Fractal Interpolation Fitness Based on BOX Dimension's Pretreatment

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Abstract. For graphs of various local complex degrees, this paper will investigate their fitting approach and conduct experiments by using the mixture processing method which is a combination of the Box dimension's pretreatment with self-affine fractal interpolation function (AFIF).

Keywords: Fractal, fitness, interpolation, AFIF.

1 Introduction

As a new tool in data fitness and interpolation, fractals are always self-similar or self-affine, which means the fractals' local complexity is same as the whole. However, this property sometimes is a restriction of further application in the data fitness.

This paper concerns a special class of fractals, AFIF. AFIF can simulate not only the graph of smooth function, but also can effectively and accurately fit rough curves and vibrating data, such as mountain range outlines, electrocardiograms ...etc [3]. It is a new interpolation tool after polynomials and splines. As for the general theory of fractal interpolation function and affine fractal interpolation function, the reader is referred to [1-6].

Based on above discussion, AFIF have same fractal dimension or same complex degree at each location. However, in the practical application, the graphic complex degrees and the sensitivities may be absolutely different when data respond to time in the disparate time periods. Thus, it is obvious that the fitting may not be effective if we directly use AFIF, which has the same complexity everywhere.

Due to the defect of using only one fractal interpolation function in the data fitting, in this paper we will use the mixture processing method, which is a combination of Box dimension's pretreatment with AFIF, to conduct experimentation and analysis. Clustering analysis is adopted according to the each sub-graph's box dimension. Then the sub-graphs are reconstructed together according to adjacent Box dimension, and fractal interpolation is adopted separately. Finally, it is readjusted and resumed.

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2 Fractal and Interpolation

2.1 Fractal Dimension

Definition 1. Let E be a compact subset of $\mathbf{R}^2$. Given $\delta > 0$, let $N_\delta(E)$ be the smallest cardinality of family of solid squares with side length δ such that the union of these squares covers E.

If the limit $\lim_{\delta\to 0} \dfrac{\log N_\delta(E)}{-\log\delta}$ exists, then

$$\dim_B E = \lim_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta} \tag{1}$$

is called the Box dimension of E.

Example 1. Smooth curve
The graph of any smooth function has Box dimension 1. This result means that polynomials and splines are too smooth to approximate the graph with high complexity.

Example 2. C×C

Let $C = \{\sum_{i=1}^{\infty} a_i 3^{-i} : a_i = 0 \text{ or } 2 \text{ for each } i\}$, which is called standard Cantor set.

Then $C \times C$ can be covered by 4^n square of side 3^{-n}, and we can check that $N_{3^{-n}}(C\times C) = 4^n$.

Therefore,

$$\dim_B (C\times C) = \lim_{n\to\infty} \frac{\log N_{3^{-n}}(E)}{-\log 3^{-n}} = \frac{\log 4}{\log 3}.$$

Example 3. Graph of continuous function

For the graph E of a continuous function f, we let $\delta = 2^{-n}$, and

$$M_\delta(E) = \sum_{k=1}^{2^n} [2^n \omega(f, [\frac{k-1}{2^n}, \frac{k}{2^n}])],$$

where $\omega(f,[a,b]) = \max_{x_1\in[a,b]} f(x_1) - \max_{x_1\in[a,b]} f(x_2)$ is the oscillation of f restricted on the interval $[a,b]$, and $[x]$ denotes the smallest integer greater than or equal to x, for example, [3]= [2.5]=3. Then there is a constant $C > 1$ such that $C^{-1}M_\delta \le N_\delta \le CM_\delta$, and thus

$$\dim_B E = \lim_{\delta\to 0} \frac{\log M_\delta(E)}{-\log\delta} \tag{2}$$

It is easy to show that for any *smooth* function, we have $D^{-1}\delta \le M_\delta \le D\delta$ for any δ, where $D > 1$ is a constant, therefore the graph has Box dimension 1.

