Fuzzy Flip-Flops Revisited

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Abstract. J-K flip-flops are elementary digital units providing sequential features/memory functions. Their definitive equation is used both in the minimal disjunctive and conjunctive forms. Fuzzy connectives do not satisfy all Boolean axioms, thus the fuzzy equivalents of these equations result in two non-equivalent definitions, "reset and set type" fuzzy flip-flops (F^3) by Hirota *& al.* when introducing the concept of F^3. There are many alternatives for "fuzzifying" digital flip-flops, using standard, algebraic or other connectives. The paper gives an overview of some of the most famous F^3-s by presenting their definitions and presenting graphs of the inner state for a typical state value situation. Then a pair of non-associative operators is introduced, and the properties of the respective F^3 are discussed. The investigation of possible fuzzy flip-flops is continued by examining Türkşen's IVFS, its midpoint values, and by introducing "minimized IVFS" (MIVFS), along with the MIVFS midpoints.

Keywords: Fuzzy logic, J-K flip-flop, logical normal forms, IVFS.

1 Introduction

As a possible starting point for defining fuzzy logical sequential circuits and for effective processing of fuzzy information, [11] introduced the notion of the fuzzy J-K flip-flop. They introduced two types of fundamental characteristic equations of fuzzy flip-flops (F^3), "reset type and set type" equations, both being fuzzy extensions of the binary J-K flip-flop. While they proposed standard operations, the algebraic fuzzy flip-flop was introduced later by [10], as an example of the general F^3 concept also defined in the same paper.

While briefly reviewing these definitions, several new examples will be given, using further norms. Next, the concept will be extended for non-associative fuzzy operations (cf. [2]). We prove the surprising result, that comparing the equations of the next state of the set and reset type [4] version of a modified version of the Fodor fuzzy flip-flop (F^4) [2, 8], i.e. there is only one single F^4 and the two formulas are equivalent. Equivalence does not hold for any other F^3 defined as far in the literature.

Eventually interval valued fuzzy flip-flops will be overviewed both in sense of Türkşen's *IVFS* and according to a new concept of minterm-maxterm *IVFS* (*MIVFS*).

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2 Binary (Boolean) and Fuzzy Flip-Flops

2.1 Binary Logic J-K Flip-Flops

All types of traditional binary flip-flop circuits, such as the most general J-K flip-flop store a single bit of information. These circuits are also the elementary components of synchronous sequential digital circuits. The next state $Q(t+1)$ of a J-K flip-flop is characterized as a function of both the present state $Q(t)$ and the two present inputs $J(t)$ and $K(t)$. In the next, (t) will be omitted. The minterm expression (Disjunctive Normal Form, DNF) of $Q(t+1)$ can be written as

$$Q(t+1) = \overline{J}\,\overline{K}Q + J\overline{K}\,\overline{Q} + J\overline{K}Q + JK\overline{Q} \ , \tag{1}$$

and this can be simplified to the minimal disjunctive form

$$Q(t+1) = J\overline{Q} + \overline{K}Q \ . \tag{2}$$

This latter is well-known as the characteristic equation of J-K flip-flops. The equivalent maxterm expression (Conjunctive Normal Form, CNF) is given by

$$Q(t+1) = (J+K+Q)(J+\overline{K}+Q)\left(J+\overline{K}+\overline{Q}\right)\left(\overline{J}+\overline{K}+\overline{Q}\right) \ , \tag{3}$$

in a similar way the minimized conjunctive form is

$$Q(t+1) = (J+Q)(\overline{K}+\overline{Q}) \ . \tag{4}$$

All four expressions, (1), (2), (3) and (4), are equivalent in Boolean logic, but there are no such fuzzy operation triplets where these forms are necessarily equivalent. Which of these four, or any other, should be considered as the most proper fuzzy extension of the definitive equation of the very fundamental concept of fuzzy J-K flip-flop? There is no justifiable argumentation that prefers any of these four to the other and there is no unambiguous way to introduce the concept of fuzzy J-K flip-flop. While normal forms are especially important for theoretical reasons both minimal forms are equally important in the practice. Thus [4, 5] proposed two dual definitions of fuzzy flip-flops (F^3). (2) was interpreted as the definition of "reset type F^3":

