Discrete Fuzzy Numbers Defined on a Subset of Natural Numbers*

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Abstract. We introduce an alternative method to approach the addition of discrete fuzzy numbers when the application of the Zadeh's extension principle does not obtain a convex membership function.

1 Introduction

Many authors have studied Fuzzy Numbers and the fuzzy arithmetic operations since many years (see, for instance, [5,6,7]). More recently, the discrete fuzzy numbers, understood as fuzzy numbers with discrete support, are present in papers of several authors [12,13]. In both cases, several definitions and allowed shapes of the membership functions have been considered. For the fuzzy numbers, in this paper we will consider convex membership functions with an interval as "support", LR-definitions and a "core" that is also a subinterval of the support.

The discrete fuzzy numbers whose support is a subset of natural numbers arise mainly when a fuzzy cardinality of a fuzzy set [4] or a fuzzy multiset [3,8,14] is considered. In both cases, the membership function of the involved discrete fuzzy numbers is decreasing. However, we can consider a wider kind of discrete fuzzy numbers in order to implement generalizations of the multiset concept [1,8,9,10,11]. So, a discrete fuzzy number $\tilde{5}$ means "about 5" occurrences of x_1 in M but with the constraint: "it is a natural number". Even, we can suppose to have an information like: "about 5 occurrences, but they are not 4".

In general, the arithmetic operations on fuzzy numbers can be approached either by the direct use of the membership function(by the Zadeh's extension principle) or by the equivalent use of the *r-cuts* representation. Nevertheless, in the discrete case, this process can yield a fuzzy subset that does not satisfy the conditions to be a discrete fuzzy number [2,13].

In a previous work [2] we have presented an approach to a closed addition of discrete fuzzy numbers after associating suitable non-discrete fuzzy numbers, which can be used like a carrier to obtain the desired addition. In this paper we prove that a suitable carrier can be a discrete fuzzy number whose support is an arithmetic sequence and even a subset of consecutive natural numbers.

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2 Preliminaries

Definition 1 [7]. A fuzzy subset u of $\mathbb{R}$ with membership mapping $u:\mathbb{R}\rightarrow[0,1]$ is called *fuzzy number* if its support is an interval $[a,b]$ and there exist real numbers s,t with $a\leq s\leq t\leq b$ and such that:

1. $u(x_i)=1$ with $s\leq x\leq t$
2. $u(x)\leq u(y)$ with $a\leq x\leq y\leq s$
3. $u(x)\geq u(y)$ with $t\leq x\leq y\leq b$
4. $u(x)$ is upper semi-continuous.

We will denote the set of fuzzy numbers by FN.

Definition 2 [12]. A fuzzy subset u of $\mathbb{R}$ with membership mapping $u:\mathbb{R}\rightarrow[0,1]$ is called *discrete fuzzy number* if its support is finite, i.e., there are $x_1,...,x_n\in\mathbb{R}$ with $x_1<x_2<...<x_n$ such that $supp(u)=\{x_1,...,x_n\}$, and there are natural numbers s,t with $1\leq s\leq t\leq n$ such that:

1. u(x_i)=1 for any natural number and i with $s\leq i\leq t$ (core)
2. $u(x_i)\leq u(x_j)$ for each natural number i,j with $1\leq i\leq j\leq s$
3. $u(x_i)\geq u(x_j)$ for each natural number i,j with $t\leq i\leq j\leq n$

From now on, we will denote the set of discrete fuzzy numbers by DFN and a discrete fuzzy number by a d.f.n.

2.1 Addition of Fuzzy Numbers [5,6,7]

Given two fuzzy numbers $u,v\in FN$, according to the Zadeh's extension principle, their addition is the fuzzy number $u\oplus v$, pointwise defined as follows:

$$(u\oplus v)(z)=\sup_{z=x+y}(\min(u(x),v(y))),\forall z\in\mathbb{R}$$

Equivalently, their addition is the fuzzy number whose r-cuts are defined:

$$[u\oplus v]^r=[u]^r+[v]^r,\forall r\in(0,1]$$

and, consequently, their membership function is:

$$(u\oplus v)(z)=\sup\left\{r\in[0,1]\text{ such that }z\in[u\oplus v]^r\right\}$$

