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Vitali D. Milman
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Preface

Since the mid-1980's the following volumes containing collections of papers reflecting the activity of the Israel Seminar in Geometric Aspects of Functional Analysis appeared:

1983-1984	Published privately by Tel Aviv University
1985-1986	Springer Lecture Notes, Vol. 1267
1986-1987	Springer Lecture Notes, Vol. 1317
1987-1988	Springer Lecture Notes, Vol. 1376
1989-1990	Springer Lecture Notes, Vol. 1469
1992-1994	Operator Theory: Advances and Applications, Vol. 77, Birkhauser
1994-1996	MSRI Publications, Vol. 34, Cambridge University Press
1996-2000	Springer Lecture Notes, Vol. 1745
2001-2002	Springer Lecture Notes, Vol. 1807
2002-2003	Springer Lecture Notes, Vol. 1850.

Of these, the first six were edited by Lindenstrauss and Milman, the seventh by Ball and Milman and the last three by the two of us.

As in the previous volumes, the current one reflects general trends of the Theory. Most of the papers deal with different aspects of Asymptotic Geometric Analysis, ranging from classical topics in the geometry of convex bodies, to inequalities involving volumes of such bodies or, more generally, log-concave measures, to the study of sections or projections of convex bodies. In many of the papers Probability Theory plays an important role; in some, limit laws for measures associated with convex bodies, resembling Central Limit Theorems, are derived and in others, probabilistic tools are used extensively. There are also papers on related subjects, including a survey on the behavior of the largest eigenvalue of random matrices and some topics in Number Theory.

All the papers here are original research papers (and one invited expository paper) and were subject to the usual standards of refereeing.

As in previous proceedings of the GAFA Seminar, we also list here all the talks given in the seminar as well as talks in related workshops and conferences. We believe this gives a sense of the main directions of research in our area.

We are grateful to Diana Yellin for taking excellent care of the typesetting aspects of this volume.

Vitali Milman
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Contents

Theory of Valuations on Manifolds, IV. New Properties of the Multiplicative Structure

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Summary. This is the fourth part in the series of articles [A4], [A5], [AF] (see also [A3]) where the theory of valuations on manifolds is developed. In this part it is shown that the filtration on valuations is compatible with the product. Then it is proved that the Euler–Verdier involution on smooth valuations is an automorphism of the algebra of valuations. Then an integration functional on valuations with compact support is introduced, and a property of selfduality of valuations is proved. Next a space of generalized valuations is defined, and some basic properties of it are proved. Finally a canonical imbedding of the space of constructible functions on a real analytic manifold into the space of generalized valuations is constructed, and various structures on valuations are compared with known structures on constructible functions.

Contents

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0 Introduction

In convexity there are many geometrically interesting and well known examples of valuations on convex sets: Lebesgue measure, the Euler characteristic, the surface area, mixed volumes, the affine surface area. For a description of older classical developments on this subject we refer to the surveys [MS], [M2]. For the general background on convexity we refer to the book [S].

Approximately during the last decade there was a significant progress in this classical subject which has led to new classification results of various classes of valuations, to discovery of new structures on them. This progress has shed a new light on the notion of valuation which allowed to generalize it to more general setting of valuations on manifolds and on not necessarily convex sets (which do not make sense on a general manifold). On the other hand author's feeling is that the notion of valuation equips smooth manifolds with a new general structure. The development of the theory of valuations on manifolds was started in three previous parts of the series of articles: [A4], [A5] by the author and [AF] by J. Fu and the author. This article in the forth part in this series.

In [A5] the notion of smooth valuation on a smooth manifold was introduced. Roughly speaking a smooth valuation can be thought as a finitely additive $\mathbb{C}$-valued measure on a class of nice subsets; this measure is requested to satisfy some additional assumptions of continuity (or rather smoothness) in some sense. The basic examples of smooth valuations on a general manifold X are smooth measures on X and the Euler characteristic. Moreover the well known intrinsic volumes of convex sets can be generalized to provide examples of smooth valuations on an arbitrary *Riemannian* manifold; these valuations are known as Lipschitz-Killing curvatures.

Let X be a smooth manifold of dimension n. The space of smooth valuations on X is denoted by $V^\infty(X)$. It has a canonical linear topology with respect to which it becomes a Fréchet space.

The space $V^\infty(X)$ carries a canonical multiplicative structure. This structure seems to be of particular interest and importance. When X is an affine space it was defined in [A4] (in even more specific situation of valuations polynomial with respect to translations it was defined in [A3]). For a general manifold X the multiplicative structure was defined in [AF]. The construction in [AF] uses the affine case [A4] and additional tools from the geometric measure theory.

It was shown in [AF] that the product $V^\infty(X) \times V^\infty(X) \to V^\infty(X)$ is a continuous map, and $V^\infty(X)$ becomes a commutative associative algebra with the unit (which is the Euler characteristic). The goal of this article is to study further properties of the multiplicative structure and apply one of them (which we call the Selfduality property) to introduce a new class of generalized valuations.

In [A5] a filtration of $V^\infty(X)$

$$V^\infty(X) = W_0 \supset W_1 \supset \cdots \supset W_n \tag{0.1.1}$$

by closed subspaces was introduced. The first main result of this article (Theorem 3.1.1) says that this filtration is compatible with the product, namely $W_i \cdot W_j \subset W_{i+j}$ (where $W_k = 0$ for $k > n$).

In [A5] the author has introduced a continuous involution $\sigma\colon V^\infty(X) \to V^\infty(X)$ called the Euler–Verdier involution. The second main result of this article says that σ is an algebra automorphism (Theorem 4.1.4).

Let us denote by $V_c^\infty(X)$ the space of compactly supported smooth valuations. Next we introduce in this article the integration functional $\int\colon V_c^\infty(X) \to \mathbb{C}$. Slightly oversimplifying, it is defined by $[\phi \mapsto \phi(X)]$. The third main result is as follows.

Theorem 0.1.1. *Consider the bilinear form*

$$V^\infty(X) \times V_c^\infty(X) \to \mathbb{C}$$

given by $(\phi, \psi) \mapsto \int \phi \cdot \psi$.

This bilinear form is a perfect pairing. More precisely the induced map

$$V^\infty(X) \to \left(V_c^\infty(X)\right)^*$$

is injective and has a dense image with respect to the weak topology on $(V_c^\infty(X))^*$.

This is Theorem 6.1.1 in the paper. Its proof uses the Irreducibility Theorem from [A2] in full generality. Roughly Theorem 0.1.1 can be interpreted as a selfduality property of the space of valuations (at least when the manifold X is compact).

Let us denote $V^{-\infty}(X) := (V_c^\infty(X))^*$. We call $V^{-\infty}(X)$ the space of generalized valuations. We show (Proposition 7.1.3) that $V^{-\infty}(X)$ has a canonical structure of $V^\infty(X)$-module.

In [A5] it was shown that the assignment to any open subset $U \subset X$

$$U \mapsto V^\infty(U)$$

with the natural restriction maps is a sheaf denoted by $\mathcal{V}_X^\infty$. Here we show that

$$U \mapsto V^{-\infty}(U)$$

with the natural restriction maps is also a sheaf which we denote by $\mathcal{V}_X^{-\infty}$. Moreover $\mathcal{V}_X^{-\infty}$ is a sheaf of $\mathcal{V}_X^\infty$-modules (Proposition 7.2.4).

Remind that by [A5] the last term W_n of the filtration (0.1.1) coincides with the space $C^\infty(X, |\omega_X|)$ of smooth densities on X (where $|\omega_X|$ denotes the line bundle of densities on X), and $V^\infty(X)/W_1$ is canonically isomorphic to the space of smooth functions $C^\infty(X)$. In Subsection 7.3 of this article we extend the filtration $\{W_\bullet\}$ to generalized valuations by taking the closure of W_i in the weak topology on $V^{-\infty}(X)$:

$$V^{-\infty}(X) = W_0\big(V^{-\infty}(X)\big) \supset W_1\big(V^{-\infty}(X)\big) \supset \cdots \supset W_n\big(V^{-\infty}(X)\big).$$

We show that $W_n(V^{-\infty}(X))$ is equal to the space $C^{-\infty}(X, |\omega_X|)$ of generalized densities on X (Proposition 7.3.5). It is also shown that $V^{-\infty}(X)/W_1(V^{-\infty}(X))$ is canonically isomorphic to the space $C^{-\infty}(X)$ of generalized valuations on X (Proposition 7.3.6).

The Euler–Verdier involution is extended by continuity in the weak topology to the space of generalized valuations (Subsection 7.4). Also the integration functional extends (uniquely) by continuity in an appropriate topology to generalized valuations with compact support (Subsection 7.4).

In Section 8 we consider valuations on a real analytic manifold X. On such a manifold one has the algebra of constructible functions $\mathcal{F}(X)$ which is a quite well known object (see [KS], Ch. 9). We construct a canonical imbedding of the space $\mathcal{F}(X)$ to the space of generalized valuations $V^{-\infty}(X)$ as a dense subspace. It turns out to be possible to interpret some properties of valuations in more familiar terms of constructible functions. Thus we show that the canonical filtration on $V^{-\infty}(X)$ induces on $\mathcal{F}(X)$ the filtration by codimension of support (Proposition 8.2.2). The restriction of the integration functional to the space of compactly supported constructible functions coincides with the well known functional of integration with respect to the

Euler characteristic (Proposition 8.3.1). The restriction of the Euler–Verdier involution on $V^{-\infty}(X)$ to $\mathcal{F}(X)$ coincides (up to a sign) with the well known Verdier duality operator (Proposition 8.4.1).

Acknowledgement. I express my gratitude J. Bernstein for numerous stimulating discussions. I thank V.D. Milman for his attention to this work. I thank A. Bernig for sharing with me the recent preprint [BB], J. Fu for very helpful explanations on the geometric measure theory, P.D. Milman for useful correspondences regarding subanalytic sets, and P. Schapira for useful discussions on constructible sheaves and functions.

1 Background

In this section we fix some notation and remind various known facts. This section does not contain new results.

In Subsection 1.1 we fix some notation and remind the notions of normal and characteristic cycles of *convex* sets. In Subsection 1.2 we review basic facts on subanalytic sets. Subsection 1.3 collects facts on normal and characteristic cycles. In Subsection 1.4 we review some notions on valuations on manifolds following mostly [A4], [A5], [AF]. Subsection 1.5 is also on valuations, and it reviews the canonical filtration on valuations following [A5].

1.1 Notation

Let V be a finite dimensional real vector space.

- Let $\mathcal{K}(V)$ denote the family of convex compact subsets of V.
- Let $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{>0}$) denote the set of non-negative (resp. positive) real numbers.
- For a manifold X let us denote by $|\omega_X|$ the line bundle of densities over X.
- For a smooth manifold X let $\mathcal{P}(X)$ denote the family of all simple subpolyhedra of X. (Namely $P \in \mathcal{P}(X)$ iff P is a compact subset of X locally diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^{n-k}_{\geq 0}$ for some $0 \leq k \leq n$. For a precise definition see [A5], Subsection 2.1.)
- We denote by $\mathbb{P}_+(V)$ the *oriented projectivization* of V. Namely $\mathbb{P}_+(V)$ is the manifold of oriented lines in V passing through the origin.
- For a vector bundle E over a manifold X let us denote by $\mathbb{P}_+(E)$ the bundle over X whose fiber over any point $x \in X$ is equal to $\mathbb{P}_+(E_x)$ (where E_x denotes the fiber of E over x).
- For a convex compact set $A \in \mathcal{K}(V)$ let us denote by h_A the *supporting functional* of A, $h_A \colon V^* \to \mathbb{R}$. It is defined by

$$h_A(y) := \sup \big\{ y(x) | x \in A \big\}.$$

- Let L denote the (real) line bundle over $\mathbb{P}_+(V^*)$ such that its fiber over an oriented line $l \in \mathbb{P}_+(V^*)$ is equal to the dual line l^*.
- For a smooth vector bundle E over a manifold X and k being a non-negative integer or infinity, let us denote by $C^k(X,E)$ the space of C^k-smooth sections of E. We denote by $C^k_c(X,E)$ the space of C^k-smooth sections with compact support. Let us denote by $C^{-\infty}(X,E)$ the space of generalized sections of E which is equal by definition to the dual space $(C^\infty_c(X, E^* \otimes |\omega_X|))^*$. We have the canonical imbedding $C^k(X,E) \hookrightarrow C^{-\infty}(X,E)$ (see e.g. [GuS], Ch. VI §1).

Let $K \in \mathcal{K}(V)$. Let $x \in K$.

Definition 1.1.1. *A tangent cone to K at x is the set denoted by T_xK which is equal to the closure of the set $\{y \in V | \exists \varepsilon > 0 \ x + \varepsilon y \in K\}$.*

It is easy to see that T_xK is a closed convex cone.

Definition 1.1.2. *A normal cone to K at x is the set*

$$(T_xK)^o := \big\{ y \in V^* |\ y(x) \geq 0 \forall x \in T_xK \big\}.$$

Thus $(T_xK)^o$ is also a closed convex cone.

Definition 1.1.3. *Let $K \in \mathcal{K}(V)$. The characteristic cycle of K is the set*

$$CC(K) := \cup_{x \in K} (T_xK)^o.$$

It is easy to see that $CC(K)$ is a closed n-dimensional subset of $T^*V = V \times V^*$ invariant with respect to the multiplication by non-negative numbers acting on the second factor. For some references on the characteristic and normal cycles of various sets see Remark 1.3.1 below.

1.2 Subanalytic Sets

In this subsection we review some basic facts from the theory of subanalytic sets of Hironaka. For more information see [Hi1], [Hi2], [H1], [H2], [BiM], [T], and §8.2 of [KS]. Let X be a real analytic manifold.