$$Q_R(t+1) = (J \wedge \neg Q) \vee (\neg K \wedge Q) \ , \tag{5}$$

where the logic operation symbols stand for standard fuzzy conjunction, disjunction and negation. In a similar way the definition of "set type F^3" was obtained by reinterpreting (4) with fuzzy operations:

$$Q_S(t+1) = (J \vee Q) \wedge (\neg K \vee \neg Q) \ . \tag{6}$$

Of course, it is possible to substitute standard operations by any other fuzzy operation triplet (preferably De Morgan triplet), this way obtaining a multitude of various fuzzy flip-flop (F^3) pairs, such as the algebraic F^3 introduced in [10,11].

3 Behavior of F^3s Based on Various Fuzzy Operations

Many different norms play important roles in applications or have nice mathematical properties. Only few have been investigated from the point of view of the F^3-s generated by deploying them in (5) and (6): the standard and algebraic norms and a pair of non-associative operations. (For the three see [4, 5], [10, 11] and [2], respectively.) Reset and set type F^3-s sometimes do have rather different behavior even though they are both extensions of the same binary circuit. They are identical at the borders (*J, K* = 0, 1), thus the pairs are generalizations, but they are disturbingly non-dual. The negation used in the whole paper will be the standard negation.

3.1 Standard Fuzzy Flip-Flops

Equations (5) (reset type F^3) and (6) (set type F^3) can be expressed respectively as

$$Q_R(t+1) = \max\left[\min(J, 1-Q), \min(1-K, Q)\right] \tag{7}$$

and

$$Q_S(t+1) = \min\left[\max(J, Q), \max(1-K, 1-Q)\right] \ . \tag{8}$$

Standard F^3-s are characterized by piecewise linear functions with breakpoints in the projections, consequently break lines in three dimensions. Calculation with standard F^3-s is fast and easy, but characteristic functions are not smooth (see Fig. 1).

3.2 Algebraic Fuzzy Flip-Flops

If deploying standard complementation, algebraic product and algebraic sum for negation, t-norm and t-conorm, respectively, equations (5) and (6) can be rewritten as

$$Q_R(t+1) = J + Q - 2JQ - KQ + JQ^2 + JQK - JQK^2 \ , \tag{9}$$

$$Q_S(t+1) = J + Q - JQ - JKQ - KQ^2 + JKQ^2 \ . \tag{10}$$

These equations show the results of transformation into a simplified form by using the definitions of algebraic product and sum. Combining equations (9) and (10) we obtain the unified equation of reset and set type:

$$Q(t+1) = J + Q - JQ - KQ \ . \tag{11}$$

This equation is considered the *fundamental equation of the algebraic* F^3. It is remarkable how simple this combined equation is. In addition to its simplicity it represents a symmetrical, dual solution.

Evaluating the curves in Fig. 1, they clearly demonstrate the relation between $Q_R(t+1)$ and $Q_S(t+1)$:

$$Q_S(t+1) - Q_R(t+1) = \left[J(1-K) + K(1-J)\right]Q(1-Q) \geq 0 \tag{12}$$

$$Q_R(t+1) \le Q_S(t+1). \tag{13}$$

Reset type curves always go below set ones. Algebraic operations produce smooth (differentiable) curves and surfaces with no breakpoints or lines at all. This is true also for the combined F^3 which is not shown here.

3.3 Drastic Fuzzy Flip-Flops

Fuzzy unions and intersections satisfying the axiomatic skeleton of boundary conditions, commutativity, monotony, and associativity are bounded by the inequalities (cf. [6])

$$\max(a,b) \le u(a,b) \le u_{\max}(a,b), \tag{14}$$

$$u_{\max}(a,b) = \begin{cases} a & \text{when } b = 0, \\ b & \text{when } a = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{15}$$

$$i_{\min}(a,b) \le i(a,b) \le \min(a,b), \tag{16}$$

$$i_{\max}(a,b) = \begin{cases} a & \text{when } b = 1, \\ b & \text{when } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Operations $u_{\max}(a,b)$ and $i_{\min}(a,b)$ are called “drastic sum” and “drastic product”. Together they represent a pair of fuzzy union and fuzzy intersection which are extremal in the sense that for all $a,b \in [0,1]$, inequality

$$u_{\max}(a,b) - i_{\min}(a,b) \ge u(a,b) - i(a,b) \tag{18}$$

is satisfied for an arbitrary pair of fuzzy union u and intersection i.

