On Covering Rough Sets

Keyun Qin[1], Yan Gao[2], and Zheng Pei[3]

[1] Department of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 610031, China
[2] College of Computer Science and Technology, Henan Polytechnic University, Jiaozuo, Henan, 454000, China
[3] School of Mathematics & Computer Science, Xihua University, Chengdu, Sichuan, 610039, China
keyunqin@263.net
pqyz@263.net

Abstract. This paper is devoted to the discussion of extended covering rough set models. Based on the notion of neighborhood, five pairs of dual covering approximation operators were defined with their properties being discussed. The relationships among these operators were investigated. The main results are conditions with which these covering approximation operators are identical.

Keywords: Rough set, covering rough set, covering lower approximation, covering upper approximation, representative element.

1 Introduction

Rough set theory(RST), proposed by Pawlak [2], [3], is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. It provides a systematic approach for the study of indiscernibility of objects. Typically, indiscernibility is described using equivalent relations. When objects of a universe are described by a set of attributes, one may define the indiscernibility of objects based on their attribute values. When two objects have the same value over a certain group of attributes, we say they are indiscernible with respect to this group of attributes, or have the same description with respect to the indiscernibility relation. Objects of the same description consist of an equivalence class and all the equivalence classes form a partition of the universe. With this partition, rough set theory approximates any subset of objects of the universe by two sets, called the lower and upper approximations. They can be formally described by a pair of unary set-theoretic operators. It is noticed that equivalence relation or partition, as the indiscernibility relation in Pawlak's original rough set theory, is restrictive for many applications. To address this issue, several interesting and meaningful extensions to equivalent relation have been proposed in the past, such as tolerance relation [4,12], similarity relation [13], and others [14,15,16,17]. This leads to various approximation operators. By adopting the notion of neighborhood systems from topological space, Lin[6,7] proposed a more general framework for the study of approximation operators. Zakowski [20] have used coverings of a

J.T. Yao et al. (Eds.): RSKT 2007, LNAI 4481, pp. 34–41, 2007.

universe for establishing the covering generalized rough set theory and an extensive body of research works have been developed [2,3,11]. In [22], the concept of reducts of coverings is introduced and the conditions for two coverings to generate the same covering lower approximation or the same covering upper approximation were given. In[5], the indiscernibility relation is generalized to any binary reflexive relation and some generalized approximation operators were introduced. The following problems worth paying attention to. Some important properties of Pawlak's lower and upper approximation do not hold for the covering lower and upper approximation, such as Duality, Multiplication and Addition [22]. For covering upper approximation, even Monotonicity does not hold.

This paper is devoted to the discussion of extended covering rough set models. Based on the concept of neighborhood, five pairs of dual covering approximation operators were defined with their properties being discussed. The relationship among these operators were investigated. Some equivalent conditions for covering approximation operators coinciding with each other were given. Furthermore, if each element of the universe is a representative element [2], the Multiplication for Zakowski's lower approximation operator holds.

2 Preliminaries

This section presents a review of some fundamental notions of Pawlak's rough sets and covering rough sets. We refer to [2,9,10] for details.