Definition 1.2.1. *Let Z be a subset of the manifold X. Z is called subanalytic at a point $x \in X$ if there exists an open neighborhood U of x, compact real analytic manifolds Y^i_j, $i = 1,2$, $j = 1, \dots, N$, and real analytic maps*

$$f^i_j \colon Y^i_j \to X$$

such that

$$Z \cap U = U \cap \cup_{j=1}^N \big(f^1_j(Y^1_j) \backslash f^2_j(Y^2_j) \big).$$

Z is called subanalytic in X if Z is subanalytic at every point of X.

Proposition 1.2.2. (i) *Let Z be a subanalytic subset of the manifold X. Then the closure and the interior of Z are subanalytic subsets.*
(ii) *The connected components of a subanalytic set are locally finite and subanalytic.*
(iii) *Let Z_1 and Z_2 be subanalytic subsets of the manifold X. Then $Z_1 \cup Z_2$, $Z_1 \cap Z_2$, and $Z_1 \backslash Z_2$ are subanalytic.*

Definition 1.2.3. *Let Z be a subanalytic subset of the manifold X. A point $x \in Z$ is called regular if there exists an open neighborhood U of x in X such that $U \cap Z$ is a submanifold of X.*

The set of regular points is denoted by Z_{reg}. Define the set of *singular* points by $Z_{\text{sing}} := Z \backslash Z_{\text{reg}}$.

Proposition 1.2.4. *The sets Z_{reg} and Z_{sing} are subanalytic, and $Z \subset \bar{Z}_{\text{reg}}$.*

If $x \in Z_{\text{reg}}$ then the dimension of Z at x is well defined; it is denoted by $\dim_x Z$. Define

$$\dim Z := \sup_{x \in Z_{\text{reg}}} \dim_x(Z).$$

Clearly $\dim Z \leq \dim X$.

Proposition 1.2.5. *Let $Z \subset X$ be a subanalytic subset. Then*

(i) $\dim(Z \backslash Z_{\text{reg}}) < \dim Z$*;*
(ii) $\dim(\bar{Z} \backslash Z) < \dim Z$.

Definition 1.2.6 ([KS], §9.7). *An integer valued function $f \colon X \to \mathbb{Z}$ is called constructible if*

1) *for any $m \in \mathbb{Z}$ the set $f^{-1}(m)$ is subanalytic;*
2) *the family of sets $\{f^{-1}(m)\}_{m \in \mathbb{Z}}$ is locally finite.*

Clearly the set of constructible $\mathbb{Z}$-valued functions is a ring with pointwise multiplication. As in [KS] we denote this ring by $CF(X)$. Define

$$\mathcal{F} := CF(X) \otimes_{\mathbb{Z}} \mathbb{C}. \qquad (1.2.1)$$

Thus $\mathcal{F}$ is a subalgebra of the $\mathbb{C}$-algebra of complex valued functions on X. In the rest of the article the elements of $\mathcal{F}$ will be called *constructible functions.*

Let $\mathcal{F}_c(X)$ denote the subspace of $\mathcal{F}(X)$ of *compactly supported* constructible functions. Clearly $\mathcal{F}_c(X)$ is a subalgebra of $\mathcal{F}(X)$ (without unit if X is non-compact).

For a subset $P \subset X$ let us denote by $\mathbb{1}_P$ the indicator function of P, namely

$$\mathbb{1}_P(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{if } x \notin P. \end{cases}$$

Proposition 1.2.7. (i) *Any function $f \in \mathcal{F}(X)$ can be presented locally as finite linear combination of functions of the form $1\!\!1_Q$ where Q is a closed subanalytic subset.*
(ii) *Any function $f \in \mathcal{F}_c(X)$ can be presented as finite linear combination of functions of the form $1\!\!1_Q$ where Q is a compact subanalytic subset.*

Proof. Both statements are proved similarly. Let prove say the second one. Let $f \in \mathcal{F}_c(X)$. We prove the statement by the induction on $\dim(\mathrm{supp}\, f)$ (note that $\mathrm{supp}\, f$ is a subanalytic subset). If $\dim(\mathrm{supp}\, f) = 0$ then there is nothing to prove. Let us assume that we have proven the results for all constructible functions with the dimension of support strictly less than k. Let us prove it for k. Clearly f is a finite linear combination of functions of the form $1\!\!1_Q$ where Q is relatively compact subanalytic subset with $\dim Q \leq k$. But

$$1\!\!1_Q = 1\!\!1_{\bar{Q}} - 1\!\!1_{\bar{Q}\backslash Q}.$$

By Proposition 1.2.2 the set $\bar{Q}\backslash Q$ is subanalytic, and by Proposition 1.2.5(ii) $\dim(\bar{Q}\backslash Q) < k$. The induction assumption implies the result. □

1.3 Characteristic and Normal Cycles

In Subsection 1.1 we have reminded the notion of characteristic cycle of convex compact sets. In this subsection we remind the notion of characteristic cycle and very similar notion of normal cycles of sets either from the class $\mathcal{P}(X)$ on a smooth manifold X, or the class of subanalytic subsets of a real analytic manifold X (in fact in the real analytic situation these notions will be discussed more generally for constructible functions on X following [KS]). The notions of characteristic and normal cycles of various classes of sets coincide on the pairwise intersections of these classes.

Remark 1.3.1. The notion of the characteristic cycle is not new. First an almost equivalent notion of normal cycle (see below) was introduced by Wintgen [W], and then studied further by Zähle [Z] by the tools of geometric measure theory. Characteristic cycles of subanalytic sets of real analytic manifolds were introduced by J. Fu [F2] using the tools of geometric measure theory and independently by Kashiwara (see [KS], Chapter 9) using the tools of the sheaf theory. J. Fu's article [F2] develops a more general approach to define the normal cycle for more general sets than subanalytic or convex ones (see Theorem 3.2 in [F2]). Applications of the method of normal cycles to integral geometry can be found in [F1].

For simplicity of the exposition, in the rest of this subsection we will assume that the manifold X is oriented. Then characteristic (resp. normal) cycle is a cycle in T^*X (resp. $\mathbb{P}_+(T^*X)$. Nevertheless the characteristic and normal cycles can be defined on non-oriented (even non-orientable) manifolds; then they are cycles taking values is the local system p^*o where o is the orientation bundle over X and $p\colon T^*X \to X$ is the canonical projection. We refer to [KS], §9.3, for the details on that. Though in our applications to valuations of these notions we will need the general case of not necessarily orientable manifolds, we will ignore here this subtlety. Thus here we discuss the notions of characteristic and normal cycles for oriented manifolds, but apply it below for general manifolds.

Let us assume first that X is a smooth oriented manifold. Set $n = \dim X$. Let $P \in \mathcal{P}(X)$. For any point $x \in P$ let us define the *tangent cone* to P at x, denoted by T_xP, the set

$$T_xP := \big\{\xi \in T_xX |\text{ there exists a } C^1 - \text{map } \gamma\colon [0,1] \to P \\ \text{such that } \gamma(0) = x \text{ and } \gamma'(0) = \xi\big\}.$$

It is easy to see that T_xP coincides with the usual tangent space if x is an interior point of P. In general T_xP is a closed convex polyhedral cone in T_xX. Define

$$CC(P) := \cup_{x\in P}(T_xP)^o \tag{1.3.1}$$

where for a convex cone C in a linear space W one denotes C^o its dual cone in W^*:

$$C^o := \{y \in W^* |\, y(x) \geq 0 \text{ for any } x \in C\}.$$

Clearly $CC(P)$ is invariant under the group $\mathbb{R}_{>0}$ of positive real numbers acting on the cotangent bundle T^*X by multiplication along the fibers. It is easy to see that $CC(P)$ is an n-dimensional Lagrangian submanifold of T^*X with singularities. A choice of orientation on X induces an orientation on $CC(P)$. Then $CC(P)$ becomes a cycle, i.e. $\partial(CC(P)) = 0$.

Let us assume now that X is a *real analytic* manifold. Again we assume that X is oriented. Let $CF(X)$ be the ring of integer valued constructible functions as in Definition 1.2.6, and let $\mathcal{F}$ denote the algebra of (complex valued) constructible functions as in (1.2.1).

In [KS], §9.7, there was constructed a group homomorphism, also called characteristic cycle,

$$CC\colon CF(X) \to \mathcal{L}(X)$$

where $\mathcal{L}(X)$ denotes the group of Lagrangian conic subanalytic cycles (with values in p^*o in the non-oriented case). For the formal definitions we refer to [KS], §§9.7, 9.2. Here we describe $\mathcal{L}(X)$ in a somewhat informal way when X is oriented. An arbitrary element $\lambda \in \mathcal{L}(X)$ is an n-cycle on T^*X (i.e. $\partial\lambda = 0$) which locally over X can be written as a finite sum $\lambda = \sum_j m_j[\Lambda_j]$ where m_j are integers, Λ_j are subanalytic oriented Lagrangian locally closed submanifolds of T^*X which are conic, i.e. invariant under the action of the group of positive real numbers $\mathbb{R}_{>0}$ on T^*X, and $[\Lambda_j]$ denotes the chain class of Λ_j.

Let us summarize some basic properties of CC which will be used later. First CC commutes with restrictions of functions to open subsets of X.

Let $P \subset X$ be a compact subanalytic subset. Assume in addition that $P \in \mathcal{P}(X)$. Then $CC(1\!\!1_P)$ coincides with the characteristic cycle $CC(P)$ defined above in (1.3.1). Thus for a subanalytic closed subset Q we will also denote by $CC(Q)$ the characteristic cycle $CC(1\!\!1_Q)$.

For a (locally closed) submanifold $S \subset X$ let us denote by $T^*_S X$ the conormal bundle of S. If S is subanalytic then $T^*_S X$ is a subanalytic subset of T^*X (Proposition 8.3.1 in [KS]).

Lemma 1.3.2. *Let $Q \subset X$ be a relatively compact subanalytic subset. Then the closure $\bar{Q}$ can be presented as a finite union $\bar{Q} = \cup_j Q_j$ of (locally closed) subanalytic submanifolds such that*

$$\text{supp}\ \big(CC(1\!\!1_Q)\big) \subset \cup_j T^*_{Q_j} X.$$

Proof. Using induction in $\dim Q$ and Propositions 1.2.4, 1.2.5 we may replace Q by Q_{reg} and thus assume that Q is a (locally closed) submanifold of X.

Let us consider the subanalytic covering $X = Q \bigsqcup (X \backslash Q)$. By Theorem 8.3.20 of [KS] there exists a μ-stratification $X = \bigsqcup_\beta X_\beta$ which is a refinement of the above covering (for the definition of μ-stratification see Definition 8.3.19 of [KS]).

Let us denote by $j \colon Q \to X$ the identity imbedding. Let $\underline{\mathbb{C}}_Q$ denote the constant sheaf on Q (with complex coefficients). Let $T_Q := j_! \underline{\mathbb{C}}_Q$ be the extension of $\underline{\mathbb{C}}_Q$ by zero. By the definition of the characteristic cycle ([KS], §9.7)

$$CC(1\!\!1_Q) = CC(T_Q) \tag{1.3.2}$$

where in the right hand side stays the characteristic cycle of the *sheaf* T_Q (see §9.4 of [KS]). Note that T_Q is obviously constructible with respect to the μ-stratification $\{X_\beta\}$. It follows from the definition of the characteristic cycle of a sheaf that

$$\text{supp}\ CC(T_Q) \subset SS(T_Q) \tag{1.3.3}$$

where $SS(\cdot)$ denotes the singular support of a sheaf (see §5.1 of [KS]). Proposition 8.4.1 of [KS] implies that $SS(T_Q) \subset \bigsqcup_\beta T^*_{X_\beta} X$. But since $T_Q|_{X \backslash \bar{Q}} = 0$ one has

$$SS(T_Q) \subset \bigsqcup_{\beta \colon X_\beta \subset \bar{Q}} T^*_{X_\beta} X. \tag{1.3.4}$$

Let us choose the covering $\bar{Q} = \cup_\alpha Q_\alpha$ where each Q_α is equal to one of the sets X_β contained in $\bar{Q}$. Thus (1.3.2)-(1.3.4) imply

$$CC(Q) \subset \cup_\alpha T^*_{Q_\alpha} X.$$

Lemma is proved. □

Let us remind the definition of a normal cycle. We will treat all the cases of subanalytic, convex, $\mathcal{P}(X)$-sets, and constructible functions simultaneously since in all these cases we already have the notion of characteristic cycle.

Let f be an element of one of these families. Let $CC(f)$ be its characteristic cycle. Let us denote by $\underline{CC}(f)$ the intersection of $CC(f)$ with the open subset of T^*X obtained by removing the zero section $\underline{0}$. Then $\underline{CC}(f)$ is an n-cycle in $T^*X\backslash\underline{0}$ invariant under the multiplication by positive real numbers. Let $q\colon T^*X\backslash\underline{0} \to \mathbb{P}_+(T^*X)$ denote the canonical quotient map. (Remind that $\mathbb{P}_+(T^*X)$ denotes the bundle over X whose fiber over a point $x \in X$ is equal to the manifold of oriented lines in T^*_xX passing through the origin.)

It is easy to see that there exists unique $(n-1)$-cycle in $\mathbb{P}_+(T^*X)$ denoted by $\tilde{CC}(f)$ such that $CC(f) = q^{-1}(\tilde{CC}(f))$. Consider the (antipodal) involution $a\colon \mathbb{P}_+(T^*X) \to \mathbb{P}_+(T^*X)$ changing the orientation of each line. Then by definition the normal cycle $N(f)$ is equal to $a_*(\tilde{CC}(f))$. It is easy to see that if $CC(f)$ is a subanalytic cycle then $N(f)$ is a subanalytic cycle, in particular if f is a constructible function then $N(f)$ is a subanalytic cycle. Also it is known that $N(f)$ is a Legendrian cycle when $\mathbb{P}_+(T^*X)$ is equipped with the canonical contact structure.

