12

Reduction by Stages with Topological Conditions

In this chapter we will use the distribution theoretical approach to formulate a reduction by stages theorem that only requires an easily verifiable point set topological condition. This condition is satisfied by a large class of Lie groups, for example, compact ones. Notice that this statement could not have been made had we followed exclusively the purely algebraic approach in § 5.2. Having said that, we will analyze the relation between the stages theorem in this chapter and that in the previous one.

12.1 Reduction by Stages III

In this section we will study a very general condition that also implies the possibility of reducing by stages. We state it in the following result:

12.1.1 Proposition. *Let M^0 be the connected component of the identity of M. Suppose that the symplectic manifold (P, Ω) is Lindelöf and paracompact. Let $\sigma \in \mathbf{J}_M(P) \subset \mathfrak{m}^*$, $\nu := i^*\sigma$, $\mathbf{J}_M^{-1}(\sigma)_C$ be one of the connected components of $\mathbf{J}_M^{-1}(\sigma)$ included in $\mathbf{J}_N^{-1}(\nu)_C$, and θ be the map introduced in* (11.3.2)*. If the orbit $M^0 \cdot \nu \subset \mathfrak{n}^*$ is closed as a subset of $\mathfrak{n}^*$, then*

$$\theta(\mathbf{J}_M^{-1}(\sigma)_C) = (\mathbf{J}_\nu^C)^{-1}(\rho)_C. \tag{12.1.1}$$

Before we proceed with the proof of this proposition we state and prove an important corollary.

12.1.2 Theorem (Reduction by Stages III)**.** *Let M^0 be the connected component of the identity of M. Suppose that the symplectic manifold (P, Ω)*

is Lindelöf and paracompact. Let $\sigma \in \mathbf{J}_M(P) \subset \mathfrak{m}^*$ *and* $\nu := i^*\sigma$*. Suppose that the orbit* $M^0 \cdot \nu \subset \mathfrak{n}^*$ *is closed as a subset of* $\mathfrak{n}^*$*. Let* $\mathbf{J}_M^{-1}(\sigma)_C$ *be one of the connected components of* $\mathbf{J}_M^{-1}(\sigma)$ *included in* $\mathbf{J}_N^{-1}(\nu)_C$*. Then the symplectic reduced spaces*

$$P_\sigma^C := \mathbf{J}_M^{-1}(\sigma)_C / M_\sigma^C \quad \text{and} \quad (P_\nu)_\rho^C := (\mathbf{J}_\nu^C)^{-1}(\rho)_C / (M_\nu^C / N_\nu^C)_\rho^C$$

are symplectomorphic.

An important particular case in which the closedness hypothesis on the orbit $M^0 \cdot \nu$ in the previous corollary is always satisfied is when the group M is compact. Consequently, whenever the manifold P is Lindelöf and paracompact and the group M is compact, the reduction by stages procedure is always viable.

The closedness hypothesis of the coadjoint orbits in the statement of Proposition 12.1.1 is needed in the proof in relation to the existence of the extensions of certain functions.

Proof of the Theorem. Consider the following variation of the diagram (11.3.2):

$$\begin{array}{ccc} \mathbf{J}_M^{-1}(\sigma)_C & \xrightarrow{\theta} & (\mathbf{J}_\nu^C)^{-1}(\rho)_C \\ {\scriptstyle \pi_\sigma^C}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\rho^C} \\ \mathbf{J}_M^{-1}(\sigma)_C / M_\sigma^C & \xrightarrow{\Theta} & (\mathbf{J}_\nu^C)^{-1}(\rho)_C / (M_\nu^C / N_\nu^C)_\rho^C, \end{array}$$

where M_σ^C is the subgroup of M_σ that leaves the connected component $\mathbf{J}_M^{-1}(\sigma)_C$ invariant, and $\pi_\sigma : \mathbf{J}_M^{-1}(\sigma)_C \to \mathbf{J}_M^{-1}(\sigma)_C / M_\sigma^C$ is the canonical projection. The equality $\theta(\mathbf{J}_M^{-1}(\sigma)_C) = (\mathbf{J}_\nu^C)^{-1}(\rho)_C$ guaranteed by the previous proposition implies that θ, and consequently Θ, are surjective. Mimicking the proof of Theorem 11.1.3 it can be shown that Θ is also an injective symplectic immersion, and therefore a symplectomorphism. ■

Proof of Proposition 12.1.1.. We start the proof by stating several lemmas and propositions. For future reference a standard result in manifold theory is stated below. For a proof see Theorem 5.5.9 in [MTA].

12.1.3 Proposition. *Let* P *be a paracompact smooth manifold and* $A \subset P$ *be a closed submanifold of* P*. Any smooth function* $f \in C^\infty(A)$ *admits an extension to a smooth function* $F \in C^\infty(P)$*.*

We now study a distribution that will be of much use.

