Lattice Ordered Monoids and Left Continuous Uninorms and t-norms

Marta Takacs

Budapest Tech, John von Neumann Faculty of Informatics 1036 Budapest,
Bécsi út 96/b, Hungary
takacs.marta@nik.bmf.hu

Abstract. The proposition about generalization of the existence of the residuum for left continuous uninorm *U* on a commutative, residuated *l*-monoid, with a neutral element is proved. The question raised previously was whether there are general operation groups which satisfy the residuum-based approximate reasoning, but at the same time are easily comprehensible and acceptable to application-oriented experts. The basic backgrounds of this research are the distance-based operators.

Keywords: uninorms, residuum, lattices.

1 Introduction

In the theory of approximate reasoning introduced by Zadeh in 1979 [15], the knowledge of system behavior and system control can be stated in the form of if-then rules. In Mamadani-based sources [7] it was suggested to represent an "if *x* is *A* then *y* is *B*" rule simply as a connection (for example as a t-norm, $T(A,B)$ or as *min*) between the so called rule premise: *x* is *A* and rule consequence: *y* is *B*.

Engineering applications based on Fuzzy Logic Systems (FLS) are mostly based on minimum and maximum operators, but from a mathematical point of view it is interesting to study the behavior of other operators in inference mechanism, which will give new possibilities for further applications. This gives as the link with the theory of pseudo-analysis, which serves also as a universal base for the system theory [6], [8]. Uninorm operator groups [17], researched in the past decade, have yielded exceptionally good results.

Applying the uninorm operators with the changeable parameter *e* in fuzzy approximate reasoning systems is born in mind that the underlying notions of soft-computing systems are flexibility and the human mind. The choice of the fuzzy environment must support the efficiency of the system and it must comply to the real world. This is more important than trying to fit the real world into the inflexible models. [1], [15], [16].

In recent years there have been numerous forms of research which analyze the application and theoretical place of the uninorms. As there is a near-complete theoretical basis for the t-norm based residuums, in a similarly detailed theoretical basis being constructed for uninorms.[4], [5], [14]

O. Castillo et al. (Eds.): Theor. Adv. and Appl. of Fuzzy Logic, ASC 42, pp. 565–572, 2007.
springerlink.com

In the paper the basic definitions for uninorm, distance based operators and its residuum are given. Having results from [3], we introduce the residuum-based inference mechanism using uninorms, especially distance-based uninorms.

2 Uninorms and Distance Based Operators

Both the neutral element 1 of a t-norm and the neutral element 0 of a t-conorm are boundary points of the unit interval. However, there are many important operations whose neutral element is an interior point of the underlying set. The fact that the first three axioms (commutativity, associativity, montonocity) coincide for t-norms and for t-conorms, i.e., the only axiomatic difference lies in the location of the neutral element, has led to the introduction of a new class of binary operations closely related to t-norms and t-conorms.

A *uninorm* is a binary operation U on the unit interval, i.e., a function $U : [0,1]^2 \to [0,1]$ which satisfies the following properties for all $x, y, z \in [0,1]$

(U1) $U(x,y) = U(y,x)$, i.e. the uninorm is commutative,

(U2) $U(U(x,y),z) = U(x,U(y,z))$, i.e. the uninorm is associative,

(U3) $x \le y \Rightarrow U(x,z) \le U(y,z)$, i.e. the uninorm monotone,

(U4) $U(e,x) = x$, i.e., a neutral element exists, which is $e \in [0,1]$.

On the one hand the practical motivations for the introduction of uninorms were the applications from multicriteria decision making, where the aggregation is one of the key issues. Some alternatives are evaluated from several points of view. Each evaluation is a number from the unit interval. Let the level of satisfaction be $e \in]0,1[$. If all criteria are satisfied to at least e-extent then we would like to assign a high aggregated value to this alternative. The opposite of that is if all evaluations are below e then we would like to assign a low aggregated value to this alternative. But if there are evaluations below and above e, an aggregated value ought to be assigned somewhere in between. Such situations can be modelled by uninorms, leading to the particular classes introduced by [13].