2.1 Fundamentals of Pawlak's Rough Sets

Let U be a finite set, the universe of discourse, and R an equivalence relation on U, called an indiscernibility relation. The pair (U, R) is called a Pawlak approximation space. R will generate a partition $U/R = \{[x]_R; x \in U\}$ on U, where $[x]_R$ is the equivalence class with respect to R containing x. $\forall X \subseteq U$, the upper approximation $\overline{R}(X)$ and lower approximation $\underline{R}(X)$ of X are defined as [9,10] $\overline{R}(X) = \{x; [x]_R \cap X \neq \emptyset\}$, $\underline{R}(X) = \{x; [x]_R \subseteq X\}$ Alternatively, in terms of equivalence classes of R, the pair of lower and upper approximation can be defined by $\overline{R}(X) = \cup\{[x]_R; [x]_R \cap X \neq \emptyset\}$, $\underline{R}(X) = \cup\{[x]_R; [x]_R \subseteq X\}$. Let $\emptyset$ be the empty set and $\sim X$ the complement of X in U, the following conclusions have been established for Pawlak's rough sets: (1)$\underline{R}(U) = U = \overline{R}(U)$. (2)$\underline{R}(\emptyset) = \emptyset = \overline{R}(\emptyset)$. (3)$\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$. (4)$\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(X)$, $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$. (5)$\underline{R}(\underline{R}(X)) = \underline{R}(X)$, $\overline{R}(\overline{R}(X)) = \overline{R}(X)$. (6)$\underline{R}(X) = \sim \overline{R}(\sim X)$, $\overline{R}(X) = \sim \underline{R}(\sim X)$. (7)$X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$, $\overline{R}(X) \subseteq \overline{R}(Y)$. (8)$\underline{R}(\sim \underline{R}(X)) = \sim \underline{R}(X)$, $\overline{R}(\sim \overline{R}(X)) = \sim \overline{R}(X)$. (9)$\overline{R}(\underline{R}(X)) \subseteq X \subseteq \underline{R}(\overline{R}(X))$.

It has been shown that (3), (4) and (8) are the characteristic properties of the lower and upper approximations [8,23,18].

2.2 Concepts and Properties of Covering Rough Sets

Definition 1. *Let U be a universe of discourse, C a family of subsets of U. If no subsets in C is empty, and $\cup C = U$, C is called a covering of U.*

It is clear that a partition of U is a covering of U, so the concept of a covering is an extension of the concept of a partition. In the following discussion, the universe of discourse U is considered to be finite. We will list some definitions and results about covering rough set.

Definition 2. *[2] Let U be a non-empty set, C a covering of U. The pair (U, C) is called a covering approximation space.*

Definition 3. *[2] Let (U, C) be a covering approximation space and $x \in U$. The family of sets $Md(x) = \{K \in C; x \in K \wedge (\forall S \in C)(x \in S \wedge S \subseteq K \Rightarrow K = S)\}$ is called the minimal description of x.*

Definition 4. *[2] Let (U, C) be a covering approximation space and $X \subseteq U$. The family $C_*(X) = \{K \in C; K \subseteq X\}$ is called the covering lower approximation set of X. Set $X_* = \cup C_*(X)$ is called the covering lower approximation of X. Set $X_*^* = X - X_*$ is called the covering boundary of X. The family $Bn(X) = \cup\{Md(x); x \in X_*^*\}$ is called the covering boundary approximation set family of X. The family $C^*(X) = C_*(X) \cup Bn(X)$ is called the covering upper approximation family of X. Set $X^* = \cup C^*(X)$ is called the covering upper approximation of X. If $C^*(X) = C_*(X)$, X is said to be exact, otherwise inexact.*

Theorem 1. *[2] Let (U, C) be a covering approximation space, then $\forall X, Y \subseteq U$ and $\forall x \in U$, (1) $C_*(\emptyset) = C^*(\emptyset) = \emptyset$, $C_*(U) = C^*(U) = C$. (2) $C_*(X) \subseteq C^*(X)$. (3) $C_*(X_*) = C_*(X) = C^*(X_*)$. (4) $X \subseteq Y \Rightarrow C_*(X) \subseteq C_*(Y)$, $X \subseteq Y \Rightarrow X_* \subseteq Y_*$. (5) $C_*(X_*)^* = \emptyset$. (6) $C_*(\{x\}) \neq \emptyset \Leftrightarrow \{x\} \in C$. (7) $C^*(\{x\}) = Md(x)$. (8) $\cap Md(x) = \cap\{K \in C; x \in K\}$.*

Theorem 2. *[2] Let (U, C) be a covering approximation space and $X, Y \subseteq U$, then (1) $U_* = U = U^*$. (2) $\emptyset_* = \emptyset = \emptyset^*$. (3) $X_* \subseteq X \subseteq X^*$. (4) $(X_*)_* = X_*$, $(X^*)^* = X^*$.*

By providing some examples, Zhu [22] shown that the following properties do not hold for the covering lower and upper approximations: (1) $(X \cap Y)_* = X_* \cap Y_*$, $(X \cup Y)^* = X^* \cup Y^*$. (2) $X_* = \sim (\sim X)^*$, $X^* = \sim (\sim X)_*$. (3) $X \subseteq Y \Rightarrow X^* \subseteq Y^*$. (4) $(\sim X_*)_* = \sim X_*$, $(\sim X^*)^* = \sim X^*$.