12.1.4 Lemma. *Let* D *be the generalized distribution on* P *given by* $D = A + E_N$*, where, for any* $z \in P$

$$A(z) = T_z(M \cdot z) = \{\xi_P(z) \mid \xi \in \mathfrak{m}\},$$

and

$$E_N(z) = \operatorname{span}\{X_{g\circ\pi_N}(z) \mid g \in C_c^\infty(P/N)\},$$

where $\pi_N : P \to P/N$ *is the projection onto the orbit space. Then:*

(i) *If* F_t *is the flow of the infinitesimal generator vector field* ξ_P, $\xi \in \mathfrak{m}$, *and* G_t *is the flow of the Hamiltonian vector field* $X_{g\circ\pi_N}$, $g \in C_c^\infty(P/N)$, *then, for any* $t_1, t_2 \in \mathbb{R}$, *we obtain,*

$$G_{t_1} \circ F_{t_2} = F_{t_2} \circ H_{t_1},$$

where H_t *is the flow of the Hamiltonian vector field associated to the function* $h := g \circ \pi_N \circ F_{t_2} \in C^\infty(P)^N$ *that can also be written as* $g \circ \bar{F}_{t_2} \circ \pi_N$. *The map* $\bar{F}_{t_2}$ *is the diffeomorphism of* P/N *uniquely determined by the relation* $\pi_N \circ F_{t_2} = \bar{F}_{t_2} \circ \pi_N$ *and* $g \circ \bar{F}_{t_2} \in C_c^\infty(P/N)$.

(ii) D *is integrable.*

(iii) *The maximal integral leaves of the distribution* D *are given by the orbits*

$$G_D \cdot z = G_A \cdot G_{E_N} \cdot z = M^0 \cdot (\mathbf{J}_N^{-1}(\nu)_C),$$

where $\mathbf{J}_N(z) = \nu$, $\mathbf{J}_N^{-1}(\nu)_C$ *is the connected component of* $\mathbf{J}_N^{-1}(\nu)$ *that contains* z, *and* M^0 *is the connected component of the identity of* M.

Proof. To prove (i), first note that for any time $t \in \mathbb{R}$ and any $z \in P$, $F_t(z) = \exp t\xi \cdot z$. Also, since N is a normal subgroup of M, for any $n \in N$ and $z \in P$ there exists an element $n' \in N$ such that $F_t(n \cdot z) = \exp t\xi n \cdot z = n' \exp t\xi \cdot z = n' \cdot F_t(z)$. Consequently, the map F_t induces a diffeomorphism $\bar{F}_t$ of P/N uniquely determined by the relation $\bar{F}_t \circ \pi_N = \pi_N \circ F_t$. Also, the function $g \circ \pi_N \circ F_t \in C^\infty(M)$ can be written as $g \circ \bar{F}_t \circ \pi_N$ which guarantees that it is an element of $C^\infty(M)^N$ and that, by Theorem 11.2.4 (i), the Hamiltonian vector field $X_{g\circ\pi_N\circ F_t} = X_{g\circ\bar{F}_t\circ\pi_N}$ is complete. Now, since the M-action on P is canonical the map F_t is Poisson and therefore

$$TF_t \circ X_{g\circ\pi_N\circ F_t} = X_{g\circ\pi_N} \circ F_t.$$

Moreover, if G_t is the flow of $X_{g\circ\pi_N}$ and H_t that of $X_{g\circ\pi_N\circ F_{t_2}}$, then it follows that $G_{t_1} \circ F_{t_2} = F_{t_2} \circ H_{t_1}$. Since all the vector fields involved in this expression are complete, this equality is valid for any $t_1, t_2 \in \mathbb{R}$.

Now we turn to the proof of (ii). According to Theorem 11.2.1 it is enough to show that the distribution D is invariant under the action of the diffeomorphisms group G_D generated by the family of vector fields that spans the distribution D, namely,

$$\{\xi_P \mid \xi \in \mathfrak{m}\} \cup \{X_{g\circ\pi_N} \mid g \in C_c^\infty(P/N)\}.$$

More specifically, we have to show that $T_z\mathcal{F}_T(D(z)) = D(\mathcal{F}_T(z))$, for each $\mathcal{F}_T \in G_D$ and any $z \in P$. Actually, it suffices to show the inclusion

$$T_z\mathcal{F}_T(D(z)) \subset D(\mathcal{F}_T(z)) \tag{12.1.2}$$

given that since (12.1.2) is valid for any element in G_D and any point in P, we get $T_{\mathcal{F}_T(z)}(\mathcal{F}_T)^{-1}(D(\mathcal{F}_T(z))) \subset D(z)$. Applying $T_z\mathcal{F}_T$ to both sides of this inclusion we obtain that $D(\mathcal{F}_T(z)) \subset T_z\mathcal{F}_T(D(z))$, as required.

Hence, we now verify that (12.1.2) holds when $\mathcal{F}_T = F^1_{t_1} \circ \cdots \circ F^n_{t_n}$ with $F^i_{t_i}$ the flow of a vector field either of the form ξ_P, with $\xi \in \mathfrak{m}$, or of the form $X_{g\circ\pi_N}$, with $g \in C^\infty_c(P/N)$. We consider both cases separately.

