Parametric Fuzzy Linear Systems

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Abstract. Systems of linear equations with elements being affine linear functions of fuzzy parameters are relevant to many practical problems. A method for solving such systems is proposed. It consists of two steps. First a finite number of parametric interval linear systems is solved using the direct method. Then membership functions of fuzzy solution elements are interpolated. Parameters are modeled by arbitrary fuzzy numbers with convex membership function and compact support. Conditions for existence of the fuzzy solution are given. The performance of the proposed method is presented using an illustrative example of truss structure.

1 Introduction

Many economical, financial, physical, engineering and electrical problems boil down to solution of linear systems of equations. When the problem characteristics are imprecise, then the linear system of equations is no longer crisp. Imprecise or unknown values of the system parameters can be modeled using probability distributions, intervals or fuzzy values.

Several methods for solving linear interval system [13], [17] have been developed since '60s [11] when interval arithmetic became more and more popular. Most of these methods assume implicitly that system coefficients vary independently within the lower and upper bound of the corresponding intervals (such system will be called classical). Usually this is not the case. Moreover, the assumption of coefficients independence makes many problem to be unsolvable [10]. This beget the need to develop methods for solving parametric linear systems [5], [6-9], [14], [18], [19] – parametrisation is used to eliminate drawbacks of interval arithmetic [13], especially the problem of dependency [3], [10]. Using the direct method proposed in [20] large systems of parametric linear equations with a lot of interval parameters can be solved efficiently. Since the solution set of interval system is usually non-convex and has a very complicated shape, instead of the solution set itself, an outer and inner interval solutions are computed. The best outer solution coincides with an interval hull solution.

Some investigations have also been reported in literature on the solution of classical fuzzy linear systems [1], [3], [12]. However, based on the extension principle, the solution of fuzzy linear equations can be viewed as a result of series of solutions of a nested family of interval systems. Hence the problem of dependency is also relevant to fuzzy linear systems. An optimization-based scheme for numerical solution of parametric fuzzy linear systems has been proposed in [16]. However, the approach appears to be difficult and inefficient for solving large systems of equations. The

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search-based algorithm for solving parametric fuzzy linear systems was presented in [15]. In both paper considerations were restricted to triangular fuzzy numbers.

In this paper an efficient method for solving parametric linear systems with elements linearly dependent on a vector of fuzzy parameters is proposed. In this context to solve means to compute an outer interval solution. Two major steps constitute the framework of the proposed methodology: 1) solve a finite number of linear systems with interval parameters corresponding to the α-cuts non-uniformly distributed on [0, 1] interval; 2) interpolate membership functions of fuzzy solution elements with cubic splines. Non-uniform distribution of the α-cuts reflects the shape of the membership function of fuzzy parameters. Interpolation is used to compute approximations of the intermediate values of the membership functions of the fuzzy solution elements.

Problem parameters are modeled by arbitrary fuzzy numbers with continous convex membership function and compact support. Conditions for existence of the fuzzy solution are given. The performance of the proposed method is presented using an illustrative example of parametric fuzzy linear system that arise in analysis of truss structures.

2 Preliminaries

A real compact interval $\boldsymbol{a} = [\underline{a}, \overline{a}] = \left\{ x \in R \mid \underline{a} \le x \le \overline{a} \right\}$. A family of real compact intervals will be denoted by IR. For an interval $\boldsymbol{a}$ define a midpoint $\check{a} = \mathrm{mid}(\boldsymbol{a}) = (\overline{a} + \underline{a})/2$ and a radius $\check{a} = \mathrm{rad}(\boldsymbol{a}) = (\overline{a} - \underline{a})/2$.

A square interval matrix $\boldsymbol{A}$ is an array of intervals:

$$\boldsymbol{A} = \{\boldsymbol{a}_{ij}\}_{i,j=1}^{n}, \ \boldsymbol{a}_{ij} \in IR, \ i, j = 1, \ldots, n \ .$$

One column ($n\times1$) interval matrix is just an interval vector. IR^n, $IR^{n\times n}$ will denote a family of interval vectors, respectively, square interval matrices.

An interval matrix $\boldsymbol{A} \in IR^{n\times n}$ is regular (or non-singular) if all real matrices $A \in \boldsymbol{A}$ are non-singular.