2.2 Addition of Discrete Fuzzy Numbers

Let $u,v\in DFN$ be two discrete fuzzy numbers. If we consider them as fuzzy subsets of $\mathbb{R}$ we can apply the extension principle to obtain their extended sum. But we can see in [2,13] that it is possible a result which does not fulfill the conditions stated above in the Definition 2. In order to overcome this drawback several authors [2,13] have proposed modifications of the process:

- [13] By means of the r-cuts for one of d.f.n. $u, v \in DFN$, the author defines the r-cuts for a new d.f.n., which will be denoted by $u \bigoplus_W v$, such that $[u \bigoplus_W v]^r = \{x \in supp(u) + supp(v) : \min([u]^r + [v]^r) \leq x \leq \max([u]^r + [v]^r)\}$ where $min([u]^r + [v]^r) = min\{x + y : x \in [u]^r, y \in [v]^r\}$, $max([u]^r + [v]^r) = max\{x + y : x \in [u]^r, y \in [v]^r\}$ and $(u \bigoplus_W v)(x) = \sup\{r \in [0,1] \text{ such that } x \in [u \bigoplus_W v]^r\}$. Moreover, this author proves that for each $u, v \in DFN$, if $u \oplus v \in DFN$ (obtained through the Zadeh's extension principle) then $u \oplus v = u \bigoplus_W v$.
- [2] The authors propose a method that can be seen as a generalization of Wang's one and it is based on the concept of association of a fuzzy number to a discrete fuzzy number, understanding by association a mapping $A : DFN \to FN$ that fulfills determined conditions. Thus, from an association A, for each couple $u, v \in DFN$, we define the discrete fuzzy number $u \bigoplus_A v$, such that:

$$supp(u \bigoplus_A v) = \begin{Bmatrix} z \in \mathbb{R} \text{ such that } z = x + y, \\ x \in supp(u), y \in supp(v) \end{Bmatrix}$$

$$(u \bigoplus_A v)(z) = (A(u) \oplus A(v))(z) \ \forall\, z \in supp(u \bigoplus_A v)$$

where $A(u) \oplus A(v)$ represents the addition of the fuzzy numbers $A(u)$ and $A(v)$ defined in the subsection 2.1 .

3 Addition of d.f.n. Whose Support Is an Arithmetic Sequence of Natural Numbers

Let's call $DFN(\mathbb{N})$ to the set of d.f.n. whose support is a subset of the set of Natural Numbers.

Let $\mathcal{A}_r$ be the set { $f \in DFN(\mathbb{N})$, such that $supp(f)$ is the set of terms of an arithmetic sequence with r as common difference}.

Proposition 3. If $f, g \in \mathcal{A}_r$ .The following facts:

1. $f \oplus g \in DFN(\mathbb{N})$
2. $f \oplus g \in \mathcal{A}_r$

hold.

Proof: Let f, g two d.f.n. belonging to $\mathcal{A}_r$ such that:

$$supp(f) = \{a, a+r, a+2\cdot r, ..., a+p\cdot r\} \text{ and } core(f) = \{a+l\cdot r, ..., a+j\cdot r\}$$
$$supp(g) = \{b, b+r, b+2\cdot r, ..., b+q\cdot r\} \text{ and } core(g) = \{b+m\cdot r, ..., b+n\cdot r\}$$

It is easy to see that:

$$supp(f \oplus g) = \{a+b+i\cdot r, \text{ where } i = 0, ..., p+q\} \quad (1)$$

$$core(f \oplus g) = \{a+b+k\cdot r, \text{ where } k = l+m, ..., j+n\} \quad (2)$$

If we call $\tilde{i} = a+b+i\cdot r$ where $i \in \{0,1,..., p+q\}$ then:

$$(f \oplus g)(\tilde{i}) = \sup_{\substack{x+y=i \\ x\in\{0,...,p\} \\ y\in\{0,...,q\}}} (f(a+x\cdot r) \wedge g(b+y\cdot r)) = f(a+x^i\cdot r) \wedge g(b+y^i\cdot r) \quad (3)$$

with $x^i \in \{0,..., p\}$, $y^i \in \{0,...,q\}$, $x^i + y^i = i$
Without lack of generality, we can suppose that:

$$(f \oplus g)(\tilde{i}) = f(a+x^i\cdot r) \quad (4)$$

By means of a tedious but easy study of all possible cases for x^i and y^i we can prove that:

