Ontological and Epistemological Grounding of Fuzzy Theory

I. Burhan Türkşen

Head Department of Industrial Engineering, TOBB-Economy and Technology University, Sögütözü Cad. No:43, Sögütözü 06560 Ankara/Turkey,
Tel.:+90 312 292 4068
bturksen@etu.edu.tr

Abstract. An Ontological and Epistemological foundation of Fuzzy Set and Logic Theory is reviewed in comparison to Classical Set and Logic Theory. It is shown that basic equivalences of classical theory breakdown but are re-established as weak equivalences as a containment relation in fuzzy theory. It is also stressed that the law of conservation of information is still upheld within fuzzy theory.

1 Introduction

About 40+ years ago (1965), L. A. Zadeh published "Fuzzy Sets"[13]. Later, he has further, exposed the fact that there is the theory of "Possibility" in contrast to the theory of "Probability"[9,12]. After 1978, "Fuzzy System Models" began to influence "Decision-Support System Models" in a substantial manner with the introduction of "Possibility" as opposed to "Probability" and as a new dimension of uncertainty in system modelling.

On the bases of philosophical foundations, fuzzy sets, logics and systems in comparison to classical sets, logics and systems may be contrasted in analogy to a comparison of the philosophical underpinnings of "modernism" ver sus "post-modernism" (Dan Simon, [4]).It is important, however, to point out that such a comparison is limited. While "modernism", in stressing crisp, black- white occurences, may be anologous to Classical theory, "post-modernism", in stressing only uncertainty could be partly analogous to fuzzy theory. In reality, fuzzy theory contains both the uncertainty interval of]0,1[in anology to "post-modernism" and the certainty boundaries of {0,1} in anology to "modernism", i.e., in fuzzy theory , a membership function, μ, maps occurences, X, to [0,1] interval; $\mu : X \rightarrow [0,1]$.

In another perspective, Zadeh`s "Computing With Words" [8] approach may be considered analogous to Turing`s [3] philosophy of man and machine, and Popper`s [2] repudiation of the classical observationalist-inductivist account of scientific method. It is clear that both of these two fine philosophers of science expressed their ideas within the classical perspective. Hence, once again the analogy would be limited. Zadeh`s approach is essentially stressing the fact that there is a "meaning" of words

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that can at best be represented by fuzzy sets as “a matter of degree” and that it is context dependent. For these reasons and others, Hodge [1] states that Zadeh`s approach generates deeper roots and new understanding. In order to really comprehend these deeper insights, one needs to properly investigate the ontological and epistemological underpinnings of fuzzy theory Türkşen [5].

Ontology: 1) A branch of metaphysics concerned with the nature and relations of being. 2) A particular theory about the nature of being or the kinds of existents.

Ontology lays the ground for the structural equivalences in classical theory; whereas it lays the ground for the structural breakdown of classical equivalences in fuzzy theory. On the other hand it reveals essential Laws of Conservation based on assumptions of existences in a different manner in both theories.

At this ground level of inquiry, one ought to ask:

“What linguistic expressions capture our positions to realty?”.

“What PNL, Precisiated Natural Language, (Zadeh, [8]) expressions capture our positions to reality?”

“What are the basic equivalences or the lack of them and the Laws of Conservation that capture our position to reality?”

Epistemology: The study or theory of the nature and grounds of knowledge, esp., with reference to its limits and validity.

Epistemology lays the ground work for the assessment of consistency and believability of a set of propositions by either a priory or evidentiary basis. Evidentiary basis could be subjective and/or objective.

At this second level, one ought to state general epistemological questions:

“What linguistic encoding allows us to access truth or knowledge?
“What linguistic expressions cause the assessment of truth and knowledge?”.

As well, one must to ask:

What accounts as good, strong, supportive evidence for Belief? What is the degree of Belief?

This requires that we have to come up with a “good” in terms of a “matter of degree”, i.,e., “Explication” of criteria of evidence or its justification.

What is the connection between a belief being well-supported by good evidence, and the likelihood that it is true? What is the degree of likelihood and its degree of truth?

