On Transitive Uncertainty Mappings

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Abstract. This paper is devoted to the discussion of transitive uncertainty mapping in general approximation space. It is proved that the best low-approximation mapping exist if the uncertainty mapping is transitive. Furthermore, the best low-approximation mapping is defined and its properties are discussed.

Keywords: Rough set, covering rough set, covering lower approximation, covering upper approximation, representative element.

1 Introduction

Rough set theory(RST), proposed by Pawlak [2], [3], is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. It provides a systematic approach for the study of indiscernibility of objects. Typically, indiscernibility is described using equivalent relations. When objects of a universe are described by a set of attributes, one may define the indiscernibility of objects based on their attribute values. When two objects have the same value over a certain group of attributes, we say they are indiscernible with respect to this group of attributes, or have the same description with respect to the indiscernibility relation. Objects of the same description consist of an equivalence class and all the equivalence classes form a partition of the universe. With this partition, rough set theory approximates any subset of objects of the universe by two sets, called the lower and upper approximations. They can be formally described by a pair of unary set-theoretic operators. It is noticed that equivalence relation or partition, as the indiscernibility relation in Pawlak's original rough set theory, is restrictive for many applications. To address this issue, several interesting and meaningful extensions to equivalent relation have been proposed in the past, such as tolerance relation [4,12], similarity relation [13], and others [14,15,16,17]. This leads to various approximation operators. By adopting the notion of neighborhood systems from topological space, Lin[6,7] proposed a more general framework for the study of approximation operators.

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Zakowski [20] have used coverings of a universe for establishing the covering generalized rough set theory and an extensive body of research works have been developed [2,3,11]. A. Gomolinska [5] provided a new approach for the study of rough approximations where the starting point is a generalized approximation space. The rough approximation operator was regarded as set-valued mapping, called approximation mapping. Two pairs of basic approximation mappings were defined typically and generalized approximation mappings were constructed by the compositions of these basic approximation mappings. Some axioms for approximation mappings were proposed. Based on these axioms, the best low-approximation mapping was studied.

This paper is devoted to the discussion of transitive uncertainty mapping. The motivation is to construct the best, in accordance with Gomolinska's axioms, approximation operators in general approximation space. It is proved that the best low-approximation mapping exist if the uncertainty mapping is transitive. Furthermore, the best low-approximation mapping is defined and its properties are discussed.

2 Preliminaries

This section presents a review of some fundamental notions of Pawlak's rough sets. We refer to [2,9,10] for details.

Let U be a finite set, the universe of discourse, and R an equivalence relation on U, called an indiscernibility relation. The pair (U, R) is called a Pawlak approximation space. R will generate a partition $U/R = \{[x]_R; x \in U\}$ on U, where $[x]_R$ is the equivalence class with respect to R containing x. For each $X \subseteq U$, the upper approximation $\overline{R}(X)$ and lower approximation $\underline{R}(X)$ of X are defined as [9,10]

$$\overline{R}(X) = \{x; [x]_R \cap X \neq \emptyset\}, \tag{1}$$

$$\underline{R}(X) = \{x; [x]_R \subseteq X\}. \tag{2}$$

Alternatively, in terms of equivalence classes of R, the pair of lower and upper approximation can be defined by

$$\overline{R}(X) = \cup\{[x]_R; [x]_R \cap X \neq \emptyset\}, \tag{3}$$

$$\underline{R}(X) = \cup\{[x]_R; [x]_R \subseteq X\}. \tag{4}$$

Let $\emptyset$ be the empty set and $\sim X$ the complement of X in U, the following conclusions have been established for Pawlak's rough sets:

(1)$\underline{R}(U) = U = \overline{R}(U)$.
(2)$\underline{R}(\emptyset) = \emptyset = \overline{R}(\emptyset)$.
(3)$\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$.
(4)$\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(X)$, $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$.
(5)$\underline{R}(\underline{R}(X)) = \underline{R}(X)$, $\overline{R}(\overline{R}(X)) = \overline{R}(X)$.