Fig. 1 illustrates also the behavior of these two extremal F^3-s. The characteristic lines are piecewise linear again, often following the 0 or 1 line. Obviously these lines (and the surfaces they represent) realize extremal cases even for the possible F^3-s.

3.4 Łukasiewicz Fuzzy Flip-Flops

The Łukasiewicz norm and co-norm are defined as follows:

$$i_L(a,b) = \max[a+b-1, 0], \tag{19}$$

$$u_L(a,b) = \min[a+b, 1]. \tag{20}$$

Based on them, the two Łukasiewicz F^3-s are given by

$$Q_R(t+1) = \min\left[\max(J-Q,0)+\max(Q-K,0),1\right] \ , \tag{21}$$

$$Q_S\left(t+1\right) = \max\left[\min(J+Q,1)+\min(2-K-Q,1)-1,0\right] \ . \tag{22}$$

3.5 Non-associative Fuzzy Flip-Flops

In [2] a pair of non-associative (non-dual) operations for a new class of fuzzy flip-flops was proposed. It was stated there that any F^3 satisfying:

P1: F (0, 0, Q) = Q,
P2: F (0, 1, Q) = 0,
P3: F (1, 0, Q) = 1,
P4: F (1, 1, Q) = n (Q),
P5: F (*e*, *e*, Q) =*e*,
P6: F (D, n (D), Q) = D for D$\in \left(0,1\right)$,

where *e* = n (*e*) is the equilibrium belonging to n and n is a strong negation. P2, P3 and P6 can be merged into the single property P6’:

P6’: F (D, n (D), Q) =D, D$\in \left[0,1\right]$.

[2] states a Theorem according to which $Q_R\left(t+1\right)$ fulfils P6’ if and only if there exists an automorphism ϕ of the unit interval such that

$$Q_R\left(t+1\right) = \phi^{-1}\left[\phi(J)(1-\phi(Q))+\phi(Q)(1-\phi(K))\right] \ . \tag{23}$$

Similarly, $Q_S\left(t+1\right)$ fulfils the same if and only if there exists an automorphism ψ satisfying

$$Q_S\left(t+1\right) = \psi^{-1}\left[\psi(J)(1-\psi(Q))+\psi(Q)\left(1-\psi\left(K\right)\right)\right] \cdot \tag{24}$$

The following equation satisfies all P_i-s (i = 1, 4, 5, 6’)

$$\begin{aligned} Q_R\left(t+1\right) &= \min\left[T\left(J,1-Q\right)+T\left(1-K,Q\right),1\right] = \\ &= T(J,1-Q)+T(1-K,Q) \ . \end{aligned} \tag{25}$$

The formula for the set type F^4 in [2] is however not the proper dual pair of (25), moreover, it is problematic in the sense of closeness for the unit interval. Thus in [8] we proposed a corrected definition for the set type formula as follows:

$$\begin{aligned} Q_S\left(t+1\right) &= \max\left[S\left(J,Q\right)+S\left(1-K,1-Q\right)-1,0\right] = \\ &= S(J,Q)+S(1-K,1-Q)-1 \ . \end{aligned} \tag{26}$$

This correction does not change the validity of the above referred Theorem concerning ψ. In (25) and (26) *T* and *S* denote the so called Łukasiewicz norms. From here:

$$Q_R(t+1) = \frac{\min\left(J,1-Q\right)+\max\left(J-Q,0\right)}{2} + \frac{\min\left(Q,1-K\right)+\max\left(Q-K,0\right)}{2}\,, \tag{27}$$

$$Q_S\left(t+1\right) = \frac{\max(J,Q)+\max\left(1-K,1-Q\right)-1}{2} + \frac{\min\left(J+Q,1\right)+\min\left(2-Q-K,1\right)-1}{2}\,. \tag{28}$$

These equations were obtained by combining the standard and the Łukasiewicz norms by the arithmetic mean in the inner part of the formula. The other parts use Łukasiewicz operations. Let us briefly denote this corrected pair of [2] type fuzzy flip-flops (referring to the name of the first author J. Fodor) by F^4.