This inquires into a new definition of “Validation” criteria and their assessment to a degree. In particular, we should investigate “Belief”, “Plausibility”, and “Probability”, related assessments for “Validation”.

In Table 1, ontological and epistemological levels of theoretical inquiry are shown.

Table 1. Ontological and Epistemological Levels of a Theoretical Inquiry

EPISTEMOLOGICAL LEVEL	4. How do we validate our knowledge? How do we know it is true? What criteria do we use to assess its truth-value? "What PNL expressions cause the assessment of truth and knowledge?" 3. What is our access to truth and knowledge in general? Where is knowledge and its truth to be found? How or from what are they constituted? "What PNL encoding allows us to assess truth or knowledge?"
ONTOLOGICAL LEVEL	2. What is our position or relation to that Reality (if we do assume that it exists on level 1 below)? "What PNL expressions capture our positions to reality?" 1. Is there any reality independent or partially independent of us? Does any absolute truth exist? Does fuzziness exists? "What PNL explicates reality?" "Are they crisp or fuzzy representations of linguistic variables and their linguistic connectives?" "What are the basic equivalences generated by PNL expressions for crisp or fuzzy representations?"

In Table 2, the same two levels of inquiry are shown for classical theory which is based on classical axioms that are shown in Table 3.

Table 2. Levels of Inquiry on Classical Set and Logic Theory

EPISTEMOLOGICAL LEVEL	4. Correspondence theory of Validity only Objective 3. Objectivist, empiricists, certain
ONTOLOGICAL LEVEL	2. sRo Cartesian dualism 1. Realism, crisp meaning representation of linguistic Variables and connectives are defined with two-valued sets and logic theory. Equivalences in "normal forms" together with classical laws of conservation, as well as formulae for Belief, Plausibility, Probability, etc.

Table 3. Axioms of Classical & Set & Logic Theory, where A, B are crisp, two valued, sets and c(.) is the complement, X is the universal set and ϕ is the empty set

Involution	$c(c(A)) = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotency	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and ϕ	$A \cup X = X$
	$A \cap \phi = \phi$
Identity	$A \cup \phi = A$
	$A \cap X = A$
Law of contradiction	$A \cap c(A) = \phi$
Law of excluded middle	$A \cup c(A) = X$
De Morgan Laws	$c(A \cap B) = c(A) \cup c(B)$
	$c(A \cup B) = c(A) \cap c(B)$

The positions taken by some of the fuzzy sets and logic theorists on these two levels of theoretical inquiry are shown in Table 4 which is based on the Meta-Linguistic interpretations of the Classical axioms which are shown in Table 5.

Table 4. Position Taken by some of Fuzzy Set and Logic Theorists on the Hierarchy of Levels of Theoretical Inquiry

GENERAL EPISTEMOLOGICAL LEVEL	4.Correspondence theory of Validity both objective and subjective.Approximate Reasoning 3.Subjective-objective, experimental and empiricist, e.g., expert and fuzzy data mining based.
ONTOLOGICAL LEVEL	2. $\mathbf{S} \xleftrightarrow{\mathbf{R}} \mathbf{O}$ schema gives credence both on the Level of subject and the object interaction 1. Realism - fuzzy and uncertain Generation of “Fuzzy Canonical Forms” that are not equivalent to each other in contrast to “Classical Normal Forms”. Generation of new Laws of Conservation for t-norms, co-norms, Belief, Plausibility, Probability, etc.

At this point, we ought to ask :“what are the basic axioms of CWW?”, and introduce “Meta-Linguistic Axioms” as a foundation for CWW which are shown in Table 5 below.

