigma)$ and $\mathcal{P}_\tau$ for all σ of length $n+1$ and each $\tau = T(\sigma)$, in such a way that if A and B extend $\tau_0 = T(\sigma), \tau_1 = T(\sigma')$ respectively for distinct strings σ and σ' of length $n+1$ and $A \in \mathcal{P}_{\tau_0}, B \in \mathcal{P}_{\tau_1}$, then A is not LR reducible to B via V_n. Since for each n there exist an infinite number of n' with $V_n = V_{n'}$, this completes the proof of the theorem.

5 Computably Enumerable *LR* Degrees

In this section we study the structure of the c.e. LR degrees and their relationship with the Turing reducibility. The results have been chosen so that they demonstrate how to transfer selected basic techniques from the c.e. Turing degrees (like Sacks coding and restraints) to the c.e. LR degrees. We note that the relationship between $\leq_{LR}$ and $\leq_T$ is nontrivial and goes beyond what we discuss here. For example there is a half of a minimal pair in the c.e. Turing degrees which is LR-complete [2,3]. The first author, using methods similar to those in [2], has shown that there is a noncuppable c.e. Turing degree which is LR-complete. This implies that every c.e. set which is computable by all LR-complete c.e. sets must be noncuppable. It is unknown if there are such noncomputable sets. For background in the theory of c.e. degrees we refer the reader to [17]. The following theorem demonstrates how infinitary Sacks coding can be handled in the LR degrees.

Theorem 10. *If* W *is an incomplete c.e. set, i.e.* $\emptyset' \not\leq_T W$*, then (uniformly in* W*) there is a c.e. set* B *such that* $B \leq_{LR} W$ *and* $B \not\leq_T W$.

We sketch the proof. A relativisation of the classic non-computable low for random argument of [8] (also see [4]) merely gives that for all A there exists B c.e. in A such that $B \not\leq_T A$ and $A \oplus B \leq_{LR} A$. If we assumed that W is low we could prove theorem 10 with a finitary argument similar to [8] by using a lowness technique (namely Robinson's trick). To prove the full result we need infinitary coding combined with cost efficiency considerations (see [12] for examples of cost-function arguments). We need to construct a c.e. operator V and a c.e. set B such that $U^B \subseteq V^W$ where U is a member of the universal oracle Martin-Löf test of lemma 1 (so that $\mu(U) < 2^{-1}$), $\mu(V^W) < 1$, and the following requirements are satisfied

$$P_e : \Phi_e^W = B \Rightarrow \Gamma_e^W = \emptyset'$$

where (Φ_e) is an effective enumeration of all Turing functionals and Γ_e are Turing functionals constructed by us. It is useful if we assume the *hat trick* for the functionals as well as the c.e. operators U, V (see [17] for more on this). This means, for example, that there will be infinitely many stages where (the current approximation to) U^B contains only permanent strings. The operator V can be defined ahead of the construction and it essentially enumerates into V the strings of U^B with large use. We can also make sure that V is enumerated in a prefix-free way. By such a definition of V we immediately get that $U^B \subseteq V^W$ is satisfied. So the main conflict we face is that on the one hand we want a Sacks coding for each of the P_e requirements (enumerations into $\emptyset'$ may trigger B-enumerations infinitely often) and on the other hand B-enumerations may force $\mu(V^W) = 1$ (via the the way that V is defined). The connection between B-enumerations and superfluous measure in V^W (in the sense that it does not serve $U^B \subseteq V^W$, it corresponds to intervals which are not in U^B) is roughly as in the noncomputable low for random construction of [8]: some interval σ is enumerated into U^B with use u, it enters V^W with use v and subsequently $B \restriction u$ changes thus ejecting σ from U^B. Then $W \restriction v$ could freeze, thus capturing a useless interval in V^W. We already have $\mu(V^W) \geq \mu(U^B)$ so we want to make sure that the measure corresponding to useless strings is bounded by 2^{-1}.

Here, however, we have an advantage over the classic argument in [8] as W may also change, thus extracting the useless string from V^W. We use this fact in order to make infinitary coding into B possible while satisfying $\mu(V^W) < 1$. The full proof can be found in [1]. This approach works even if we require $U^{B \oplus W} \subseteq V^W$ instead of $U^B \subseteq V^W$. In that case we obtain $B \oplus W \equiv_{LR} W$, $W <_T B \oplus W$ and hence the following theorem, given that there are T-incomplete sets in the complete LR degree and the known embedding results for the c.e. Turing degrees (an antichain is embeddable in every nontrivial interval).

Theorem 11. *Every c.e. LR degree contains infinitely many c.e. Turing degrees (in the form of chains and antichains) and every incomplete c.e. LR degree has no maximal c.e. Turing degree.*

As far as the global structure is concerned, we can get a similar result by relativising known constructions of low for random degrees. In particular, the relativised noncomputable low for random construction [8] gives that for every B there is A which is B-c.e. and $A \oplus B \equiv_{LR} B$, $B <_T A \oplus B$; and a slight extension of the argument gives that every B is T-below an antichain of T degrees in the same LR degree, hence the following theorem.

Theorem 12. *Every LR degree contains infinitely many Turing degrees (in the form of chains and antichains) and no maximal Turing degree.*

Next we show a splitting theorem which also shows how Sacks restraints work in the LR degrees.