3 The Extension of the Covering Approximation Operators

Let (U, C) be a covering approximation space. For each $x \in U$, $N(x) = \cap\{K \in C; x \in K\}$ is called the neighborhood of x. By (8) of Theorem 1, $N(x) = \cap Md(x)$. We know that Pawlak's approximation operators can be defined in two different, but equivalent, ways. Similarly, we consider five pairs of dual approximation operators defined by means of neighborhoods as follows: for each $X \subseteq U$, (I1) $\underline{C_1}(X) = X_* = \cup\{K \in C; K \subseteq X\}$, $\overline{C_1}(X) = \sim \underline{C_1}(\sim X) = \cap\{\sim K; K \in C, K \cap X = \emptyset\}$. (I2) $\underline{C_2}(X) = \{x \in U; N(x) \subseteq X\}$, $\overline{C_2}(X) = \{x \in U; N(x) \cap X \neq \emptyset\}$. (I3) $\underline{C_3}(X) = \{x \in U; \exists u(u \in N(x) \wedge N(u) \subseteq X)\}$, $\overline{C_3}(X) =$

$\{x \in U; \forall u(u \in N(x) \rightarrow N(u) \cap X \neq \emptyset)\}$. (I4) $\overline{C_4}(X) = \cup\{N(x); N(x) \cap X \neq \emptyset\}$, $\underline{C_4}(X) = \{x \in U; \forall u(x \in N(u) \rightarrow N(u) \subseteq X)\}$. (I5) $\overline{C_5}(X) = \cup\{N(x); x \in X\}$, $\underline{C_5}(X) = \{x \in U; \forall u(x \in N(u) \rightarrow u \in X)\}$.

Remark 1. The operator $\underline{C_1}$ is just Zakowski's lower approximation[20] and is studied in [2,3,11]. $\overline{C_5}$ is Zhu's upper approximation operator[22].

3.1 On Approximation Operators (I1)

By the definition, $\underline{C_1}(X) = X_*$ for any $X \subseteq U$. Consequently, by Theorem 1, Theorem 2 and the duality of $\overline{C_1}$ and $\underline{C_1}$, we have:

Theorem 3. *Let (U, C) be a covering approximation space and $X, Y \subseteq U$, then (1) $\underline{C_1}(U) = U = \overline{C_1}(U)$, $\underline{C_1}(\emptyset) = \emptyset = \overline{C_1}(\emptyset)$. (2) $\underline{C_1}(X) \subseteq X \subseteq \overline{C_1}(X)$. (3) $X \subseteq Y \Rightarrow \underline{C_1}(X) \subseteq \underline{C_1}(Y) \wedge \overline{C_1}(X) \subseteq \overline{C_1}(Y)$. (4)$\underline{C_1}(\underline{C_1}(X)) = \underline{C_1}(X)$, $\overline{C_1}(\overline{C_1}(X)) = \overline{C_1}(X)$.*

Definition 5. *[2] Let (U, C) be a covering approximation space and $K \in C, x \in K$. x is called a representative element of K if $\forall S \in C(x \in S \Rightarrow K \subseteq S)$.*

By Fact 7 in [2], x is a representative element of K if and only if $Md(x) = \{K\}$, and if and only if $N(x) = K$. We denote by C_0 the set of all representative elements of sets of the covering C, that is $C_0 = \{x \in U; \exists K \in C(x \in K \wedge \forall S \in C(x \in S \Rightarrow K \subseteq S))\}$.