Comparing the modified form of the set type F^4 we come to the surprising result that there is only one F^4 in this particular case, as the two formulas are equivalent.

Statement 3.1

${Q_R}^{F^4} = {Q_S}^{F^4}$, thus there is *only one* (symmetrical, corrected) F^4.
In [9] we proved that

$$Q_R\left(t+1\right) = Q_S\left(t+1\right) \tag{29}$$

We found no straightforward technique for the proof other than analyzing the equations theoretically for every possible combination of J,K,Q. For the 6 values *J, K, Q,* $\neg J$, $\neg K$, $\neg Q$, the total number of all possible sequence combinations is $3!*2^3 = 48$ as any variable and its negated are symmetrical to the equilibrium $e = 0.5$. These 48 cases are, however, not all essentially different. We succeeded to identify 13 essentially different cases and we could prove the equality for each of them separately. The deduction of these proofs is omitted here for the sake of saving space.

Thus the new F^4 proposed in [9] is indeed a single F^3 with nice dual and symmetrical behavior (see also Fig. 1).

4 The DNF-CNF Interval

The Disjunctive and Conjunctive Normal Forms (DNF and CNF) play very special roles in classical Boolean logic. They represent those standard forms, which do not contain any redundancy in the sense that they cannot be further reduced by applying the idempotence law; however they consist of complete members containing all variables in question. Thus usually they can be simplified by merging and eliminating, this way obtaining the corresponding minimal forms (DMF and CMF). In fuzzy logic there are no standard forms in this sense as idempotence itself does usually not hold. Any repeated member would usually change the value of the expression.

Despite this latter fact several authors consider the fuzzy extensions of DNF and CNF as having special significance. Especially, Türkşen [12] examined the special properties of the extended normal forms and came to the very convenient conclusion that in fuzzy logic the equivalents of the CNF and DNF forms represent the extremes of all possible expressions that correspond to forms being equivalent in binary logic. Thus he showed that for any fuzzy connective the value of every other form lies in the interval formed by the CNF and DNF values. In [13], this result was proven for logical operations with an arbitrary number of variables.

4.1 Interval Valued Fuzzy Sets

While the theory of the interval valued fuzzy sets (*IVFS*) is much more general and can be considered as a special case of L-fuzzy sets [3], Türkşen proposed the interval determined by the Disjunctive and Conjunctive Normal Forms as the interval associated with the value belonging to an expression obtained by the fuzzy extension of some classic binary concept. Indeed, any theoretically possible formulation of the same concept would result in a value lying within the interval thus proposed.

According to Türkşen's statement DNF is always included in the corresponding CNF, i.e., $DNF(\cdot) \subseteq CNF(\cdot)$ where $(\cdot)$ represents any particular expression [12]. The fundamental result that every DNF is contained in its corresponding CNF is true for min-max operators and for algebraic triplets as well. Türkşen [12] proposed the definition of the interval-valued fuzzy set representing a Boolean expression as follows:

$$IVFS(\cdot) = \left[DNF(\cdot), CNF(\cdot) \right] . \tag{30}$$

4.2 Fuzzy Flip-Flops Based on Türkşen's *IVFS* and *MIVFS*

The DNF and CNF forms for J-K flip-flops are expressed in 2.1. By applying the usual denotations for fuzzy negation, t-norm and t-conorm, (1) is re-written as follows

$$Q_{DNF}(t+1) = \big((1-J) \wedge (1-K) \wedge Q\big) \vee \big(J \wedge (1-K) \wedge (1-Q)\big) \vee \big(J \wedge (1-K) \wedge Q\big) \vee \big(J \wedge K \wedge (1-Q)\big), \tag{31}$$

further, in the same way, (3) becomes

$$Q_{CNF}(t+1) = (J \vee K \vee Q) \wedge \big(J \vee (1-K) \vee Q\big) \wedge \big(J \vee (1-K) \vee (1-Q)\big) \wedge \big((1-J) \vee (1-K) \vee (1-Q)\big). \tag{32}$$

This way we obtained two more definitions of min-max fuzzy flip-flops. It is questionable, of course, whether these new definitions play any more important role in the practice than the previous set type and reset type equations. These latter ones may be called "normalized reset type and set type" F^3-s.

