Elementary Differences Among Jump Hierarchies

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Abstract. It is shown that $\mathrm{Th}(\mathbf{H}_1) \neq \mathrm{Th}(\mathbf{H}_n)$ holds for every $n > 1$, where $\mathbf{H}_m$ is the upper semi-lattice of all high_m computably enumerable (c.e.) degrees for $m > 0$, giving a first elementary difference among the highness hierarchies of the c.e. degrees.

1 Introduction

Let $n \geq 0$. We say that a *computably enumerable* (c.e.) degree $\mathbf{a}$ is high_n (or low_n), if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$ (or $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$), where $\mathbf{x}^{(n+1)} = (\mathbf{x}^{(n)})'$, $\mathbf{x}^{(0)} = \mathbf{x}$, $\mathbf{y}'$ is the *Turing jump* of $\mathbf{y}$. Let $\mathbf{H}_n$ ($\mathbf{L}_n$) be the set of all high_n (low_n) c.e. degrees. For $n = 1$, we also call an element of $\mathbf{H}_1$ (or $\mathbf{L}_1$) high (or low).

Sacks [1963] showed a (Sacks) *Jump Theorem* that for any degrees $\mathbf{s}$ and $\mathbf{c}$, if $\mathbf{s}$ is c.e.a in $\mathbf{0}'$ and $\mathbf{0} < \mathbf{c} \leq \mathbf{0}'$, then there exists a c.e. degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{s}$ and $\mathbf{c} \not\leq \mathbf{a}$, and that there exists a non-trivial high c.e. degree. Note that an easy priority injury argument gives a nonzero low c.e. degree. By relativising the construction of high and low c.e. degrees to $\mathbf{0}^{(n)}$ and using the Sacks Jump Theorem, it follows that for all n, $\mathbf{H}_n \subset \mathbf{H}_{n+1}$ and $\mathbf{L}_n \subset \mathbf{L}_{n+1}$. And Martin [1966a], Lachlan [1965] and Sacks [1967] each proved that the union of the high/low hierarchies does not exhaust the set $\mathcal{E}$ of the c.e. degrees. And Sacks [1964] proved the (Sacks) *Density Theorem* of the c.e. degrees. While early researches were aiming at characterisations of the high/low hierarchy. The first result on this aspect is the *Martin* [1966b] *Characterisation of High Degrees*: A set A satisfies $\emptyset^{''} \leq_{\mathrm{T}} A^{'}$ iff there is a function $f \leq_{\mathrm{T}} A$ such that f dominates all computable functions. And Robinson [1971a] proved a *Low Splitting Theorem* that if $\mathbf{c} < \mathbf{b}$ are c.e. degrees and $\mathbf{c}$ is low, then there are c.e. degrees $\mathbf{x}, \mathbf{y}$ such that $\mathbf{c} < \mathbf{x}, \mathbf{y} < \mathbf{b}$ and $\mathbf{x} \vee \mathbf{y} = \mathbf{b}$. In the proof of this theorem, a characterisation of low c.e. degrees was given. The lowness is necessary, because Lachlan [1975] proved a *Nonsplitting Theorem* that for some c.e. degrees $\mathbf{c} < \mathbf{b}$, $\mathbf{b}$ is

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not splittable over $\mathbf{c}$. The strongest nonsplitting along this line has been given by Cooper and Li [2002]: there exists a low$_2$ c.e. degree above which $\mathbf{0}'$ is not splittable.

Extending both the Sacks Jump Theorem and the Sacks Density Theorem, Robinson [1971b] proved an *Interpolation Theorem*: given c.e. degrees $\mathbf{d} < \mathbf{c}$ and a degree $\mathbf{s}$ c.e. in $\mathbf{c}$ with $\mathbf{d}' \leq \mathbf{s}$, there is a c.e. degree $\mathbf{a}$ such that $\mathbf{d} < \mathbf{a} < \mathbf{c}$ and $\mathbf{a}' = \mathbf{s}$. Using this theorem, we can transfer some results from lower levels to higher levels of the high/low hierarchy. For instance, every high c.e. degree bounds a properly high$_n$, and a properly low$_n$ c.e. degree for each $n > 0$, so any ideal I of $\mathcal{E}$ contains an element of $\mathbf{H}_1$ will contain elements of $\mathbf{H}_{n+1} - \mathbf{H}_n$, $\mathbf{L}_{n+1} - \mathbf{L}_n$ for all $n > 0$. However the transfer procedure is constrained by the non-uniformity of the Robinson Interpolation Theorem.