An interval matrix $\boldsymbol{A} \in IR^{n\times n}$ is called an H-matrix [13] (should not be confused with a Hadamard matrix) iff there exist $u \in R^n$, $u > 0$ such that $\langle \boldsymbol{A} \rangle u > 0$. Here $\langle \boldsymbol{A} \rangle$ is a real $n\times n$ matrix with entries:

$$\langle \boldsymbol{a} \rangle_{ii} = \begin{cases} \min\{|\underline{a}_{ii}|, |\overline{a}_{ii}|\}, & 0 \notin \boldsymbol{a}_{ii} \\ 0 & 0 \in \boldsymbol{a}_{ii} \end{cases}, \ \langle \boldsymbol{a} \rangle_{ij} = -\max\{|\underline{a}_{ii}|, |\overline{a}_{ii}|\}, \ i \ne j \ .$$

and is called an Ostrovsky matrix [13].

Theorem 1 (Neumaier [13]). If $\boldsymbol{A} \in IR^{n\times n}$ is an H-matrix and $\boldsymbol{B} \subseteq \boldsymbol{A}$, $\boldsymbol{B} \in IR^{n\times n}$, then $\boldsymbol{B}$ is an H-matrix.

For an arbitrary set $X \subset R^n$ an interval hull X is defined as

$$X = \bigcap \left\{ Y \in IR^n \mid X \subseteq Y \right\} \ .$$

A convex fuzzy number $\boldsymbol{a}$ with compact support is defined by its membership function $\mu_{\boldsymbol{a}}:R \rightarrow [0, 1]$, such that:

- there exist a unique $m \in R$ with $\mu_{\boldsymbol{a}}(m)=1$,
- a support $\text{supp}(\boldsymbol{a}) = Cl\left\{x \in R \mid \mu_{\boldsymbol{a}}(x) > 0\right\}$ is bounded in R,
- $\mu_{\boldsymbol{a}}$ is fuzzy convex on $\text{supp}(\boldsymbol{a})$,
- $\mu_{\boldsymbol{a}}$ is upper semi-continous on R.

The function $\mu_{\boldsymbol{a}}$ is called fuzzy convex on $\text{supp}(\boldsymbol{a})$ if

$$\mu_{\boldsymbol{a}}(\lambda x_1+(1-\lambda)x_2) \geq \min\{\mu_{\boldsymbol{a}}(x_1),\mu_{\boldsymbol{a}}(x_2)\} \ ,$$

for $0 \leq \lambda \leq 1$ and $x_1 \neq x_2$. A family of convex fuzzy numbers with continous membership function and compact support will be denoted by F.

For $\boldsymbol{a} \in F$, an interval corresponding to α-value of a membership function

$$\boldsymbol{a}(\alpha) = \{\ x \mid \mu_{\boldsymbol{a}}(x) \geq \alpha\ \} \in IR,\ \alpha \in (0, 1] \ ,$$

is called a (weak) α-cut; a 0-cut is defined separately as $\boldsymbol{a}(0) = \text{supp}(\boldsymbol{a})$.

For each $\alpha \in [0, 1]$, $\boldsymbol{a}(\alpha)$ is compact and connected. A fuzzy number $\boldsymbol{a} \in F$ can be viewed as a nested family of α-cuts in a sense that

$$\boldsymbol{a}(\alpha) \subseteq \boldsymbol{a}(\beta), \quad \text{for } \alpha > \beta \ .$$

A square fuzzy matrix $\boldsymbol{A}$ is an array of fuzzy numbers:

$$\boldsymbol{A} = \{\boldsymbol{a}_{ij}\}_{i,j=1}^{n},\ \boldsymbol{a}_{ij} \in F,\ i, j = 1, \ldots, n \ .$$

Fuzzy vector is a one column ($n\times 1$) fuzzy matrix. F^n, $F^{n\times n}$ will denote a family of fuzzy vectors, respectively square fuzzy matrices.

Let $\boldsymbol{A} = \{\boldsymbol{a}_{ij}\}_{i,j=1}^{n}$ be a fuzzy matrix, then an α-cut

$$\boldsymbol{A}(\alpha) = \{\boldsymbol{a}_{ij}(\alpha)\}_{i,j=1}^{n}$$

is computed componentwise. $\boldsymbol{A}(\alpha)$, $0 \leq \alpha \leq 1$, is an interval matrix.

Corollary 1. Let $\boldsymbol{A} \in F^{n\times n}$. If $\boldsymbol{A}(0)$ is regular (an H-matrix), then $\boldsymbol{A}(\alpha)$ is regular (an H-matrix) for each $\alpha \in [0, 1]$.