Table 5. Meta-Linguistic Expression of the Axioms for CWW

Involution:	NOT(NOT(A)) vs A
Commutativity:	A AND B vs B AND A
	A OR B vs B OR A
Associativity:	(A AND B) AND C vs A AND (B AND C)
	(A OR B) OR C vs A OR (B OR C)
Distributivity:	A AND (B OR C)vs(A AND B) OR (A AND C)
	A OR (B AND C)vs(A OR B) AND (A OR C)
Idempotency:	A AND A vs A
	A OR A vs A
Absorption :	A OR (A AND B) VS A
	A AND (A OR B) VS A
Absorption by X and ∅:	A OR X vs X
	A AND ∅ vs∅
Identity:	A OR ∅ vs A
	A AND X vs A
Law of contradiction:	∅ vs A AND NOT(A)
Law of excluded middle:	A OR NOT(A) vs X
De Morgan’s Laws:	NOT(A AND B) vs NOT(A) AND NOT(B)
	NOT(A OR B) vs NOT(A) OR NOT(B)

2 Classical Theory

On the bases of Tables 2 and 3, we hava the well known Classical equivalences between the Disjunctive and Conjunctive Normal Form, DNF(.) and CNF(.). For example, on the ontological levels, we hava the Classical equivalences such as:

DNF(A OR c(A)) ≡ CNF(A OR c(A)),
DNF(A AND c(A)) ≡ CNF(A AND c(A)),

and the associated law of conservation:
μ[DNF(A OR c(A)) ≡ CNF(A OR c(A))]
+ μ[DNF(A AND c(A)) ≡ CNF(A AND c(A))] = 1

We have their equivalences such as:
DNF(A AND B) ≡ CNF(A AND B), and
DNF(A OR B) ≡ CNF(A OR B),

Furthermore for T-norms and co-norms, Belief, Plausibility, Probability, etc., we receive the well known formulae:

T(a,b) = 1 – S(n(a), n(b)),
Bel(A) + Pl(c(A)) = 1
Pl(A) + Bel(c(A)) = 1
Pr(A) +Pr(c(A)) = 1

On the Epistemological Level, we find various developments of system models with application technologies known as statistical methods, such as multi-variate regression equations, programming methods such as linear and non-linear optimization algorithms or optimal control schemas developed on objective data that are obtained by measurement devises and depend on description, D, and validation, V, frameworks on the two valued theory, {D{0,1}, V{0,1}}.

As well the validation of the models are assessed with domain-specific test data that are assumed to be standing on descriptive and verified framework of {D{0,1}, V{0,1}}. The validations of the domain specific models are executed with the classical inference schemas such as Modus Ponens. This may entail a re-computation of, say, regression, or programming or control models with test data. Results obtained from such models are assumed to be on {D{0,1}, V{0,1}} framework based on some level of statistical risk.

Furthermore, "validation" of such system models is assumed in terms of certain criteria that are developed by classical statistical methods. For example, some of these are known as

RMSE – Root Mean Square Error
R2 – How successful the fit is in explaining the variation in the data
Accuracy of Prediction
Power of Prediction

These are defined as:

$$RMSE = \sqrt{MSE} \quad MSE = \frac{SSE}{n} \quad R^2 = 1 - \frac{SSE}{SST} \quad SST = \sum_{i=1}^{n} (y_i - \overline{y}_i)^2$$

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

MSE: Mean Square Error
SSE: Sum of Square Errors
SST: Total Sum of Squares

Accuracy t (%) = X / P
Power t (%) = X / A

X: Total number of predicted values that are hit correctly in the interval t
P: Total number of predicted values at interval t
A: Total number of actual values at interval t

It should be noted that these are crisp definitions. We need to develop their fuzzy versions for the fuzzy theory.

3 Fuzzy Theory

However, in most theoretical and applied investigations of fuzzy theory, only a part of the Classical axioms are considered as shown in Table 6. Since, they are crisp axioms, we suggest that this is a myopic adaptation.

Recall that in fuzzy theory, briefly, every element belongs to a concept class, say A, to a partial degree, i.e., $\mu A: X \to [0,1]$, $\mu A(x)=a \in [0,1]$, $x \in X$, where $\mu A(x)$ is the membership assignment of an element $x \in X$ to a concept class A in a proposition. Thus most of all concepts are definable to be true to a degree.