Based on Martin's Characterisation of High Degrees, Cooper [1974a] proved that every high$_1$ c.e. degree bounds a minimal pair. And Lachlan [1979] showed that there exists a nonzero c.e. degree which bounds no minimal pair. And Cooper [1974b] and Yates proved a *Noncupping Theorem*: there exists nonzero c.e. degree $\mathbf{a}$ such that for any c.e. degree $\mathbf{x}$, $\mathbf{a} \vee \mathbf{x} = \mathbf{0}'$ iff $\mathbf{x} = \mathbf{0}'$. This result was further extended by *Harrington* [1976] *Noncupping Theorem*: for any high$_1$ c.e. degree $\mathbf{h}$, there exists a high$_1$ c.e. degree $\mathbf{a} \leq \mathbf{h}$ such that for any c.e. degree $\mathbf{x}$, if $\mathbf{h} \leq \mathbf{a} \vee \mathbf{x}$, then $\mathbf{h} \leq \mathbf{x}$. In contrast Harrington [1978] also proved a *Plus Cupping Theorem* that there exists c.e. degree $\mathbf{a} \neq \mathbf{0}$ such that for any c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{0} < \mathbf{x} \leq \mathbf{a} \leq \mathbf{y}$, then there is a c.e. degree $\mathbf{z} < \mathbf{y}$ such that $\mathbf{x} \vee \mathbf{z} = \mathbf{y}$. And remarkably, Nies, Shore and Slaman [1998] have shown that $\mathbf{H}_n$, $\mathbf{L}_{n+1}$ are definable in $\mathcal{E}$ for each $n > 0$.

A basic question about the high/low hierarchies is the following:

Question 1.1. (i) Are there any $m \neq n$ such that $\mathrm{Th}(\mathbf{L}_m) = \mathrm{Th}(\mathbf{L}_n)$?

(ii) Are there any $m \neq n$ such that $\mathrm{Th}(\mathbf{H}_m) = \mathrm{Th}(\mathbf{H}_n)$?

Since this paper was written in 2001, part (i) has been answered negatively. Jockusch, Li, and Yang [2004] proved a nice join theorem for the c.e. degrees: For any c.e. degree $\mathbf{x} \neq \mathbf{0}$, there is a c.e. degree $\mathbf{a}$ such that $(\mathbf{x} \vee \mathbf{a})'' = \mathbf{0}'' = \mathbf{a}''$. Cholak, Groszek and Slaman [2001] showed that there is a nonzero c.e. degree $\mathbf{a}$ which joins to a low degree with any low c.e. degree. By combining the two theorems above, we have that for any $n > 1$, $\mathrm{Th}(\mathbf{L}_1) \neq \mathrm{Th}(\mathbf{L}_n)$. While the remaining case was resolved by Shore [2004] by using coding of arithmetics.

For part (ii) of question 1.1, we know nothing, although Cooper proved that every high c.e. degree bounds a minimal pair, and Downey, Lempp and Shore [1993] (and independently both Lerman and Kučera) could construct a high$_2$ c.e. degree which bounds no minimal pair.

In this paper, we show that

Theorem 1.2. There exists a high$_2$ c.e. degree $\mathbf{a}$ such that for any c.e. degrees $\mathbf{x}, \mathbf{y}$, if $\mathbf{0} < \mathbf{x} \leq \mathbf{a} \leq \mathbf{y}$, then there is a c.e. degree $\mathbf{z}$ such that $\mathbf{z} < \mathbf{y}$ and $\mathbf{x} \vee \mathbf{z} = \mathbf{y}$. Then we have:

Theorem 1.3. For each $n > 1$, $\mathrm{Th}(\mathbf{H}_1) \neq \mathrm{Th}(\mathbf{H}_n)$.

Proof. Let P be the following

$$\forall \mathbf{x} \exists \mathbf{a} \leq \mathbf{x} \forall \mathbf{y} (\mathbf{a} \vee \mathbf{y} = \mathbf{0}' \leftrightarrow \mathbf{y} = \mathbf{0}').$$

By Harrington's Noncupping Theorem, P holds for $\mathbf{H}_1$. Note that for any incomplete c.e. degree $\mathbf{a}$, there is an incomplete high c.e. degree $\mathbf{h} \geq \mathbf{a}$. So by theorem 1.2, for each $n > 1$, P fails to hold for $\mathbf{H}_n$.