(6)$\underline{R}(X) = \sim \overline{R}(\sim X)$, $\overline{R}(X) = \sim \underline{R}(\sim X)$.
(7)$X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$, $\overline{R}(X) \subseteq \overline{R}(Y)$.
(8)$\underline{R}(\sim \underline{R}(X)) = \sim \underline{R}(X)$, $\overline{R}(\sim \overline{R}(X)) = \sim \overline{R}(X)$.
(9)$\overline{R}(\underline{R}(X)) \subseteq X \subseteq \underline{R}(\overline{R}(X))$.

It has been shown that (3), (4) and (8) are the characteristic properties of the lower and upper approximations [8,23,18].

3 A General Notion of Rough Approximation Mapping

A general approximation space is a triple $A = (U, I, k)$, where U is a non-empty set called the universe, $I : U \to P(U)$ is an uncertainty mapping, and $k : P(U) \times P(U) \to [0, 1]$ is a rough inclusion function.

In general approximation space $A = (U, I, k)$, $w \in I(u)$ is understood as w is in some sense similar to u and it is reasonable to assume that $u \in I(u)$ for every $u \in U$. Then $\{I(u); u \in U\}$ forms a covering of the universe U. The role of the uncertainty mapping may be played by a binary relation on U.

We consider mappings $f : P(U) \to P(U)$. We can define a partial ordering relation, $\leq$, on the set of all such mappings as follows: $f \leq g$ if and only if $\forall \subseteq U(f(x) \subseteq g(x))$, for every $f, g : P(U) \to P(U)$. By id we denote the identity mapping on $P(U)$. $g \circ f : P(U) \to P(U)$ defined by $g \circ f(x) = g(f(x))$ for every $x \subseteq U$, is the composition of f and g. We call g dual to f, written $g = f^d$, if $g(x) = \sim f(\sim x)$. The mapping f is monotone if and only if for every $x, y \subseteq U$, $x \subseteq y$ implies $f(x) \subseteq f(y)$.

3.1 Axioms for Rough Approximation Mappings

Theoretically speaking, every rough approximation operator is a mapping from $P(U)$ to $P(U)$, we call it approximation mapping. [5] proposed some fundamental properties that any reasonable rough approximation mapping $f : P(U) \to P(U)$ should possibly possess. They are the following axioms:

($a1$) Every low-mapping f is decreasing, $i.e.$, $f \leq id$.
($a2$) Every upp-mapping f is increasing, $i.e.$, $id \leq f$.
($a3$) If f is a low-mapping, then $(*)\forall x \subseteq U \forall u \in f(x)(I(u) \subseteq x)$.
($a4$) If f is a upp-mapping, then $(**)\forall x \subseteq U \forall u \in f(x)(I(u) \cap x \neq \emptyset)$.
($a5$) For each $x \subseteq U$, $f(x)$ is definable in A, $i.e.$, there exists $y \subseteq U$ such that $f(x) = \cup\{I(u); u \in y\}$.
($a6$) For each $x \subseteq U$ definable in A, $f(x) = x$.

The motivation behind these axioms was analyzed in[5]. Also, it is noticed that finding appropriate candidates for low- and upp-mappings satisfying these axioms is not an easy matter in general case.

3.2 The Structure of Rough Approximation Mappings

Let $A = (U, I, k)$ be a general approximation space. The approximation mappings $f_0, f_1 : P(U) \to P(U)$ were defined as[5]: for every $x \subseteq U$,

$$f_0(x) = \bigcup\{I(u); u \in x\}, \tag{5}$$

$$f_1(x) = \{u; I(u) \cap x \neq \emptyset\}. \tag{6}$$

Observe that f_0^d and f_1^d satisfy:

$$f_0^d(x) = \{u; \forall w(u \in I(w) \Rightarrow w \in x)\}, \tag{7}$$

$$f_1^d(x) = \{u; I(u) \subseteq x\}. \tag{8}$$

If $\{I(u); u \in U\}$ is a partition of U, then $f_0 = f_1$, $f_0^d = f_1^d$ and they are the classical rough approximation operators.