Proof: Obvious.

3 Fuzzy Linear Systems

The following matrix equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \ , \tag{1}$$

with $\boldsymbol{a}_{ij}$, $\boldsymbol{b}_i \in F$ ($i, j = 1, \ldots, n$), is called a fuzzy linear system. For a fuzzy linear systems different solution sets have been considered [3], [12]. Finally, three types of the solutions can be distinguished: classical solution $\boldsymbol{S}_C$, marginal solutions $\boldsymbol{S}_E$ and $\boldsymbol{S}_I$, and joint (or vector) solution $\boldsymbol{S}_J$. Classical solution $\boldsymbol{S}_C$ employs α-cuts and interval arithmetic in order to work out the solution. Taking α-cuts of Eq. (1) the system

$$\sum_{j=1}^{n}[\underline{a}_{ij}(\alpha),\overline{a}_{ij}(\alpha)][\underline{x}_j(\alpha),\overline{x}_j(\alpha)]=[\underline{b}_i(\alpha),\overline{b}_i(\alpha)] \ , \tag{2}$$

for $0 \le \alpha \le 1$, $1 \le i \le n$, is obtained. Interval multiplication and addition is used to evaluate the left-hand side of Eq. (2). Hence, for each $\alpha \in [0, 1]$, the original $n\times n$ system is transformed into a $2n\times 2n$ system. Eq. (2) is solved for $\underline{x}_j(\alpha)$, $\overline{x}_j(\alpha)$ hoping they produce the α-cuts of fuzzy numbers $\boldsymbol{x}_j$. Buckley et al. [2], [3] point out that very often the classical solution doesn't exist, which is mainly due to the dependency problem. $\boldsymbol{S}_E$ and $\boldsymbol{S}_I$ solutions are obtained by solving the corresponding crisp system using Cramer's rules, and then evaluating the solution using extension principle, respectively, interval arithmetic for each α-cut. The joint solution $\boldsymbol{S}_J$, on which the paper is focused, is a fuzzy vector that coincides with united solution set [13] of linear interval systems for each α-cut.

Set

$$\boldsymbol{S}(\alpha) = \{x \in R^n \mid Ax = b, A \in \boldsymbol{A}(\alpha), b \in \boldsymbol{b}(\alpha)\} \tag{3}$$

and define $\boldsymbol{S}_J$, a fuzzy subset of R^n, by its membership function

$$\mu_{\boldsymbol{S}_J}(x) = \begin{cases} \sup\{\alpha \mid x \in \boldsymbol{S}(\alpha)\}, & x \in \boldsymbol{S}(0) \\ 0 & x \notin \boldsymbol{S}(0) \end{cases} . \tag{4}$$

Theorem 2 (Buckley [3]). If $\boldsymbol{A}(0)$ is regular, then $\boldsymbol{S}_J$ is a fuzzy vector.

To eliminate the dependency problem, parametric fuzzy linear systems are considered.

4 Parametric Fuzzy Linear Systems

A system

$$\boldsymbol{A}(\boldsymbol{p})x = \boldsymbol{b}(\boldsymbol{p}) \ , \tag{5}$$

where $\boldsymbol{p} \in F^k$ is a vector of fuzzy numbers, is called a parametric fuzzy linear system. The problem of solving the fuzzy system (5) can be transformed [15] into an equivalent problem of solving a nested family of parametric linear interval systems

$$\boldsymbol{A}(\boldsymbol{p}(\alpha))\mathrm{x} = \boldsymbol{b}(\boldsymbol{p}(\alpha)), \quad \alpha \in [0, 1] \ . \tag{6}$$

Interval solution set of a parametric interval linear system $A(\boldsymbol{p})x = b(\boldsymbol{p})$, $\boldsymbol{p} \in IR^k$ is defined as

$$S(p) = \{x \in R^n \mid Ax = b, A \in A(p), b \in b(p)\} \ . \tag{7}$$

Theorem 3. If $A(p)$ is regular, then interval solution of parametric interval linear system exists.

Proof: Obvious.