Theorem 1.3 follows. □

This gives a partial solution to question 1.1 (ii), while the remaining case of it is still an intriguing open question.

We now outline the proof of theorem 1.2. In section 2, we formulate the conditions of the theorem by requirements, and describe the strategies to satisfy the requirements; in section 3, we arrange all strategies on nodes of a tree, the *priority tree* T and analyse the consistency of the strategies.

Our notation and terminology are standard and generally follow Soare [1987].

2 Requirements and Strategies

In this section, we formulate the conditions of theorem 1.2 by requirements, and describe the strategies to satisfy the requirements.

The Requirements. To prove theorem 1.2, we construct a c.e. set A, a Turing functional Γ to satisfy the following properties and requirements,

(1) For any x, y, z, $\Gamma(A; x, y, z)$ is defined.
(2) For any x, y, $\lim_z \Gamma(A; x, y, z)$ exists.
(3) For any x, $\lim_y \lim_z \Gamma(A; x, y, z)$ exists.
$\mathcal{P}_x$: $\emptyset'''(x) = \lim_y \lim_z \Gamma(A; x, y, z)$
$\mathcal{R}_e$: $W_e = \Phi_e(A) \rightarrow (\exists X_e, \Omega_e)[X_e \leq_{\mathrm{T}} V_e \oplus A \,\&\, V_e \oplus A = \Omega_e(W_e, X_e) \,\&\, (\text{a.e.}i)\mathcal{S}_{e,i}]$
$\mathcal{S}_{e,i}$: $[W_e = \Phi_e(A) \;\&\; A = \Psi_i(X_e)] \rightarrow W_e \leq_{\mathrm{T}} \emptyset$

where $x, y, z, e, i \in \omega$, $\{(W_e, \Phi_e, V_e) \mid e \in \omega\}$ is an effective enumeration of all triples (W, Φ, V) of c.e. sets W, V and of Turing functionals Φ, $\{\Psi_i \mid i \in \omega\}$ is an effective enumeration of all Turing functionals Ψ, X_e is a c.e. set built by us, Ω_e is a Turing functional built by us for each $e \in \omega$.

Clearly meeting the requirements is sufficient to prove the theorem. We assume that the use function ϕ of a given Turing functional Φ is increasing in arguments, nondecreasing in stages. We now look at the strategies to satisfy the requirements.

A $\mathcal{P}$-Strategy. Since $\emptyset''' \in \Sigma_3$, we can choose a c.e. set J such that for all x, both (i) and (ii) below hold,

(i) $x \in \emptyset'''$ iff $(\text{a.e.}y)[J^{[\langle y,x\rangle]} = \omega^{[\langle y,x\rangle]}]$.
(ii) $x \notin \emptyset'''$ iff $(\forall y)[J^{[\langle y,x\rangle]} =^* \emptyset]$.

To satisfy $\mathcal{P}_x$, we introduce infinitely many subrequirements $\mathcal{Q}_{x,y}$ for all $y \in \omega$. $\mathcal{Q}$-strategies will define and rectify the Turing functional Γ. Before describing the $\mathcal{Q}$-strategies, we look at some properties of Γ.

Γ-Rules. We ensure that the Turing functional Γ will satisfy the following properties, which are called Γ-*rules*.

(i) Whenever we define $\Gamma(A; x, y, z)$, we locate it at a node ξ say.

Let $\Gamma(A; x, y, z)[s]$ be located at ξ.

(ii) $\gamma(x, y, z)[s] \downarrow \neq \gamma(x, y, z)[s+1]$ iff $\gamma(x, y, z)[s]$ is enumerated into $A_{s+1} - A_s$ iff there is a strategy $\xi' <_{\mathrm{L}} \xi$ which is visited at stage $s + 1$.

Therefore for all x, y, z, the permanent computation $\Gamma(A; x, y, z)$ is the computation which is located at a node, ξ say, at a stage, s say, such that there is no $\alpha <_{\mathrm{L}} \xi$ which can be visited at any stage $v > s$.

A $\mathcal{Q}$-Strategy. Given a $\mathcal{Q}_{x,y}$-strategy σ, we use J^σ to denote the set $J^{[\langle y,x\rangle]}$ which is measured by σ. We say that s is σ-*expansionary*, if $J^\sigma[v] \subset J^\sigma[s]$ for all $v < s$ at which some $\alpha \supseteq \sigma$ is visited. Then σ will proceed as follows.