1.4 Some Valuation Theory

First let us remind some results from [A1]. Let V be an n-dimensional real vector space. Let $\bar{K} = (K_1, K_2, \dots, K_s)$ be an s-tuple of compact convex subsets of V. Let $r \in \mathbb{N} \cup \{\infty\}$. For any $\mu \in C^r(V, |\omega_V|)$ consider the function $M_{\bar{K}}F : \mathbb{R}^s_+ \to \mathbb{C}$, where $\mathbb{R}^s_+ = \{(\lambda_1, \dots, \lambda_s) \,|\, \lambda_j \geq 0\}$, defined by

$$(M_{\bar{K}}\mu)(\lambda_1, \dots, \lambda_s) = \mu\left(\sum_{i=1}^{s} \lambda_i K_i\right).$$

Theorem 1.4.1 ([A1]). (1) *$M_{\bar{K}}\mu \in C^r(\mathbb{R}^s_+)$ and $M_{\bar{K}}$ is a continuous operator from $C^r(V, |\omega_V|)$ to $C^r(\mathbb{R}^s_+)$.*
(2) *Assume that a sequence $\mu^{(m)}$ converges to μ in $C^r(V, |\omega_V|)$. Let $K_i^{(m)}$, K_i, $i = 1, \dots, s$, $m \in \mathbb{N}$, be convex compact sets in V, and for every $i = 1, \dots, s$ $K_i^{(m)} \to K_i$ in the Hausdorff metric as $m \to \infty$. Then $M_{\bar{K}^{(m)}}\mu^{(m)} \to M_{\bar{K}}\mu$ in $C^r(\mathbb{R}^s_+)$ as $m \to \infty$.*

Definition 1.4.2. a) *A function $\phi : \mathcal{K}(V) \to \mathbb{C}$ is called a valuation if for any K_1, $K_2 \in \mathcal{K}(V)$ such that their union is also convex one has*

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

b) *A valuation ϕ is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}(V)$.*

For the classical theory of valuations we refer to the surveys McMullen–Schneider [MS] and McMullen [M2]. For the general background from convexity we refer to Schneider [S].

In [A4] one has introduced a class $SV(V)$ of valuations called *smooth valuations*. We refer to [A4] for an axiomatic definition. Here we only mention that $SV(V)$ is a $\mathbb{C}$-linear space (with the obvious operations) with a natural Fréchet topology. In this article we will need a description of $SV(V)$ which is Theorem 1.4.3 below.

Let us denote by ${}^{\mathbf{C}}L$ the (complex) line bundle over $\mathbb{P}_+(V^*)$ whose fiber over $l \in \mathbb{P}_+(V^*)$ is equal to $l^* \otimes_{\mathbb{R}} \mathbb{C}$ (where l^* denotes the dual space to l).

Note that for any convex compact set $A \in \mathcal{K}(V)$ the supporting functional h_A is a continuous section of ${}^{\mathbf{C}}L$, i.e $h_A \in C(\mathbb{P}_+(V^*), {}^{\mathbf{C}}L)$.

Theorem 1.4.3 ([A4], Corollary 3.1.7). *There exists a continuous linear map*

$$\mathcal{T}\colon \oplus_{k=0}^n C^\infty\Big(V \times \mathbb{P}_+(V^*)^k, |\omega_V| \boxtimes {}^{\mathbf{C}}L^{\boxtimes k}\Big) \to SV(V)$$

which is uniquely characterized by the following property: for any $k=0,1,\dots,n$, any $\mu \in C^\infty(V, |\omega_V|)$, any strictly convex compact sets $A_1, \dots, A_k$ with smooth boundaries, and any $K \in \mathcal{K}(V)$ one has

$$\mathcal{T}(\mu \boxtimes h_{A_1} \boxtimes \cdots \boxtimes h_{A_k})(K) = \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \mu\Big(K + \sum_{i=1}^k \lambda_i A_i\Big)$$

where $\lambda_i \geq 0$ in the right hand side.

Moreover the map $\mathcal{T}$ is an epimorphism.

In [A5] one has introduced for any smooth manifold X a class of finitely additive measures on the family of simple subpolyhedra $\mathcal{P}(X)$. This class is denoted by $V^\infty(X)$. It is a $\mathbb{C}$-linear space (with the obvious operations). Then $V^\infty(X)$ has a natural Fréchet topology. Moreover in the case of linear Fréchet space V any element $\phi \in V^\infty(V)$ being restricted to $\mathcal{K}(V) \cap \mathcal{P}(V)$ has a (unique) extension by continuity in the Hausdorff metric to $\mathcal{K}(V)$, and this extension belongs to $SV(V)$. Thus one gets a linear map

$$V^\infty(V) \to SV(V).$$

In [A5], Proposition 2.4.10, the following result was proved.

Proposition 1.4.4. *The above constructed map $V^\infty(V) \to SV(V)$ is an isomorphism of Fréchet spaces.*

We will also need the following description of $V^\infty(X)$ obtained in [A5] (based on some results on normal cycles from Section 2 of [AF]). Let us denote by T^*X the cotangent bundle of X. Let $p\colon T^*X \to X$ be the canonical projection. Let Ω^n denote the vector bundle of n-forms over T^*X. Let us denote by o the orientation bundle over X. Let us denote by $\tilde{C}^\infty(T^*X, \Omega^n \otimes p^*o)$ the space of C^∞-smooth sections of the bundle $\Omega^n \otimes p^*o$ such that the restriction of the projection p to the support of such section is a proper map.

Theorem 1.4.5 ([A5], Theorem 0.1.3). (i) *Let $\omega \in \tilde{C}^\infty(T^*X, \Omega^n \otimes p^*o)$. The functional $\mathcal{P}(X) \to \mathbb{C}$*

$$P \mapsto \int_{CC(P)} \omega \tag{1.4.1}$$

is a smooth valuation.
(ii) *Conversely, any smooth valuation $\phi \in V^\infty(X)$ has the form (1.4.1), i.e. there exists a form $\omega \in \tilde{C}^\infty(T^*X, \Omega^n \otimes p^*o)$ such that $\phi(P) = \int_{CC(P)} \omega$ for any $P \in \mathcal{P}(X)$.*

Remark 1.4.6. (1) The integration (1.4.1) is well defined since a choice of orientation of the manifold X induces an orientation of $CC(P)$.
(2) A presentation of a valuation ϕ in the form (1.4.1) is highly non-unique.

Let us describe the multiplicative structure on $V^\infty(X)$ following [AF]. It was shown in [A5] that the assignment to any open subset $U \subset X$

$$U \mapsto V^\infty(U)$$

with the natural restriction maps is a sheaf. The product on smooth valuations commutes with the restrictions to open subsets. Hence it is enough to describe the product locally, say under the assumption that X is diffeomorphic to $\mathbb{R}^n$. Let us fix a diffeomorphism $X \tilde{\to} \mathbb{R}^n$. Proposition 1.4.4 provides an isomorphism $V^\infty(\mathbb{R}^n) \tilde{\to} SV(\mathbb{R}^n)$. In [A4] the author has described the product on $SV(\mathbb{R}^n)$ which we will remind below. The main point of [AF] was to show that the obtained product on $V^\infty(X)$ does not depend on the choices of diffeomorphisms.

Thus it remains to describe the product on $SV(\mathbb{R}^n)$ following [A4]. The product

$$SV(\mathbb{R}^n) \times SV(\mathbb{R}^n) \to SV(\mathbb{R}^n)$$

is a continuous map which is uniquely defined by the distributivity and the following property: let $\phi, \psi \in SV(\mathbb{R}^n)$ have the form

$$\phi(K) = \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \mu\Big(K + \sum_{i=1}^k \lambda_i A_i\Big), \tag{1.4.2}$$

$$\psi(K) = \frac{\partial^l}{\partial\mu_1 \dots \partial\mu_l}\Big|_0 \mu\Big(K + \sum_{j=1}^l \mu_j B_j\Big) \tag{1.4.3}$$

where $0 \leq k, l \leq n$; μ, ν are smooth densities on $\mathbb{R}^n$; $A_1, \dots, A_k, B_1, \dots, B_l$ are strictly convex compact sets with smooth boundaries, K is an arbitrary convex compact subset in $\mathbb{R}^n$. Then

$$(\phi\cdot\psi)(K)=\frac{\partial^{k+l}}{\partial\lambda_1\dots\partial\lambda_k\partial\mu_1\dots\partial\mu_l}\Big|_0(\mu\boxtimes\nu)\\ \left(\Delta(K)+\left(\left(\sum_{i=1}^k\lambda_iA_i\right)\times\left(\sum_{j=1}^l\mu_jB_j\right)\right)\right)\quad(1.4.4)$$

where $\Delta\colon\mathbb{R}^n\hookrightarrow\mathbb{R}^n\times\mathbb{R}^n$ is the diagonal imbedding, $\mu\boxtimes\nu$ denotes the usual exterior product of densities. Note that in (1.4.2)-(1.4.4) the derivatives exist due to Theorem 1.4.1.

Equipped with this product, the space $V^\infty(X)$ becomes a commutative associative algebra with unit (the unit is the Euler characteristic).

Let us describe the Euler–Verdier involution σ on $V^\infty(X)$ following [A5]. Let $a\colon T^*X\to T^*X$ be the involution of multiplication by -1 in each fiber of the projection $p\colon T^*X\to X$. It induces the involution

$$a^*\colon\tilde{C}^\infty(T^*X,\Omega^n\otimes p^*o)\to\tilde{C}^\infty(T^*X,\Omega^n\otimes p^*o).$$

We have the following proposition.

Proposition 1.4.7 ([A5], Proposition 3.3.1). *The involution $(-1)^na^*$ factorizes (uniquely) to the involution of $V^\infty(X)$ which is denoted by σ. Moreover σ commutes with the restrictions to open subsets and thus induces an involution of the sheaf $\mathcal{V}_X^\infty$ which is also denoted by σ.*

1.5 Filtration on Valuations

In [A5] we have introduced on the space of smooth valuations $V^\infty(X)$ a canonical finite filtration by closed subspaces:

$$V^\infty(X)=W_0(X)\supset W_1(X)\supset\dots\supset W_n(X)\qquad(1.5.1)$$

where $n=\dim X$. Let us remind some of the main properties of this filtration.

Proposition 1.5.1 ([A5], Proposition 3.1.2). *The assignment to each open subset $U\subset X$*

$$U\mapsto W_i(U)$$

is a subsheaf of $\mathcal{V}_X^\infty$. (This sheaf is denoted by $\mathcal{W}_i$.)

It turns out that the associated graded sheaf $gr_{\mathcal{W}}\mathcal{V}_X^\infty:=\bigoplus_{i=0}^n\mathcal{W}_i/\mathcal{W}_{i+1}$ admits a simple description in terms of translation invariant valuations. To state it let us denote by $\mathrm{Val}(TX)$ the (infinite dimensional) vector bundle over X such that its fiber over a point $x\in X$ is equal to the space $\mathrm{Val}^{\mathrm{sm}}(T_xX)$ of smooth translation invariant valuations on the tangent space T_xX. By McMullen's theorem [M1] the space $\mathrm{Val}^{\mathrm{sm}}(T_xX)$ has natural grading by the degree of homogeneity which must be an integer between 0 and n. Thus $\mathrm{Val}(TX)$ is a graded vector bundle. Let us denote by $\underline{\mathrm{Val}}(TX)$ the sheaf $U\mapsto C^\infty(U,\mathrm{Val}(TX))$ where the last space denotes the space of infinitely smooth sections of $\mathrm{Val}(TX)$ over U.

Theorem 1.5.2 ([A5], Theorem 0.1.2 and Section 3). *There exists a canonical isomorphism of graded sheaves*

$$gr_{\mathcal{W}}\mathcal{V}_X^\infty \simeq \underline{\mathrm{Val}}(TX).$$

Moreover for any open subset $U \subset X$ the induced isomorphism on global sections is isomorphism of linear topological spaces.

This theorem provides a description of smooth valuations since translation invariant valuations are studied much better.

Remark 1.5.3. Interpreted appropriately, Theorem 1.5.2 says in particular that the last term of the filtration $\mathcal{W}_n$ is canonically isomorphic to the sheaf of C^∞-smooth measures (=densities) on X, and the first quotient $\mathcal{V}_X^\infty/\mathcal{W}_1$ is canonically isomorphic to the sheaf of C^∞-smooth functions on X.

The filtration $\{W_\bullet\}$ on valuations can be interpreted in terms of Theorem 1.4.5 as follows. First remind the general construction of a filtration differential forms on a total space of a bundle.

Let X be a smooth manifold. Let $p : P \to X$ be a smooth bundle. Let $\Omega^N(P)$ be the vector bundle over P of N-forms. For a vector space R we denote by $Gr_N(R)$ the Grassmannian of N-dimensional linear subspaces in R. Let us introduce a filtration of $\Omega^N(P)$ by vector subbundles $W_i(P)$ as follows. For every $y \in P$ set

$$\big(W_i(P)\big)_y := \Big\{\omega \in \wedge^N T_y^*P \big|\, \omega|_F \equiv 0 \text{ for all } F \in Gr_N(T_yP) \text{ with } \dim\big(F \cap T_y\big(p^{-1}p(y)\big)\big) > N-i\Big\}.$$

Clearly we have

$$\Omega^N(P) = W_0(P) \supset W_1(P) \supset \cdots \supset W_N(P) \supset W_{N+1}(P) = 0.$$

Let us discuss this filtration in greater detail following [A4].

