Path Embedding on Folded Hypercubes

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Abstract. We analyze some edge-fault-tolerant properties of the folded hypercube, which is a variant of the hypercube obtained by adding an edge to every pair of nodes with complementary address. We show that an n-dimensional folded hypercube is $(n-2)$-edge-fault-tolerant Hamiltonian-connected when $n(\geq 2)$ is even, $(n-1)$-edge-fault-tolerant strongly Hamiltonian-laceable when $n(\geq 1)$ is odd, and $(n-2)$-edge-fault-tolerant hyper Hamiltonian-laceable when $n(\geq 3)$ is odd.

1 Introduction

Because of the hypercube's importance, many variants of it have been proposed (for example, see [3,6,7,16]). One variant that has been the focus of a great deal of research is the *folded hypercube*, an extension of the hypercube constructed by adding an edge to every pair of nodes that are the farthest apart, i.e., two nodes with complementary addresses. It has been shown that, compared to a regular hypercube, the folded hypercube can improve the system's performance in many measurements, such as diameter, mean inter-node distance, and traffic density [3,20].

A *graph* $G = (V, E)$ is a pair of two sets composed of a *node set* V and an *edge set* E, where V is a finite set and E is a subset of $\{(u, v)|\ (u, v)$ is an unordered pair of $V\}$. We also use $V(G)$ and $E(G)$ to denote the node set and edge set of G, respectively. A *path*, $P[v_0, v_k] = \langle v_0, v_1, \ldots, v_k\rangle$, is a sequence of distinct nodes in which any two consecutive nodes are adjacent. We call v_0 and v_k the *end-nodes* of the path. A path with end-nodes u and v is said to be a *u-v path.* A path may contain a *subpath*, denoted as $\langle v_0, v_1, \ldots, v_i, P[v_i, v_j], v_j, v_{j+1}, \ldots, v_k\rangle$, where $P[v_i, v_j] = \langle v_i, v_{i+1}, \ldots, v_{j-1}, v_j\rangle$. A *cycle* is a path with $v_0 = v_k$ and $k \geq 3$. When the Hamiltonicity of a graph G is being investigated, it is necessary to determine whether G is Hamiltonian or Hamiltonian-connected. A cycle (respectively, path) in G is called a *Hamiltonian cycle* (respectively, *Hamiltonian path*) if it contains every node of G exactly once. G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and *Hamiltonian-connected* if there exists a Hamiltonian path between every two nodes of G.

A graph $G = (V_0 \bigcup V_1, E)$ is *bipartite* if $V_0 \cap V_1 = \emptyset$ and $E \subseteq \{(x, y)|\ x \in V_0$ and $y \in V_1\}$. We say V_0 and V_1 are *partite sets* of G, and $V_0 \bigcup V_1$ a *bipartition.* Two well-known interconnection networks, hypercubes [6,14] and star

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graphs [1,12], are both bipartite. However, because a bipartite graph is not Hamiltonian-connected, except for K_1 or K_2, Simmons [17] introduced the concept of Hamiltonian-laceability for such graphs. A Hamiltonian bipartite graph $G = (V_0 \bigcup V_1, E)$ is *Hamiltonian-laceable* if there is a Hamiltonian path between any two nodes x and y, where $x \in V_0$ and $y \in V_1$. Hsieh *et al.* [11] extended this concept and proposed the concept of *strong Hamiltonian-laceability.* A graph $G = (V_0 \bigcup V_1, E)$ is *strongly Hamiltonian-laceable* if there is a simple path of length $|V_0| + |V_1| - 2$ between any two nodes of the same partite set. Lewinter *et al.* [15] introduced another concept, called hyper Hamiltonian-laceability. A bipartite graph $G = (V_0 \bigcup V_1, E)$ is *hyper Hamiltonian-laceable* if for any node $f \in V_i$, $i \in \{0, 1\}$, there is a Hamiltonian path of $G - f$ between any two nodes of V_{1-i}.[1]

The edge-fault-tolerant Hamiltonicity proposed by Hsieh *et al.* [10] measures Hamiltonicity in interconnection networks with faulty edges. A Hamiltonian graph G is *k-edge-fault-tolerant Hamiltonian* if $G - F$ remains Hamiltonian for every $F \subset E(G)$ with $|F| \leq k$. A Hamiltonian-laceable graph G is *k-edge-fault-tolerant Hamiltonian-laceable* if $G - F$ remains Hamiltonian-laceable for every $F \subset E(G)$ with $|F| \leq k$. A strongly Hamiltonian-laceable graph G is *k-edge-fault-tolerant strongly Hamiltonian-laceable* if $G - F$ remains strongly Hamiltonian-laceable for every $F \subset E(G)$ with $|F| \leq k$. A hyper Hamiltonian-laceable graph G is *k-edge-fault-tolerant hyper Hamiltonian-laceable* if $G - F$ remains hyper Hamiltonian-laceable for every $F \subset E(G)$ with $|F| \leq k$.