2.1 Distance Based Operators

The distance-based operators can be expressed by means of the min and max operators as follows:

the *maximum distance minimum operator with respect* to $e \in [0,1]$ is defined as

$$\max_e^{\min} = \begin{cases} \max(x,y), & \text{if } y > 2e - x \\ \min(x,y), & \text{if } y < 2e - x \\ \min(x,y), & \text{if } y = 2e\text{-}x \end{cases} \tag{1}$$

the *minimum distance minimum operator* with respect to $e \in [0,1]$ is defined as

$$min_e^{min} = \begin{cases} min(x,y), & \text{if } y > 2e - x \\ max(x,y), & \text{if } y < 2e - x \\ min(x,y), & \text{if } y = 2e\text{-}x \end{cases} \quad (2)$$

the *maximum distance maximum operator with respect to* $e \in [0,1]$ is defined as

$$max_e^{max} = \begin{cases} max(x,y), & \text{if } y > 2e - x \\ min(x,y), & \text{if } y < 2e - x \\ max(x,y), & \text{if } y = 2e\text{-}x \end{cases} \quad (3)$$

the *minimum distance maximum operator with respect to* $e \in [0,1]$ is defined as

$$min_e^{max} = \begin{cases} min(x,y), & \text{if } y > 2e - x \\ max(x,y), & \text{if } y < 2e - x \\ max(x,y), & \text{if } y = 2e\text{-}x \end{cases} \quad (4)$$

The modified distance based operators described above are changed in the boundary condition for neutral element *e*:

- th*e* maximum distance minimum operator and the the maximum distance maximum operator with respect to $e \in]0,1]$,
- the minimum distance minimum operator and the minimum distance maximum operator with respect to $e \in [0,1[$.

More about distance based operators we can find in [9], [10].

The distance-based operators have the following properties: max_e^{min} *and* max_e^{max} are uninorms, the dual operator of the uninorm max_e^{min} *is* max_{1-e}^{max} ,and the dual operator of the uninorm max_e^{max} *is* max_{1-e}^{min}.

Based on results from [3] we conclude:

Operator $max_{0.5}^{min}$ is a conjunctive left-continuous idempotent uninorm with neutral element $e \in]0,1]$ with the super-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

Operator $min_{0.5}^{max}$ is a disjunctive right-continuous idempotent uninorm with neutral element $e \in]0,1]$ with the sub-involutive decreasing unary operator $g(x) = 2e - x = 2 \cdot 0.5 - x \Rightarrow g(x) = 1 - x$.

3 Lattice Ordered Monoids and Left Continuous Uninorms and t-norms

Let *L* be a non-empty set. Lattice is a partially (totally) ordered set which for any two elements $x, y \in L$ also contains their *join* $x \vee y$ (i.e., the least upper bound of the set

$\{x, y\}$), and their *meet* $x \wedge y$ (i.e., the greatest lower bound of the set $\{x, y\}$), denoted by $(L, \preceq)$. Secondly, $(L, *)$ is a semi-group with the neutral element. Following [2] let the following be introduced:

Definition 1.
Let $(L, \preceq)$ be a lattice and $(L, *)$ a semi-group with the neutral element.[6]
The triple $(L, *, \preceq)$ is called a *lattice-ordered monoid* (or an *l-monoid*) if for all $x, y, z \in L$ we have

($LM1$) $x * (y \vee z) = (x * y) \vee (x * z)$ and
($LM2$) $(x \vee y) * z = (x * z) \vee (y * z)$.

An $(L, *, \preceq)$ *l*- monoid is said to be *commutative* if the semi-group $(L, *)$ is commutative.

A commutative $(L, *, \preceq)$ *l*- monoid is said to be *commutative, residuated l-monoid* if there exists a further binary operation $\rightarrow_*$ on L, i.e., a function $\rightarrow_*: L^2 \rightarrow L$ (*the * residuum*), such that for all $x, y, z \in L$ we have

$$x * y \preceq z \text{ if and only if } x \preceq (y \rightarrow_* z) \tag{5}$$

An *l*-monoid $(L, *, \preceq)$ is called an *integral* if there is a greatest element in the lattice $(L, \preceq)$ (often called the universal upper bound) which coincides with the neutral element of the semi-group $(L, *)$.