2.2 Affine Fractal Interpolation Function

Given points $\{(x_i, y_i)\}_{i=0}^N$ in the plane, we suppose $\{\omega_1, \omega_2, \dots, \omega_N\}$ is an *iterated function system* satisfying

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_i \\ h_i \end{pmatrix} \tag{3}$$

$$\omega_i \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix}, \quad \omega_i \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \tag{4}$$

where $|d_i| < 1$ and $a_i \in (0,1)$ for any i with $1 \le i \le N$. By (3) and (4), we notice that $\{\omega_i\}_{i=1}^N$ is determined by $\{(x_i, y_i)\}_{i=0}^N$ and. We always call $\{d_i\}_{i=1}^N$ vertical factors and $\{(x_i, y_i)\}_{i=0}^N$ interpolation points respectively.

Definition 2. Suppose $f(x)$ is a continuous function on the interval $[x_0, x_N]$. Let

$$\Gamma = \{(x, f(x)) : x \in [x_0, x_N])\}$$

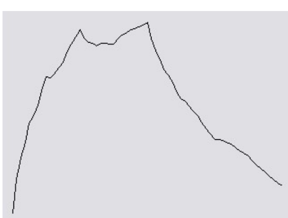
be the graph of $f(x)$. We say that $f(x)$ is an **affine fractal interpolation function**, if

$$\Gamma = \cup_{i=1}^N \omega_i(\Gamma) \tag{5}$$

Example 4

For AFIF defined by (3)-(5), the dimension $\dim_B \Gamma$ of the graph Γ satisfies the following dimension formula ([1]):

$$\sum_{i=1}^N |d_i| \cdot |a_i|^{\dim_B \Gamma - 1} = 1 \tag{6}$$



Interpolation points	(0,0.1),(0.5,0.8),(1,0.2)
Vertical factors	d_1=0.5, d_2= - 0.2

Fig. 1. An Example of AFIF

Remark 1: Formula (6) holds when the interpolation points do not lie in a line simultaneously and $\sum_{i=1}^{N} |d_i| > 1$. Any connected part of the graph Γ of AFIF has the same dimension $\dim_B \Gamma$.

3 Algorithm

- **Step 1:** We divide the interval into several subintervals and thus obtain some sub-graph.
- **Step 2:** By using formula (2) to estimate the dimension of each sub-graph.
- **Step 3:** Clustering the sub-graphs according to their dimensions, we reconstruct some new graph $\Gamma_1, \Gamma_2, \cdots \Gamma_k$, each of which is composed of some sub-graphs with adjacent values of Box dimension. In the process of reconstruction, we should make translation for each sub-graph along y-axis to ensure the *connectedness* of graph.
- **Step 4:** For each new graph Γ_i in Step 3, we use AFIF to approximate it.
- **Step 5:** Reconstruct these AFIFs to obtain an approximation of the original graph.

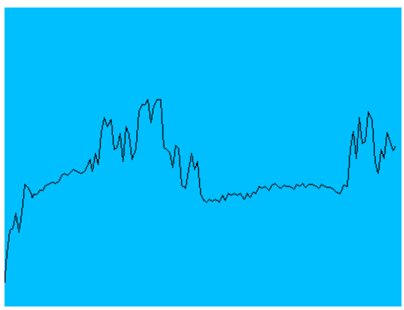


Fig. 2. Original graph

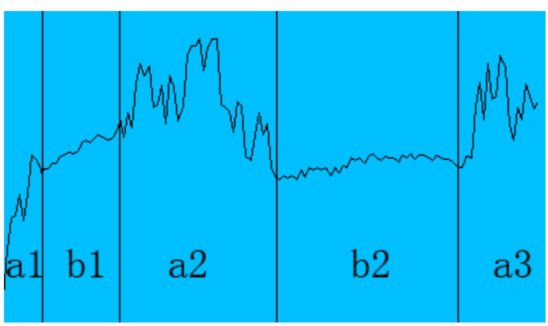


Fig. 3. Partition

4 Experimental Results

The graph is a piece of the tendency picture of captial stock certificate. We use it as an example for fractal interpolation.