Latifi *et al.* [13] showed that an n-dimensional hypercube is $(n-2)$-edge-fault-tolerant Hamiltonian. Tsai *et al.* [18] further showed that an n-dimensional hypercube is $(n - 2)$-edge-fault-tolerant strongly Hamiltonian-laceable, and $(n - 3)$-edge-fault-tolerant hyper Hamiltonian-laceable. Wang [20] showed that the n-dimensional folded hypercube is $(n - 1)$-edge-fault-tolerant Hamiltonian. It is known that the n-dimensional folded hypercube is bipartite (non-bipartite) when n is odd (even) [23]. In this paper, we show that an n-dimensional folded hypercube is $(n - 2)$-edge-fault-tolerant Hamiltonian-connected when $n(\geq 2)$ is even, $(n - 1)$-edge-fault-tolerant strongly Hamiltonian-laceable when $n(\geq 1)$ is odd, and $(n - 2)$-edge-fault-tolerant hyper Hamiltonian-laceable when $n(\geq 3)$ is odd.

2 Preliminaries

When using undirected graphs to model interconnection networks, our fundamental graph terminologies follow those in [21]. A *subgraph* of $G = (V, E)$ is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. Given a set $V' \subseteq V$, the subgraph of $G = (V, E)$ *induced* by V' is the graph $G' = (V', E')$, where $E' = \{(u, v) \in E|\ u, v \in V'\}$. In a graph G, the *neighbors* of a node v are the nodes adjacent to it in G, and the *degree* of v is the number of edges incident

[1] Let S be a set of edges and/or nodes of a graph G. Throughout this paper, the notation $G - S$ represents the resulting graph obtained by deleting those elements in S from G.

to it. A graph is said to be *regular* if all nodes have a common degree. Two graphs G_1 and G_2 are *isomorphic* if there is a one-to-one function π from $V(G_1)$ onto $V(G_2)$ such that $(u,v) \in E(G_1)$ iff $(\phi(u), \phi(v)) \in E(G_2)$.

An n-dimensional hypercube (n-cube for short) can be represented as an undirected graph $Q_n = (V, E)$, where V consists of 2^n nodes that are labelled as binary numbers of length n from $\underbrace{00\ldots0}_{n}$ to $\underbrace{11\ldots1}_{n}$, and E is the set of edges that connects two nodes iff their labels differ by exactly one bit. Note that Q_n is regular because the degree of each node is equal to n, and $|E| = n2^{n-1}$. An edge $e = (u, v) \in E$ is said to be of *dimension* i if $u = b_n b_{n-1} \ldots b_i \ldots b_1$ and $v = b_n b_{n-1} \ldots \overline{b_i} \ldots b_1$, where $b_j \in \{0, 1\}$ for $j = 1, 2, \ldots, n$, and $\overline{b_i}$ is the *one's complement of* b_i, i.e., $\overline{b_i} = 1 - b_i$. Note that there are 2^{n-1} edges in each dimension.

Let $x = x_n x_{n-1} \ldots x_1$ be an n-bit binary string. For $1 \le k \le n$, we use x^k to denote the binary strings $y_n y_{n-1} \ldots y_1$ such that $y_k = 1 - x_k = \overline{x_k}$ and $x_i = y_i$ for all $i \ne k$. The *Hamming weight* $hw(x)$ of x is the number of i's that make $x_i = 1$. Let $x = x_n x_{n-1} \ldots x_1$ and $y = y_n y_{n-1} \ldots y_1$ be two n-bit binary strings. The *Hamming distance* $h(x, y)$ between two nodes x and y is the number of different bits in the corresponding strings of both nodes. Note that Q_n is a bipartite graph with a bipartition $\{x|\ hw(x)$ is odd$\}$ and $\{x|\ hw(x)$ is even$\}$. Let $d_G(x, y)$ be the distance of the shortest path between two nodes x and y in graph G. It is known that $d_{Q_n}(x, y) = h(x, y)$.