Obviously, each *l*-monoid $(L, *, \preceq)$ is a partially ordered semi-group, and in the case of commutativity the axioms ($LM1$) and ($LM2$) are equivalent.

In the following investigations the focus will be on the lattice $([0,1], \leq)$, we will usually work with a complete lattice, i.e., for each subset A of L its join $\bigvee A$ and its $\bigwedge A$ exist and are contained in L. In this case, L always has a greatest element, also called the *universal upper bound.*

Example 1. If we define $*: [0,1]^2 \rightarrow [0,1]$ by

$$x * y = \begin{cases} min(x, y) & if \quad x + y \leq 1 \\ max(x, y) & otherwise \end{cases} \tag{6}$$

then $([0,1], *, \leq)$ is a commutative, residuated *l*-monoid, and the *-residuum is given by

$$x \rightarrow_* y = \begin{cases} max(1 - x, y) & if \quad x \leq y \\ min(1 - x, y) & otherwise \end{cases} \tag{7}$$

It is not an integral, since the neutral element is 0.5.

The operation * results in an *uninorm*, and special types of distance based operators (see [12] and [6]).

The following result is on important characterization of left-continuous uninorms.

Theorem 1.

For each function $U:[0,1]^2 \to [0,1]$ the following are equivalent:

(*i*) $([0,1],U,\leq)$ is a commutative, residuated *l*-monoid, with a neutral element

(*ii*) *U* is a left continuous uninorm.

In this case the *U*-residuum $\to_U$ is given by

$$x \to_U y = sup\{z \in [0,1] | U(x,z) \leq y\} \tag{8}$$

Proof.

It is easy to see, that $([0,1],U,\leq)$ is a commutative, residuated *l*-monoid with a neutral element if and only if *U* is a uninorm.

Therefore, in order to prove that (*i*)⇒(*ii*), assume that $([0,1],U,\leq)$ is residuated, fix and arbitrary sequence $(x_n)_{n\in N}$ in $[0,1]$ and put $x_0 = \sup_{n\in N} x_n$.

Let $y_0 \in [0,1]$, and $z_0 = \sup_{n\in N} U(x_n, y_0)$.

Obviously $z_0 \leq U(x_0, y_0)$, and (8) implies $(y_0 \to_U z_0) \geq x_n$ for all $n \in N$, subsequently, $(y_0 \to_U z_0) \geq x_0$.

Applying again (8) in the opposite direction, we obtain $U(x_0, y_0) \leq z_0$. Because of the monotonicity of uninorm U, (U3), and based on *Proposition 1.22.* from [6], we have $U(x_0, y_0) = z_0$, i.e., $\sup_{n\in N} U(x_n, y_0) = U\left(\sup_{n\in N}(x_n), y_0\right)$

Conversely, if the uninorm *U* is left-continuous, define the operation $\to_U$ by (8). Then it is clear that for all $x,y,z \in [0,1]$, $x \leq (y \to_U z)$ whenever $U(x,y) \leq z$. The left-continuity of U then implies $U(y \to_U z, y) \leq z$, which together with the monotonicity (U3), ensures that $\to_U$ is indeed the *U*-residuum. ◊

The work of De Baets, B. and Fodor, J. [3] presents general theoretical results related to residual implicators of uninorms, based on residual implicators of t-norms and t-conorms.

Residual operator R_U, considering the uninorm U, can be represented in the following form:

$$R_U(x,y) = sup\{z | z \in [0,1] \wedge U(x,z) \leq y\} \tag{9}$$

Uninorms with the neutral elements $e=0$ and $e=1$ are t-norms and t-conorms, respectively, and related residual operators are widely discussed, we also find suitable definitions for uninorms with neutral elements $e \in]0,1[$.

If we consider a uninorm *U* with the neutral element $e \in]0,1[$, then the binary operator R_U is an implicator if and only if $(\forall z \in]e,1[)(U(0,z)=0)$. Furthermore R_U is an implicator if U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying $(\forall z \in [0,1])(g(z) = 0 \Leftrightarrow z = 1)$.