"Fuzzy Truth Tables" and in turn the derivation of the combination of concepts for any two fuzzy sets A and B, when they are represented by a Type 1 fuzzy sets, turn to reveal two canonical terms as Fuzzy Disjunctive and Conjunctive Canonical Forms. For example for "AND", "OR" and "IMP" linguistic expressions, we receive the following two forms for each combination of concepts:

$$\text{"A AND B"} = \begin{cases} \text{FDCF(A AND B)} = A \cap B \\ \text{FCCF(A AND B)} = (A \cup B) \cap (c(A) \cup B) \cap (A \cup c(B)), \end{cases} \quad \text{and}$$

$$\text{"A OR B"} = \begin{cases} \text{FDCF(A OR B)} = (A \cap B) \cup (c(A) \cap B) \cup (A \cap c(B)) \\ \text{FCCF(A AND B)} = A \cup B, \end{cases} \quad \text{and}$$

$$\text{"A IMP B"} = \begin{cases} \text{FDCF(A IMP B)} = (A \cap B) \cup (c(A) \cap B) \cup (c(A) \cap c(B)) \\ \text{FCCF(A IMP B)} = c(A) \cup B, \end{cases}$$

etc., in analogy to the two-valued set and logic theory where FDCF(.)=DNF(.) and FCCF(.)=CNF(.) in form only.

Furthermore, as it is shown in Türksen [5,6,7], the equivalence, DNF(.)≡CNF(.), breaks down, i.e., we have FDCF(.)not=FCCF(.) and in particular we get FDCF(.)⊆FCCF(.) for certain classes of t-norms and t-conorms that are strict and nilpotent Archimedean.

For example, particular consequences that we receive are:

(1) FDCF(A OR NOT A)⊆FCCF(A OR NOT A)

which is the realization of the law of "Fuzzy Middle" as opposed to the Law of Excluded Middle and

(2) the Law of "Fuzzy Contradiction, FDCF(A AND NOT A) ⊆FCCF(A AND NOT A)

as opposed to the Law of Crisp Contradiction.

As a consequence of these, we obtain new Laws of Conservation in fuzzy theory as:

m[FDCF(A AND c(A))] + m[FCCF(A OR c(A))] = 1;

which is well known but re-interpreted for fuzzy sets and

m[FDCF(A OR c(A))] + m[FCCF(A AND c(A))] = 1.

which now exists for fuzzy sets only!

Hence we once again observe that the "Principle of Invariance" is re-established in Interval-Valued Type 2 fuzzy set theory, but as two distinct Laws of Conservation.

This means that linguistic connectives "AND", "OR", "IMP", etc., are not interpreted in a one-to-one correspondence, i.e., non-isomorphic, to be equal to "∩", "∪", "c(.), ∪", etc. That is the imprecise and varying meanings of linguistic connectives are not precisiated in an absolute manner and there is no absolute precisiation of the meaning of words nor is there an absolute precisiation of the meaning of connectives. This provides a framework for the representation of uncertainty in the combination of words and hence in reasoning with them as a foundation for CWW.

The break down of the equivalences in Fuzzy theory, i.e., FDCF(.)not=FCCF(.), in turn generates new additional formulae for t-norm-conorms, Belief, Plausibility and Probability. That is, we now obtain:

Two T-Norm-Conorms Formulae for fuzzy sets in fuzzy theory:

(1) T(a,b) = 1 – S(n(a),n(b))

which is well known but re-interpreted in fuzzy theory; and

(2) T[T(S(a,b), S(n(a),b)), S(a,n(b))]
= 1 – S[S(T(n(a), n(b)), T(a,n(b))], T(n(a),b)]

which is new in fuzzy theory!

Two Belief and Plausibility measures over fuzzy sets at particular α-cuts:

(1) Pl[FDCF(A AND B)] + [Bel[FCCF(c(A) OR c(B)] = 1
Pl[(A∩B) + Bel(c(A) ∪ c(B)) = 1

which is well known but re-interpreted for fuzzy theory; and

(2) Pl[FCCF(A AND B)] + Bel[FDCF(c(A) OR c(B))] = 1
Pl[(A∪B) ∩ (c(A) ∪ B) ∩ (A ∪ c(B))] + Bel[(c(A) ∩ c(B)) ∪ (A ∩ c(B)) ∪ (c(A) ∩ B)] = 1

which is new in fuzzy theory!