An *n-dimensional folded hypercube* (*folded n-cube* for short) FQ_n is a regular n-dimensional hypercube augmented by adding more edges between its nodes. More specifically, a folded n-cube is obtained by adding an edge between two nodes whose addresses are complementary, i.e., a node whose address is $b = b_n b_{n-1} \ldots b_1$ now has one more edge to node $\overline{b} = \overline{b_n b_{n-1}} \ldots \overline{b_1}$, in addition to its original n edges. Hence, FQ_n has 2^{n-1} more edges than Q_n. We call these augmented edges *skips*, to distinguish them from regular edges.

For convenience, FQ_n can be represented by $\underbrace{**\ldots**}_{n} = *^n$, where $* \in \{0, 1\}$ means "*don't care*". An *i-partition on $FQ_n = *^n$* partitions FQ_n along dimension i for some $i \in \{1, 2, \ldots, n\}$, into two *subcubes*, $Q^0_{n-1} = *^{n-i}0*^{i-1}$ and $Q^1_{n-1} = *^{n-i}1*^{i-1}$, where Q^0_{n-1} and Q^1_{n-1} are the subgraphs of FQ_n induced by $\{x_n \ldots x_i \ldots x_1 \in V(FQ_n)|\ x_i = 0\}$ and $\{x_n \ldots x_i \ldots x_1 \in V(FQ_n)|\ x_i = 1\}$), respectively. Note that each Q^j_{n-1}, where $j \in \{0, 1\}$, is isomorphic to an $(n-1)$-cube. An *i-partition of an n-cube Q_n* can be defined similarly.

The following lemmas are useful in our method.

Lemma 1. [8] *Let x and y be two distinct nodes in Q_n; and let $h(x, y) = d$. There are x-y paths in Q_n whose lengths are $d, d+2, d+4, ..., c$, where $c = 2^n - 1$ if d is odd, and $c = 2^n - 2$ if d is even.*

Lemma 2. [18] *The following two statements hold:*

1. *Q_n is $(n-2)$-edge-fault-tolerant strongly Hamiltonian-laceable.*
2. *Q_n is $(n-3)$-edge-fault-tolerant hyper Hamiltonian-laceable.*

It is well known that Q_n (FQ_n for odd n) is bipartite. Thus, the following proposition is used in each proof presented in this paper.

Proposition 1. *For two distinct nodes x and y in Q_n (or FQ_n for odd n), $h(x, y)$ is odd (even) iff x and y are in different partite sets (the same partite set).*[2]

Two paths are said to be *node-disjoint* if they have no common node.

Lemma 3. [19] *Let V_0 and V_1 be partite sets of a fault-free Q_n, where $n \geq 2$. Let a and b be two distinct nodes of V_0, and a' and b' be two distinct nodes of V_1. Then, there exist two node-disjoint paths $P[a, a']$ and $P[b, b']$ spanning $V(Q_n)$, i.e., $V(P[a, a']) \cup V(P[b, b']) = V(Q_n)$.*

3 Three Edge-Fault-Tolerant Properties

Let $Q^0_{n-1}(= *^{n-i}0*^{i-1})$ and $Q^1_{n-1} = (*^{n-i}1*^{i-1})$ be two subcubes after executing an i-partition on FQ_n. We define the set of *crossing edges* as $E_C = \{(u, v) \in E(FQ_n)|\ u \in V(Q^0_{n-1}), v \in V(Q^1_{n-1}),\ \text{and}\ h(u, v) \neq n\}$, and the set of skips as $E_S = \{(u, v) \in E(FQ_n)|\ u \in V(Q^0_{n-1}), v \in V(Q^1_{n-1}),\ \text{and}\ h(u, v) = n\}$. Moreover, let F be the set of faulty edges of FQ_n; $F_0 = F \cap E(Q^0_{n-1})$; $F_1 = F \cap E(Q^1_{n-1})$; $F_C = F \cap E_C$; and let $F_S = F \cap E_S$. For a node $u = u_n u_{n-1} \ldots u_1$ in FQ_n, recall that $\overline{u} = \overline{u_n} \ldots \overline{u_1}$.

3.1 Edge-Fault-Tolerant Hamiltonian-Connectivity

In this subsection, we demonstrate the edge-fault-tolerant Hamiltonian-connectivity of the folded hypercube.

Lemma 4. *FQ_n is $(n-2)$-edge-fault-tolerant Hamiltonian-connected when $n(\geq 2)$ is an even integer.*

Proof. Since FQ_2 is a complete graph comprised of four nodes, it is clearly Hamiltonian-connected. We now consider FQ_n, where $n \geq 4$ is an even integer. In the following, we attempt to construct a fault-free Hamiltonian path between two arbitrary distinct nodes x and y when $|F| = n-2$. We consider the following two cases.