The residual implicator R_U of uninorm U can be denoted by Imp_U.

Consider a uninorm U, then R_U is an implicator in the following cases:

- U is a conjunctive uninorm,
- U is a disjunctive representable uninorm,

U is a disjunctive right-continuous idempotent uninorm with unary operator g satisfying $(\forall z \in [0,1])(g(z) = 0 \Leftrightarrow z = 1)$.

Theorem 1. implies in a special case Proposition 2.47. from [6]:

Corollary 1

For each function $T : [0,1]^2 \rightarrow [0,1]$ the following are equivalent:

(i) $([0,1], T, \leq)$ is a commutative, residuated integral l-monoid,

(ii) T is a left continuous t-norm.

In this case the T-residuum $\rightarrow_T$ is given by (*ResT*)

$$x \rightarrow_T y = sup\{z \in [0,1] | T(x,z) \leq y\}.$$

Because of its interpretation in [0,1]-valued logics, the T-residuum $\rightarrow_T$ is also called *residual implication* (or briefly *R-implication*). [12]

3.1 Residual Implicators of Distance Based Operators

According to Theorem 8. in [3] we introduce implicator of distance based operator $max_{0.5}^{min}$.

Consider the conjunctive left-continuous idempotent uninorm $max_{0.5}^{min}$ with the unary operator $g(x) = 1 - x$, then its residual implicator $Imp_{max_{0.5}^{min}}$ is given by

$$Imp_{max_{0.5}^{min}} = \begin{cases} max(1-x, y) & if \quad x \leq y \\ min(1-x, y) & elsewhere \end{cases} \tag{10}$$

3.2 Residuum-Based Approximate Reasoning with Distance Based Operator

In many sources it was suggested to represent an "if x is A then y is B" rule simply as a connection (for example as a t-norm, *T(A,B)* or any conjunctive operator) between the so called rule premise: x is A and rule consequence: y is B. (Let x be from universe X, y from universe Y, and let x and y be linguistic variables. Fuzzy set A on $X \subset R$ finite universe is characterized by its membership function μA: $x \rightarrow [0,1]$, and fuzzy set B on Y universe is characterized by its membership function μB: $y \rightarrow [0,1]$). If the rule output B' in a fuzzy rule base for one rule if A then B and the system input A' is modeled and calculated by the expression in the form

$$B'(y)=\sup_{x\in X}\left(ConjunctiveOperator\left(A'(x),Imp\left(A(x),B(y)\right)\right)\right) \tag{11}$$

$$B'(y)=\sup_{x\in X}\left(ConjunctiveOperator\left(A'(x),Imp\left(A(x),B(y)\right)\right)\right)$$

Let we consider the uninom residuum-based approximate reasoning and inference mechanism. Hence, and because of the results from above we can consider the general rule consequence for *i*-th rule from a rule system as

$$B_i{}'(y)=\sup_{x\in X}\left(max_{0.5}^{min}\left(A'(x),Imp_{max_{0.5}^{min}}\left(A_i(x),B_i(y)\right)\right)\right) \tag{12}$$

$$B_i{}'(y)=\sup_{x\in X}\left(max_{0.5}^{min}\left(A'(x),Imp_{max_{0.5}^{min}}\left(A_i(x),B_i(y)\right)\right)\right),$$

or, using (10)

$$B_i{}'(y)=\sup_{x\in X}\begin{cases} max_{0.5}^{min}\left(A'(x),max\left(1-A_i(x),B_i(y)\right)\right) & if \quad A_i(x)\le B_i(y) \\ max_{0.5}^{min}\left(A'(x),min\left(1-A_i(x),B_i(y)\right)\right) & elsewhere \end{cases} \tag{13}$$

$$B_i{}'(y)=\sup_{x\in X}\begin{cases} max_{0.5}^{min}\left(A'(x),max\left(1-A_i(x),B_i(y)\right)\right) & if \quad A_i(x)\le B_i(y) \\ max_{0.5}^{min}\left(A'(x),min\left(1-A_i(x),B_i(y)\right)\right) & elsewhere \end{cases}$$

Details see in [12].