Based on f_0, f_1 and their dual mappings, several approximation mappings were defined[5] by means of operations of composition and duality as follows: for every $x \subseteq U$,

$f_2 \doteq f_0 \circ f_1^d$: *i.e.*, $f_2(x) = \bigcup\{I(u); I(u) \subseteq x\}$,

$f_3 \doteq f_0 \circ f_1$: *i.e.*, $f_3(x) = \bigcup\{I(u); I(u) \cap x \neq \emptyset\}$,

$f_4 \doteq f_0^d \circ f_1 = f_2^d$: *i.e.*, $f_4(x) = \{u; \forall w(u \in I(w) \Rightarrow I(w) \cap x \neq \emptyset)\}$,

$f_5 \doteq f_0^d \circ f_1^d = f_3^d$: *i.e.*, $f_5(x) = \{u; \forall w(u \in I(w) \Rightarrow I(w) \subseteq x)\}$,

$f_6 \doteq f_1^d \circ f_1^d$: *i.e.*, $f_6(x) = \{u; \forall w(w \in I(u) \Rightarrow I(w) \subseteq x)\}$,

$f_7 \doteq f_0 \circ f_6 = f_0 \circ f_1^d \circ f_1^d = f_2 \circ f_1^d$: *i.e.*, $f_7(x) = \bigcup\{I(u); \forall w(w \in I(u) \Rightarrow I(w) \subseteq x)\}$,

$f_8 \doteq f_1^d \circ f_1$: *i.e.*, $f_8(x) = \{u; \forall w(w \in I(u) \Rightarrow I(w) \cap x \neq \emptyset)\}$,

$f_9 \doteq f_0 \circ f_8 = f_0 \circ f_1^d \circ f_1 = f_2 \circ f_1$: *i.e.*, $f_9(x) = \bigcup\{I(u); \forall w(w \in I(u) \Rightarrow I(w) \cap x \neq \emptyset)\}$.

Theorem 1. *[5] Consider any* $f : P(U) \to P(U)$.

(1) $f(x)$ *is definable for any* $x \subseteq U$ *iff there is a mapping* $g : P(U) \to P(U)$ *such that* $f = f_0 \circ g$.

(2) The condition $(*)$ *is satisfied iff* $f \leq f_1^d$.

(3) The condition $(**)$ *is satisfied iff* $f \leq f_1$.

Theorem 2. *[5] For any sets* $x, y \subseteq U$, *we have that:*

(1) $f_i(\emptyset) = \emptyset$ *and* $f_i(U) = U$ *for* $i = 0, 1, \cdots, 9$. $f_i^d(\emptyset) = \emptyset$ *and* $f_i^d(U) = U$ *for* $i = 0, 1$.

(2) f_i *and* f_j^d *are monotone for* $i = 0, 1, \cdots, 9$ *and* $j = 0, 1$.

(3) $f_i(x \cup y) = f_i(x) \cup f_i(y)$ *for* $i = 0, 1, 3$.

(4) $f_i(x \cap y) = f_i(x) \cap f_i(y)$ *and* $f_j^d(x \cap y) = f_j^d(x) \cap f_j^d(y)$ *for* $i = 5, 6$ *and* $j = 0, 1$.

Theorem 3. *[5] For any sets* $x, y \subseteq U$, *we have that:*

(1) $f_5 \leq f_1^d \leq f_2 \leq id \leq f_4 \leq f_1 \leq f_3$.

(2) $f_5 \leq f_0^d \leq id \leq f_0 \leq f_3$.

(3) $f_6 \leq f_7 \leq f_1^d$.

(4) $f_8 \leq f_9 \leq f_1$.

(5) $f_i \circ f_i = f_i$ *for* $i = 2, 4$.

In view of the previous results and in accordance with the axioms, any low- or upp-mapping should have the form $f_0 \circ g$, where $g : P(U) \to P(U)$ satisfies $f_0 \circ g \circ f_0 = f_0$ and, moreover, $f_0 \circ g \leq f_1^d$ in the lower case, while $id \leq$

$f_0 \circ g \leq f_1$ in the upper case[5]. Clearly, $\leq -$maximal among the low-mappings and $\leq -$minimal among the upp-mappings would be the best approximation operators. The greatest element among the low-mappings just described is the mapping $h : P(U) \rightarrow P(U)$ where for any $x \subseteq U$,

$$h(x) = \cup\{(f_0 \circ g)(x); g : P(U) \rightarrow P(U) \wedge f_0 \circ g \circ f_0 = f_0 \wedge f_0 \circ g \leq f_1^d\}. \quad (9)$$

It is noticed that an analogous construction, using $\cap$, does not provide us with the least element of the family of upp-mappings[5].