Case 1. $h(x, y)$ is odd. As FQ_n is constructed from Q_n by adding skips, $FQ_n - F$ contains a subgraph G that is isomorphic to Q_n with at most $n-2$ faulty edges. Since $h(x, y)$ is odd, x and y are in different partite sets in H. By Lemma 2(1), G contains a fault-free Hamiltonian path between x and y, and so as $FQ_n - F$.

Case 2. $h(x, y)$ is even. We have the following scenarios.

[2] Hereafter, the terms "$h(x, y)$ is odd" and "x and y are in different partite sets" are used interchangeably; and "$h(x, y)$ is even" and "x and y are in the same partite set" are used similarly.

Case 2.1. There is at least one faulty skip, i.e., $F_S \neq \emptyset$. We can execute an i-partition on FQ_n for some $i \in \{1, 2, \ldots, n\}$ such that x and y are in different subcubes. Without loss of generality, we assume that $x \in V(Q^0_{n-1})$ and $y \in V(Q^1_{n-1})$. Since $F_S \neq \emptyset$, we have $|F_C \cup F_S| \geq 1$. Recall that both F_C and F_S are two set of edges located between Q^0_{n-1} and Q^1_{n-1} after executing an i-partition. Therefore, the number of faulty edges remaining in each of $\{Q^0_{n-1}, Q^1_{n-1}\}$ is at most $(n-2)-1 = n-3$, i.e., $|F_0| \leq n-3$, and $|F_1| \leq n-3$. Let $u(\neq x)$ be a node in Q^0_{n-1} such that $h(x,u)$ is odd, $\overline{u} \neq y$, and $(u, \overline{u})$ is fault-free. (If such a node u does not exist, then $|F_S| \geq 2^{n-2} - 1 > n-2$ for $n \geq 4$, which is a contradiction.) Clearly, $h(x, \overline{u}) = n - h(x,u)$ is odd because n is even, i.e., x and $\overline{u}$ are in different partite sets in the subgraph Q_n according to Proposition 1.[3] Moreover, since x and y are in the same partite set in Q_n because of even $h(x,y)$, y and $\overline{u}$ are in different partite sets in Q_n. Since $|F_j| \leq n-3$ for $j = 0, 1$, by Lemma 2(1), $Q^0_{n-1} - F_0$ ($Q^1_{n-1} - F_1$) contains a fault-free Hamiltonian path $P_0[x,u]$ ($P_1[\overline{u}, y]$). A desired Hamiltonian x-y path can be constructed as $P_0[x,u] \oplus (u, \overline{u}) \oplus P_1[\overline{u}, y]$, where $\oplus$ denotes a path-concatenation operation.[4]

Case 2.2. There are no faulty skips, i.e., $F_S = \emptyset$. In this case, no faulty edges are skips. Let e be an arbitrary faulty edge whose dimension is i, where $i \in \{1, 2, \ldots, n\}$. We can execute an i-partition on FQ_n such that $|F_C \cup F_S| = |F_C| \geq 1$. Using an argument similar to that in Case 2.1, we have $|F_0| \leq n-3$ and $|F_1| \leq n-3$.

Case 2.2.1. x and y are in the same subcube. Without loss of generality, we assume that $x, y \in V(Q^0_{n-1})$.

Case 2.2.1.1. $|F_j| = n-3$ for some $j \in \{0, 1\}$. Without loss of generality, we assume that $|F_0| = n-3$. Then, $|F_C| = 1$ and $|F_1| = 0$. We first select an arbitrary node w in Q^0_{n-1} such that w is in a different partite set to the partite set that $\{x, y\}$ belongs to. We then select one arbitrary faulty edge $(u,v) \in F_0$. By Lemma 2(2), $Q^0_{n-1} - w - (F_0 - (u,v))$ contains a Hamiltonian path $P_0[x,y]$. We then have the following scenarios.

Case 2.2.1.1.1. $P_0[x,y]$ contains (u,v). Path $P_0[x,y]$ can be represented by $P_0[x,u] \oplus (u,v) \oplus P_0[v,y]$. Consider four nodes $\overline{u}, \overline{v}, \overline{w}$, and w^i. Since $\overline{w} \neq \overline{u}$ and $\overline{w} \neq \overline{v}$, we have $|\{\overline{u}, \overline{v}\} \cap \{\overline{w}, w^i\}| \leq 1$.