Lemma 1. *Let (U, C) be a covering approximation space and $x \in U$. Then, $x \in C_0$ if and only if $|Md(x)| = 1$.*

Proof. Assume that $x \in C_0$. It follows that $Md(x) = \{N(x)\}$ and $|Md(x)| = 1$. Conversely, assume that $|Md(x)| = 1$. We suppose that $Md(x) = \{K\}$, this means that K is the unique minimal element of $\{S \in C; x \in S\}$ and x is a representative element of K, and consequently $x \in C_0$.

Theorem 4. *Let (U, C) be a covering approximation space. Then, $C_0 = U$ if and only if for any $X, Y \subseteq U$, $\underline{C_1}(X \cap Y) = \underline{C_1}(X) \cap \underline{C_1}(Y)$.*

Proof. ($\Rightarrow$) Suppose that $C_0 = U$. For each $X, Y \subseteq U$ and $x \in \underline{C_1}(X) \cap \underline{C_1}(Y)$, there exist $K_1, K_2 \in C$ such that $x \in K_1, K_1 \subseteq X$ and $x \in K_2, K_2 \subseteq Y$. By $C_0 = U$, there exist $K \in C$ such that x is a representative element of K, it follows that $K \subseteq K_1$, $K \subseteq K_2$ and hence $x \in K \subseteq K_1 \cap K_2 \subseteq X \cap Y$, that is $x \in \underline{C_1}(X \cap Y)$. It follows that $\underline{C_1}(X \cap Y) \supseteq \underline{C_1}(X) \cap \underline{C_1}(Y)$, and hence $\underline{C_1}(X \cap Y) = \underline{C_1}(X) \cap \underline{C_1}(Y)$ by (3) of Theorem 4. Conversely, if $C_0 \neq U$, then there exists $x \in U$ such that $x \notin C_0$. It follows that $|Md(x)| > 1$. Suppose that $K_1, K_2 \in Md(x)$ and $K_1 \neq K_2$, it follows that $x \in K_1 \cap K_2 = \underline{C_1}(K_1) \cap \underline{C_1}(K_2)$. On the other hand, for each $K \in C$ such that $K \subseteq K_1 \cap K_2$, $x \notin K$ and hence $x \notin \cup\{K \in C; K \subseteq K_1 \cap K_2\} = \underline{C_1}(K_1 \cap K_2)$, this contradicts $\underline{C_1}(K_1 \cap K_2) = \underline{C_1}(K_1) \cap \underline{C_1}(K_2)$.

By the duality, we have the following corollary:

Corollary 1. *Let (U, C) be a covering approximation space. The following conditions are equivalent: (1) $C_0 = U$. (2)For each $X, Y \subseteq U$, $\overline{C_1}(X \cup Y) = \overline{C_1}(X) \cup \overline{C_1}(Y)$.*

Theorem 5. *Let (U, C) be a covering approximation space such that $C_0 = U$. Then for each $X \subseteq U$, $\underline{C_1}(X) = \underline{C_2}(X)$, $\overline{C_1}(X) = \overline{C_2}(X)$. This means that approximation operators $(I1)$ and $(I2)$ are equivalent, when $C_0 = U$.*

Proof. By the duality, we need only to prove $\underline{C_1}(X) = \underline{C_2}(X)$. If $x \in \underline{C_1}(X)$, then there exists $K \in C$ such that $K \subseteq X$ and $x \in K$, it follows that $N(x) \subseteq K \subseteq X$ and hence $x \in \underline{C_2}(X)$. Conversely, if $x \in \underline{C_2}(X)$, then $N(x) \subseteq X$, suppose that x is a representative element of S, it follows that $S = N(x) \subseteq X$ and hence $x \in \underline{C_1}(X)$ by $x \in S$.