4 The Best Approximation Operators

In this section, we discuss the condition with which the approximation mapping h exist. We noticed that (9) make sense provided $S \neq \emptyset$ where
$S = \{g; g : P(U) \rightarrow P(U), f_0 \circ g \circ f_0 = f_0, f_0 \circ g \leq f_1^d\}$.

Theorem 4. *[5] Consider any $f : P(U) \rightarrow P(U)$. f satisfies $(a5)$ and $(a6)$ if and only if there is a mapping $g : P(U) \rightarrow P(U)$ such that $f = f_0 \circ g$ and $f_0 \circ g \circ f_0 = f_0$.*

Proof. ($\Rightarrow$) Assume f satisfies $(a5)$ and $(a6)$. By Theorem 1, there is a mapping $g : P(U) \rightarrow P(U)$ such that $f = f_0 \circ g$. Consider any $x \subseteq U$, by definability of $f_0(x)$, we have $f_0 \circ g \circ f_0(x) = f(f_0(x)) = f_0(x)$. Hence $f_0 \circ g \circ f_0 = f_0$.

($\Leftarrow$) Assume $f = f_0 \circ g$ and $f_0 \circ g \circ f_0 = f_0$ for some $g : P(U) \rightarrow P(U)$. By Theorem 1, f satisfies $(a5)$. If $x \subseteq U$ is definable, then there is $y \subseteq U$ such that $x = \cup\{I(u); u \in y\} = f_0(y)$. Consequently $f(x) = f_0 \circ g(x) = f_0 \circ g(f_0(y)) = f_0(y) = x$. Hence f satisfies $(a6)$.

Theorem 5. *If $S \neq \emptyset$,then $h = f_0 \circ G$ is the greatest element among the low-mappings which satisfies $(a1), (a3), (a5)$ and $(a6)$, where $G : P(U) \rightarrow P(U)$ satisfies: for every $x \subseteq U$,*
$G(x) = \cup\{g(x); g \in S\}$.

The proof of this theorem is trivial.

Theorem 6. *If*

$$\forall u \in U \forall v \in U (u \in I(v) \Rightarrow I(u) \subseteq I(v)) \quad (10)$$

is satisfied, then
(1) $f_0 \circ f_1^d = f_1^d$,
(2) $f_1^d \circ f_0 = f_0$.

Proof. Assume (10). Consider any $x \subseteq U$ and $u \in U$.

(1) If $u \in f_0 \circ f_1^d(x) = \cup\{I(v); v \in f_1^d(x)\}$, then there exist $v \in f_1^d(x)$ such that $u \in I(v)$. Hence $I(u) \subseteq I(v) \subseteq x$. By definition, $u \in f_1^d(x)$ and $f_0 \circ f_1^d \leq f_1^d$. It follows that $f_0 \circ f_1^d = f_1^d$ by $f_0 \geq id$.

(2) If $u \in f_0(x)$, then there exist $v \in x$ such that $u \in I(v)$. Hence $I(u) \subseteq I(v) \subseteq f_0(x)$ and $u \in f_1^d \circ f_0(x)$. In other words, $f_1^d \circ f_0 \geq f_0$. It follows that $f_1^d \circ f_0 = f_0$.

Theorem 7. *$S \neq \emptyset$ if and only if* (10) *is satisfied.*

Proof. Suppose that $S \neq \emptyset$. It follows that there exists $g : P(U) \to P(U)$ such that $f_0 \circ g \circ f_0 = f_0$ and $f_0 \circ g \leq f_1^d$. By

$$f_0 = f_0 \circ g \circ f_0 \leq f_1^d \circ f_0 \leq f_0,$$

$f_1^d \circ f_0 = f_0$ followed. For every $u, v \in U$ with $u \in I(v)$, by

$$I(v) = f_0(\{v\}) = f_1^d \circ f_0(\{v\}) = f_1^d(I(v)) = \{w; I(w) \subseteq I(v)\},$$

it follows that $I(u) \subseteq I(v)$.