Let us make some elementary observations from linear algebra. Let L be a finite dimensional vector space. Let $E \subset L$ be a linear subspace. For a non-negative integer i set

$$W(L,E)_i := \big\{\omega \in \wedge^N L^* \big|\, \omega|_F \equiv 0 \text{ for all } F \subset L \text{ with } \dim(F \cap E) > N-i\big\}.$$

Clearly

$$\wedge^N L^* = W(L,E)_0 \supset W(L,E)_1 \supset \cdots \supset W(L,E)_N \supset W(L,E)_{N+1} = 0.$$

Lemma 1.5.4 ([A4], Lemma 5.2.3). *There exists canonical isomorphism of vector spaces*

$$W(L,E)_i/W(L,E)_{i+1} = \wedge^{N-i}E^* \otimes \wedge^i (L/E)^*.$$

Let us apply this construction in the context of integration with respect to the characteristic cycle. Let X be a smooth manifold of dimension n. Let $P := T^*X$ be the cotangent bundle. Let $p : P \to X$ be the canonical projection. Let us denote by o the orientation bundle on X. The above construction gives a filtration of $\Omega^n(P)$ by subbundles

$$\Omega^n(P) = W_0\big(\Omega^n(P)\big) \supset \cdots \supset W_n\big(\Omega^n(P)\big).$$

Twisting this filtration by p^*o we get a filtration of $\Omega^n(P) \otimes p^*o$ by subbundles denoted by $W_i(\Omega^n(P) \otimes p^*o)$.

Let us denote by $\tilde{C}^\infty(P, W_i(\Omega^n \otimes p^*o))$ the space of infinitely smooth sections of the bundle $W_i(\Omega^n \otimes p^*o)$ such that the restriction of the projection p to the support of these sections is proper. The we have the following result.

Theorem 1.5.5 ([A5], Proposition 3.1.9). *For any valuation $\phi \in W_i(X)$ there exists $\omega \in \tilde{C}^\infty(T^*X, W_i(\Omega^n \otimes p^*o))$ such that for any $P \in \mathcal{P}(X)$*

$$\phi(P) = \int_{CC(P)} \omega.$$

Conversely any such valuation belongs to $W_i(X)$.

2 A Technical Lemma

In this section we will prove a technical lemma which will be used later on in this article.

Lemma 2.1.1. *Let $i = 0, 1, \ldots, n$. Let $\phi \in W_i(V_c^\infty(X))$. Then there exists a compactly supported form $\omega \in C_c^\infty(T^*X, W_i(\Omega^n \otimes p^*o))$ such that*

$$\phi(P) = \int_{CC(P)} \omega \quad \text{for any} \quad P \in \mathcal{P}(X).$$

Remark 2.1.2. A version of this lemma for smooth valuations without the assumption on the compactness of support was proved in [A5], Proposition 3.1.9; it will be used in the proof of Lemma 2.

Proof of Lemma 2. As in [A5] consider the sheaves on X

$$\mathcal{W}_i(U) = W_i\big(V^\infty(U)\big), \tag{2.1.1}$$
$$\mathcal{W}'_i(U) = \tilde{C}^\infty(T^*U, W_i\big(\Omega^n \otimes p^*o)\big) \tag{2.1.2}$$

for any open subset $U \subset X$; in equality (2.1.2) the symbol $\tilde{C}^\infty$ denotes the space of infinitely smooth sections of a vector bundle over T^*U such that the restriction of the canonical projection $p\colon T^*U \to U$ to the support of such sections is proper. We have the obvious inclusions:

$$\mathcal{W}'_n \subset \mathcal{W}'_{n-1} \subset \cdots \subset \mathcal{W}'_0;$$
$$\mathcal{W}_n \subset \mathcal{W}_{n-1} \subset \cdots \subset \mathcal{W}_0 = \mathcal{V}_X^\infty.$$

The integration over the the characteristic cycle gives a morphism of sheaves

$$T_i \colon \mathcal{W}'_i \to \mathcal{W}_i. \tag{2.1.3}$$

By Proposition 3.1.9 of [A5] T_i is an epimorphism of sheaves. Clearly the restriction of T_i to $\mathcal{W}'_{i+1}$ is equal to T_{i+1}. Define the sheaves

$$\mathcal{K}_i := \operatorname{Ker} T_i. \tag{2.1.4}$$

We obviously have

$$\mathcal{K}_n \subset \mathcal{K}_{n-1} \subset \cdots \subset \mathcal{K}_0 \subset \mathcal{W}'_0.$$

Let us consider the associated graded sheaves

$$\mathcal{F} := \oplus_{i=o}^n \mathcal{W}_i/\mathcal{W}_{i+1}, \tag{2.1.5}$$
$$\mathcal{F}' := \oplus_{i=0}^n \mathcal{W}'_i/\mathcal{W}'_{i+1}. \tag{2.1.6}$$

The epimorphism $T_0 \colon \mathcal{W}'_0 \twoheadrightarrow \mathcal{W}_0$ induces the epimorphism

$$T \colon \mathcal{F}' \twoheadrightarrow \mathcal{F}. \tag{2.1.7}$$

Define $\mathcal{T} := \operatorname{Ker} T$. Clearly

$$\mathcal{T} = \oplus_{i=0}^n \mathcal{K}_i/\mathcal{K}_{i+1}. \tag{2.1.8}$$

Let us denote by $\mathcal{O}_X$ the sheaf of C^∞-smooth functions on X. It was shown in [A5] (see the proof of Proposition 3.1.9) that $\mathcal{T}$ is naturally isomorphic to a sheaf of $\mathcal{O}_X$-modules. Hence by Section 3.7 of Ch. II in Godement's book [G] one has

$$H_c^j(X, \mathcal{T}) = 0 \text{ for } j > 0.$$

Hence $H_c^j(X, \mathcal{K}_i/\mathcal{K}_{i+1}) = 0$ for $j > 0$, $i = 0, 1, \ldots, n$. By the long exact sequence we deduce

$$H_c^j(X, \mathcal{K}_i) = 0 \text{ for } j > 0,\ i = 0, 1, \ldots, n. \tag{2.1.9}$$

For the short exact sequence of sheaves

$$0 \to \mathcal{K}_i \to \mathcal{W}'_i \overset{T_i}{\to} \mathcal{W}_i \to 0$$

consider the beginning of the long exact sequence in cohomology with compact support

$$H^0_c(X,\mathcal{W}'_i)\to H^0_c(X,\mathcal{W}_i)\to H^1_c(X,\mathcal{K}_i). \tag{2.1.10}$$

But the last space in (2.1.10) vanishes due to (2.1.9). Hence the map $H^0_c(X,\mathcal{W}'_i)\to H^0_c(X,\mathcal{W}_i)$ is surjective. But

$$H^0_c(X,\mathcal{W}_i)=W_i\big(V^\infty_c(X)\big);$$
$$H^0_c(X,\mathcal{W}'_i)=C^\infty_0(T^*X,\Omega^n\otimes p^*o).$$

Thus lemma is proved. □

3 Compatibility of the Filtration with the Product

The main results of this section are Theorems 3.1.1 and 3.1.2 below.

Recall that in Subsection 1.5 we have discussed the canonical filtration by closed subspaces

$$V^\infty(X)=W_0(X)\supset W_1(X)\supset\dots\supset W_n(X).$$

It will be convenient to extend this filtration infinitely by putting

$$W_i(X)=0 \text{ for } i>n.$$

Theorem 3.1.1. *For any $i,j\geq 0$ one has*

$$W_i(X)\cdot W_j(X)\subset W_{i+j}(X).$$

Proof. By Corollary 4.1.4 of [AF] $\mathcal{V}^\infty_X$ is a sheaf of algebras, i.e. the product commutes with the restriction to open subsets. Hence we may assume that X is diffeomorphic to $\mathbb{R}^n$. Let us fix a diffeomorphism $X\tilde{\to}\mathbb{R}^n$. Let us consider the induced isomorphism of linear topological spaces

$$V^\infty(X)\tilde{\to}SV(\mathbb{R}^n)$$

from Proposition 1.4.4. By Proposition 3.1.3 of [A5] the subspace $W_i(X)$ is isomorphic under this isomorphism to a closed subspace of $SV(\mathbb{R}^n)$. Let us denote this subspace by $\hat{W}_i$; it was explicitly defined in Section 3 of [A4] in slightly different notation. Moreover by Theorem 4.1.2(4) of [A4]

$$\hat{W}_i\cdot\hat{W}_j\subset\hat{W}_{i+j}. \tag{3.1.1}$$

Hence our result follows from (3.1.1) and the construction of the product on $V^\infty(X)$ described in Subsection 1.4. □

Recall that by Theorem 1.5.2 there exists a canonical isomorphism of graded linear topological spaces

$$gr_W V^\infty(X):=\oplus_{i=0}^n W_i(X)/W_{i+1}(X)\tilde{\to}C^\infty\big(X,\mathrm{Val}(TX)\big) \tag{3.1.2}$$

where the vector bundle $\mathrm{Val}(TX)$ over X was defined in Subsection 1.5.

Observe that $gr_W V^\infty(X)$ is a graded algebra with the product induced from $V^\infty(X)$. Note also that $C^\infty(X, \mathrm{Val}(TX))$ is also a graded algebra with the product defined pointwise. Namely if $f, g \in C^\infty(X, \mathrm{Val}(TX))$ then for any point $x \in X$

$$(f \cdot g)(x) = f(x) \cdot g(x) \in \mathrm{Val}^{sm}(T_x X).$$

We are going to prove

Theorem 3.1.2. *The isomorphism (3.1.2) is an isomorphism of algebras.*

Proof. As in the proof of Theorem 3.1.1, the statement is local. Thus we may assume that X is diffeomorphic to $\mathbb{R}^n$. Now the result follows from the construction of the product described in Subsection 1.4 and Theorem 4.1.3 of [A4] where the corresponding statement was proved for valuations on convex subsets of a linear space. □

4 The Automorphism Property of the Euler–Verdier Involution.

The main result of this section is Theorem 4.1.4.

Lemma 4.1.1. *Let $\phi \in V^\infty(X)$. Let $P \in \mathcal{P}(X)$. Then*

$$(\sigma\phi)(P) = (-1)^{\dim P}\big(\phi(P) - \phi(\partial P)\big).$$

Proof. Equality (15) in [A5] says that for any $\omega \in \tilde{C}^\infty(T^*X, \Omega^n \otimes p^*o)$ and any $P \in \mathcal{P}(X)$ one has

$$\int_{CC(P)} a^*\omega = (-1)^{n-\dim P}\left(\int_{CC(P)} \omega - \int_{CC(\partial P)} \omega\right) \qquad (4.1.1)$$

where $\partial P := P \backslash \mathrm{int} P$ and $\mathrm{int} P$ denotes the relative interior of P. The result follows immediately from Proposition 1.4.7 and (4.1.1). □

From Lemma 4.1.1 we immediately deduce that the Euler–Verdier involution commutes with restriction to submanifolds. More precisely we have the following lemma.

Lemma 4.1.2. *Let Y be a smooth submanifold of a manifold X. Let $\phi \in V^\infty(X)$. Then*

$$(\sigma\phi)|_Y = \sigma(\phi|_Y).$$

Lemma 4.1.3. *Let $\phi \in V^\infty(\mathbb{R}^n)$ be a smooth valuation such that for any $K \in \mathcal{K}(\mathbb{R}^n)$ one has*

$$\phi(K) = \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \mu\left(K + \sum_{i=1}^k \lambda_i A_i\right)$$

where μ is a smooth density on $\mathbb{R}^n$, and $A_1, \dots, A_k$ are strictly convex compact subsets with smooth boundaries and containing the origin in the interior. Then

$$(\sigma\phi)(K) = (-1)^{n-k} \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \mu\Big(K + \sum_{i=1}^k \lambda_i(-A_i)\Big). \tag{4.1.2}$$

Proof. For $k = 0$ the lemma is obvious. Let us assume that $k > 0$. It is enough to prove (4.1.2) under the assumption that K has non-empty interior and strictly convex smooth boundary. For any $\lambda_1, \dots, \lambda_k > 0$ the map

$$\Xi_{\lambda_1 \dots \lambda_k} \colon V \times \mathbb{P}_+(V^*) \times (0,1] \to V$$

given by $(p, n, t) \mapsto p + t\sum_{i=1}^k \lambda_i \nabla h_{A_i}(n)$ induces a homeomorphism of $N(K) \times (0,1]$ onto its image $(K + \sum_{i=1}^k \lambda_i A_i)\backslash K$ (this is well known; see e.g. Proposition 3.1.2 of [AF] where this statement is proved under some more general assumptions). Hence

$$\phi(K) = \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \int_{N(K)\times[0,1]} \Xi^*_{\lambda_1 \dots \lambda_k}\mu.$$

Let us denote by $\tilde{a} \colon \mathbb{P}_+(V^*) \to \mathbb{P}_+(V^*)$ the involution of changing an orientation of a line. Then

$$\begin{aligned}(\sigma\phi)(K) &= (-1)^n \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \tilde{a}^*(\Xi^*_{\lambda_1 \dots \lambda_k}\mu) \\ &= (-1)^n \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 ((\Xi_{\lambda_1 \dots \lambda_k} \circ \tilde{a})^*\mu).\end{aligned} \tag{4.1.3}$$

Observe that

$$(\Xi_{\lambda_1 \dots \lambda_k} \circ a)(p, n, t) = p + t\sum_{i=1}^k \lambda_i (\nabla h_{A_i})(-n).$$

But $h_{-A}(n) = h_A(-n)$. Hence

$$(\Xi_{\lambda_1 \dots \lambda_k} \circ a)(p, n, t) = p - t\sum_{i=1}^k \lambda_i (\nabla h_{-A_i})(n). \tag{4.1.4}$$

Note that

$$\begin{aligned}&\frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \mu(K + \sum_{i=1}^k \lambda_i\big(-A_i)\big) \\ &\qquad = \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \int_{N(K)\times[0,1]} \tilde{\Xi}_{\lambda_1 \dots \lambda_k}\mu\end{aligned} \tag{4.1.5}$$

where $\tilde{\Xi}_{\lambda_1 \dots \lambda_k} \colon V \times \mathbb{P}_+(V^*) \times [0,1] \to V$ is defined by $\tilde{\Xi}_{\lambda_1 \dots \lambda_k}(p, n, t) = p + t\sum_{i=1}^k \lambda_i \nabla h_{-A_i}(n)$. Now Lemma 4.1.3 follows from (4.1.3),(4.1.4),(4.1.5). □

Theorem 4.1.4. *The Euler–Verdier involution* $\sigma\colon \mathcal{V}_X^\infty \to \mathcal{V}_X^\infty$ *is an algebra automorphism. Moreover it preserves the filtration* $\mathcal{W}_\bullet$*, namely* $\sigma(\mathcal{W}_i) = \mathcal{W}_i$ *for any* $i = 0, \dots, n$.