1. If s is σ-expansionary, then
   – let $\langle y', z' \rangle$ be the least pair $\langle m, n \rangle$ such that $m \geq y$ and $\Gamma(A; x, m, n)$ is not defined,
   – define $\Gamma(A; x, y', z') \downarrow = 1$ with $\gamma(x, y', z')$ fresh in the sense that it is the least natural number greater than any number mentioned so far, and
   – locate $\Gamma(A; x, y', z')$ at $\sigma\hat{\ }\langle 0 \rangle$.
2. Otherwise, then
   – let z' be the least n such that $\Gamma(A; x, y, n) \uparrow$,
   – define $\Gamma(A; x, y, z') \downarrow = 0$ with $\gamma(x, y, z')$ fresh, and locate it at $\sigma\hat{\ }\langle 1 \rangle$.

So the *possible outcomes of* σ are $0 <_{\mathrm{L}} 1$ to denote infinite and finite actions respectively. By the strategy, if there are infinitely many σ-expansionary stages, then for almost every pair $\langle y', z' \rangle$ with $y' \geq y$, $\Gamma(A; x, y', z') \downarrow = 1$ is defined and located at $\sigma\hat{\ }\langle 0 \rangle$. In this case, $\lim_y \lim_z \Gamma(A; x, y, z) \downarrow = 1$, and by the choice of J, $x \in \emptyset'''$. $\mathcal{P}_x$ is satisfied. Otherwise, then by the $\mathcal{Q}_{x,y}$-strategy σ, we have that for almost every z, $\Gamma(A; x, y, z) \downarrow = 0$ is defined and located at $\sigma\hat{\ }\langle 1 \rangle$, so that $\lim_z \Gamma(A; x, y, z) \downarrow = 0$, giving $\lim_y \lim_z \Gamma(A; x, y, z) \downarrow = 0$. Therefore in any case, $\mathcal{P}_x$ is satisfied.

An $\mathcal{R}$-Strategy. First we define the notion of α-*believable computation.* Given a node α, we say that $\Phi(A; w) \downarrow = v$ is α-*believable*, if for any x, y, z, if $\Gamma(A; x, y, z)$ is defined and located at some node ξ with $\alpha <_{\mathrm{L}} \xi$, then $\phi(w) < \gamma(x, y, z)$.

An $\mathcal{R}$-strategy, α say, will satisfy an $\mathcal{R}$-requirement, $\mathcal{R}$ say (we drop the index), we define the *length function of agreement* $l(\alpha) = (W, \Phi(A))$ as usual, of course α uses only α-believable computations. We say that s is α-*expansionary*, if $l(\alpha)[s] > l(\alpha)[v]$ for all $v < s$ at which α is visited.

If there are only finitely many α-expansionary stages, then either $l(W, \Phi(A))[s]$ is bounded over the construction, or there is a fixed w say such that there are infinitely many stages at which α is visited and at which $\Phi(A; w) \downarrow$ is not α-believable, and by the Γ-rules, at which some elements $\leq \phi(w)$ are enumerated into A. In this case, $\phi(w)[s]$ will be unbounded over the construction. Therefore in either case, $W \neq \Phi(A)$, $\mathcal{R}$ is satisfied.

Suppose that there are infinitely many α-expansionary stages. Then we will build a c.e. set X, two Turing functionals Ξ and Ω such that both (a) and (b) below hold.

(a) $X = \Xi(V, A)$,
(b) $V \oplus A = \Omega(W, X)$.

For Ξ, whenever we define $\Xi(V, A; x)$, we define $\Xi(V, A; x) \downarrow = X(x)$ with $\xi(x) = x$. And once $V \upharpoonright (x + 1)$ or $A \upharpoonright (x + 1)$ changes, we set $\Xi(V, A; x)$ to be undefined. We ensure that an element x is enumerated into X, only if $\Xi(V, A; x)$ is currently undefined. So if $\Xi(V, A)$ is total, then $\Xi(V, A) = X$.

For Ω, whenever we define $\Omega(W, X; x)$, we define $\Omega(W, X; x) \downarrow = (V \oplus A)(x)$ with $\omega(x)$ fresh. And if $\Omega(W, X; x) \downarrow \neq (V \oplus A)(x)$, we enumerate $\omega(x)$ into X. This ensures that if $\Omega(W, X)$ is total, then $\Omega(W, X) = V \oplus A$.