Case 2.2.1.1.1.1. $|\{\overline{u}, \overline{v}\} \cap \{\overline{w}, w^i\}| = 0$. Clearly, $\overline{u}$ and $\overline{v}$ belong to different partite sets in Q^1_{n-1}. Moreover, since $h(x, w^i)$ is even and $h(x, \overline{w}) = n - h(x,w)$ is odd, w^i and $\overline{w}$ belong to different partite sets in Q^1_{n-1}; Hence, there are two nodes, one derived from $\{\overline{u}, \overline{v}\}$ and the other from $\{w^i, \overline{w}\}$,

[3] For convenience, we adopt the notation Q_n to represent a subgraph in FQ_n that is isomorphic to an n-dimensional hypercube.

[4] Throughout the paper, we use the notation "$\oplus$" to represent a path-concatenation operation in order to distinguish it from an ordinary addition "+" operation.

which come from different partite sets. Without loss of generality, we assume that $\overline{u}$ and $\overline{w}$ are in different partite sets, and $\overline{v}$ and w^i are in different partite sets. By Lemma 3, Q^1_{n-1} contains two node-disjoint paths $P_1[\overline{u}, \overline{w}]$ and $P_1[\overline{v}, w^i]$ spanning $V(Q^1_{n-1})$. A desired Hamiltonian x-y path can be constructed as $P_0[x, u] \oplus (u, \overline{u}) \oplus P_1[\overline{u}, \overline{w}] \oplus (\overline{w}, w) \oplus (w, w^i) \oplus P_1[w^i, \overline{v}] \oplus (\overline{v}, v) \oplus P_0[v, y]$.

Case 2.2.1.1.1.2. $|\{\overline{u}, \overline{v}\} \cap \{\overline{w}, w^i\}| = 1$. Without loss of generality, we assume that $\overline{v} = w^i$. Recall that $\overline{u}$ and $\overline{v}$ are in different partite sets, and w^i and $\overline{w}$ are in different partite sets. Therefore, $\overline{u}$ and $\overline{w}$ are in the same partite set in Q^1_{n-1}. By Lemma 2(2), $Q^1_{n-1} - w$ contains a fault-free Hamiltonian path $P_1[\overline{u}, \overline{w}]$. A desired Hamiltonian x-y path can be constructed as $P_0[x, u] \oplus (u, \overline{u}) \oplus P_1[\overline{u}, \overline{w}] \oplus (\overline{w}, w) \oplus (w, \overline{v}) \oplus (\overline{v}, v) \oplus P_0[v, y]$.

Case 2.2.1.1.2. $P_0[x, w]$ does not contain (u, v). In this case, we can select an arbitrary edge in place of (u, v). A desired Hamiltonian x-y path can then be constructed using a method similar to that in Case 2.2.1.1.1.

Case 2.2.1.2. $|F_0| \leq n-4$ and $|F_1| \leq n-4$. We first select a node $w \in V(Q^0_{n-1})$ such that (w, w^i) is fault-free and w is in a different partite set to the partite set that x and y belong to. (If such a w does not exist, then $|F_c| > 2^{n-2} > n-2$ for $n \geq 4$, which is a contradiction.) By Lemma 2(2), $Q^0_{n-1} - w - F_0$ contains a Hamiltonian path $P_0[x, y]$. Let v be a unique node in $P_0[x, y]$ such that $\overline{v} = w^i$, and let $u \in P_0[x, y]$ be a unique neighbor of v such that $P_0[x, y] = P_0[x, u] \oplus (u, v) \oplus P_0[v, y]$. Using an argument similar to that applied in Case 2.2.1.1.1.1, we know that $\overline{u}$ and $\overline{w}$ are in different partite sets. Again, by Lemma 2(2), $Q^1_{n-1} - w - F_0$ contains a fault-free Hamiltonian path $P_1[\overline{u}, \overline{w}]$. Therefore, a desired Hamiltonian x-y path can be constructed as $P_0[x, u] \oplus (u, \overline{u}) \oplus P_1[\overline{u}, \overline{w}] \oplus (\overline{w}, w) \oplus (w, \overline{v}) \oplus (\overline{v}, v) \oplus P_0[v, y]$.