Accordingly to (4) define the joint vector solution $S_J(p)$ of the parametric system (5) by its membership function

$$\mu_{S_J(p)}(x) = \mu(x \mid S_J(p)) = \begin{cases} \sup\{\alpha \mid x \in S(p(\alpha))\}, & x \in S(p(0)) \\ 0 & x \notin S(p(0)) \end{cases} . \tag{8}$$

Theorem 4. If $A(p(0))$ is regular, then the join solution of parametric fuzzy linear exists.

Proof: See theorem (2).

In what follows parametric linear systems with elements linearly dependent on elements of a vector of fuzzy parameters $p \in F^k$

$$a_{ij}(p) = \omega(i, j)^{\mathrm{T}} p, \quad b_j(p) = \omega(0, j)^{\mathrm{T}} p \tag{9}$$

are considered, where $\omega \in (R^k\})^{n\times n}$ is a matrix of real k-dimensional vectors. Elements of the induced family of parametric interval linear systems can be expressed as

$$a_{ij}(p(\alpha)) = \omega(i, j)^{\mathrm{T}} p(\alpha), \quad b_j(p(\alpha)) = \omega(0, j)^{\mathrm{T}} p(\alpha) \ , \tag{10}$$

for $\alpha \in [0, 1]$. Systems (6) will be solved using a direct method [20]. A brief description of the method is given in the next section.

5 Description of the Direct Method

An efficient direct method (DM for short) for solving parametric linear systems with elements linearly dependent on a set of interval parameters have been proposed in [20]. The method can be used to solve large systems with a lot number of interval parameters. The method is based on the following.

Theorem 5 (Skalna [20]). Let $A(p)x = b(p)$ with $p \in IR^k$, $R \in R^{n\times n}$, and $\tilde{x} \in R^n$. If $C \in IR^{n\times n}$ given by formula

$$C_{\mathrm{ij}} = \sum_{k=1}^{n} R_{ik}\,\omega(k, j)^{\mathrm{T}} p \,.., \tag{11}$$

is an H-matrix then

$$S(p) \subseteq \tilde{x} + \langle C\rangle^{-1} \mid Z \mid [-1, 1] \tag{12}$$

with

$$\boldsymbol{Z}_i = \sum_{j=1}^{n} R_{ij}\left(\omega(0, j) - \sum_{k=1}^{n} \tilde{x}\,\omega(j,k)\right)^{\mathrm{T}} \boldsymbol{p} \tag{13}$$

It is recommended to choose $R = A^{-1}(\breve{p})$ and $\tilde{x} = A^{-1}(\breve{p})b(\breve{p})$ so that $\boldsymbol{C}$ and $\boldsymbol{Z}$ are of small norms (see theorem 4.1.10 [13]).

6 Illustrative Example

Consider a 20-floor cantilever truss structure depicted in Fig. 1. There are 42 nodes, 81 beams, full support at node 1 and partial support (sliding along Y-axis) at node 2. This results in 81 variables and 81 fuzzy parameters. Set Young's modulus $E = 2.0\times10^{11}$[Pa], cross section area $A = 0.005$[m^2], and the length of the vertical beams $L = 1$[m].

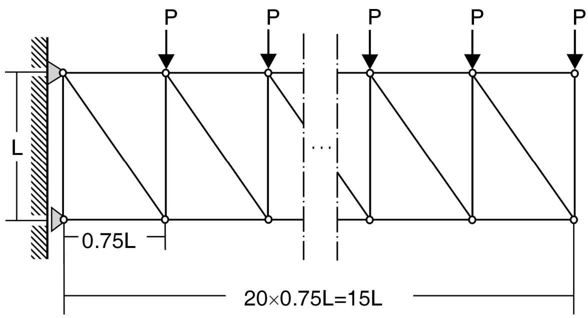

Fig. 1. 20-floor cantilever truss structure

To compute the displacements of the nodes, the following parametric fuzzy linear system

$$\boldsymbol{K}(\boldsymbol{s})d = \boldsymbol{Q}(\boldsymbol{s}) \;,$$

has to be solved, where $\boldsymbol{K}(\boldsymbol{s})$ is a fuzzy stiffness matrix, $\boldsymbol{Q}(\boldsymbol{s})$ is a fuzzy vector of forces, $\boldsymbol{d}$ is unknown vector of fuzzy displacements, and $\boldsymbol{s}$ is a vector of fuzzy parameters.