3.2 On Approximation Operators (I2∼I5)

By Theorem 4.4 of [5], we have

Theorem 6. *Let (U, C) be a covering approximation space and $X, Y \subseteq U$, then (1) $\underline{C_i}(U) = U = \overline{C_i}(U)$, $\underline{C_i}(\emptyset) = \emptyset = \overline{C_i}(\emptyset)$ for $i = 2, 3, 4, 5$. (2) $\underline{C_i}(X) \subseteq X \subseteq \overline{C_i}(X)$ for $i = 2, 4, 5$. (3) $X \subseteq Y \Rightarrow \underline{C_i}(X) \subseteq \underline{C_i}(Y) \wedge \overline{C_i}(X) \subseteq \overline{C_i}(Y)$ for $i = 2, 3, 4, 5$. (4)$\underline{C_i}(X \cap Y) = \underline{C_i}(X) \cap \underline{C_i}(Y)$, $\overline{C_i}(X \cup Y) = \overline{C_i}(X) \cup \overline{C_i}(Y)$ for $i = 2, 4, 5$.*

Theorem 7. *Let (U, C) be a covering approximation space and $X \subseteq U$, then (1) $\underline{C_i}(\underline{C_i}(X)) = \underline{C_i}(X)$, $\overline{C_i}(\overline{C_i}(X)) = \overline{C_i}(X)$ for $i = 2, 5$. (2) $X \subseteq \underline{C_4}(\overline{C_4}(X))$, $\overline{C_4}(\underline{C_4}(X)) \subseteq X$.*

Proof. We only prove (1) for $i = 2$. By (2) of Theorem 6 , $\underline{C_2}(\underline{C_2}(X)) \subseteq \underline{C_2}(X)$. Conversely, suppose that $x \in \underline{C_2}(X)$. It follows that $N(x) \subseteq X$. Consequently, $N(y) \subseteq N(x) \subseteq X$ for any $y \in N(x)$, that is $y \in \underline{C_2}(X)$ and hence $N(x) \subseteq \underline{C_2}(X)$, $x \in \underline{C_2}(\underline{C_2}(X))$.

The following properties do not hold in general: (1) $X \subseteq \underline{C_2}(\overline{C_2}(X))$. (2) $X \subseteq \underline{C_5}(\overline{C_5}(X))$. (3) $\underline{C_4}(\underline{C_4}(X)) = \underline{C_4}(X)$, $\overline{C_4}(\overline{C_4}(X)) = \overline{C_4}(X)$. (4) $\underline{C_3}(X) \subseteq X \subseteq \overline{C_3}(X)$.

Example 1. Let $U = \{x, y, z\}$, $K_1 = \{x, y\}$, $K_2 = \{y, z\}$, $C = \{K_1, K_2\}$. Clearly, C is a covering of U, $N(x) = \{x, y\}$,$N(y) = \{y\}$, $N(z) = \{y, z\}$. (1) For $X = \{x\}$, $\overline{C_4}(X) = \cup\{N(u); x \in N(u)\} = N(x) = \{x, y\}$, $\overline{C_4}(\overline{C_4}(X)) = \overline{C_4}(\{x, y\}) = N(x) \cup N(y) \cup N(z) = U$. (2) For $X = \{y\}$, $\underline{C_5}(\overline{C_5}(X)) = \overline{C_5}(N(y)) = (\sim N(x)) \cap (\sim N(z)) = \emptyset$. $\underline{C_3}(X) = \{v \in U; \exists u(u \in N(v) \wedge N(u) \subseteq X)\} = U$. (3) For $X = \{z\}$, $\underline{C_2}(\overline{C_2}(X)) = \underline{C_2}(\{z\}) = \emptyset$.

4 Connections of Covering Approximation Operators

Theorem 8. *Let (U, C) be a covering approximation space. For each $X \subseteq U$, (1) $\underline{C_4}(X) \subseteq \underline{C_2}(X)$. (2) $\underline{C_2}(X) \subseteq \underline{C_3}(X)$. (3) $\underline{C_4}(X) \subseteq \underline{C_5}(X)$.*

Proof. (1)Suppose that $x \in \underline{C_4}(X)$. By the definition, for every $u \in U$ such that $x \in N(u)$, $N(u) \subseteq X$ followed. Consequently, $N(x) \subseteq X$ and $x \in \underline{C_2}(X)$. (2)Suppose that $x \in \underline{C_2}(X)$. It follows that $N(x) \subseteq X$. Hence $x \in \underline{C_3}(X)$ by $x \in N(x)$. (3)can be proved similarly.