1. If $\tilde{i} \in \{a+b, ..., a+b+(l+m-1)\cdot r\}$ then $(f \oplus g)(\tilde{i}) \le (f \oplus g)(\widetilde{i+1})$
2. If $\tilde{i} \in \{a+b+(j+n+1)\cdot r, ..., a+b+(p+q)\cdot r\}$ then $(f \oplus g)(\tilde{i}) \ge (f \oplus g)(\widetilde{i+1})$
3. If $\tilde{i} \in \{a+b+(l+m)\cdot r, ..., a+b+(j+n)\cdot r\}$ then $(f \oplus g)(\tilde{i}) = 1$

Therefore, $f \oplus g \in DFN(\mathbb{N})$. On the other hand, if we consider the previous relations (1) and (2) we will obtain $f \oplus g \in \mathcal{A}_r$

Corollary 1. Let $\mathcal{C}$ be the set { $f \in DFN(\mathbb{N})$; such that for all $i, j \in supp(f)$, if $i \le k \le j$, with $k \in \mathbb{N}$ then $k \in supp(f)$ }.
If $f, g \in \mathcal{C}$, then $f \oplus g \in \mathcal{C}$.

Proof: It is a direct consequence of the previous Proposition, because $\mathcal{C} = \mathcal{A}_1$.

3.1 Remarks

1. The previous proposition does not hold if the supports of the considered d.f.n. f and g, as arithmetic progressions, do not have the same common difference. For instance, let $u = \{0.2/1, 0.4/3, 1/5, 0.7/7, 0.5/9\}$ and $v = \{0.2/6, 1/10, 0.8/14, 0.7/18\}$ be two d.f.n.. In both cases the points in their support form an arithmetic sequence with a common difference, namely 2 and 4 respectively. If we apply the extension

principle to calculate their sum, we obtain the fuzzy subset $u \oplus v = \{0.2/7, 0.2/9, 0.2/11, 0.4/13, 1/15, 0.7/17, 0.8/19, 0.7/21, 0.7/23, 0.7/25, 0.5/27\}$ that does not belong to DFN, because $(u \oplus v)(17) = 0.7 < (u \oplus v)(19) = 0,8$ and so for this reason the condition 3 of definition 2 doesn't hold .

2. The previous proposition cannot be generalized to the set:

   $\mathcal{G}_r = \{ f \in DFN(\mathbb{N})$ such that $supp(f)$ is a geometric sequence with common ratio r }

   For instance, if $u = \{0.5/1, 1/2, 0.7/4\}$ and $v = \{0.8/8, 1/16, 0.7/32\}$.

   Then $u, v \in , \mathcal{G}_r$ but the fuzzy subset

   $u \oplus v = \{0.5/9, 0.8/10, 0.7/12, 0.5/17, 1/18, 0.7/20, 0.5/33, 0.7/34, 0.7/36\}$ that does not belong to DFN, because $(u \oplus v)(10) = 0.8 > (u \oplus v)(12) = 0,7$

   and so for this reason the condition 2 of definition 2 doesn't hold.

4 Addition of Discrete Fuzzy Numbers Whose Support Is Any Subset of the Natural Numbers

The two examples included in the above remark and the following one show that the extension principle applied to a couple of discrete fuzzy numbers, whose support is any subset of the natural numbers, does not determine in general a d.f.n.

Let's consider $u = \{0.3/1, 1/2, 0.5/3\}$ and $v = \{0.4/4, 1/6, 0.8/8\}$. Then: $u \oplus v = \{0.3/5, 0.4/6, 0.4/7, 1/8, 0.5/9, 0.8/10, 0.5/11\}$ which does not satisfy the conditions stated in Definition 2.

In order to overcome this drawback, we will associate a d.f.n. belonging to the set $\mathcal{C}$ to each d.f.n..

This association will be a carrier to obtain an approach to the addition of two d.f.n. that will be a d.f.n.

Definition 4. Let $u \in DFN(\mathbb{N})$ be a discrete fuzzy number such that its support is $supp(f) = \{x_1, ..., x_s, ..., x_t, ..., x_n\}$ with $x_1 < \ldots < x_s < \ldots < x_t < \ldots < x_n$ and $u(x_p) = 1$ for all natural number p, $s \le p \le t$.