Proof. The second part of the theorem was proved in [A5]. Thus it remains to show that σ is an algebra automorphism. The statement is local thus we may and will assume that $X = \mathbb{R}^n$. Let $\phi, \psi \in V^\infty(\mathbb{R}^n)$. We may assume that for any $K \in \mathcal{K}(\mathbb{R}^n)$

$$\phi(K) = \frac{d^k}{d\varepsilon^k}\Big|_0 \mu(K + \varepsilon A),\ \psi(K) = \frac{d^l}{d\delta^l}\Big|_0 \nu(K + \delta B)$$

where μ, ν are smooth densities on $\mathbb{R}^n$, and A, B are strictly convex compact subsets with smooth boundaries and containing the origin the interior. Then

$$(\phi \cdot \psi)(K) = \frac{\partial^{k+l}}{\partial^k \varepsilon \cdot \partial^l \delta}\Big|_0 (\mu \boxtimes \nu)\big(\Delta(K) + (\varepsilon A, \delta B)\big)$$

where $\Delta\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$ is the diagonal imbedding. By Lemma 4.1.3 one has

$$\begin{aligned}(\sigma\phi)(K) &= (-1)^{n-k} \frac{d^k}{d\varepsilon^k}\Big|_0 \mu\big(K + \varepsilon(-A)\big),\ (\sigma\psi)(K) \\ &= (-1)^{n-l} \frac{d^l}{d\delta^l}\Big|_0 \nu\big(K + \delta(-B)\big) \\ (\sigma(\phi \cdot \psi))(K) &= (-1)^{2n-(k+l)} \frac{\partial^{k+l}}{\partial^k \varepsilon \cdot \partial^l \delta}\Big|_0 (\mu \boxtimes \nu)\Big(\Delta(K) + \big(\varepsilon(-A), \delta(-B)\big)\Big).\end{aligned}$$

Hence we have

$$\begin{aligned}(\sigma\phi \cdot \sigma\psi)(K) &= (-1)^{2n-(k+l)} \frac{\partial^{k+l}}{\partial^k \varepsilon \cdot \partial^l \delta}\Big|_0 (\mu \boxtimes \nu)\Big(\Delta(K) + \big(\varepsilon(-A), \delta(-B)\big)\Big) \\ &= \big(\sigma(\phi \cdot \psi)\big)(K).\end{aligned}$$

□

5 The Integration Functional on Valuations

In Subsection 5.1 we describe canonical linear topology on the space $V_c^\infty(X)$ of compactly supported smooth valuations. In Subsection 5.2 we construct a canonical continuous linear functional $V_c^\infty(X) \to \mathbb{C}$ called the *integration functional*.

5.1 Valuations with Compact Support

In this subsection we introduce the space of valuations $V_c^\infty(X)$ with compact support and establish some of the simplest properties of it.

Let $\phi \in V^\infty(X)$. We say that a point $x \in X$ does not belong to the support of ϕ if there exists a neighborhood U of x such that $\phi|_U \equiv 0$. The set of all points which does not belong to support of ϕ is an open subset of X. Its complement is called the support of ϕ and is denoted by supp ϕ. Thus supp ϕ is a closed subset of X. The following lemma is obvious.

Lemma 5.1.1. *For any $\phi, \psi \in V^\infty(X)$*

$$\mathrm{supp}\ (\phi \cdot \psi) \subset \mathrm{supp}\ \ \phi \cap \mathrm{supp}\ \ \psi.$$

The space of all valuations with compact support will be denoted by $V_c^\infty(X)$. Also for any subset $S \subset X$ let us denote

$$V_S^\infty(X) := \big\{ \phi \in V^\infty(X) \,|\, \mathrm{supp}\ \ \phi \subset S \big\}.$$

By Lemma 5.1.1 $V_S^\infty(X)$ is a subalgebra of $V^\infty(X)$ (without unit, unless $S = X$). If S is closed then $V_S^\infty(X)$ is a closed subalgebra in $V^\infty(X)$. Also

$$V_c^\infty(X) = \bigcup_{S \text{ compact}} V_S^\infty(X) = \varinjlim_{S \text{ compact}} V_S^\infty(X). \tag{5.1.1}$$

Let us equip $V_c^\infty(X) = \varinjlim_{S \text{ compact}} V_S^\infty(X)$ with the linear topology of inductive limit when each space $V_S^\infty(X)$ is equipped with the topology induced from $V^\infty(X)$. It is easy to see that $V_c^\infty(X)$ is a locally convex Hausdorff linear topological space. The identical imbedding $V_c^\infty(X) \hookrightarrow V^\infty(X)$ is continuous.

For any subset $S \subset X$ let us denote

$$W_{i,S} := V_S^\infty(X) \cap W_i(X),$$
$$W_{i,c} := V_c^\infty(X) \cap W_i(X).$$

If S is closed then $W_{i,S} \subset W_i(X)$ is a closed subspace. We will need the following lemma.

Lemma 5.1.2. *Let S be a closed subset of X. Then for any $j = 0, \dots, n$*

$$H_S^i(X, \mathcal{W}_j) = 0 \textit{ for } i > 0$$

where H_S^i denotes the i-th cohomology with support in S.

Proof. The sheaf $\mathcal{W}_j$ has the descending filtration

$$\mathcal{W}_j \supset \mathcal{W}_{j+1} \supset \dots \supset \mathcal{W}_n.$$

It is enough to show that for any p $H_S^i(X, \mathcal{W}_p/\mathcal{W}_{p+1}) = 0$ for $i > 0$. Let us denote by $\mathcal{O}_X$ the sheaf of C^∞-smooth functions on X. Then $\mathcal{W}_p/\mathcal{W}_{p+1}$ is a sheaf of $\mathcal{O}_X$-modules. It is well known (see e.g. [G], Section 3.7 of Ch. II) that on any smooth manifold X, for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, and any closed subset $S \subset X$ one has

$$H_S^i(X, \mathcal{F}) = 0 \text{ for } i > 0.$$

This implies the lemma. ◻

Lemma 5.1.3. (1) *For any closed subset $S \subset X$ the canonical isomorphism*

$$W_i/W_{i+1} \tilde{\to} C^\infty\big(X, \mathrm{Val}_i^{\mathrm{sm}}(TX)\big) \tag{5.1.2}$$

induces isomorphism

$$W_{i,S}/W_{i+1,S} \tilde{\to} C_S^\infty\big(X, \mathrm{Val}_i^{\mathrm{sm}}(TX)\big)$$

where C_S^∞ stays for the space of infinitely smooth sections with support in S.
(2) *Similarly the isomorphism (5.1.2) indices isomorphism*

$$W_{i,c}/W_{i+1,c} \tilde{\to} C_c^\infty\big(X, \mathrm{Val}_i^{\mathrm{sm}}(TX)\big).$$

Proof. Part (2) follows from part (1) by passing to direct limit. Thus let us prove part (1). Equality (3.1.2) implies that we have a short exact sequence of sheaves on X

$$0 \to \mathcal{W}_{i+1} \to \mathcal{W}_i \to \underline{\mathrm{Val}}_i(TX) \to 0.$$

Taking sections with the support in S we obtain the following exact sequence

$$0 \to W_{i+1,S} \to W_{i,S} \to C_S^\infty\big(X, \mathrm{Val}_i^{\mathrm{sm}}(TX)\big) \to H_S^1(X, \mathcal{W}_{i+1}).$$

But by Lemma 5.1.2 $H_S^1(X, \mathcal{W}_{i+1}) = 0$. Hence the result follows. □

5.2 The Integration Functional

In this subsection we are going to introduce a canonical linear functional

$$\int : V_c^\infty(X) \to \mathbb{C}$$

which we call the *integration functional*. With slight oversimplification

$$\int \phi = \phi(X)$$

for any $\phi \in V_c^\infty(X)$. This definition is formally correct if X is compact. Otherwise $X \notin \mathcal{P}(X)$, and the above definition requires an explanation.

Let us construct the integration functional formally for general manifold X. First fix a compact subset $S \subset X$. Let us choose a compact subset S' with smooth boundary and such that S in contained in the interior of S'. Then $S' \in \mathcal{P}(X)$. For any $\phi \in V_S^\infty(X)$ define

$$^S\!\!\int \phi := \phi(S'). \tag{5.2.1}$$

Lemma 5.2.1. (1) $^S\!\int : V_S^\infty(X) \to \mathbb{C}$ *is a continuous linear functional.*
(2) *For fixed S, the right hand side in (5.2.1) is independent of S' containing S.*

Proof. Part (1) is obvious. Let us prove part (2). Let S'' be another compact subset with smooth boundary containing S in the interior. We have to show that $\phi(S') = \phi(S'')$. Choosing a larger subset if necessary one may assume that S' is contained in the interior of S''. Then

$$\phi(S'') = \phi(S') + \phi\big(\overline{S''\backslash S'}\big) - \phi(\partial S') = \phi(S')$$

where the last equality is due to the fact that supp $\phi \subset S \subset \mathrm{int} S'$. □

As in the proof of Lemma 5.2.1 it is easy to see that if $S_1 \subset S_2$ then the restriction of ${}^{S_2}\!\int$ to $V^\infty_{S_1}(X)$ is equal to ${}^{S_1}\!\int$. Thus we obtain a *continuous* linear functional

$$\int : V^\infty_c(X) \to \mathbb{C}.$$

Remark 5.2.2. The space of smooth compactly supported densities is a subspace of $V^\infty_c(X)$; it is equal to $W_{n,c}$. The restriction of the above constructed integration functional $\int$ to this subspace coincides with the usual integration of densities.

6 The Selfduality Property of Valuations

The goal of this section is to establish the selfduality property of valuations (Theorem 6.1.1, Subsection 6.1). Subsection 6.2 contains a technical result on partition of unity in valuations.

6.1 The Selfduality Property

Probably the most interesting property of the multiplicative structure on valuations is Theorem 6.1.1 below. Its prove heavily uses the Irreducibility Theorem for translation invariant valuations from [A2].

Theorem 6.1.1. *Consider the bilinear form*

$$V^\infty(X) \times V^\infty_c(X) \to \mathbb{C}$$

given by $(\phi, \psi) \mapsto \int \phi \cdot \psi$.

This bilinear form is a perfect pairing. More precisely the induced map

$$V^\infty(X) \to \big(V^\infty_c(X)\big)^*$$

is injective and has a dense image with respect to the weak topology on $(V^\infty_c(X))^*$.

Theorem 6.1.1 follows from the next more precise statement by application of the Hahn–Banach theorem.

Theorem 6.1.2. (1) *For any $\phi \in W_i \backslash W_{i+1}$ there exists $\psi \in W_{n-i,c}$ such that* $\int \phi \cdot \psi \neq 0$.
(2) *Similarly for any $\phi \in W_{i,c} \backslash W_{i+1,c}$ there exists $\psi \in W_{n-i}$ such that* $\int \phi \cdot \psi \neq 0$.

Proof. The proves of these two statements are very similar. Thus let us prove only the first one. Let $\phi \in W_i \backslash W_{i+1}$. Let us denote by $\tilde{\phi}$ the image of ϕ is $W_i/W_{i+1} = C^\infty(X, \mathrm{Val}_i^{\mathrm{sm}}(TX))$. Thus $\tilde{\phi} \neq 0$. We will show that there exists $\psi \in W_{n-i,c}$ such that $\int \phi \cdot \psi \neq 0$. Since $W_{i+1} \cdot W_{n-i} = 0$ and $W_i \cdot W_{n-i+1} = 0$, the product $\phi \cdot \psi$ depends only on $\tilde{\phi}$ and on the image $\tilde{\psi}$ of ψ in $W_{n-i,c}/W_{n-i+1,c} = C_c^\infty(X, \mathrm{Val}_{n-i}^{\mathrm{sm}}(TX))$ (where the last equality is due to Lemma 5.1.3(2).

Thus it is enough to show that for any $\tilde{\phi} \in C^\infty(X, \mathrm{Val}_i^{\mathrm{sm}}(TX))$ there exists $\tilde{\psi} \in C_c^\infty(X, \mathrm{Val}_{n-i}^{\mathrm{sm}}(TX))$ such that

$$\int_X \tilde{\phi} \cdot \tilde{\psi} \neq 0$$

where the product $\tilde{\phi} \cdot \tilde{\psi}$ is understood pointwise in the tangent space of each point, $\tilde{\phi} \cdot \tilde{\psi} \in C_c^\infty(X, \mathrm{Val}_n^{\mathrm{sm}}(TX)) = C_c^\infty(X, |\omega_X|)$, and the integration is understood in the sense of the usual integration of densities.