Corollary 2. *Let (U, C) be a covering approximation space. For each $X \subseteq U$, (1) $\underline{C_4}(X) \subseteq \underline{C_2}(X) \subseteq X \subseteq \overline{C_2}(X) \subseteq \overline{C_4}(X)$. (2) $\underline{C_4}(X) \subseteq \underline{C_5}(X) \subseteq X \subseteq \overline{C_5}(X) \subseteq \overline{C_4}(X)$. (3)$\underline{C_4}(X) \subseteq \underline{C_2}(X) \subseteq \underline{C_3}(X)$, $\overline{C_3}(X) \subseteq \overline{C_2}(X) \subseteq \overline{C_4}(X)$.*

Generally, we cannot substitute $=$ for $\subseteq$ due to the following example.

Example 2. Let $U = \{x, y, z\}$, $K_1 = \{x, y\}$, $K_2 = \{y, z\}$, $C = \{K_1, K_2\}$. Clearly, C is a covering of U, $N(x) = \{x, y\}$,$N(y) = \{y\}$, $N(z) = \{y, z\}$. For $X = \{y\}$, $\underline{C_4}(X) = \cap\{\sim N(u); N(u) \nsubseteq X\} = (\sim N(x)) \cap (\sim N(z)) = \emptyset$, $\underline{C_2}(X) = \{u; N(u) \subseteq X\} = \{y\}$. $\underline{C_3}(X) = \{v \in U; \exists u(u \in N(v) \wedge N(u) \subseteq X)\} = U$. For $X = \{x, z\}$, $\underline{C_4}(X) = \cap\{\sim N(u); N(u) \nsubseteq X\} = (\sim N(x)) \cap (\sim N(y)) \cap (\sim N(z)) = \emptyset$, $\underline{C_5}(X) = \cap\{\sim N(u); u \in (\sim X)\} =\sim N(y) = \{x, z\}$, $\underline{C_2}(X) = \{u; N(u) \subseteq X\} = \emptyset$. $\underline{C_3}(X) = \{v \in U; \exists u(u \in N(v) \wedge N(u) \subseteq X)\} = \emptyset$.

Lemma 2. *Let (U, C) be a covering approximation space. Then, $\{N(x); x \in U\}$ forms a partition of U if and only if for each $x, y \in U$, $x \in N(y) \Rightarrow y \in N(x)$.*

Proof. Suppose that $\{N(x); x \in U\}$ forms a partition of U. For each $x, y \in U$, if $x \in N(y)$, then $N(x) \subseteq N(y)$, and $N(x) \cap N(y) = N(x) \neq \emptyset$, it follows that $N(x) = N(y)$ and $y \in N(y) = N(x)$. Conversely, suppose that $x \in N(y) \Rightarrow y \in N(x)$ for each $x, y \in U$. If $N(x) \cap N(y) \neq \emptyset$, then there exist $z \in U$ such that $z \in N(x)$ and $z \in N(y)$, it follows that $x \in N(z)$ and $y \in N(z)$, consequently, $N(x) = N(z)$, $N(y) = N(z)$, and $N(x) = N(y)$. That is to say, $\{N(x); x \in U\}$ forms a partition of U.