Of course we have to ensure that W-change will never make $\Omega(W, X)$ partial, in fact, we ensure that Ω and Ξ will have the following properties,

(i) if $\Omega(W, X)$ is total, then $\Xi(V, A)$ is total, and
(ii) if $\Omega(W, X)$ is partial, then either $\Phi(A)$ is partial or $W \leq_{\mathrm{T}} \emptyset$.

Finally we define the *possible outcomes of an $\mathcal{R}$-strategy* to be $0 <_{\mathrm{L}} 1$ to denote infinite and finite actions respectively.

An $\mathcal{S}$-Module. An $\mathcal{S}_{e,i}$-module assumes that an $\mathcal{R}_e$-strategy α, say, is building a Turing functional Ω. It will try to satisfy its $\mathcal{S}$-requirement, $\mathcal{S}_{e,i}$. For simplicity, we drop the indices e, i in the following discussion.

Suppose that β is an $\mathcal{S}$-module. Let $\alpha\hat{}\langle 0\rangle \subseteq \beta$. Then β will have to deal with the injury from the building of $\Omega(W, X)$. It will work with a fixed *threshold* k say. Whenever we define the threshold, we define it as fresh. If $(V \oplus A) \upharpoonright k$ changes, then any previous action of β is cancelled but keep the threshold k unchanged, in which case, we say that β is *reset*. Clearly β is reset only finitely many times. Then the $\mathcal{S}$-module β will build a Turing function f and will proceed as follows.

1. Define an *agitator* a to be fresh.
   [Note that if both a and $\omega(k)$ are defined, then $a < \omega(k)$, where k is the threshold of β.]
2. (Create a Link (α, β)) Wait for a stage, v say, at which
   (2a) $\Psi(X; a) \downarrow = 0 = A(a)$,
   (2b) $W \upharpoonright (\omega(k) + 1) = \Phi(A) \upharpoonright (\omega(k) + 1)$ via β-believable computations.
   Then:
   – define $r = -1$ to be the A-restraint of β,
   – enumerate a into A, and
   – create a *link* (α, β).

3. (Travel the Link (α, β)) Wait for the next α-expansionary stage at which $W \upharpoonright (\omega(k)+1) = \Phi(A) \upharpoonright (\omega(k)+1)$ via α-believable computations. Then travel the link (α, β) through one of the following cases.
   Case 3a. $W_v \upharpoonright (\omega(k)+1) \neq W \upharpoonright (\omega(k)+1)$. Then
   – set $\omega(k)$ to be undefined,
   – remove the link (α, β) and stop.
   [Now we have created and preserved an inequality $\Psi(X; a) \downarrow = 0 \neq 1 = A(a)$. $\mathcal{S}$ is satisfied.]
   Case 3b. Otherwise, and $\Phi(A) \upharpoonright (\omega(k)+1)$ are β-believable. Then:
   – remove the link (α, β),
   – for each $x \leq \omega(k)$, if $f(x) \uparrow$, then define $f(x) = W(x)$,
   – enumerate $\omega(k)$ into X,
   – define an agitator a as fresh, and
   – define $r = \phi(\omega(k))$ to be the A-restraint of β.
   [The enumeration of a into A at stage v created a $(V \oplus A) \upharpoonright \omega(k)$-permission via Ω, which has been kept by the link (α, β). So we can enumerate $\omega(k)$ into X at this stage.]
   Case 3c. Otherwise, then do nothing.

The Possible Outcomes

The *possible outcomes of the $\mathcal{S}$-module* are as follows.

g: Case 3b occurs infinitely many times.

In this case, $\omega(k)[s]$ will be unbounded, so that f is defined to be a computable function. We prove that for every x, if $f(x) \downarrow = y$, then $W(x) = y$. Given x, let s_1 be the stage at which $f(x)$ is defined for the first time, then $f(x) = W_{s_1}(x)$. Let v_1 be minimal greater than s_1 at which step 2 of the module occurs. By the A-restraint $r[s] = r[s_1]$ for all $s \in [s_1, v_1)$, $f(x) = W_{v_1}(x)$. Let s_2 be the least stage greater than v_1 at which case 3b of β occurs. By the choice of s_2, $W_{s_2}(x) = f(x)$. Suppose by induction that $s_n \geq s_2$, that case 3b of β occurs at stage s_n, and that $W_{s_n}(x) = f(x)$. Let v_n be the least stage $> s_n$ at which step 2 of β occurs. Then for each $s \in [s_n, v_n)$, $r[s] = r[s_n]$, which ensures that $W_{v_n}(x) = f(x)$. Let s_{n+1} be the least stage greater than v_n at which case 3b of β occurs. By the choice of s_{n+1}, we have that $W_{s_{n+1}}(x) = f(x)$. It follows that there are infinitely many stages at which $W(x) = f(x)$, giving $W(x) = f(x)$. Since x is arbitarily given we have that $f = W$. $\mathcal{R}$ is satisfied.

u: Otherwise, and case 3c occurs infinitely many times.