Firstly, let F_t be the flow of $X_{g\circ\pi_N}$, with $g \in C^\infty_c(P/N)$, and $X_{f\circ\pi_N}$ be another Hamiltonian vector field with $f \in C^\infty_N(P/N)$. Then, since F_t is a Poisson map, we see that for any $z \in P$

$$\begin{aligned} T_zF_t(X_{f\circ\pi_N}(z)) &= T_zF_t(X_{f\circ\pi_N\circ F_{-t}\circ F_t}(z)) \\ &= X_{f\circ\pi_N\circ F_{-t}}(F_t(z)) \\ &= X_{f\circ\bar{F}_{-t}\circ\pi_N}(F_t(z)), \end{aligned}$$

where $\bar{F}_{-t}$ is the diffeomorphism of P/N uniquely determined by the equality $\bar{F}_{-t} \circ \pi_N = \pi_N \circ F_{-t}$. Given that $f \circ \bar{F}_{-t} \circ \pi_N \in C^\infty(P)^N$ and $f\circ\bar{F}_{-t} \in C^\infty_c(P/N)$, we obtain $X_{f\circ\pi_N\circ F_{-t}}(F_t(z)) \in E_N(F_t(z)) \subset D(F_t(z))$.

Secondly, let ξ_P, be the vector field on P constructed using the infinitesimal generators associated to the element $\xi \in \mathfrak{m}$. The flow of this vector field is given by the map $G_t := \Phi_{\exp t\xi}$. Consequently,

$$\begin{aligned} T_zF_t(\xi_P(z)) = \left.\frac{d}{ds}\right|_{s=0} F_t(\exp s\xi \cdot z) &= \left.\frac{d}{ds}\right|_{s=0} \exp s\xi \cdot F_t^{g\circ\pi_N\circ\Phi_{\exp s\xi}}(z) \\ &= \xi_P(F_t(z)) + \left.\frac{d}{ds}\right|_{s=0} F_t^{g\circ\pi_N\circ\Phi_{\exp s\xi}}(z), \end{aligned} \tag{12.1.3}$$

where $F_t^{g\circ\pi_N\circ\Phi_{\exp s\xi}}$ is the flow of $X_{g\circ\pi_N\circ\Phi_{\exp s\xi}}$ which, by part (i), is a N-equivariant vector field. Note that the smooth curve $c(s) := F_t^{g\circ\pi_N\circ\Phi_{\exp s\xi}}(z)$ is such that $c(0) = F_t(z)$ and, since $g \circ \pi_N \circ \Phi_{\exp s\xi} \in C^\infty(P)^N$ for all the values of the parameter s then, by Noether's Theorem, $c(s) \in \mathbf{J}_N^{-1}(\nu)$ with $\nu = \mathbf{J}_N(z)$. Therefore,

$$\left.\frac{d}{ds}\right|_{s=0} F_t^{g\circ\Phi_{\exp s\xi}}(z) \in \ker T_{F_t(z)}\mathbf{J}_N = E_N(F_t(z))$$

which, substituted in (12.1.3) allows us to conclude that $T_zF_t(\xi_P(z)) \in D(F_t(z))$.

Thirdly, consider the case in which $G_t := \Phi_{\exp t\xi}$ is the flow of ξ_P, $\xi \in \mathfrak{m}$, and let $\eta \in \mathfrak{m}$ be another arbitrary element in the Lie algebra of M. It is easy to check that

$$T_zG_t(\eta_P(z)) = \eta_P(\exp t\xi \cdot z) + \left.\frac{d}{ds}\right|_{s=0} \exp(t\,\mathrm{Ad}_{\exp -s\eta}\,\xi)\cdot z.$$

Let $g(s) = \exp(t\,\mathrm{Ad}_{\exp -s\eta}\,\xi)$. This curve in M is such that $g(0) = \exp t\xi$, hence, there exists some element $\rho \in \mathfrak{m}$ such that $T_zG_t(\eta_P(z)) = \eta_P(\exp t\xi \cdot z) + \rho_P(\exp t\xi \cdot z) \in A(G_t(z)) \subset E(G_t(z))$, as required.

Finally, let $g\circ\pi_N \in C^\infty(P)^N$, $g \in C_c^\infty(P/N)$, with N-equivariant Hamiltonian flow F_t. Part **(i)** allows us to write

$$T_zG_t(X_{g\circ\pi_N}(z)) = \left.\frac{d}{ds}\right|_{s=0} G_t\circ F_s(z) = \left.\frac{d}{ds}\right|_{s=0} H_s\circ G_t(z),$$

with H_s the flow of the Hamiltonian vector field associated to the N-invariant smooth function $g\circ\pi_N\circ\Phi_{\exp -t\xi}$, hence

$$\begin{aligned}T_zG_t(X_{g\circ\pi_N}(z)) &= X_{g\circ\pi_N\circ G_{-t}}(G_t(z))\\ &= X_{g\circ\bar{G}_{-t}\circ\pi_N}(G_t(z)) \in E_N(G_t(z)) \subset D(G_t(z)).\end{aligned}$$

The four cases studied allow us to conclude that the distribution D is integrable.

Turning to (iii), the integrability of D proved in the previous point and the general theory summarized in Theorem 11.2.1 establish that the maximal integral leaves of D are given by the G_D-orbits. Clearly, $G_A\cdot G_{E_N} \subset G_D$. Part (i) implies the reverse inclusion and therefore $G_A\cdot G_{E_N} = G_D$. Now, by Theorem 11.2.4, $G_{E_N}\cdot z = \mathbf{J}_N^{-1}(\nu)_C$, where $\mathbf{J}_N(z) = \nu$ and $\mathbf{J}_N^{-1}(\nu)_C$ is the connected component of $\mathbf{J}_N^{-1}(\nu)$ that contains z. Consequently,