Case 2.2.2. x and y are in different subcubes. Without loss of generality, we assume that $x \in Q^0_{n-1}$ and $y \in Q^1_{n-1}$. Let $w (\neq x)$ be a node in Q^0_{n-1} such that $h(x, w)$ is odd. Note that $h(x, \overline{w}) = n - h(x, w)$ is also odd. Moreover, since x and y are in the same partite set, $\overline{w}$ and y are in different partite sets (restricted to Q_n). Since both $|F_0|$ and $|F_1|$ are at most $n-3$, $Q^0_{n-1} - F_0$ ($Q^1_{n-1} - F_1$) contains a fault-free Hamiltonian path $P_0[x, w]$ ($P_1[\overline{w}, y]$) by Lemma 2(1). Therefore, a desired Hamiltonian x-y path can be constructed as $P_0[x, w] \oplus (w, \overline{w}) \oplus P_1[\overline{w}, y]$.

By combining the above cases, we complete the proof.

Due to the space limitation, we omit the proof for FQ_n being $(n-1)$-edge-fault-tolerant strongly Hamiltonian-laceable when $n (\geq 1)$ is odd.

3.2 Edge-Fault-Tolerant Hyper Hamiltonian-Laceability

In this subsection, we demonstrate the edge-fault-tolerant hyper Hamiltonian-laceability of the folded hypercube.

Lemma 5. *FQ_n is $(n-2)$ edge-fault-tolerant hyper Hamiltonian-laceable, where $n(\geq 3)$ is an odd integer.*

Proof. Suppose that $FQ_n = (V_0 \cup V_1, E)$, where $n(\geq 3)$ is odd. Let f be any node in V_i, $i \in \{0, 1\}$. In the following, for two arbitrary distinct nodes $x, y \in V_{1-i}$, we attempt to construct a fault-free Hamiltonian x-y path in $FQ_n - F - f$, where $|F| = n - 2$. We consider the following two cases.

Case 1. $F_S \neq \emptyset$. Since $FQ_n - F$ contains a subgraph isomorphic to Q_n with at most $n - 3$ faulty edges, the result holds by Lemma 2(2).

Case 2. $F_S = \emptyset$. In this case, all faulty edges are not skips. We can execute an i-partition on FQ_n, for some $i \in \{1, 2, \ldots, n\}$, such that $|F_C \cup F_S| = |F_C| \geq 1$, $|F_0| \leq n - 3$, and $|F_1| \leq n - 3$. There are the following scenarios.

Case 2.1. $|F_C| = 1$ and $|F_j| = n - 3$ for some $j \in \{0, 1\}$. Without loss of generality, we assume that $|F_0| = n - 3$. Then, $|F_1| = 0$.

Case 2.1.1. $x, y \in V(Q^0_{n-1})$ and $f \in V(Q^1_{n-1})$.

By Lemma 2(1), $Q^0_{n-1} - F_0$ contains a fault-free Hamiltonian cycle $C_0 = \langle u_0, u_1, \ldots u_{2^{n-1}-1}, u_0 \rangle$ of length 2^{n-1}, where $x = u_0$ and $y = u_k$ for some $k \in \{1, \ldots, 2^{n-1} - 1\}$. As $h(x, y)$ is even, $k \geq 2$. Let $x' = u_{k+1 \pmod{2^{n-1}}}$ and $y' = u_1$. Clearly, $Q^0_{n-1} - F_0$ contains two fault-free paths, $P_0[x, x'] = \langle u_0, u_{2^{n-1}-1}, u_{2^{n-1}-2}, \ldots, u_{k+1 \pmod{2^{n-1}}} \rangle$ and $P_0[y', y] = \langle u_1, u_2, \ldots, u_k \rangle$, which spans $V(Q^0_{n-1})$. Since x and y are in the same partite set and x' and y are in different partite sets, x and x' are in different partite sets, i.e., $h(x, x')$ is odd. Similarly, y and y' are in different partite sets, i.e., $h(y, y')$ is odd. Moreover, $h(x, \overline{x'}) = n - h(x, x')$ and $h(y, \overline{y'}) = n - h(y, y')$ are both even, i.e., $x, y, \overline{x'}$, and $\overline{y'}$ are in the same partite set and thus $h(\overline{x'}, \overline{y'})$ is even. Note that f and $\{\overline{x'}, \overline{y'}\}$ are in different partite sets. Since Q^1_{n-1} is fault-free, by Lemma 2(2), $Q^1_{n-1} - f$ contains a fault-free Hamiltonian path $P_1[\overline{x'}, \overline{y'}]$ of length $2^{n-1} - 2$. A desired x-y path can be constructed as $P_0[x, x'] \oplus (x', \overline{x'}) \oplus P_1[\overline{x'}, \overline{y'}] \oplus (\overline{y'}, y') \oplus P_0[y', y]$, which has length $2^{n-1} + 2^{n-1} - 2 = 2^n - 2$.