Let us fix a point $x_0 \in X$ such that $\tilde{\phi}(x_0) \neq 0$. By the Poincaré duality for the translation invariant valuations (Theorem 0.8 in [A3]) there exists $\xi_0 \in \mathrm{Val}_{n-i}^{\mathrm{sm}}(T_{x_0}X)$ such that $\tilde{\phi}(x_0) \cdot \xi_0 \neq 0$. Let $\xi \in C^\infty(X, \mathrm{Val}_{n-i}^{\mathrm{sm}}(TX))$ be a section such that $\xi(x_0) = \xi_0$.

Consider the following C^∞-smooth density on X

$$\tau := \tilde{\phi} \cdot \xi.$$

Thus $\tau(x_0) \neq 0$. Hence we can find a smooth compactly supported function $\delta \in C_c^\infty(X)$ such that $\int_X \tau \cdot \delta \neq 0$. Take $\tilde{\psi} := \xi \cdot \delta$. Then

$$\int \tilde{\phi} \cdot \tilde{\psi} = \int_X \tau \cdot \delta \neq 0. \qquad \square$$

From Theorem 6.1.2 we immediately deduce the following corollary.

Corollary 6.1.3.

$$W_i = \Big\{ \phi \in V^\infty(X) \,\Big|\, \int \phi \cdot \psi = 0 \quad \textit{for any} \quad \psi \in W_{n-i+1,c} \Big\},$$
$$W_{i,c} = \Big\{ \phi \in V_c^\infty(X) \,\Big|\, \int \phi \cdot \psi = 0 \quad \textit{for any} \quad \psi \in W_{n-i+1} \Big\}.$$

6.2 Partition of Unity in Valuations

Proposition 6.2.1. *Let $\{U_\alpha\}_\alpha$ be a locally finite open covering of a manifold X. Then there exist $\{\phi_\alpha\}_\alpha \subset V^\infty(X)$ such that*

$$\text{supp }(\phi_\alpha) \subset U_\alpha \textit{ and } \sum_\alpha \phi_\alpha \equiv \chi$$

where the sum is locally finite, and χ *denotes the Euler characteristic.*

Proposition 6.2.1 is an immediate consequence of the fact that the sheaf $\mathcal{V}_X^\infty$ of smooth valuations is soft (by Proposition 3.1.8 of [A5]) and the following general result.

Proposition 6.2.2 ([G], Theorem 3.6.1, Ch. II). *Let* X *be a paracompact topological space. Let* $\{U_i\}_{i\in I}$ *be a locally finite open covering of* X*. Let* $\mathcal{L}$ *be a soft sheaf over* X*. Then for any section* $s \in \mathcal{L}(X)$ *there exists a collection of sections* $\{s_i\}_{i\in I} \subset \mathcal{L}(X)$ *such that*

(1) supp $s_i \subset U_i$*;*
(2) *the family of subsets* $\{\text{supp } s_i\}_{i\in I}$ *is locally finite;*
(3) $s = \sum_{i\in I} s_i$*.*

7 Generalized Valuations

In this section we introduce and study the space $V^{-\infty}(X)$ of *generalized* valuations. It is defined in Subsection 7.1. In Subsection 7.2 it is shown that generalized valuations form naturally a sheaf on X; it is a sheaf of modules over the sheaf of algebras of smooth valuations. In Subsection 7.3 a canonical filtration on generalized valuations is introduced and studied; it extends in a sense the canonical filtration on smooth valuations. In Subsection 7.4 we extend the Euler–Verdier involution from smooth valuations to generalized ones.

7.1 The Space of Generalized Valuations

Definition 7.1.1. *Define the space of generalized valuations by*

$$V^{-\infty}(X) := \left(V_c^\infty(X)\right)^*$$

equipped with the usual weak topology on the dual space.

Remark 7.1.2. It is important to observe that by Theorem 6.1.1 we have a canonical imbedding

$$V^\infty(X) \hookrightarrow V^{-\infty}(X)$$

with the image dense in the weak topology. Thus we can consider the space of generalized valuations as a completion of the space of smooth compactly supported valuations with respect to the weak topology.

Let us describe on $V^{-\infty}(X)$ the canonical structure of $V^\infty(X)$-module. Let $\xi \in V^\infty(X)$, $\psi \in V^{-\infty}(X)$. Define their product $\xi \cdot \psi$ by

$$< \xi \cdot \psi, \phi > = < \phi, \xi \cdot \phi >$$

for any $\phi \in V_c^\infty(X)$. Clearly this defines a map

$$\mu \colon V^\infty(X) \times V^{-\infty}(X) \to V^{-\infty}(X).$$

Proposition 7.1.3. *The map μ is a separately continuous bilinear map. It defines a structure of $V^\infty(X)$-module on $V^{-\infty}(X)$. Moreover $V^\infty(X)$ is a submodule of $V^{-\infty}(X)$, and the induced structure of $V^\infty(X)$-module on it is the standard one.*

Proof. The bilinearity is obvious from the definition. Let us check the continuity. We have to check that for any $\phi \in V_c^\infty(X)$ the map

$$V^\infty(X) \times V^{-\infty}(X) \to \mathbb{C}$$

given by $(\xi, \psi) \mapsto < \psi, \xi \cdot \phi >$ is separately continuous. But this is an immediate consequence of the continuity of the map $V^\infty(X) \to V_c^\infty(X)$ given by $\xi \mapsto \xi \cdot \phi$ and separate continuity of the canonical pairing $V_c^\infty(X) \times V^{-\infty}(X) \to \mathbb{C}$.

Let us check now that the above map $\mu \colon V^\infty(X) \times V^{-\infty}(X) \to V^{-\infty}(X)$ defines the standard $V^\infty(X)$-module structure on $V^\infty(X) \hookrightarrow V^{-\infty}(X)$. Namely we have to show that for $\xi, \psi \in V^\infty(X)$ one has $\mu(\xi, \psi) = \xi \cdot \psi$ where the last product is understood in the usual sense. Let $\phi \in V_c^\infty(X)$. Then we have

$$< \mu(\xi, \psi), \phi > = < \psi, \xi \cdot \phi > =$$
$$\int \psi \cdot (\xi \cdot \phi) = \int (\xi \cdot \psi) \cdot \phi = < \xi \cdot \psi, \phi > .$$

Hence $\mu(\xi, \phi) = \xi \cdot \psi$.

Since $V^\infty(X)$ is dense in $V^{-\infty}(X)$ and μ is continuous it follows that μ defines $V^\infty(X)$-module structure on $V^{-\infty}(X)$. ☐

7.2 The Sheaf Property of Generalized Valuations

In this subsection we describe the canonical sheaf structure on generalized valuations.

First observe that for two open subsets $U_1 \subset U_2$ of a manifold X we have the identity imbedding

$$V_c^\infty(U_1) \hookrightarrow V_c^\infty(U_2). \tag{7.2.1}$$

Hence by duality we have a *continuous* map

$$V^{-\infty}(U_2) \to V^{-\infty}(U_1). \tag{7.2.2}$$

Lemma 7.2.1. *The map (7.2.2) being restricted to* $V^\infty(U_2) \subset V^{-\infty}(U_2)$ *coincides with the usual restriction map* $V^\infty(U_2) \to V^\infty(U_1)$.

Proof. Let us denote temporarily the imbedding (7.2.1) by τ, and its dual (7.2.2) by τ^*. Let $\phi \in V^\infty(U_2)$. Then for any $\psi \in V_c^\infty(U_1)$ one has

$$< \tau^*(\phi), \psi >= \big(\phi \cdot \tau(\psi)\big)(U_2) = (\phi|_{U_1} \cdot \psi)(U_1) =< \phi|_{U_1}, \psi > .$$

Hence $\tau^*(\phi) = \phi|_{U_1}$. □

Proposition 7.2.2. *The assignment*

$$U \mapsto V^{-\infty}(U)$$

to any open subset $U \subset X$ *with the above restriction maps defines a sheaf on* X *denoted by* $\mathcal{V}_X^{-\infty}$.

Remark 7.2.3. Given this proposition, it is clear that $\mathcal{V}_X^\infty$ is a subsheaf of $\mathcal{V}_X^{-\infty}$.

Proof of Proposition 7.2.2. Let $\{U_\alpha\}_\alpha$ be an open covering of an open subset U. Let $\phi \in V^{-\infty}(V)$ such that $\phi|_{U_\alpha} = 0$ for any α. Let us show that $\phi = 0$. Replacing $\{U_\alpha\}$ by a refinement we may assume that $\{U_\alpha\}$ is locally finite. Let us choose a partition of unity $\{\phi_\alpha\}$ subordinate to this covering using Proposition 6.2.1. For any $\psi \in V_c^\infty(X)$ we have

$$< \phi, \psi >=< \phi, \sum_\alpha \phi_\alpha \cdot \psi >=$$

$$\sum_\alpha < \phi, \phi_\alpha \cdot \psi >= \sum_\alpha < \phi|_{U_\alpha}, (\phi_\alpha \cdot \psi)_{U_\alpha} >= 0.$$

Hence $\phi = 0$.

Now let us assume that we are given an open covering $\{U_\alpha\}_\alpha$ of an open subset $U \subset X$, and for any α we are given a generalized valuation $\psi_\alpha \in V^{-\infty}(U_\alpha)$ such that $\psi_\alpha|_{U_\alpha \cap U_\beta} = \psi_\beta|_{U_\alpha \cap U_\beta}$ for any α, β. Let us show that there exists $\psi \in V^{-\infty}(U)$ such that $\psi|_{U_\alpha} = \psi_\alpha$. Again by choosing a refinement we may assume that the covering $\{U_\alpha\}$ is locally finite. Let us fix a partition of unity $\{\phi_\alpha\}$ subordinate to it. Define ψ by

$$< \psi, \phi >:= \sum_\alpha < \psi_\alpha, (\phi_\alpha \cdot \phi)|_{U_\alpha} >$$

for any $\phi \in V_c^\infty(U)$. It is easy to see that $\psi \in V^{-\infty}(U)$ and $\psi|_{U_\alpha} = \psi_\alpha$. □

Proposition 7.2.4. *Being equipped with the above restriction maps and the defined above product of generalized valuations by smooth ones,* $\mathcal{V}_X^{-\infty}$ *is a sheaf of* $\mathcal{V}_X^\infty$*-modules.*

Proof. For an open subset $U \subset X$ let us denote by

$$\mu_U \colon V^\infty(U) \times V^{-\infty}(U) \to V^{-\infty}(U)$$

the canonical product. We have to check that for any open subsets $U \subset V \subset X$, any $\xi \in V^\infty(V)$, $\psi \in V^{-\infty}(V)$ one has

$$\big(\mu_V(\xi,\psi)\big)|_U = \mu_U(\xi|_U, \psi|_U). \tag{7.2.3}$$

Let $\phi \in V_c^\infty(U)$. Let us denote the identity imbedding $V_c^\infty(U) \hookrightarrow V_c^\infty(V)$ by τ.

Then we have

$$< \big(\mu_V(\xi,\psi)\big)|_U, \phi >=< \mu_V(\xi,\psi), \tau(\phi) >=< \psi, \xi \cdot \tau(\phi) >=$$
$$< \psi, \tau(\xi|_U \cdot \phi) >=< \psi|_U, \xi|_U \cdot \phi >=< \mu_U(\xi|_U, \psi|_U), \phi > .$$

Hence (7.2.3) follows. □

7.3 Filtration on Generalized Valuations

Definition 7.3.1. *Define $W_i(V^{-\infty}(X))$ to be the closure of $W_i(X)(\subset V^\infty(X) \subset V^{-\infty}(X))$ in the space $V^{-\infty}(X)$ with respect to the weak topology.*

Clearly one has

$$V^{-\infty}(X) = W_0\big(V^{-\infty}(X)\big) \supset W_1\big(V^{-\infty}(X)\big) \supset \cdots \supset W_n\big(V^{-\infty}(X)\big).$$

In this subsection we will also use the following notation. The subspace $W_i(X)$ of $V^\infty(X)$ will also be denoted by $W_i(V^\infty(X))$. Set

$$W_i\big(V_c^\infty(X)\big) := W_i\big(V^\infty(X)\big) \cap V_c^\infty(X),$$
$$W_i\big(V_c^{-\infty}(X)\big) := W_i\big(V^{-\infty}(X)\big) \cap V_c^{-\infty}(X).$$

It is easy to see (using the separate continuity of the product $V^\infty(X) \times V^{-\infty}(X) \to V^{-\infty}(X)$) that

$$W_i\big(V^\infty(X)\big) \cdot W_j\big(V^{-\infty}(X)\big) \subset W_{i+j}\big(V^{-\infty}(X)\big). \tag{7.3.1}$$

Proposition 7.3.2. *For any $i = 0, 1, \dots, n$*

$$W_i\big(V^{-\infty}(X)\big) = \big\{\phi \in V^{-\infty}(X) |\ < \phi, \psi >= 0 \text{ for any } \psi \in W_{n-i+1}\big(V_c^\infty(X)\big)\big\}. \tag{7.3.2}$$

Proof. Let us denote by $W_i'(V^{-\infty}(X))$ the space in the right hand side of (7.3.2). The equality (7.3.1) implies that

$$W_i\big(V^{-\infty}(X)\big) \subset W_i'\big(V^{-\infty}(X)\big).$$

Let us prove the converse inclusion. Let us assume in the contrary that there exists $\psi \in W_i'(V^{-\infty}(X))\backslash W_i(V^{-\infty}(X))$. Since $W_i(V^{-\infty}(X))$ is a closed subspace of $V^{-\infty}(X)$ in the weak topology, the Hahn–Banach theorem implies that there exists $\phi \in V_c^{\infty}(X)$ such that $< \psi, \phi > \neq 0$ and for any $\xi \in W_i(V^{-\infty}(X))$

$$< \xi, \phi >= 0.$$

Since $W_i(V^{\infty}(X)) \subset W_i(V^{-\infty}(X))$ Corollary 6.1.3 implies that $\phi \in W_{n-i+1}$ $(V_c^{\infty}(X))$. But then (7.3.1) implies that $< \psi, \phi >= 0$. This is a contradiction. □

Corollary 7.3.3.

$$W_i\big(V^{-\infty}(X)\big) \cap V^{\infty}(X) = W_i\big(V^{\infty}(X)\big).$$

Proof. This immediately follows from Proposition 7.3.2 and Corollary 6.1.3. □

For a subset $S \subset X$ let us denote by $V_S^{-\infty}(X)$ the space of generalized valuations with support contained in S. Clearly $V_S^{-\infty}(X)$ is a $V^{\infty}(X)$-submodule of $V^{-\infty}(X)$. If S is a closed subset of X then $V_S^{-\infty}(X)$ is a closed subspace of $V^{-\infty}(X)$ in the weak topology. It is easy to see that

$$V_c^{-\infty}(X) = \varinjlim_{S \text{ compact}} V_S^{-\infty}(X).$$

Let us equip $V_c^{-\infty}(X)$ with the topology of inductive limit when each of $V_S^{-\infty}(X)$ is equipped with the topology induced from $V^{-\infty}(X)$. Then $V_c^{-\infty}(X)$ is a locally convex Hausdorff linear topological space.