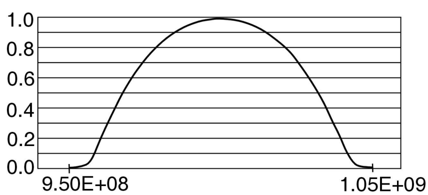

Fig. 2. Membership function of the fuzzy parameters

Assume the fuzzy parameters have the membership function (Fig. 2) given by formula

$$\mu_{s_{ij}}(x) = \begin{cases} e^{4(m_1-x)(x-m_2)-(m_2-m_1)^2/4(m_2-x)(x-m_1)}, & m_1 < x < m_2 \\ 0, & \text{elsewhere} \end{cases}$$

where $m_1 = E \cdot A \cdot (1 - \varepsilon)$, $m_2 = E \cdot A \cdot (1 + \varepsilon)$, $\varepsilon = 5\%$.

The results produced by DM method (10 α-cuts), for two chosen coordinates x_{41}, y_{41}, are presented in Table 1.

Table 1. Results of the DM method: 10 α-cuts

α	x_{41}	y_{41}
0.0	[0.00568536, 0.00927714]	[-0.331072, -0.0422026]
0.1	[0.00633381, 0.00862869]	[-0.260963, -0.112312]
0.2	[0.00648488, 0.00847762]	[-0.247216, -0.126059]
0.3	[0.00661057, 0.00835193]	[-0.236577, -0.136698]
0.4	[0.00672633, 0.00823617]	[-0.227446, -0.145829]
0.5	[0.00683787, 0.00812463]	[-0.219279, -0.153996]
0.6	[0.0069482, 0.0080143]	[-0.211839, -0.161436]
0.7	[0.00705947, 0.00790303]	[-0.205015, -0.16826]
0.8	[0.007174, 0.0077885]	[-0.198758, -0.174517]
0.9	[0.00729645, 0.00766605]	[-0.193018, -0.180257]
1.0	[0.00748125, 0.00748125]	[-0.186638, -0.186637]

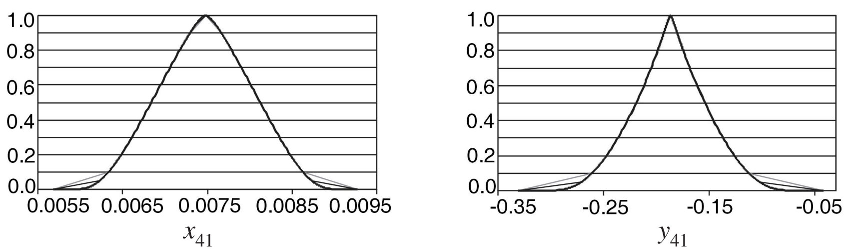

Fig. 3. Comparison of the results of the DM method obtained for 10 α-cuts (gray thin line), 20 α-cuts (black thin line) and 100 α-cuts (black thick line)

The differences between the shapes of the membership functions (based on 10, 20 and 100 α-cuts), depicted in the Fig. 3, are significant, especially for small values of α. The differences between the results of the DM method (451 α-cuts) and interpolation, based on 25 points (α-cuts) of non-uniform distribution (Fig. 4), is quite neligible. The benefit of the interpolation is that once the coefficients of cubic functions are computed, the approximation of the fuzzy solution can be easily computed for any α-value.

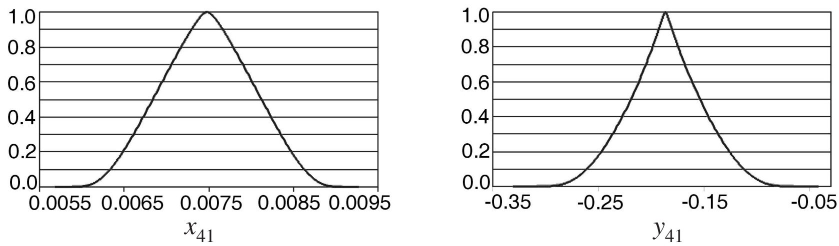

Fig. 4. Comparison of the results of the DM method for 451α-cuts with the results of interpolation based on 25 points non-uniformly distributed on [0, 1] interval

6 Conclusions

Parametric linear systems with coefficients being affine linear functions of convex fuzzy parameters with compact support have been studied. A method for approximating the vector solution of such systems has been introduced. Conditions for existence of the vector solution have been given. It has been shown, using an illustrative example of 20-floor cantilever truss structure, that the method can be applied to large systems with a lot of fuzzy parameters. The method is very efficient and is easy to implement, which is very important for practical use.

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