Case 2.1.2. $f, x, y \in V(Q^0_{n-1})$. Let (u, v) be an arbitrary faulty edge in F_0. Note that $(u, \overline{u})$ and $(v, \overline{v})$ are both fault-free. By Lemma 2(2), $Q^0_{n-1} - f - (F_0 - (u, v))$ contains a path $P_0[x, y]$ of length $2^{n-1} - 2$.

Case 2.1.2.1. $P_0[x, y]$ contains (u, v). Thus $P_0[x, y] = P_0[x, u] \oplus (u, v) \oplus P_0[v, y]$. By Lemma 1, Q^1_{n-1} contains a fault-free Hamiltonian path $P_1[\overline{u}, \overline{v}]$ of length $2^{n-1} - 1$. A desired x-y path can be constructed as $P_0[x, u] \oplus (u, \overline{u}) \oplus P_1[\overline{u}, \overline{v}] \oplus (\overline{v}, v) \oplus P_0[v, y]$, which has length $(2^{n-1} - 2) - 1 + 2 + (2^{n-1} - 1) = 2^n - 2$.

Case 2.1.2.2. $P_0[x, y]$ does not contain (u, v). In this case, we select an arbitrary edge in $P_0[x, y]$ instead of (u, v) in Case 2.1.2.1. The construction of a desired path is similar to that of Case 2.1.2.1.

Case 2.1.3. $f, x \in V(Q_{n-1}^0)$ and $y \in V(Q_{n-1}^1)$. By Lemma 2(1), $Q_{n-1}^0 - F_0$ contains a fault-free Hamiltonian path $P_0[x, f]$. Let $u \in P_0[x, f]$ be the neighbor of f. Thus $P_0[x, f] = P_0[x, u] \oplus (u, f)$. Note that u and x are in the same partite set, and $\overline{u}$ and y are in different partite sets. By Lemma 1, Q_{n-1}^1 contains a fault-free Hamiltonian path $P_1[\overline{u}, y]$. A desired x-y path can be constructed as $P_0[x, u] \oplus (u, \overline{u}) \oplus P_1[\overline{u}, y]$, which has length $2^n - 2$.

Case 2.1.4. $x \in V(Q_{n-1}^0)$ and $f, y \in V(Q_{n-1}^1)$. Let $w(\neq x)$ be the node in Q_{n-1}^0 such that $h(x, w)$ is odd and $\overline{w} \notin \{f, y\}$. Since $h(\overline{w}, x) = n - h(x, w)$ is even and $h(x, y)$ is even, y and $\overline{w}$ are in the same partite set. Since $|F_0| = n - 3$, by Lemma 2(1), $Q_{n-1}^0 - F_0$ contains a fault-free Hamiltonian path $P_0[x, w]$. Moreover, by Lemma 2(2), $Q_{n-1}^1 - f - F_1$ contains a fault-free Hamiltonian path $P_1[\overline{w}, y]$. A desired x-y path can be constructed as $P_0[x, w] \oplus (w, \overline{w}) \oplus P_1[\overline{w}, y]$, which has length $2^n - 2$.

Case 2.1.5. $f, x, y \in V(Q_{n-1}^1)$. By Lemma 2(2), $Q_{n-1}^1 - f$ contains a fault-free Hamiltonian path $P_1[x, y]$. Let (u, v) be an edge in $P_1[x, y]$. Thus $P_1[x, y] = P_1[x, u] \oplus (u, v) \oplus P_1[v, y]$. By Lemma 2(1), $Q_{n-1}^0 - F_0$ contains a fault-free Hamiltonian path $P_0[\overline{u}, \overline{v}]$. A desired x-y path can be constructed as $P_1[x, u] \oplus (u, \overline{u}) \oplus P_0[\overline{u}, \overline{v}] \oplus (\overline{v}, v) \oplus P_1[v, y]$, which has length $2^n - 2$.