We will associate to u any d.f.n. $C(u) \in \mathcal{C}$ (defined in Corollary 1) fulfilling the following properties:

1. If $x_i \in supp(u)$ then $C(u)(x_i) = u(x_i)$ for each $i = 1, \ldots, n$
2. $u(x_i) \le C(u)(x) \le u(x_{i+1})$, $\forall x \in [x_i, x_{i+1}]$ with $1 \le i \le i+1 \le s$.
3. $C(u)(x_i) = 1$, $\forall x \in [x_i, x_{i+1}]$ with $s \le i \le i+1 \le t$.
4. $u(x_i) \ge C(u)(x_i) \ge u(x_{i+1})$, $\forall x \in [x_i, x_{i+1}]$ with $t \le i \le i+1 \le n$.

4.1 Examples

If $u \in DFN(\mathbb{N})$ we will call $\alpha-$association, to the d.f.n. $C_\alpha(u) \in \mathcal{C}$ defined in the following way

$$C_\alpha(u)(x) = \begin{cases} u(x_i) & \text{if} \quad x \in [x_i, x_{i+1}), \text{with } x_{i+1} < x_s \\ 1 & \text{if} \quad x \in [x_s, x_t] \\ u(x_{i+1}) & \text{if} \quad x \in (x_i, x_{i+1}], \text{with } x_i > x_t \end{cases}$$

For instance:

If $v = \{0.4/2, 1/5, 1/6, 0.8/9\}$. Its $\alpha-$associated will be the following discrete fuzzy number

$$C_\alpha(v)(x) = \begin{cases} 0.4 & \text{if} \quad x \in \{2,3,4\} \\ 1 & \text{if} \quad x \in \{5,6\} \\ 0.8 & \text{if} \quad x \in \{7,8,9\} \end{cases}$$

If $u = \{0.3/1, 1/3, 0.5/7\}$ its $\alpha-$associated will be the following discrete fuzzy number

$$C_\alpha(u)(x) = \begin{cases} 0.3 & \text{if} \quad x \in \{1,2\} \\ 1 & \text{if} \quad x \in \{3\} \\ 0.5 & \text{if} \quad x \in \{4,5,6,7\} \end{cases}$$

If $u \in DFN(\mathbb{N})$ we will call $\omega-$association, to the d.f.n. $C_\omega(u) \in \mathcal{C}$ defined in the following way:

$$C_\omega(u)(x) = \begin{cases} u(x) & \text{if} \quad x \in supp(u) \\ u(x_{i+1}) & \text{if} \quad x \in (x_i, x_{i+1}), \text{with } x_{i+1} \leq x_s \\ 1 & \text{if} \quad x \in (x_s, x_t) \\ u(x_i) & \text{if} \quad x \in (x_i, x_{i+1}), \text{with } x_i \geq x_t \end{cases}$$

For instance:

If $v=\{0.4/2, 0.6/4, 1/5, 1/6, 0.9/7, 0.8/9\}$ its $\omega-$associated will be the d.f.n. defined in the following way: $C_{\omega}(v)(x)=\begin{cases} 0.4 & \text{if} \quad x\in\{2\} \\ 0.6 & \text{if} \quad x\in\{3,4\} \\ 1 & \text{if} \quad x\in\{5,6\} \\ 0.9 & \text{if} \quad x\in\{7,8\} \\ 0.8 & \text{if} \quad x\in\{9\} \end{cases}$

If $u\in DFN(\mathbb{N})$ we will call linear-association, to the d.f.n. $C_l(u)\in\mathcal{C}$ defined in the following way: $C_l(x)=\begin{cases} u(x) & \text{if} \quad x\in supp(u) \\ \frac{u(x_{i+1})-u(x_i)}{x_{i+1}-x_i}(x-x_i)+u(x_i) & \text{if} \quad x\in(x_i,x_{i+1}) \end{cases}$

It is straightforward to prove that C_{α}, C_{ω} and C_l satisfy the properties of an association as have been established in Definition 4.

4.2 Remark

If $C: DFN(\mathbb{N})\to\mathcal{C}$ is an association, then the relation $C_{\alpha}(u)\le C(u)\le C_{\omega}(u)$ holds for all $u\in DFN(\mathbb{N})$.

Proof :
Straightforward.