Using the algebraic operations, a similar pair of normalized flip-flops may be obtained. Here we omitted the explicit formulas as it is rather easy to determine them but they are somewhat lengthy.

Although the examination of these new F^3-s may be interesting itself, in this paper we go one step further. Reset and set type behaviors are generally different and none of them possess the ˝nice˝ symmetrical behavior of the original J-K flip-flop. This is why a symmetrical F^3 was proposed earlier by combining the two minimal forms in the equilibrium point [10]. Only one exceptional combination of operations has been found as far where the two types completely coincided (see Section 3.5).

If we intend to combine two different extensions of the original definition we might also choose a single representative point of each interval corresponding to the *IVFS* obtained from the two normal forms. As the most obvious representative, the midpoint is proposed here. Fig. 1 depicts the graphs belonging to DNF and CNF flip-flops. The graph presents the behavior of the F^3 for the general case *J=0.45,* and *K=0.80* (in steps of 0.25 from 0 to 1).

In all earlier publications on F^3-s, however, *minterm* and *maxterm* expressions played important roles rather then the normal forms. Thus, we propose here a new idea, the interval limited by the *minterm* and the *maxterm* expressions:

$$MIVFS(\cdot) = \left[DMF(\cdot), CMF(\cdot) \right] \tag{33}$$

MIVFS gives a narrower interval according to the inequality $DNF(\cdot) \subseteq DMF(\cdot) \subseteq CMF(\cdot) \subseteq CNF(\cdot)$. Properties of *MIVFS* based F^3-s will be examined later.

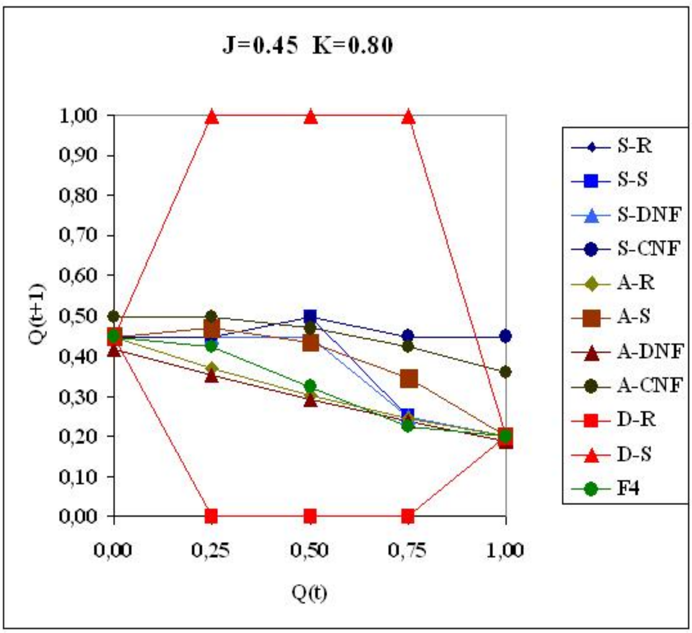

Fig. 1. The behavior of various F^3-s for a general JK value pair

5 Conclusions

After an overview of previous definitions of F^3-s, extensions for further connectives were presented. Then non-associative F^3-s were investigated and a surprising result was presented. Next Türkşen's *IVFS*-s were introduced as derived from DNF and CNF in Boolean logic. The extremal F^3-s generated by *IVFS* was discussed. Considering that the flip-flops thus defined were described by an "opening" pair of graphs, the midpoint of the *IVFS* was suggested as the definitive point for a new symmetrical F^3. Similarly to *IVFS* minterm-maxterm interval based fuzzy sets (*MIVFS*) were also proposed for defining a narrower band of F^3-s, whose midpoint offers another alternative for a symmetrical F^3.

In the future we intend to investigate the behavior of complex fuzzy sequential circuits based on *IVFS*, *MIVFS* and midpoint F^3-s. We expect the behavior of such fuzzy networks would be interesting from the viewpoint of the possible convergence/divergence behavior. We also intend to find the optimal interval or point valued F^3-s for practical applications, such as adaptive behavior and learning.

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