In this case, there is a link (α, β) which was created and which will neither be cancelled nor be removed, and which is called *a permanent link*. We note that $\lim_s \omega(k)[s] \downarrow = v < \omega$ for some v, and that there are infinitely many stages at which $\Phi(A; v)$ is not β-believable, and at which some elements $\gamma(x, y, z) \leq \phi(v)$ are enumerated into A, by the Γ-rules. Therefore $\Phi(A)$ is partial. Both $\mathcal{R}$ and $\mathcal{S}$ are satisfied.

However every ξ strictly between α and β is *covered* by β in the sense that ξ is visited only finitely many times. The solution is the following observation:

(1) If ξ is either an $\mathcal{R}$- or a $\mathcal{P}$-strategy, then ξ's requirement has lower priority than that of α, we can introduce a *backup strategy* below $\beta\hat{}\langle u\rangle$ for the requirement of ξ. Therefore the injury of ξ from β is harmless.

(2) If ξ is a $\mathcal{Q}$- or an $\mathcal{S}$-strategy which works on a subrequirement whose global requirement has lower priority than that of α, then we can neglect this ξ, because, for a $\mathcal{P}$-, or an $\mathcal{R}$-requirement, we are allowed to give up finitely many subrequirements $\mathcal{Q}$ or $\mathcal{S}$.

(3) Otherwise and $\xi = \sigma$ is a $\mathcal{Q}$-strategy. Then we have that $\sigma\hat{}\langle 1\rangle \subseteq \beta$ holds. Now in case 3c of β, we may allow σ to act if the current stage is σ-expansionary.

(4) Otherwise and $\xi = \beta'$ is an $\mathcal{S}$-strategy. Then $\beta'\hat{}\langle w\rangle \subseteq \beta$ holds. In this case, whenever case 3c of β occurs, we may allow β' to act, if β' is ready to create a link (or to open an A-gap), in the sense that step 2 of strategy β' appears.

w: Otherwise. Now it is easy to see that one of the following cases occurs.

Case 1. Case 3a of β occurs. Then $\Psi(X;a)\downarrow = 0 \neq 1 = A(a)$ is created and preserved for some fixed a.

Case 2. Otherwise, and (2a) in step 2 fails to hold infinitely often. This means that $\Psi(X;a) \neq 0 = A(a)$.

Case 3. Otherwise, then there are infinitely many stages at which if $W \upharpoonright (\omega(k)+1) = \Phi(A) \upharpoonright (\omega(k)+1)$, then $\Phi(A;\omega(k))$ is not β-believable, in which case, by the Γ-rules, some elements $\gamma(x,y,z) \leq \phi(\omega(k))$ are enumerated into A infinitely many times. We have that $W \neq \Phi(A)$.

So in any case, we have that either $\Psi(X) \neq A$ or $W \neq \Phi(A)$, $\mathcal{S}$ is satisfied.

We define the priority ordering of the possible outcomes of β by $g <_{\mathrm{L}} u <_{\mathrm{L}} w$.

And a general $\mathcal{S}$-strategy is just an modification of the $\mathcal{S}$-module according to the observations in (1)–(4) above.

3 The Priority Tree *T*

In this section, we build the priority tree T and analyse some basic properties about the priority tree. First we define *the priority ranking of the requirements.*

Definition 3.1. Given a sequence $\mathcal{L} = (X_0, X_1, \cdots, X_n)$ of requirements, let m be the greatest $j \leq n$ such that X_j is a $\mathcal{P}$- or an $\mathcal{R}$-requirement. Then:

(i) We say that $\mathcal{P}_x$ *is complete in* $\mathcal{L}$ if there is a k such that $m < k \leq n$ and $X_k = \mathcal{Q}_{x,y}$ for some $y \in \omega$.