Proposition 7.3.4. *For any $i = 0, 1, \dots, n$, the space $W_i(V_c^{\infty}(X))$ is dense in $W_i(V_c^{-\infty}(X))$ in the above topology of inductive limit.*

Proof. Fix $\phi \in W_i(V_c^{-\infty}(X))$. Set $S :=$ supp ϕ be the support of ϕ. S is a compact set. Let U be an open relatively compact neighborhood of S. Since the sheaf $\mathcal{V}_X^{\infty}$ of smooth valuations is soft (by Proposition 3.1.8 of [A5]), there exists $\alpha \in V^{\infty}(X)$ such that α is equal to the Euler characteristic χ in a neighborhood of S, and $\alpha|_{X\backslash U} \equiv 0$.

Since $W_i(V^{\infty}(X))$ is dense in $W_i(V^{-\infty}(X))$ in the weak topology, there exists a net $\{\phi_\lambda\} \subset W_i(V^{\infty}(X))$ which converges to ϕ in the weak topology. But then $\{\alpha \cdot \phi_\lambda\} \subset W_i(V_U^{\infty}(X))$, and $\{\alpha \cdot \phi_\lambda\}$ converges to $\alpha \cdot \phi = \phi$ in $V_c^{-\infty}(X)$. □

Proposition 7.3.5. *Let $n = \dim X$ as previously. Then there exists a canonical isomorphism of linear topological spaces*

$$W_n\big(V^{-\infty}(X)\big) = C^{-\infty}(X, |\omega_X|).$$

Proof. By Proposition 7.3.2 one has

$$W_n\big(V^{-\infty}(X)\big) = W_1\big(V_c^\infty(X)\big)^\perp = \big(V_c^\infty(X)/W_1\big(V_c^\infty(X)\big)\big)^* \\ = \big(C_c^\infty(X)\big)^* = C^{-\infty}(X,|\omega_X|)$$

where the third equality is due to Lemma 5.1.3(2). □

Proposition 7.3.6. *There exists a canonical isomorphism of linear topological spaces*

$$V^{-\infty}(X)/W_1\big(V^{-\infty}(X)\big) = C^{-\infty}(X).$$

Proof. Using Proposition 7.3.2 one has

$$V^{-\infty}(X)/W_1\big(V^{-\infty}(X)\big) = V_c^\infty(X)^*/W_n\big(V_c^\infty(X)\big)^\perp = W_n\big(V_c^\infty(X)\big)^* \\ = \big(C_c^\infty(X,|\omega_X|)\big)^* = C^{-\infty}(X).$$ □

Recall that by Proposition 7.3.4 $V_c^\infty(X)$ is dense in $V_c^{-\infty}(X)$ (in the topology of inductive limit).

Proposition 7.3.7. *The integration functional*

$$\int : V_c^\infty(X) \to \mathbb{C}$$

extends uniquely by continuity to the functional

$$\int : V_c^{-\infty}(X) \to \mathbb{C}.$$

Proof. First observe that for any $\alpha \in V_c^\infty(X)$ the functional $V^\infty(X) \to \mathbb{C}$ given by $\phi \mapsto \int \alpha\cdot\phi$ extends (uniquely) by continuity in the weak topology to $V^{-\infty}(X)$. Indeed this extension is given by $\psi \mapsto <\psi,\alpha>$. Let us denote this functional by $\hat\alpha$. Thus $\hat\alpha\colon V^{-\infty}(X) \to \mathbb{C}$ is a continuous functional.

Let us fix an arbitrary compact subset $S \subset X$. Let us fix a smooth compactly supported valuation $\alpha \in V_c^\infty(X)$ such that α equals to the Euler characteristic in a neighborhood of S. Consider the corresponding continuous linear functional $\hat\alpha\colon V^{-\infty}(X) \to \mathbb{C}$. We claim that the restriction of $\hat\alpha$ to $V_S^{-\infty}(X)$ is the desired extension of the integration functional to $V_S^{-\infty}(X)$.

To check it let us fix a compact neighborhood S' of S such that the restriction of α to S' is still equal to the Euler characteristic. By (the proof of) Proposition 7.3.4 every valuation from $V_S^{-\infty}(X)$ can be approximated in the weak topology by a net from $V_{S'}^\infty(X)$. Hence it is enough to check that for any $\phi \in V_{S'}^\infty(X)$ one has

$$\int \alpha\cdot\phi = \int \phi.$$

But this is obvious since

$$\phi\cdot(\alpha-\chi) \equiv 0.$$ □

Lemma 7.3.8. *Let* $\{\zeta_\lambda\}_{\lambda\in\Lambda} \subset V^\infty(X)$ *be a net such that for any compact subset* $K \subset X$ *there exists* $\lambda_K \in \Lambda$ *such that for all* $\lambda \geq \lambda_K$

$$(\mathrm{supp}\ \zeta_K) \cap K = \emptyset.$$

Then

$$\lim_\Lambda \zeta_\lambda = 0 \quad \textit{in} \quad V^\infty(X).$$

Proof. Consider the map

$$T_i\colon \tilde{C}^\infty\big(T^*X, W_i(\Omega^n \otimes p^*o)\big) \to W_i\big(V^\infty(X)\big)$$

given by the integration with respect to the characteristic cycle. By Proposition 3.1.3 of [A5] T_i is an epimorphism. By the definition of the topology on $V^\infty(X)$ (see Subsection 3.2 of [A5]) T_i is a continuous map. Hence it is enough to show that for any compact subset $K \subset X$ there exists $\lambda_K \in \Lambda$ such that for any $\lambda \geq \lambda_K$ there exists $\eta_\lambda \in \tilde{C}^\infty(T^*X, \Omega^n \otimes p^*o)$ satisfying

(i) $T_i(\eta_\lambda) = \zeta_\lambda$;
(ii) η_λ vanishes in a neighborhood of $p^{-1}(K)$.

Indeed then we would have

$$\lim_\Lambda \zeta_\lambda = \lim_\Lambda T_i(\eta_\lambda) = 0.$$

For the rest of the proof of the lemma let fix a compact subset $K \subset X$. As in Section 2 consider the sheaves on X

$$\mathcal{W}_i(U) = W_i\big(V^\infty(U)\big),$$
$$\mathcal{W}'_i(U) = \tilde{C}^\infty(T^*U, W_i\big(\Omega^n \otimes p^*o)\big)$$

for any open subset $U \subset X$. The integration over the the characteristic cycle gives a morphism of sheaves

$$T_i\colon \mathcal{W}'_i \to \mathcal{W}_i$$

which is an epimorphism (we denote this morphism by the same symbol T_i). Set again $\mathcal{K}_i := \mathrm{Ker}\, T_i$. It was shown in [A5] (see the proof of Proposition 3.1.9) that the sheaves $\mathcal{K}_j/\mathcal{K}_{j+1}$ are isomorphic to the sheaves of $\mathcal{O}_X$-modules where $\mathcal{O}_X$ denotes the sheaf of C^∞-smooth functions on X. By Section 3.7 of Ch. II of [G] the sheaves $\mathcal{K}_j/\mathcal{K}_{j+1}$ are soft for any j. Hence for any closed subset $Z \subset X$ the positive cohomology groups with support in Z vanish:

$$H^i_Z(X, \mathcal{K}_j/\mathcal{K}_{j+1}) = 0 \text{ for } i > 0.$$

Using the long exact sequence we get

$$H^i_Z(X, \mathcal{K}_j) = 0 \text{ for } i > 0 \text{ and any } j.$$

Consider the short exact sequence of sheaves

$$0 \to \mathcal{K}_i \to \mathcal{W}_i' \to \mathcal{W}_i \to 0.$$

From the long exact sequence we obtain

$$H^0_Z(X, \mathcal{W}_i') \to H^0_Z(X, \mathcal{W}_i) \to H^1_Z(X, \mathcal{K}_i) = 0 \tag{7.3.3}$$

for any closed subset $Z \subset X$. Namely the map

$$H^0_Z(X, \mathcal{W}_i') \to H^0_Z(X, \mathcal{W}_i) \tag{7.3.4}$$

is surjective.

Let us choose Z as follows. Let U be an open relatively compact neighborhood of K. Set $Z := X \backslash U$. There exists $\lambda_0 \in \Lambda$ such that for any $\lambda \geq \lambda_0$ one has $(\text{supp}\, \zeta_\lambda) \cap U = \emptyset$. Then clearly $\zeta_\lambda \in H^0_Z(X, \mathcal{W}_i)$ for $\lambda \geq \lambda_0$. The surjectivity of the map (7.3.4) implies the lemma. □

Lemma 7.3.9. (1) *For any $i = 0, 1, \dots, n$ the space $W_i(V_c^\infty(X))$ is dense in $W_i(V^\infty(X))$.*
(2) *For any $i = 0, 1, \dots, n$ the space $W_i(V_c^{-\infty}(X))$ is dense in $W_i(V^{-\infty}(X))$.*

Proof. Let us prove first part (1). For any compact subset $K \subset X$ let us choose a compactly supported valuation $\tau_K \in V_c^\infty(X)$ such τ_K is equal to the Euler characteristic χ in a neighborhood of K. Let $\psi \in W_i(V^\infty(X))$. It is enough to show that

$$\lim_{K \text{ compact}} (\tau_K \cdot \psi) = \psi \text{ in } V^\infty(X).$$

Let us denote $\zeta_K := (\tau_K - \chi) \cdot \psi$. Clearly ζ_K vanishes in a neighborhood of K. By Lemma 7.3.8 $\lim_{K \text{ compact}} \zeta_K = 0$.

Let us prove part (2). Fix $\psi \in W_i(V^{-\infty}(X))$. For any compact subset $K \subset X$ let us fix a compactly supported smooth valuation $\tau_K \in V_c^\infty(X)$ which is equal to the Euler characteristic χ in a neighborhood of K. Let $\psi_K := \tau_K \cdot \psi$. Clearly $\psi_K \in W_i(V_c^{-\infty}(X))$. It suffices to show that

$$\lim_{K \text{ compact}} \psi_K = \psi \text{ in } V^{-\infty}(X).$$

Let $\phi \in V_c^\infty(X)$. We have to show that

$$\lim_{K \text{ compact}} < \psi_K, \phi > = < \psi, \phi > .$$

We have

$$\lim_{K \text{ compact}} < \psi_K, \phi > = \lim_{K \text{ compact}} < \psi, \tau_K \cdot \phi > =$$
$$< \psi, \lim_{K \text{ compact}} (\tau_K \cdot \phi) > = < \psi, \phi > .$$

Part (2) is proved too. □

Let us observe now that the bilinear map $V_c^{-\infty}(X) \times V^{\infty}(X) \to \mathbb{C}$ given by

$$(\psi, \phi) \mapsto \int \phi \cdot \psi$$

is separately continuous. Hence it defines a continuous map

$$\theta\colon V_c^{-\infty}(X) \to V^{\infty}(X)^*$$

where $V^{\infty}(X)^*$ is equipped with the weak topology, and a continuous map

$$\theta'\colon V^{\infty}(X) \to V_c^{-\infty}(X)^*$$

where $V_c^{-\infty}(X)^*$ is equipped with the weak topology.

Proposition 7.3.10. *The maps θ and θ' are isomorphisms of linear spaces.*

Proof. First observe that if the manifold X is compact then the result follows immediately from the definitions. Let us assume that X is not compact.

First let us check that θ is injective. Assume that $\psi \in \operatorname{Ker}\theta$. Since $\psi \in V_c^{-\infty}(X) \subset V^{-\infty}(X) = V_c^{\infty}(X)^*$, then for any $\phi \in V_c^{\infty}(X)$ one has $<\psi, \phi>= 0$. Hence $\psi = 0$.