$$G_D\cdot z = G_A\cdot G_{E_N}\cdot z = M^0\cdot(\mathbf{J}_N^{-1}(\nu)_C),$$

as required. ▼

12.1.5 Lemma. *Let $\nu \in \mathfrak{n}^*$ be an element in $\mathfrak{n}^*$ and M^0 be the connected component of the identity of M. Suppose that ν is such that the orbit $M^0\cdot\nu \subset \mathfrak{n}^*$ is closed as a subset of $\mathfrak{n}^*$. Then the set $\mathbf{J}_N^{-1}(M^0\cdot\nu)$ is a closed embedded submanifold of P. Moreover, if $\mathbf{J}_N^{-1}(M^0\cdot\nu)_C$ is the connected component of $\mathbf{J}_N^{-1}(M^0\cdot\nu)$ that contains $\mathbf{J}_N^{-1}(\nu)_C$, then*

$$M^0\cdot\mathbf{J}_N^{-1}(\nu)_C = \mathbf{J}_N^{-1}(M^0\cdot\nu)_C. \tag{12.1.4}$$

Proof. As we already know, since N is a normal subgroup of M, $\mathfrak{n}^*$ is a M-space, therefore a M^0-space, and hence the orbit $M^0\cdot\nu$ is an immersed submanifold of $\mathfrak{n}^*$. Moreover, we can think of $M^0\cdot\nu$ as one of the maximal integral manifolds of the singular integrable distribution D_{M^0} on $\mathfrak{n}^*$ defined by

$$D_{M^0}(\zeta) := \{\mathrm{ad}_\xi^*\,\zeta \mid \xi\in\mathfrak{m}\}, \quad \text{for all} \quad \zeta\in\mathfrak{n}^*.$$

A standard theorem (see Proposition 2.2 in Dazord [1985]) guarantees that the closed integral leaves of a generalized distribution are always imbedded. Therefore, as $M^0 \cdot \nu$ is closed, it is consequently an embedded submanifold of $\mathfrak{n}^*$. Recall now that since $\mathbf{J}_N$ is the momentum map associated to a free canonical action, it is necessarily a submersion and therefore each point of the orbit $M^0 \cdot \nu$ is one of its regular values. The Transversal Mapping Theorem guarantees in these circumstances that $\mathbf{J}_N^{-1}(M^0 \cdot \nu)$ is an embedded submanifold of P. This result also ensures that, for any $z \in \mathbf{J}_N^{-1}(M^0 \cdot \nu)$,

$$T_z(\mathbf{J}_N^{-1}(M^0 \cdot \nu)) = (T_z\mathbf{J}_N)^{-1}(T_{\mathbf{J}_N(z)}(M^0 \cdot \nu)).$$

The M^0-infinitesimal equivariance of $\mathbf{J}_N$ implies that

$$\begin{aligned} T_{\mathbf{J}_N(z)}(M^0 \cdot \nu) &= \{-\operatorname{ad}^*_\xi \mathbf{J}_N(z) \mid \xi \in \mathfrak{m}\} \\ &= \{T_z\mathbf{J}_N(\xi_P(z)) \mid \xi \in \mathfrak{m}\} \\ &= T_z\mathbf{J}_N(T_z(M^0 \cdot z)), \end{aligned}$$

and consequently,

$$T_z(\mathbf{J}_N^{-1}(M^0 \cdot \nu)) = T_z(M^0 \cdot z) + \ker T_z\mathbf{J}_N = T_z(M \cdot z) + E_N(z).$$

This equality implies that the manifold $\mathbf{J}_N^{-1}(M^0 \cdot \nu)_C$ is an integral submanifold of the distribution D introduced in Lemma 12.1.4, everywhere of maximal dimension. In that result we saw that the maximal integral submanifolds are given by the subsets of the form $M^0 \cdot (\mathbf{J}_N^{-1}(\nu)_C)$. It is clear that $M^0 \cdot (\mathbf{J}_N^{-1}(\nu)_C) \subset \mathbf{J}_N^{-1}(M^0 \cdot \nu)_C$. The maximality of $M^0 \cdot (\mathbf{J}_N^{-1}(\nu)_C)$ implies equality (12.1.4). ▼

We are now in the position to state the result on extensions that we will need in the proof of the proposition.

12.1.6 Proposition. *Let $\nu \in \mathfrak{n}^*$ and M^0 be the connected component of the identity of M. Suppose that ν is such that the orbit $M^0 \cdot \nu \subset \mathfrak{n}^*$ is closed as a subset of $\mathfrak{n}^*$. Then, every function $f \in C^\infty(\mathbf{J}_N^{-1}(\nu)_C)^{M_\nu^C}$ admits an extension to a function $F \in C^\infty(P)^{M^0}$.*

Proof. The natural injection $\varphi : \mathbf{J}_N^{-1}(\nu)_C \hookrightarrow M^0 \cdot \mathbf{J}_N^{-1}(\nu)_C = \mathbf{J}_N^{-1}(M^0 \cdot \nu)_C$ induces a smooth map $\phi : \mathbf{J}_N^{-1}(\nu)_C/M_\nu^C \to \mathbf{J}_N^{-1}(M^0 \cdot \nu)_C/M^0$ that makes the following diagram commutative

$$\begin{array}{ccc} \mathbf{J}_N^{-1}(\nu)_C & \xrightarrow{\varphi} & \mathbf{J}_N^{-1}(M^0 \cdot \nu)_C \\ {\scriptstyle \pi_{M_\nu^C}}\downarrow & & \downarrow{\scriptstyle \pi_{M^0}} \\ \mathbf{J}_N^{-1}(\nu)_C/M_\nu^C & \xrightarrow{\phi} & \mathbf{J}_N^{-1}(M^0 \cdot \nu)_C/M^0. \end{array}$$

Since in this case the identity (12.1.4) holds, it is easy to verify that ϕ is a bijection. Moreover, it is an immersion, and therefore a diffeomorphism.