Theorem 9. *Let (U, C) be a covering approximation space. Then, $\{N(x); x \in U\}$ forms a partition of U if and only if for each $X \subseteq U$, $\underline{C_4}(X) = \underline{C_2}(X)$.*

Proof. Suppose that $\{N(x); x \in U\}$ forms a partition of U, $X \subseteq U$ and $x \in U$. If $x \in \underline{C_2}(X)$, then $N(x) \subseteq X$. For each $y \in U$ such that $x \in N(y)$, $y \in N(x)$ followed and hence $N(y) = N(x) \subseteq X$. By the definition, $x \in \underline{C_4}(X)$. This means $\underline{C_4}(X) \supseteq \underline{C_2}(X)$ and hence $\underline{C_4}(X) = \underline{C_2}(X)$ by (1) of Theorem 8. Conversely, suppose that $\underline{C_4}(X) = \underline{C_2}(X)$ for each $X \subseteq U$. For each $x, y \in U$ such that $x \in N(y)$, by $x \in \underline{C_2}(N(x)) = \underline{C_4}(N(x)) = \{v \in U; \forall u(v \in N(u) \rightarrow N(u) \subseteq N(x))\}$, it follows that $N(y) \subseteq N(x)$ and hence $y \in N(x)$. By Lemma 2, $\{N(x); x \in U\}$ forms a partition of U.

Theorem 10. *Let (U, C) be a covering approximation space. Then, $\{N(x); x \in U\}$ forms a partition of U if and only if for each $X \subseteq U$, $\overline{C_4}(X) = \overline{C_5}(X)$.*

Proof. Suppose that $\{N(x); x \in U\}$ forms a partition of U, $X \subseteq U$ and $x \in U$. If $x \in \overline{C_4}(X) = \cup\{N(y); N(y) \cap X \neq \emptyset\}$, then there exists $y \in U$ such that $x \in N(y)$ and $N(y) \cap X \neq \emptyset$. Let $z \in N(y) \cap X$, it follows that $z \in X$ and $z \in N(y)$. Consequently, $y \in N(z)$, $N(y) \subseteq N(z)$ and $x \in N(z)$, that is $x \in \cup\{N(u); u \in X\} = \overline{C_5}(X)$, and $\overline{C_4}(X) \subseteq \overline{C_5}(X)$. By (2) of Theorem 8, $\overline{C_4}(X) = \overline{C_5}(X)$. Conversely, suppose that for each $X \subseteq U$, $\overline{C_4}(X) = \overline{C_5}(X)$. For each $x, y \in U$ such that $x \in N(y)$, by $y \in N(y)$, it follows that $y \in \cup\{N(z); N(z) \cap \{x\} \neq \emptyset\} = \overline{C_4}(\{x\})$, and hence $y \in \overline{C_5}(\{x\}) = N(x)$. This means $\{N(x); x \in U\}$ forms a partition of U.

Theorem 11. *Let (U, C) be a covering approximation space, then (1) $\{N(x); x \in U\}$ forms a partition of U if and only if for each $X \subseteq U$, $\overline{C_4}(X) = \overline{C_3}(X)$. (2) $\{N(x); x \in U\}$ forms a partition of U if and only if for each $X \subseteq U$, $\overline{C_2}(X) = \overline{C_3}(X)$. (3) $\{N(x); x \in U\}$ forms a partition of U if and only if for each $X \subseteq U$, $\overline{C_5}(X) = \overline{C_3}(X)$. (4) $\{N(x); x \in U\}$ forms a partition of U if and only if for each $X \subseteq U$, $\overline{C_2}(X) = \overline{C_5}(X)$.*

By Theorem 9, 10 and 11, if any two pairs of operators are identical, then they are all identical.

5 Conclusions

In this paper, five pairs of dual covering approximation operators were defined and their properties have been discussed. Some equivalent conditions about these operators were given. For a covering approximation space (U, C), define a binary relation R on U as follows: for each $x, y \in U$, $(x, y) \in R$ if and only if $\forall K \in C(x \in K \rightarrow y \in K)$. It is trivial to verify that the successor of an element with respect to R coincides with its neighborhood, that is $R_s(x) = \{y \in U; (x, y) \in R\} = N(x)$. With this definition, binary relation based covering rough set can be constructed. We will discuss this problem in our future work.

Acknowledgements

This work has been supported by the National Natural Science Foundation of China (Grant No. 60474022) and the Young Foundation of Sichuan Province (Grant no. 06ZQ026-037).

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