Theorem 13. *If A is c.e. and not low for random then there are c.e. B, C such that $B \cap C = \emptyset$, $B \cup C = A$, $B \not\leq_{LR} C$ and $C \not\leq_{LR} B$.*

Proof. Here is a sketch of the proof. The main idea is as in the classic Sacks splitting theorem. We just have to translate the main tools like *length of agreement* and *Sacks restraints* to the case of LR reductions. This will not be a problem as $\leq_{LR}$ is Σ_3^0. Fix a member U of a universal oracle Martin-Löf test; an LR reduction is defined via a c.e. operator V (as opposed to a Turing functional), a $q \in \mathbb{Q}$ and A is LR reducible to B via V, q if

$$\mu(V^B) < 1 - q \text{ and } U^A \subseteq V^B. \tag{2}$$

To define the length of agreement $\ell(U^A, V^B)$ of this possible reduction consider computable enumerations of U, V, A, B. Let M_s be the set of strings σ such that $\sigma \in U_s^{A_t}$ for some $t \leq s$ and let (σ_s) be a computable enumeration of $M = \cup_s M_s$. Now for all s we define $\ell(U^A, V^B)[s]$ to be the maximum n such the following hold:

– $\sigma_n[s]\downarrow$ (i.e. the nth member of M has been enumerated by stage s)
– $\forall i \leq m \ ([\sigma_i] \subseteq V_s^{B_s} \ \vee \ \sigma_i \notin U_s^{A_s})$
– $\mu(\{\sigma_i \mid i < n \ \wedge \ [\sigma_i] \subseteq V_s^{B_s}\}) < 1 - q$.

It is clear that reduction (2) is total iff $\liminf_s \ell(U^A, V^B)[s] = \infty$. Now in general, if we wish to destroy a given reduction like (2) where A is a given c.e. set of nontrivial LR degree and B is enumerated by us, its enough if we respect the following restraint at every stage s:

$$r(V, q, s) = \mu t \ [\forall i \leq \ell(U^A, V^B)[s] \ ([\sigma_i] \subseteq V_s^{B_s} \text{ with } B\text{-use} < t \ \vee \ \sigma_i \notin U_s^{A_s})].$$

Indeed, it can be shown that if $r(V, q, s)$ is respected for a cofinite set of stages then

$$\lim_s \ell(U^A, V^B)[s] < \infty. \tag{3}$$

So either there is a stage where the measure goes over the threshold $1 - q$, or there is some i such that σ_i is a permanent resident of U^A and σ_i is never covered by strings in V^B. In any case (2) is destroyed and the restraint comes to a limit. This is all we need in order to apply the classic Sacks splitting argument (see [17] for a presentation). For more details we refer to [1].

The Sacks restraints argument in theorem 13 works exactly as in the Turing degrees, only that the restraints are defined in a different way. Hence it is natural to ask whether given a noncomputable B we can run the restraints argument in the Turing degrees, constructing some A such that $B \not\leq_T A$, while we code part of B into A so that $B \leq_{LR} A$. It turns out that this is possible and the reason is that compared to the Turing restraints, the LR restraints are more demanding. For example a single B-enumeration below the length of agreement of some potential reduction $\Phi^A = B$ (accompanied by the existing restraint) suffices in order to destroy the reduction; but no single such enumeration suffices in order to destroy a potential LR reduction $U^B \subseteq V$.

Theorem 14. *Given noncomputable c.e. B there is a c.e. A such that $B \not\leq_T A$ and $B \leq_{LR} A$. Moreover A can be chosen such that $A <_T B$.*

References

1. Barmpalias, G., Lewis, A.E.M., Soskova, M.: Randomness, Lowness and Degrees, Draft submitted for publication, 19 pages, current version available at `http://www.maths.leeds.ac.uk/~georgeb/`.
2. Barmpalias G., Montalban, A.: A cappable almost everywhere dominating computably enumerable degree, Electronic Notes in Theoretical Computer Science, Volume **167** (2007).
3. Binns, S., Kjos-Hanssen, B., Miller, J.S., Solomon, R.: Lowness notions, measure and domination, in preparation.
4. Downey, R., Hirschfeldt, D.: Algorithmic Randomness and Complexity, in preparation. Current draft available at `http://www.mcs.vuw.ac.nz/~downey/`.
5. Downey, R., Miller, J.S.: A basis theorem for Π^0_1 classes of positive measure and jump inversion for random reals, Proceedings of the American Mathematical Society **134** (1), pages 283-288 (2006).
6. Jockusch, C.G.: Simple proofs of some theorems on high degrees of unsolvability, Canadian Journal of Mathematics **29** (1977)
7. Kjos-Hanssen, B.: Low for random reals and positive-measure domination, to appear in the Proceedings of the AMS.
8. Kučera, A., Terwijn, S.A.: Lowness for the class of random sets, Journal of Symbolic Logic, vol. **64** (1999) 1396–1402.
9. Kučera, A.: Measure, Π^0_1 classes and complete extensions of PA. In Recursion theory week (Oberwolfach, 1984), Volume **1141** of Lecture Notes in Math., pages 245-259. Springer, Berlin, 1985.
10. Lerman, M.: Degrees of Unsolvability: Local and Global Theory, Springer-Verlag (July 1983)
11. Nies, A.: Low for random sets: the story. Unpublished draft which is available at the authors webpage `http://www.cs.auckland.ac.nz/~nies/`.
12. Nies, A.: Computability and Randomness, monograph to appear. Current draft available at `http://www.cs.auckland.ac.nz/~nies/`.
13. Nies, A.: Lowness properties and randomness. Advances in Mathematics, **197**: 274-305 (2005).
14. Sacks, G.: Degrees of Unsolvability, Princeton University Press, 1963.

15. Shore R., Slaman, T.: Working below a high recursively enumerable degree, Journal of Symbolic Logic **58** (1993), 824–859.
16. Simpson, S.G.: Almost everywhere domination and superhighness. *Draft* available at `http://www.math.psu.edu/simpson/`.
17. Soare, R.I.: Recursively enumerable sets and degrees, Berlin London: Springer-Verlag, 1987