Indeed, let $[z]_{M_\nu^C} \in \mathbf{J}_N^{-1}(\nu)_C/M_\nu^C$ be arbitrary and $v_z \in T_z\mathbf{J}_N^{-1}(\nu)_C$ be such that

$$T_{[z]_{M_\nu^C}}\phi(T_z\pi_{M_\nu^C} \cdot v_z) = 0.$$

This can be rewritten as

$$T_z(\phi \circ \pi_{M_\nu^C}) \cdot v_z = T_z(\pi_{M^0} \circ \varphi) \cdot v_z = T_z\pi_{M^0}(T_z\varphi \cdot v_z) = 0.$$

The last equality implies the existence of an element $\xi \in \mathfrak{m}$ such that $T_z\varphi(v_z) = \xi_P(z)$, hence $v_z \in T_z(M \cdot z) \cap T_z(\mathbf{J}_N^{-1}(\nu)) = T(M_\nu \cdot z)$, and consequently $T_z\pi_{M_\nu^C} \cdot v_z = 0$, as required. The equality

$$T_z(M \cdot z) \cap T_z(\mathbf{J}_N^{-1}(\nu)) = T(M_\nu \cdot z)$$

follows easily after recalling that if $\xi_P(z) \in T_z(M \cdot z) \cap T_z(\mathbf{J}_N^{-1}(\nu))$, then $T_z\mathbf{J}_N \cdot \xi_P(z) = -\operatorname{ad}_\xi^* \nu = 0$.

Now take an arbitrary function $f \in C^\infty(\mathbf{J}_N^{-1}(\nu)_C)^{M_\nu^C}$, which induces a function $\bar{f} \in C^\infty\left(\mathbf{J}_N^{-1}(\nu)_C/M_\nu^C\right)$ uniquely determined by the relation $\bar{f} \circ \pi_{M_\nu^C} = f$. Let now $\bar{g} \in C^\infty\left(\mathbf{J}_N^{-1}(M^0 \cdot \nu)_C/M^0\right)$ be the smooth function defined by $\bar{g} = \bar{f} \circ \phi^{-1}$. This function induces a M^0 invariant function $g \in C^\infty(\mathbf{J}_N^{-1}(M^0 \cdot \nu)_C)^{M^0}$ on $\mathbf{J}_N^{-1}(M^0 \cdot \nu)_C$ via the equality $g = \bar{g} \circ \pi_{M^0}$. Since $\mathbf{J}_N^{-1}(M^0 \cdot \nu)_C$ is a closed embedded submanifold of P, the function g can be extended by Proposition 12.1.3 to a smooth function $F \in C^\infty(P)$. The properness of the M^0-action and the M^0-invariance of the function g and of the submanifold $\mathbf{J}_N^{-1}(M^0 \cdot \nu)_C$ guarantee that F can be chosen M^0 invariant, as required (check for instance with Proposition 2 of Arms, Cushman, and Gotay [1991]). ▼

12.1.7 Corollary. *Suppose that the coadjoint orbit $M^0 \cdot \sigma \subset \mathfrak{m}^*$ is closed in $\mathfrak{m}^*$. Then every function $f \in C^\infty(\mathbf{J}_M^{-1}(\sigma))^{M_\sigma^C}$ admits an extension to a function $F \in C^\infty(P)^{M^0}$. Also, if the coadjoint orbit $M \cdot \sigma \subset \mathfrak{m}^*$ is closed and embedded in $\mathfrak{m}^*$, then every function $f \in C^\infty(\mathbf{J}_M^{-1}(\sigma))^{M_\sigma}$ admits an extension to a function $F \in C^\infty(P)^M$.*

Proof. For the proof of the first statement just take $N = M$ in the proof of the previous proposition. As to the second one, erase the symbols C that refer to connected components and substitute M^0 by M. As to the hypothesis regarding $M \cdot \sigma \subset \mathfrak{m}^*$ being embedded in $\mathfrak{m}^*$ we need it to reproduce the argument at the very beginning of the proof of Lemma 12.1.5 where we would show, in our case, that $\mathbf{J}_M^{-1}(M \cdot \sigma)$ is a closed embedded submanifold of P. ▼