Conversely, assume (10). By Theorem 6,

$$f_0 \circ f_1^d \circ f_0 = (f_0 \circ f_1^d) \circ f_0 = f_1^d \circ f_0 = f_0,$$

$$f_0 \circ f_1^d \leq f_1^d.$$

Hence $f_1^d \in S$ and $S \neq \emptyset$.

By Theorem 6 and Theorem 7, if (10) is satisfied, f_1^d is $\leq -$maximal among the low-mappings which satisfy $(a1)$, $(a3)$, $(a5)$ and $(a6)$. Hence f_1^d is the best low-approximation mapping.

5 The Transitive Uncertainty Mapping

In view of the previous results, the condition (10) plays a central role in general approximation spaces. It is just the transitivity of uncertainty mapping. In this section, we will concentrate on properties specific for this kind of uncertainty mapping.

Theorem 8. *Assume* (10)*. For any sets $x, y \subseteq U$, we have that:*
(1) $f_i \circ f_i = f_i$ for $i = 0, 1$.
(2)$f_2 = f_6 = f_7 = f_1^d$.
(3) $f_4 = f_1$.
(4) $f_8 = f_9$.

Proof. Consider any $x \subseteq U$ and $u \in U$.

(1) If $u \in f_1 \circ f_1(x) = \{v; I(v) \cap f_1(x) \neq \emptyset\}$, then $I(u) \cap f_1(x) \neq \emptyset$. It follows that there exists $v \in I(u)$ such that $I(v) \cap x \neq \emptyset$. By $I(v) \subseteq I(u)$, $I(u) \cap x \neq \emptyset$ followed. By the definition, $u \in f_1(x)$. In other words, $f_1 \circ f_1 \leq f_1$. Consequently, $f_1 \circ f_1 = f_1$ by $f_1 \geq id$.

If $u \in f_0 \circ f_0(x) = \cup\{I(v); v \in f_0(x)\}$, then there exists $v \in U$ such that $v \in f_0(x)$ and $u \in I(v)$. By the definition, there is $w \in x$ such that $v \in I(w)$. Consequently, $u \in I(v) \subseteq I(w)$ and $u \in f_0(x)$. In other words, $f_0 \circ f_0 \leq f_0$ and $f_0 \circ f_0 = f_0$ followed by $f_0 \geq id$.

(2) By Theorem 6, $f_7 = f_0 \circ f_1^d \circ f_1^d = (f_0 \circ f_1^d) \circ f_1^d = f_1^d \circ f_1^d = f_6$, $f_6 = f_1^d \circ f_1^d = (f_1 \circ f_1)^d = f_1^d$, $f_2 = f_0 \circ f_1^d = f_1^d$.

(3) and (4) can be proved similarly.

By (1) of Theorem 8, $f_0 \circ id \circ f_0 = f_0 \circ f_0 = f_0$ and $f_0 = f_0 \circ id$, it follows that f_0 satisfies axiom $(a6)$. We summarize the approximation mappings and satisfiability of the axioms in Table 1. By $+$ (resp., $-$) we denote that a condition is (is not) satisfied, while $\perp$ denotes that the result does not count. From the Table we know that f_1^d is the best low-approximation mapping since it satisfies all axioms and f_0 is in our opinion the best candidate for a upp-mapping since it satisfies three axioms $(a1)$, $(a5)$ and $(a6)$.

Table 1. Approximation mappings and satisfiability of the axioms if (10) holds

f	Form	status	$a1$	$a2$	$a3$	$a4$	$a5$	$a6$
f_0		upp	$\perp$	$+$	$\perp$	$-$	$+$	$+$
$f_1 = f_4$		upp	$\perp$	$+$	$\perp$	$+$	$-$	$-$
f_0^d		low	$+$	$\perp$	$-$	$\perp$	$-$	$-$
$f_1^d = f_2 = f_6 = f_7$		low	$+$	$\perp$	$+$	$\perp$	$+$	$+$
f_3	$f_0 \circ f_1$	upp	$\perp$	$+$	$\perp$	$-$	$+$	$-$
f_5	$f_0^d \circ f_1^d$	low	$+$	$\perp$	$+$	$\perp$	$-$	$-$
$f_8 = f_9$	$f_1^d \circ f_1$	upp	$\perp$	$-$	$\perp$	$+$	$-$	$-$

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