Two Probability measures over fuzzy sets at particular α-cuts:

Pr(A AND B) + Pr(c(A) OR c(B) = 1
Pr[FDCF(A AND B] + Pr[FCCF(c(A) OR c(B)] = 1

which is well known but re-interpreted for fuzzy theory; and

(3) Pr[(A∩B) + Pr(c(A) ∪ c(B)) = 1
Pr[FCCF(A AND B)] + Pr[FDCF(c(A) OR c(B))] = 1

Pr[(A∪B) ∩ (c(A) ∪ B) ∩ (A ∪ c(B))]
+ Pr[(c(A) ∩ c(B)) ∪ (A ∩ c(B)) ∪ (c(A) ∩ B)] = 1

which is new in fuzzy theory!

This particular interpretation of measures at all a-cut levels of fuzzy sets together with knowledge representation and reasoning formulae form a unique foundation for Type 2 fuzzy set theory in general and in particular for Interval-Valued Type 2 fuzzy set theory generated by the combination of linguistic concepts with linguistic connectives even if the initial meaning representation of words are to be reduced to Type 1 membership representation. More general representations start with Type 2 representation schema and then form Type 2 reasoning schemas to capture both imprecision and uncertainty.

Therefore on the Epistemological level, we first have approximate reasoning models expressed in particular as Interval-Valued Type 2 fuzzy set as:

$$\text{A IMP B} = \begin{cases} \text{FDCF(A IMP B)} \\ \text{FCCF(A IMP B)} \end{cases}$$

as a descriptive model, i.e., an Interval-Valued Type 2 rule, a premise. That is {{D[0,1] V{0,1}} IMP {D[0,1] V{0,1}}}={D[0,1] V{0,1}} which is within the framework of a fuzzy inference schema such as Generalized Modus Ponens, GMP, originally proposed by Zadeh as Compositional Rule of Inference, CRI [10]. In this framework, the first premise {D[0,1] V{0,1}} for “A IMP B” combined with a second premise {D[0,1] V{0,1}} for “A′” result in a consequence {D[0,1] V{0,1}} for B*. The validation is based on a fuzzy comparison of the actual output for a given test input data and model output for the same test input data. The error is usually accepted to be a true, V{0,1}, verification but based on a risk statistically but fuzzily evaluated assessment dependent on a fuzzy test of hypothesis. It should be noted that all of the proceeding exposition which is made for the Descriptive fuzzy set paradigm. A similar exposition is applicable to the Veristic fuzzy set paradigm.

It is to be noted that we are yet to develop “Fuzzy RMSE”, “Fuzzy R2”, “Fuzzy Accuracy of Prediction”, and “Fuzzy power of Prediction” in analogy to their classical versions.

It is in these respects that many of the familiar revisions and alternatives to classical thinking, suggested by Black, Lukasiewicz, Kleene, etc., were preliminary break away strategies from the classical paradigm. With the grand paradigm shift caused by Zadeh’s seminal work and continuous stream of visionary proposals, it is now clear that most of them reflect very different stances adopted at the more fundamental levels of our proposed ontology and epistemology. Those changes, it appears, have sometimes been made only in a more tacit and implicit manner. In our studies, it became obvious that the most radical revisions are likely to be the ones that stem from modification to be made at the ontological and epistemological levels where the basic grounding of a theory takes place.

4 Conclusions

In this exposition, we have stated that there are essential properties of the fuzzy theory that comes to light at the ontological and epistemological levels of theoretical inquiry which are quite different from the properties of the classical theory. We have stated these both in linguistic expressions as well as in terms of classical axiomatic assumptions for the classical theory but in terms of Meta-Linguistic interpretations of classical axioms for the grounding of the fuzzy theory in a comparative manner.

In this comparative approach, we have also stated that there are two distinct fuzzy canonical forms in fuzzy theory that correspond to all classical normal forms which form equivalences in classical theory. This causes the generation of “Interval-Valued Type 2 Fuzzy Sets”. The law of conservation of information is still upheld within fuzzy theory but in two distinct forms.

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