We are now ready to prove the relation (12.1.1), that is,

$$\theta(\mathbf{J}_M^{-1}(\sigma)_C) = (\mathbf{J}_\nu^C)^{-1}(\rho)_C.$$

The inclusion $\theta(\mathbf{J}_M^{-1}(\sigma)_C) \subset (\mathbf{J}_\nu^C)^{-1}(\rho)_C$ is already known and is a consequence of (11.3.1). Let $\pi_{M_\nu^C/N_\nu^C} : P_\nu^C \to P_\nu^C/(M_\nu^C/N_\nu^C)$ be the canonical projection onto the orbit space. In order to show the equality take an arbitrary point $\pi_\nu^C(z) \in \theta(\mathbf{J}_M^{-1}(\sigma)_C) \subset P_\nu^C$ and consider the maximal integral leaf of the generalized distribution on P_ν^C defined by

$$E_{M_\nu^C/N_\nu^C} = \left\{ X_f \;\middle|\; f \in C^\infty\left(P_\nu^C\right)^{M_\nu^C/N_\nu^C} \quad \text{with} \right.$$
$$\left. f = F \circ \pi_{M_\nu^C/N_\nu^C},\, F \in C_c^\infty\left(P_\nu^C/(M_\nu^C/N_\nu^C)\right) \right\}$$

that goes through $\pi_\nu^C(z)$ which, by Theorem 11.2.4, is the entire $(\mathbf{J}_\nu^C)^{-1}(\rho)_C$. If we are able to show that for any $\mathcal{F}_T \in G_{E_{M_\nu^C/N_\nu^C}}$, we have

$$\mathcal{F}_T(\pi_\nu^C(z)) \in \theta(\mathbf{J}_M^{-1}(\sigma)_C) = \pi_\nu^C(\mathbf{J}_M^{-1}(\sigma)_C),$$

we will have proved the equality. For the sake of simplicity suppose that $\mathcal{F}_T = F_t$, with F_t the Hamiltonian flow associated to the function $f \in C^\infty\left(P_\nu^C\right)^{M_\nu^C/N_\nu^C}$. Let $\bar{f} \in C^\infty(\mathbf{J}_N^{-1}(\nu)_C)^{M_\nu^C}$ be the smooth function defined by $\bar{f} := f \circ \pi_\nu^C$, and let $g \in C^\infty(P)^{M^0}$ be one of its smooth M^0 invariant extensions to P, whose existence is guaranteed by Proposition 12.1.6. Let G_t be the Hamiltonian flow associated to the function g. Note that

$$F_t(\pi_\nu^C(z)) = \pi_\nu^C(G_t(z)).$$

By Theorem 11.2.4, $G_t(z) \in \mathbf{J}_M^{-1}(\sigma)_C$ and therefore

$$\mathcal{F}_T(\pi_\nu^C(z)) = \pi_\nu^C(G_t(z)) \in \theta(\mathbf{J}_M^{-1}(\sigma)_C),$$

as required. ■

12.2 Relation Between Stages II and III

The reader may be wondering if there is a relation between the versions II and III of the reduction by stages theorem. Even though it is true that both results identify sufficient conditions that allow symplectic reduction in two stages, these conditions seem to be nonequivalent. The following proposition shows that the closedness hypothesis in the version III needs to be complemented with an additional condition in order to imply the stages hypothesis II, and therefore the version II of the reduction by stages theorem.

12.2.1 Proposition. *Suppose that the hypotheses of Proposition* 12.1.1 *hold and that, additionally, the following condition is satisfied: for any* $\sigma' \in \mathfrak{m}^*$ *such that* $\nu := i^*\sigma'$*, there is at least one connected component* $\mathbf{J}_M^{-1}(\sigma')_C$ *of* $\mathbf{J}_M^{-1}(\sigma')$ *included in the given connected component* $\mathbf{J}_N^{-1}(\nu)_C$*. Then,* σ *satisfies stages hypothesis II.*

Proof. Let $\sigma' \in \mathfrak{m}^*$ be such that $\sigma'|_{\mathfrak{n}} = \sigma|_{\mathfrak{n}} = \nu$ and $\sigma'|_{\mathfrak{m}_\nu^C} = \sigma|_{\mathfrak{m}_\nu^C}$. By hypothesis, there is a connected component $\mathbf{J}_M^{-1}(\sigma')_C$ of $\mathbf{J}_M^{-1}(\sigma')$ included in $\mathbf{J}_N^{-1}(\nu)_C$. Let $z \in \mathbf{J}_M^{-1}(\sigma')$. Then, for any $[\xi] \in \mathfrak{m}_\nu^C/\mathfrak{n}_\nu^C$ we have

$$\langle \mathbf{J}_\nu^C(\pi_\nu^C(z)), [\xi]\rangle = \langle \sigma', \xi\rangle - \langle \bar{\nu}, \xi\rangle = \langle \sigma, \xi\rangle - \langle \bar{\nu}, \xi\rangle = \langle \rho, [\xi]\rangle.$$

Consequently, $\pi_\nu^C(z)$ belongs to the set $(\mathbf{J}_\nu^C)^{-1}(\rho)_C$ which by (12.1.1) equals $\theta(\mathbf{J}_M^{-1}(\sigma)_C)$. Hence, there exists $z' \in \mathbf{J}_M^{-1}(\sigma)_C$ such that $\pi_\nu^C(z) = \pi_\nu^C(z')$ and therefore $z' = n \cdot z$ for some $n \in N_\nu^C \subset M_{\nu,\rho}^C$. Applying the map $\mathbf{J}_M$ to both sides of this equality we obtain that $\sigma' = \mathrm{Ad}^*_{n^{-1}} \sigma$. Hence, σ satisfies the stages hypothesis II. ■

The following example shows that the situation is similar regarding the reverse implication. More specifically, our example will describe a situation where the stages hypothesis II holds but not the closedness hypothesis needed in the version III of the reduction by stages theorem.