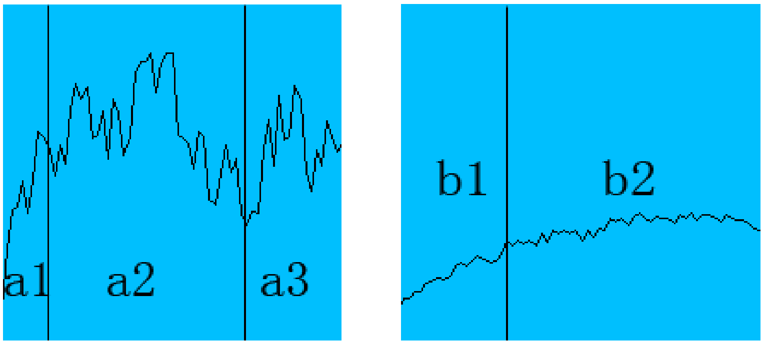
Here we give a partition of the original graph to obtain 5 parts: a1, b1, a2, b2 and a3. By formula (2), we get

$\dim_B$(Part a1) ≈0.37, $\dim_B$ (Part a2)≈0.38, $\dim_B$ (Part a3)≈0.38

and

$\dim_B$ (Part b1)≈0.14 , $\dim_B$ (Part b2)≈0.13.

And thus we reconstruct the blow graphs.



graph (a) graph (b)

Fig. 4. Reconstruction of graphs according to Box dimension

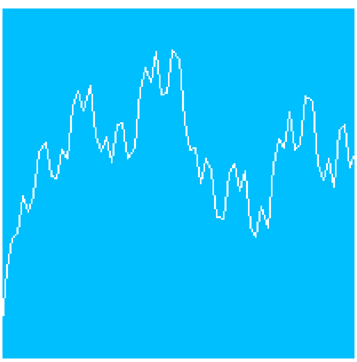


Fig. 5. AFIF (a') with respect to graph (a)

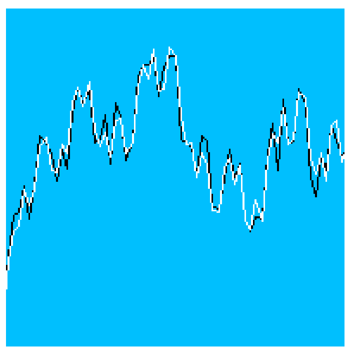


Fig. 6. AFIF (a') and graph (a)

Interpola-tion points	（0,0.1），（0.5,0.85），（1,0.6）
Vertical factors	d_1=0.6, d_2=-0.7

Fig. 7. AFIF (b') with respect to graph (b)

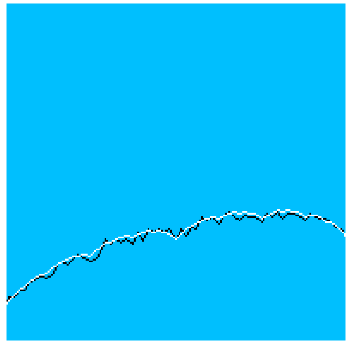


Fig. 8. AFIF (b') and graph (b)

Interpolation points	（0,0.1）（0.5,0.3）（1,0.2）
Vertical factors	d_1=0.5, d_2=0.6

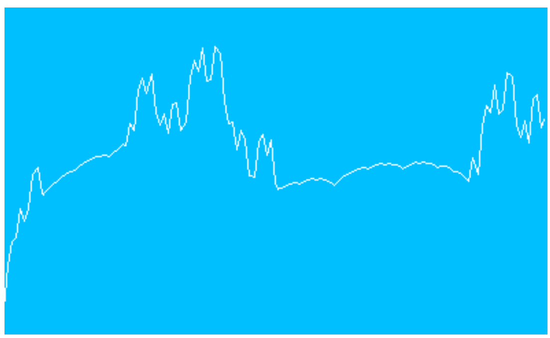


Fig. 9. Reconstruction of AFIF (a') and (b')

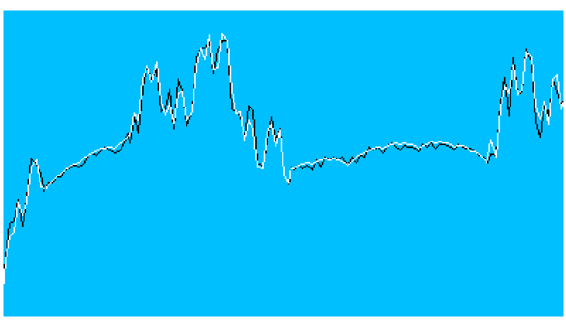


Fig. 10. Original graph (black) and graph of AFIFs with reconstruction (white)

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