Case 2.1.6. $f \in V(Q_{n-1}^0)$ and $x, y \in V(Q_{n-1}^1)$. Let $u \neq f$ be a node in Q_{n-1}^0 whose partite set is the same with the partite set to which x and y belong. Therefore, u and f are in different partite sets. Since $|F_0| = n - 3$, by Lemma 2(1), $Q_{n-1}^0 - F_0$ contains a Hamiltonian path $P_0[u, f]$. Let $v \in P_0[u, f]$ be the neighbor of f. Thus $P_0[u, f] = P_0[u, v] \oplus (v, f)$. Clearly, u and v are in the same partite set, and $\overline{u}$ and $\overline{v}$ are in the same partite set. Further, the partite set of $\{\overline{u}, \overline{v}\}$ is different from the partite set to which $\{x, y\}$ belongs. By Lemma 3, Q_{n-1}^1 contains two node-disjoint paths $P_1[x, \overline{u}]$ and $P_1[\overline{v}, y]$ spanning $V(Q_{n-1}^1)$. A desired x-y path can be constructed as $P_1[x, \overline{u}] \oplus (\overline{u}, u) \oplus P_0[u, v] \oplus (v, \overline{v}) \oplus P_1[\overline{v}, y]$, which has length $2^n - 2$.

Case 2.2. $|F_C| > 1$, $|F_0| \leq n - 4$, and $|F_1| \leq n - 4$.

Case 2.2.1. $f, x, y \in V(Q_{n-1}^j)$ for some $j \in \{0, 1\}$. Without loss of generality, we assume that $f, x, y \in V(Q_{n-1}^0)$. By Lemma 2(2), $Q_{n-1}^0 - f - F_0$ contains a fault-free Hamiltonian path $P_0[x, y]$. Let (u, v) be an arbitrary edge in $P_0[x, y]$ and thus $P_0[x, y] = P_0[x, u] \oplus (u, v) \oplus P_0[v, y]$. By Lemma 2(1), Q_{n-1}^1 contains a fault-free Hamiltonian path $P_1[\overline{u}, \overline{v}]$. A desired x-y path can be constructed as $P_0[x, u] \oplus (u, \overline{u}) \oplus P_1[\overline{u}, \overline{v}] \oplus (\overline{v}, v) \oplus P_0[v, y]$, which has length $2^n - 2$.

Case 2.2.2. $x, y \in V(Q_{n-1}^j)$ and $f \in V(Q_{n-1}^{1-j})$ for some $j \in \{0, 1\}$. The proof is similar to that of Case 2.1.1 and thus we omit here.

Case 2.2.3. $x, f \in V(Q_{n-1}^j)$ and $y \in V(Q_{n-1}^{1-j})$ for some $j \in \{0, 1\}$. Without loss of generality, we assume that $x, f \in V(Q_{n-1}^0)$ and $y \in V(Q_{n-1}^1)$. Let $w(\notin \{f, x\})$ be the node in Q_{n-1}^0 such that $h(x, w)$ is even. Since $h(x, \overline{w}) = n - h(x, w)$ is odd, $\overline{w} \neq y$ because $h(x, y)$

is even. This implies that $\overline{w}$ and y are in different partite set. By Lemma 2(2), $Q_{n-1}^0 - f - F_0$ contains a fault-free Hamiltonian path $P_0[x, w]$. Moreover, by Lemma 2(1), $Q_{n-1}^1 - F_1$ contains a fault-free Hamiltonian path $P_1[\overline{w}, y]$. A desired x-y path can be constructed as $P_0[x, w] \oplus (w, \overline{w}) \oplus P_1[\overline{w}, y]$, which has length $2^n - 2$.

By combining the above cases, we complete the proof.

We now present our main result.

Theorem 1. *There are three edge-fault-tolerant properties for FQ_n as follows:*

P1. FQ_n *is* $(n-2)$*-edge-fault-tolerant Hamiltonian-connected, where* $n(\geq 2)$ *is an even integer.*
P2. FQ_n *is* $(n-1)$*-edge-fault-tolerant strongly Hamiltonian-laceable, where* $n(\geq 1)$ *is an odd integer.*
P3. FQ_n *is* $(n-2)$*-edge-fault-tolerant hyper Hamiltonian-laceable, when* $n(\geq 3)$ *is an odd integer.*

4 Concluding Remarks

The path (linear array) is the most fundamental network for parallel and distributed computation, which is suitable for designing simple algorithms with low communication costs. Numerous efficient algorithms designed on the path for solving various algebraic problems and graph problems can be found in [2,14]. The path can also be used as control/data flow structure for distributed computation in arbitrary networks. Another application for the longest path to a practical problem was addressed in the on-line optimization of a complex flexible manufacturing system [4]. These applications motivate the embedding of paths in networks. Our result implies that those algorithms designed for paths can also be executed well on the folded hypercube with faulty edges.

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