Example. Let M be the subgroup of $SL(2, \mathbb{R})$ given by

$$M = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \middle| \, a > 0, b \in \mathbb{R} \right\}.$$

Consider now the closed normal subgroup N of M given by

$$N = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{m}$ of M is given by the matrices of the form

$$\mathfrak{m} = \left\{ \begin{bmatrix} \xi_1 & \xi_2 \\ 0 & -\xi_1 \end{bmatrix} \middle| \, \xi_1, \xi_2 \in \mathbb{R} \right\}.$$

If we choose the matrices

$$\mathbf{e}_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

as a basis for $\mathfrak{m}$, we can write its elements as two–tuples of the form $(\xi_1, \xi_2) \in \mathbb{R}^2$. In these coordinates, the adjoint action of M on $\mathfrak{m}$ can be expressed as

$$\mathrm{Ad}_g(\xi_1, \xi_2) = (\xi_1, -2ab\xi_1 + a^2\xi_2),$$

where

$$g = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in M \text{ and } (\xi_1, \xi_2) \in \mathfrak{m}.$$

If we identify $\mathfrak{m}^*$ with $\mathfrak{m} \simeq \mathbb{R}^2$ via the Euclidean inner product on $\mathbb{R}^2$, the coadjoint action of M on $\mathfrak{m}^*$ takes the following expression

$$\mathrm{Ad}^*_{g^{-1}}(\alpha_1, \alpha_2) = \left(\alpha_1 + \frac{2b}{a}\alpha_2, \frac{1}{a^2}\alpha_2 \right), \qquad (12.2.1)$$

where

$$g = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in M \text{ and } (\alpha_1, \alpha_2) \in \mathfrak{m}^*.$$

Analogously, the inclusion $i : \mathfrak{n} \hookrightarrow \mathfrak{m}$ is given by $\eta \longmapsto (0, \eta)$, and the dual projection $i^* : \mathfrak{m}^* \to \mathfrak{n}^*$ by $(\alpha_1, \alpha_2) \longmapsto \alpha_2$. Moreover, the coadjoint action of M on $\mathfrak{n}^*$ is given by

$$\mathrm{Ad}^*_{g^{-1}} \alpha = \frac{1}{a^2}\alpha, \text{ for any } g = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in M. \tag{12.2.2}$$

If we visualize $\mathfrak{m}^*$ as the (α_1, α_2)–plane, expression (12.2.1) implies that the M–coadjoint orbits in $\mathfrak{m}^*$ are the open upper and lower half planes together with the points in the α_1–axis. Analogously, by (12.2.2) we can conclude that the coadjoint action of M on $\mathfrak{n}^*$ exhibits three coadjoint orbits, namely, two open half lines, and the point where they meet. In conclusion, if $\alpha \in \mathfrak{n}^*$ is different from zero, its M–orbit is not closed in $\mathfrak{n}^*$ and, consequently, the hypothesis of version III of the Stages Theorem is not satisfied. Nevertheless, we will now consider a free canonical action of M on a symplectic manifold for which both the Stages Hypotheses I and II hold.

Consider the lifted action of M on its cotangent bundle T^*M. If we trivialize T^*M using right translations (space coordinates) we have the following expressions for this canonical action and for its associated M and N–momentum maps:

$$g \cdot (h, \alpha) = (gh, \mathrm{Ad}^*_{g^{-1}} \alpha), \quad \mathbf{J}_M(g, \alpha) = \alpha, \quad \mathbf{J}_N(g, \alpha) = \alpha_2,$$

for any $g, h \in M$ and $\alpha = (\alpha_1, \alpha_2) \in \mathfrak{m}^*$. As the level sets of $\mathbf{J}_M$ and $\mathbf{J}_N$ are connected, there is no difference between the versions I and II of the stages hypothesis. We now verify that this hypothesis holds: first of all notice that for any $\alpha \in \mathfrak{n}^*$

$$M_\alpha = \begin{cases} M, & if \quad \alpha = 0 \\ N, & if \quad \alpha \neq 0. \end{cases}$$

Therefore, if $\alpha = 0$ the stages hypothesis holds trivially. If $\alpha \neq 0$ and we have two elements $\sigma, \sigma' \in \mathfrak{m}^*$ such that $\sigma|_{\mathfrak{n}} = \sigma'|_{\mathfrak{n}} = \sigma|_{\mathfrak{m}_\alpha} = \sigma'|_{\mathfrak{m}_\alpha} = \alpha$ then there exist necessarily $\beta, \gamma \in \mathbb{R}$ such that $\sigma = (\beta, \alpha)$ and $\sigma' = (\gamma, \alpha)$, and consequently:

$$\sigma' = \mathrm{Ad}^*_{g^{-1}} \sigma, \quad \text{with} \quad g = \begin{bmatrix} 1 & \dfrac{\gamma - \beta}{2\alpha} \\ 0 & 1 \end{bmatrix} \in N_\alpha = N,$$

as required. ♦

12.3 Connected Components of Reduced Spaces

A natural question that arises when making the comparison between the distribution approach to the reduction by stages problem and the one taken in § 5.2 is how the reduced spaces obtained in both cases are related. In the following paragraphs we will show that if the coadjoint orbit $M\cdot\sigma \subset \mathfrak{m}^*$ is a closed and embedded submanifold of $\mathfrak{m}^*$, then P_σ^C is a connected component of $\mathbf{J}_M^{-1}(\sigma)/M_\sigma$. An analogous claim can be made regarding $P_{\nu,\rho}^C$. We state all of this in the following proposition.