Definition 5. Let $u,v\in DFN(\mathbb{N})$ be a couple of d.f.n. and let $C: DFN(\mathbb{N})\to\mathcal{C}$ be an association. Let's consider the fuzzy subset, that will be denoted by $u\bigoplus_C v$, defined as follows:

$$supp(u\bigoplus_C v)=\left\{\begin{matrix} z\in\mathbb{N} \text{ such that } z=x+y, \\ x\in supp(u), y\in supp(v) \end{matrix}\right\}$$

$$(u\bigoplus_C v)(z)=(C(u)\oplus C(v))(z), \forall\, z\in supp(u\bigoplus_C v)$$

Then $u\bigoplus_C v\in DFN(\mathbb{N})$ and we will call it the C-addition of the couple u,v of discrete fuzzy numbers.

4.3 Remark

We will denote the C-addition by $u \bigoplus_{\alpha} v$ when the association C is the α-association C_α defined in 4.1 and $u \bigoplus_{\omega} v$ when the association is the ω-association C_ω also defined in 4.1.

4.4 Examples

Let $u = \{0.3/1, 1/3, 0.5/7\}$ and $v = \{0.4/2, 1/5, 1/6, 0.8/9\}$ be two d.f.n.. We have:

- According to the extension principle: $u \oplus v = \{0.3/3, 0.4/5, 0.3/6, 0.3/7, 1/8, 1/9, 0.3/10, 0.8/12, 0.5/13, 0.5/16\}$ and $u \oplus v \notin DFN$
- If we consider the α-associations which have been calculated in 4.1, then: $u \bigoplus_{\alpha} v = \{0.3/3, 0.4/5, 0.4/6, 0.4/7, 1/8, 1/9, 0.8/10, 0.8/12, 0.5/13, 0.5/16\}$
- If we consider the ω-associations:
  $C_\omega(u) = \{0.3/1, 1/2, 1/3, 1/4, 1/5, 1/6, 0.5/7\}$ and
  $C_\omega(v) = \{0.4/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 0.8/9\}$ then:
  $u \bigoplus_{\omega} v = \{0.3/3, 1/5, 1/6, 1/7, 1/8, 1/9, 1/10, 1/12, 1/13, 0.5/16\}$

Proposition 6. If $u, v \in DFN(\mathbb{N})$ are two d.f.n.. Then, $u \bigoplus_{W} v = u \bigoplus_{\alpha} v$ where $u \bigoplus_{W} v$ has been defined in 2.2 and $u \bigoplus_{\alpha} v$ has been defined in 4.3.

Proof: We will see that $\left[u \bigoplus_{W} v \right]^r = \left[u \bigoplus_{\alpha} v \right]^r, \forall r \in [0,1]$

Since the definition of $C_\alpha(u)$ and $C_\alpha(v)$, for each $r \in [0,1]$:

a) $[u]^r \subseteq [u_\alpha]^r$ and $[v]^r \subseteq [v_\alpha]^r$

b) $\min [u]^r = \min [u_\alpha]^r$ and $\min [v]^r = \min [v_\alpha]^r$

c) $\max [u]^r = \max [u_\alpha]^r$ and $\max [v]^r = \max [v_\alpha]^r$

d) Besides, from the definition of $u \bigoplus_{\alpha} v$, it can be deduced that:

$$\left[u \bigoplus_{\alpha} v \right]^r \subseteq \left[u_\alpha \oplus v_\alpha \right]^r$$

e) Since 2.2: $x \in \left[u \underset{W}{\bigoplus} v \right]^r \Leftrightarrow \begin{cases} x \in \operatorname{supp}\left(u \underset{W}{\bigoplus} v \right) = \operatorname{supp}(u) + \operatorname{supp}(v) \\ \min([u]^r + [v]^r) \le x \le \max([u]^r + [v]^r) \end{cases}$

Taking into account a), b), c), d), e), the proof can be completed like in [2], where an analogous proposition for non-discrete associations is proved.

5 Conclusion

This paper has shown an alternative method to approach the addition of discrete fuzzy numbers when the application of the Zadeh's extension principle does not obtain a d.f.n. Also, we have seen that in the particular case of addition of d.f.n., whose support is a set of terms of an arithmetic sequence with r as common difference, the Zadeh's extension principle yields a d.f.n.. But, this is not true when the d.f.n. have as support either a set of terms of a geometric sequence with common ratio r or a set of terms of arithmetic sequences with different differences.

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