4 Conclusion

In fact the uninorms offer new possibilities in fuzzy approximate reasoning, because the low level of covering over of rule premise and rule input has measurable influence on rule output as well. The modified, uninorm-based Mamdani's approach, with similarity measures between rule premises and rule input, does not rely on the compositional rule inference any more, but still satisfies the basic conditions supposed for the approximate reasoning for a fuzzy rule base system.[11]

Residuum-based approximate reasoning focused on distance based operators violates needed practical axioms for the rule outputs in a fuzzy logic control system. In the cases when we have normal input *A'* the output is contained in all consequences if we have not "faired" rule. If $A\neq A'$, the rule output belongs not to the convex hull of all rule outputs B_i, ($i=1,n$).

References

1. Bellmann, R.E., Zadeh., L.A., (1977), *Local and fuzzy logic,* Modern Uses of Multiple-Valued Logic, edited by Dunn, J.M., Epstein, G., Reidel Publ., Dordrecht, The Netherlands, pp. 103-165.
2. Birkhoff, G., (1973), *Lattice theory*. American Mathematical Siciety, Providence
3. De Baets, B., Fodor, J., (1999), *Residual operators of uninorms*,, Soft Computing 3., (1999), pp. 89-100.

4. Fodor, J., Rubens, M., (1994), *Fuzzy Preference Modeling and Multi-criteria Decision Support.* Kluwer Academic Pub.,1994.
5. Fodor, J., (1996), *Fuzzy Implications,* Proc. Of International Panel Conference on Soft and Intelligent Computing, Budapest, ISBN 963 420 510 0, pp. 91-98. 21] J., Fodor, B., De Baets, T., Calvo, *Structure of uninorms with given continuous underlying t-norms and t-conorms*, Proc. of the 24th Linz Seminar on Fuzzy Sets, 2003.
6. Klement, E.P., Mesiar, R, Pap, E., *'Triangular Norms*', Kluwer Academic Publishers, 2000, ISBN 0-7923-6416-3
7. E., H., Mamdani, S., Assilian , *An experiment in linguistic syntesis with a fuzzy logic controller,* Intern.,. J. Man-Machine Stud. 7. 1-13., 1975.
8. Pap, E., *Triangular norms in modelling uncertainly, non-linearity and decision*, in Proceedings of the 2th International Symposium of Hungarian researchers Computational Intelligence, ISBN 963 7154 06 X, pp. 7-18.
9. Rudas, I., *Absorbing-Norms,* in Proceedings of the 3th International Symposium of Hungarian Researchers Computational Intelligence, Buadpest, Nov. 2002., pp. 25-43.
10. Takacs, M., Rudas I. J., (1999), *Generalized Mamdani Type Fuzzy Controllers,* in Proceedings of Eurofuse-SIC 99, Budapest, May, 1999. pp. 162-165.
11. Takacs, M., (2003) *Approximate reasoning with Distance-based Operators and degrees of coincidence*, in Principles of Fuzzy Preference Modelling and Decision Making, edited by Bernard de Baets, János Fodor, Academia Press Gent,., ISBN 90-382-0567-8
12. Takacs, M., Approximate Reasoning in Fuzzy Systems Based On Pseudo-Analysis, Phd Thesis, Univ. of Novi Sad, 2004.
13. Yager, R. R., Rybalov, A., (1996), *Uninorm aggregation operators*, Fuzzy sets and systems 80., pp. 111-120.
14. R. R. Yager, (2001), Uninorms in fuzzy system modeling, Fuzzy Ses and Systems, 122, pp:167-175.
15. Zadeh, L. A., *A Theory of approximate reasoning*, In Hayes, J., and editors, Mashine Intelligence, Vol. 9, Halstead Press, New York, 1979., pp. 149-194.
16. Zadeh, L. A., (1999), *From Computing with Numbers to Computing with Words – From Manipulation of Measurements to manipulation of Perceptions,* In Proc. Of EUROFUSE – SIC Conf. 1999., Budapest, pp. 1-3.
17. Zimmermann, H.J., (1991), *Fuzzy Sets, Decision Making and Expert Systems*. Kluwer, Boston, 1991.