12.3.1 Proposition. *Let M^0 be the connected component of the identity of M. Suppose that the symplectic manifold (P,Ω) is Lindelöf and paracompact and that the coadjoint orbit $M\cdot\sigma \subset \mathfrak{m}^*$ is a closed and embedded submanifold of $\mathfrak{m}^*$. Then P_σ^C is a connected component of $\mathbf{J}_M^{-1}(\sigma)/M_\sigma$. The same conclusion holds for $P_{\nu,\rho}^C$ whenever the orbit $(M_\nu^C/N_\nu^C)\cdot\rho$ under the affine action of (M_ν^C/N_ν^C) on $(\mathfrak{m}_\nu^C/\mathfrak{n}_\nu^C)^*$ is closed in $(\mathfrak{m}_\nu^C/\mathfrak{n}_\nu^C)^*$.*

Proof. First of all, notice that since $\mathbf{J}_M^{-1}(\sigma)_C$ is connected, so is

$$\mathbf{J}_M^{-1}(\sigma)_C/M_\sigma^C = P_\sigma^C$$

and hence the projection of the inclusion $\mathbf{J}_M^{-1}(\sigma)_C \hookrightarrow \mathbf{J}_M^{-1}(\sigma)$ provides us with an injection

$$i_C : \mathbf{J}_M^{-1}(\sigma)_C/M_\sigma^C \longrightarrow \left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C$$

of $\mathbf{J}_M^{-1}(\sigma)_C/M_\sigma^C$ into some connected component $\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C$ of P_σ. We will prove that i_C is onto. To do this, we will follow a strategy similar in spirit to the one we used to establish the surjectivity of the map Θ. As $\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C$ is a connected symplectic manifold, any two of its points can be joined by piecewise Hamiltonian paths or, more explicitly, the maximal integral leaf of the distribution on $\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C$

$$D = \left\{X_{h_\sigma} \mid h_\sigma \in C^\infty\left(\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C\right)\right\}$$

going through any point in $\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C$ is $\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C$ itself. We will show the surjectivity of i_C by proving that for any $h_\sigma \in C^\infty\left(\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C\right)$ with associated Hamiltonian flow F_t^σ, and for any $\pi_\sigma^C(z) \in P_\sigma^C$, we get

$$F_t^\sigma(i_C(\pi_\sigma^C(z))) \in i_C(P_\sigma^C).$$

Let $\bar{h}_\sigma \in C^\infty\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)$ be an extension of $h_\sigma \in C^\infty\left(\left(\mathbf{J}_M^{-1}(\sigma)/M_\sigma\right)_C\right)$ and let $h \in C^\infty\left(\mathbf{J}_M^{-1}(\sigma)\right)^{M_\sigma}$ be the function defined by $h = \bar{h}_\sigma \circ \pi_\sigma$. By Corollary 12.1.7 the function h admits an extension to a function $H \in C^\infty(P)^M$; let F_t be its associated Hamiltonian flow. Then,

$$F_t^\sigma(i_C(\pi_\sigma^C(z))) = \pi_\sigma(F_t(z)).$$

By Noether's Theorem $F_t(z) \in \mathbf{J}_M^{-1}(\sigma)_C$ and consequently $\pi_\sigma(F_t(z)) \in i_C(P_\sigma^C)$, as required. ■

Conclusions for Part II

In this part we have given a thorough treatment of the problem of regular symplectic reduction by stages. There are, however, many things left to do in this area. Amongst these, there is a need for additional concrete physical applications. Another is to make use of the stages structures in numerical applications using, for instance, variational integrators, as in Marsden and West [2001]. Finally, there is a need for additional functional analytic treatments of infinite dimensional cases, some of which were mentioned in the introduction and in the text.

Recall from the introduction that there is a parallel theory of *Lagrangian reduction by stages* developed in Cendra, Marsden, and Ratiu [2001a]. A critical difference is that the theory of Lagrangian reduction by stages is the Lagrangian analog of Poisson reduction in that no imposition of a level set of the momentum map is made. Nevertheless, there are many strong connections and parallels between the results in the present work and those in the theory of Lagrangian reduction by stages. The Lagrangian analog of point reduction in the symplectic context is that of Routh reduction studied in Marsden, Ratiu and Scheurle [2000]. Of course developing a reduction by stages theory in that context would be of interest.

One may also speculate on further relations with group theory along the lines of the orbit method. After all, the orbit method for semidirect products is closely related to the method of induced representations of Mackey. One would imagine that keeping track of representation theory parallel to reduction by stages would also be interesting.

Another important issue is how to properly generalize things to the multisymplectic context (see, for instance Marsden, Patrick, and Shkoller [1998]). As we have mentioned, in a number of examples in field theory, including complex fluids, one has a cocycle in the associated Poisson structure (Holm and Kupershmidt [1982, 1983b]). The structure of those theories strongly suggests that a reduction by stages approach would be profitable, although the analog of symplectic reduction in field theories is known to be tricky as one normally does *not* impose momentum map constraints until after a 3+1 (space + time) split has been made. This will complicate any eventual theory. On the other hand, from a Lagrangian reduction by stages standpoint, some interesting progress has been made in this direction (see Holm [2002]).