(ii) We say that $\mathcal{R}_e$ *is complete in* $\mathcal{L}$, if there is a k such that $m < k \leq n$ and $X_k = \mathcal{S}_{e,i}$ for some $i \in \omega$.

(iii) We say that $\mathcal{L} = (X_0, X_1, \cdots, X_n)$ *is complete*, if for every j, if X_j is a $\mathcal{P}$- or an $\mathcal{R}$-requirement, then X_j is complete in $\mathcal{L}$.

We now define the priority ranking $\mathcal{L}$ of the requirements inductively.

Definition 3.2. (i) Define the priority ranking of the $\mathcal{P}$- and $\mathcal{R}$-requirements such that $\mathcal{P}_e < \mathcal{R}_e < \mathcal{P}_{e+1} < \mathcal{R}_{e+1}$ holds for each $e \in \omega$.

(ii) Define $\mathcal{L} = \emptyset$.

Suppose by induction that $\mathcal{L} = (X_0, X_1, \cdots, X_n)$ has been defined.

(iii) If $\mathcal{L}$ is not complete, then let j be the least k such that X_k is a $\mathcal{P}$- or an $\mathcal{R}$-requirement which is not complete in $\mathcal{L}$. If $X_j = \mathcal{P}_x$ for some x, then let y be minimal such that $\mathcal{Q}_{x,y}$ is not in $\mathcal{L}$, and set $X_{n+1} = \mathcal{Q}_{x,y}$. If $X_j = \mathcal{R}_e$ for some e, then let i be the least i' such that $\mathcal{S}_{e,i'}$ is not in $\mathcal{L}$ and set $X_{n+1} = \mathcal{S}_{e,i}$.

Set $\mathcal{L} = (X_0, X_1, \cdots, X_n, X_{n+1})$ and go back to (iii).

(iv) Otherwise, then let X_{n+1} be the least $\mathcal{P}$- or $\mathcal{R}$-requirement as defined in (i) which is not in $\mathcal{L}$, set $\mathcal{L} = (X_0, X_1, \cdots, X_n, X_{n+1})$ and go back to (iii).

(v) Suppose that $\mathcal{L} = (X_0, X_1, \cdots)$. Then we define $X_i < X_j$ iff $i < j$, giving the *priority ranking of the requirements.*

Proposition 3.3. Suppose that $\mathcal{L}$ is the priority ranking of the requirements defined in definition 3.2. Then for all $e \in \omega$, we have:

(i) $\mathcal{P}_e < \mathcal{R}_e < \mathcal{P}_{e+1} < \mathcal{R}_{e+1}$,
(ii) $\mathcal{P}_e < \mathcal{Q}_{e,i} < \mathcal{Q}_{e,i+1}$ for all $i \in \omega$, and
(iii) $\mathcal{R}_e < \mathcal{S}_{e,i} < \mathcal{S}_{e,i+1}$ for all $i \in \omega$.

Proof. This is immediate from definitions 3.1 and 3.2. □

Definition 3.4. We define *the possible outcomes of a strategy* as the same as that in section 2.

Definition 3.5. Given a node ξ:

(i) $\mathcal{P}_x$ *is satisfied at* ξ, if there are $\mathcal{P}_x$-strategy τ and $\mathcal{Q}_{x,y}$-strategy σ for some y such that
(a) $\tau \subset \tau\hat{\ }\langle 0\rangle \subseteq \sigma \subset \sigma\hat{\ }\langle 0\rangle \subseteq \xi$,
(b) there is no $\mathcal{S}_{e,i}$-strategy β such that $\sigma\hat{\ }\langle 0\rangle \subseteq \beta \subset \beta\hat{\ }\langle u\rangle \subseteq \xi$ for any $e < x$.

(ii) $\mathcal{P}_x$ *is active at* ξ, if $\mathcal{P}_x$ is not satisfied at ξ and there is a $\mathcal{P}_x$-strategy τ such that $\tau \subset \xi$ and there is no $\mathcal{S}_{e,i}$-strategy β such that $\tau \subset \tau\hat{\ }\langle 0\rangle \subseteq \beta \subset \beta\hat{\ }\langle u\rangle \subseteq \xi$ for any $e < x$.

(iii) $\mathcal{R}_e$ *is satisfied at* ξ, if either (a) or (b) below holds,
(a) there is an $\mathcal{R}_e$-strategy α such that $\alpha\hat{\ }\langle 1\rangle \subseteq \xi$ and there is no $\mathcal{S}_{e',i'}$-strategy β such that $\alpha\hat{\ }\langle 1\rangle \subseteq \beta \subset \beta\hat{\ }\langle u\rangle \subseteq \xi$ for any $e' < e$.
(b) there is an $\mathcal{S}_{e,i}$-strategy β such that $\beta\hat{\ }\langle a\rangle \subseteq \xi$ for some $a \in \{g, u\}$ and such that there is no $\mathcal{S}_{e'i'}$-strategy β' with $\beta\hat{\ }\langle a\rangle \subseteq \beta' \subset \beta'\hat{\ }\langle u\rangle \subseteq \xi$ for any $e' < e$.

(iv) We say that $\mathcal{R}_e$ *is active at* ξ, if $\mathcal{R}_e$ is not satisfied at ξ, and there is an $\mathcal{R}_e$-strategy α such that
(a) $\alpha\hat{\ }\langle 0\rangle \subseteq \xi$,
(b) there is no $\mathcal{Q}_{x,y}$-strategy σ such that $\alpha\hat{\ }\langle 0\rangle \subseteq \sigma \subset \sigma\hat{\ }\langle 0\rangle \subseteq \xi$ for any $x \leq e$, and
(c) there is no $\mathcal{S}_{e',i'}$-strategy β such that $\alpha\hat{\ }\langle 0\rangle \subseteq \beta \subset \beta\hat{\ }\langle b\rangle \subseteq \xi$ for any $b \in \{g, u\}$ and any $e' < e$.

(v) We say that $\mathcal{Q}_{x,y}$ *is satisfied at* ξ if there is a $\mathcal{Q}_{x,y}$-strategy $\sigma \subset \xi$.

(vi) We say that $\mathcal{S}_{e,i}$ *is satisfied at* ξ if there is an $\mathcal{S}_{e,i}$-strategy $\beta \subset \xi$.

We now define the priority tree T.

Definition 3.6. Let $\mathcal{L}$ be the priority ranking of the requirements defined in definition 3.2. Then:

(i) Define the root node $\emptyset$ to be the strategy for the first requirement in $\mathcal{L}$, which is actually $\mathcal{P}_0$.

(ii) The immediate successors of a node are the possible outcomes of the corresponding strategy.

(iii) A node ξ will work on the least element in $\mathcal{L}$ which is not satisfied, and not active at ξ.

As usual, we have the following:

Proposition 3.7. (Finite Injury Along Any Path Proposition) Let f be an infinite path through T. Then for every $\mathcal{P}$- or $\mathcal{R}$-requirement X, there is a fixed n_0 such that either X is satisfied at $f \upharpoonright n$ for all $n \geq n_0$, or X is active at $f \upharpoonright n$ for all $n \geq n_0$.

Proof. By induction on the priority ranking of the requirements. □

Given an $\mathcal{S}_{e,i}$-strategy, we define *the top of* β to be the longest $\mathcal{R}_e$-strategy α such that $\alpha\hat{\ }\langle 0\rangle \subseteq \beta$, denoted by $\text{top}(\beta)$.

We also need some more properties about the structure of the priority tree T.

Proposition 3.8. Let $\beta \in T$ be an $\mathcal{S}_{e,i}$-strategy, and $\alpha = \text{top}(\beta)$. Then:

(i) If σ is a $\mathcal{Q}_{x,y}$-strategy and $\alpha \subset \alpha\hat{\ }\langle 0\rangle \subseteq \sigma \subset \sigma\hat{\ }\langle 0\rangle \subseteq \beta$, then $x > e$.

(ii) If β' is an $\mathcal{S}_{e',i'}$-strategy such that $\alpha \subset \alpha\hat{\ }\langle 0\rangle \subseteq \beta' \subset \beta'\hat{\ }\langle a\rangle \subseteq \beta$ for some $a \in \{g, u\}$, then for $\alpha' = \text{top}(\beta')$, $\alpha \subset \alpha' \subset \beta' \subset \beta$, and $e' > e$.

(iii) If α' is an $\mathcal{R}_{e'}$-strategy such that $\alpha \subset \alpha' \subset \beta$, then $e' > e$.

(iv) If τ is a $\mathcal{P}_x$-strategy such that $\alpha \subset \tau \subset \beta$, then $x > e$.

Proof. It is straightforward from definitions 3.5 and 3.6. □

The full construction and its verification is a $\mathbf{0}'''$-priority tree argument which will be given in the full version of the paper.

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