Let us check now that θ is onto. Let $\zeta \in V^{\infty}(X)^*$. Since the identity imbedding $V_c^{\infty}(X) \hookrightarrow V^{\infty}(X)$ is continuous, the restriction $\tilde{\zeta}$ of ζ to $V_c^{\infty}(X)$ is a continuous functional on $V_c^{\infty}(X)$, i.e. belongs to $V_c^{\infty}(X)^* = V^{-\infty}(X)$. Let us show that supp $\tilde{\zeta}$ is compact, i.e. $\tilde{\zeta} \in V_c^{-\infty}(X)$. Assume in the contrary that supp $\tilde{\zeta}$ is not compact. It means that for any compact subset $K \subset X$ there exists a valuation $\phi \in V_c^{\infty}(X)$ with supp $\phi \cap K = \emptyset$ such that $<\tilde{\zeta}, \phi>\neq 0$. Since we have assumed that X is not compact we can construct an open covering $\{U_\alpha\}_\alpha$ of X which does not have a finite subcovering. Since any manifold is paracompact (by definition) and locally compact, by choosing a refinement if necessary we may assume that this covering is locally finite and any U_α is relatively compact. Let us choose U_{α_1} so that supp $\tilde{\zeta} \cap U_{\alpha_1} \neq \emptyset$. Denote $K_1 := \bar{U}_{\alpha_1}$. Assume we have constructed compact sets $K_1, \dots, K_{N-1}$ with the following properties:

1. for each $i = 1, \dots, N-1$ there exists α_i such that $K_i = \bar{U}_{\alpha_i}$;
2. the interior of K_i intersects supp $\tilde{\zeta}$ non-trivially for each $i = 1, \dots, N-1$;
3. $K_i \cap K_j = \emptyset$ for $1 \leq i \neq j \leq N-1$.

Let us construct K_N such that the sequence of sets $K_1, \dots, K_{N-1}, K_N$ has the same properties. Let us fix an open relatively compact neighborhood T of the set $\cup_{i=1}^{N-1} K_i$. Since the covering $\{U_\alpha\}$ is locally finite, and supp $\tilde{\zeta}$ is not compact, there exists α_N such that $U_{\alpha_N} \cap T = \emptyset$ and $U_{\alpha_N} \cap \operatorname{supp} \tilde{\zeta} \neq \emptyset$. Set $K_N := \bar{U}_{\alpha_N}$. Then $K_N \cap (\cup_{i=1}^{N-1} K_i) = \emptyset$ and $K_N \cap \operatorname{supp} \tilde{\zeta} \neq \emptyset$. By induction we obtain an infinite sequence of pairwise disjoint compact sets $\{K_N\}_{N \in \mathbb{N}}$ with non-empty interiors such that $\operatorname{int} K_N \cap \operatorname{supp} \tilde{\zeta} \neq \emptyset$ for any $N \in \mathbb{N}$.

Since $\operatorname{int} K_N \cap \operatorname{supp} \tilde{\zeta} \neq \emptyset$ we can choose a valuation $\phi_N \in V^{\infty}(X)$ with supp $\phi_N \subset \operatorname{int} K_N$ and such that $<\tilde{\zeta}, \phi_N>= 1$. Let us define

$$\phi := \sum_{N=1}^{\infty} \phi_N.$$

This series converges in $V^\infty(X)$ by Lemma 7.3.8. Then

$$<\zeta, \phi>= \lim_{N\to\infty} <\zeta, \sum_{n=1}^{N} \phi_n>= \lim_{N\to\infty} N = \infty.$$

This is a contradiction. Hence we have shown that supp $\tilde{\zeta}$ is compact.

Let us show that $\zeta = \theta(\tilde{\zeta})$. For any $\phi \in V_c^\infty(X)$ we have

$$<\zeta, \phi>=<\tilde{\zeta}, \phi> .$$

Hence $<\zeta, \phi>=<\theta(\tilde{\zeta}), \phi>$ for any $\phi \in V_c^\infty(X)$. But by Lemma 7.3.9 $V_c^\infty(X)$ is dense in $V^\infty(X)$. Hence $\zeta = \theta(\tilde{\zeta})$. Thus we have shown that $\theta\colon V_c^{-\infty}(X) \to V^\infty(X)^*$ is an isomorphism of linear spaces.

Let us show that θ' is an isomorphism of linear spaces. First let us check that θ' is injective. Assume that $\psi \in \operatorname{Ker}\theta'$. Since $V_c^\infty(X) \subset V_c^{-\infty}(X)$ then for any $\phi \in V_c^\infty(X)$ one has

$$\int \phi \cdot \psi = 0.$$

By the Selfduality Property (Theorem 6.1.1) $\psi \equiv 0$.

Let us show that θ' is surjective. Let $\zeta \in V_c^{-\infty}(X)^*$. For any compact subset $K \subset X$ let us fix a compactly supported valuation $\gamma_K \in V_c^\infty(X)$ such that the restriction of γ_K to a neighborhood of K is equal to the Euler characteristic χ. Consider the linear functional

$$\zeta_K\colon V^{-\infty}(X) \to \mathbb{C}$$

defined by $\zeta_K(\phi) = \zeta(\gamma_K \cdot \phi)$. It is easy to see that ζ_K is a continuous functional on $V^{-\infty}(X)$ equipped with the weak topology. Hence $\zeta_K \in V_c^{-\infty}(X)^* = V_c^\infty(X)$. It is also clear that if $K_1 \subset K_2$ then the restriction of ζ_{K_2} to K_1 is equal to ζ_{K_1}. Taking limit over all compact subsets of X we get a smooth valuation on X denoted by $\tilde{\zeta}$. Clearly the restriction of $\tilde{\zeta}$ to any compact subset $K \subset X$ is equal to ζ_K. Then evidently $\zeta = \theta'(\tilde{\zeta})$. □

7.4 The Euler–Verdier Involution on Generalized Valuations

We are going to extend the Euler–Verdier involution from smooth valuations to generalized ones.

Theorem 7.4.1. (i) *There exists unique continuous in the weak topology linear map*

$$\sigma\colon V^{-\infty}(X) \to V^{-\infty}(X) \tag{7.4.1}$$

such that the restriction of it to $V^\infty(X)$ is the Euler–Verdier involution on smooth valuations.
(ii) $\sigma^2 = Id$.
(iii) *σ commutes with the restrictions to open subsets of X, and thus induces an involution of the sheaf $\mathcal{V}_X^{-infty}$ of generalized valuations (defined in Subsection 7.2).* (iv) $\sigma(W_i(V^{-\infty}(X))) = W_i(V^{-\infty}(X))$ *for any $i = 0, 1, \ldots, n$.* (v) *For any $\phi \in V^\infty(X)$, $\xi \in V^{-\infty}(X)$ one has*

$$\sigma(\phi \cdot \xi) = \sigma(\phi) \cdot \sigma(\xi). \tag{7.4.2}$$

Proof. Let us prove first part (i) The uniqueness is obvious since $V^\infty(X)$ is dense in $V^{-\infty}(X)$ in the weak topology. Let us probe the existence.

We have the Euler–Verdier involution on smooth valuations

$$\sigma \colon V^\infty(X) \to V^\infty(X).$$

Since this map commutes with restrictions to open subsets of X, it preserves support of a smooth valuation. Hence $\sigma \colon V_c^\infty(X) \to V_c^\infty(X)$ is a continuous operator (with respect to the topology of inductive limit on $V_c^\infty(X)$). Consider the dual operator

$$\sigma^* \colon V^{-\infty}(X) \to V^{-\infty}(X).$$

σ^* is continuous in the weak topology. Let us show that the restriction of σ^* to smooth valuations coincides with the Euler–Verdier involution on $V^\infty(X)$. This will finish the proof of part (i) since σ^* is the operator we need (which will be denoted again by σ).

Let $\psi \in V^\infty(X) \subset V^{-\infty}(X)$. It is enough to show that for any $\phi \in V_c^\infty(X)$ one has

$$< \sigma^*\psi, \phi > = < \sigma\psi, \phi > .$$

Using the automorphism property of the Euler–Verdier involution on smooth valuations (Theorem 4.1.4) we have

$$< \sigma^*\psi, \phi > = < \psi, \sigma\phi > = \int \psi \cdot \sigma\phi =$$
$$\int \sigma(\sigma\psi \cdot \phi) = \int \sigma\psi \cdot \phi = < \sigma\psi, \phi > .$$

Part (i) is proved. The remaining statements of the theorem follow from the continuity and the corresponding properties of the Euler–Verdier involution on smooth valuations. □

8 Valuations on Real Analytic Manifolds

The goal of this section is to make a comparison of valuations with a more familiar space of constructible functions on a real analytic manifold. Let us fix a real analytic manifold X of dimension n.

In Subsection 8.1 we construct a canonical imbedding of the space of constructible functions $\mathcal{F}(X)$ into the space of generalized valuations $V^{-\infty}(X)$ as a dense subspace. In Subsection 8.2 we show that the restriction of the canonical filtration on $V^{-\infty}(X)$ to $\mathcal{F}(X)$ is the filtration of $\mathcal{F}(X)$ by codimension of the support. In Subsection 8.3 it is proved that the restriction of the integration functional on the space of generalized valuations with compact support to the subspace $\mathcal{F}_c(X)$ of constructible functions with compact support is the integration with respect to the Euler characteristic. In Subsection 8.4 we show that the restriction of the Euler–Verdier involution on generalized valuation to $\mathcal{F}(X)$ coincides (up to a sign) with the Verdier duality operator on the latter.

8.1 Imbedding of Constructible Functions to Generalized Valuations

In this subsection we will construct a canonical $\mathbb{C}$-linear map

$$\Xi\colon \mathcal{F}(X) \to V^{-\infty}(X)$$

and prove that it is injective and has a dense image in the weak topology, where $\mathcal{F}(X)$ is the space of constructible functions on X defined in Subsection 1.2 (equality (1.2.1)).

The construction of the map Ξ is based on the notion of characteristic cycle attached to an arbitrary constructible function $f \in \mathcal{F}(X)$ denoted by $CC(f)$. This notion was discussed in Subsection 1.3.

Note in addition that the characteristic cycle satisfies

$$CC(\alpha f + \beta g) = \alpha CC(f) + \beta CC(g) \tag{8.1.1}$$

for any $\alpha, \beta \in \mathbb{C}$ (see [KS], §9.7).

Now let us describe the canonical map

$$\Xi\colon \mathcal{F}(X) \to V^{-\infty}(X) = \big(V_c^{\infty}(X)\big)^*. \tag{8.1.2}$$

Let us denote by $C_c^{\infty}(T^*X, \Omega^n \otimes p^*o)$ the space of C^{∞}-sections with *compact support* of the bundle $\Omega^n \otimes p^*o$ over T^*X. By Lemma 2.1.1 we have the canonical continuous epimorphism

$$C_c^{\infty}(T^*X, \Omega^n \otimes p^*o) \twoheadrightarrow V_c^{\infty}(X) \tag{8.1.3}$$

given by

$$\omega \mapsto \Big[P \mapsto \int_{CC(P)} \omega\Big] \tag{8.1.4}$$

for any $P \in \mathcal{P}(X)$. For any constructible function $f \in \mathcal{F}(X)$ let us define $\Xi(f)$ by

$$< \Xi(f), \phi >= \int_{CC(f)} \omega \tag{8.1.5}$$

where $\omega \in C_c^\infty(T^*X, \Omega^n \otimes p^*o)$ is an arbitrary lift of ϕ. Once we show that $\Xi(f)$ is well defined, then automatically it is a continuous linear functional on $V_c^\infty(X)$.

Thus it remains to check that Ξ is well defined. More explicitly, assume that $\omega \in C_c^\infty(T^*X, \Omega^n \otimes p^*o)$ satisfies

$$\int_{CC(P)} \omega = 0 \tag{8.1.6}$$

for any $P \in \mathcal{P}(X)$. We have to check that

$$\int_{CC(f)} \omega = 0 \tag{8.1.7}$$

for any constructible function $f \in \mathcal{F}(X)$.

Let us fix such an ω. By (8.1.1) it is enough to assume that f is the indicator function of a subanalytic subset Q.

Let us observe first of all that (obviously) every point $x \in X$ has a compact subanalytic neighborhood (and also an open subanalytic neighborhood). Hence we can choose a compact subanalytic neighborhood S of the support of ω. It is enough to check that for any subanalytic subset $Q \subset S$ one has

$$\int_{CC(Q)} \omega = 0.$$

Any point $x \in X$ has a pair of subanalytic neighborhoods $U_x \subset V_x$ such that U_x is compact, V_x is open, and there exists a real analytic diffeomorphism $g_x \colon V_x \tilde{\to} \mathbb{R}^n$. Hence one can find a finite covering of S by compact subanalytic subsets $\{U_i\}_i$, find open subanalytic subsets $\{V_i\}_i$ with $U_i \subset V_i$, and real analytic diffeomorphisms $f_i \colon V_i \tilde{\to} \mathbb{R}^n$.

By the linearity of the characteristic cycle (8.1.1), intersecting Q with each U_i we may assume that Q is relatively compact subset of V_{i_0} for some i_0. Thus it remains to prove the following statement.

Lemma 8.1.1. *Let $\omega \in \tilde{C}^\infty(T^*\mathbb{R}^n, \Omega^n \otimes p^*o)$ satisfies*

$$\int_{CC(P)} \omega = 0 \textit{ for any } P \in \mathcal{P}(X).$$

Then for any bounded subanalytic subset $Q \subset \mathbb{R}^n$ one has

$$\int_{CC(Q)} \omega = 0.$$

Proof. We will reduce the proof of the lemma to Theorem 1 of [BB]. Let us fix an orientation on $\mathbb{R}^n$. Let ψ denote the restriction of ω to the zero section $\underline{0}$ of T^*X. Thus $\psi \in C^\infty(\mathbb{R}^n, \Omega^n)$. Let

$$q\colon T^*\mathbb{R}^n\backslash\underline{0} \to \mathbb{P}_+(T^*\mathbb{R}^n)$$

be the canonical projection. Let $\tilde{\omega} := q_*\omega$ be the integration of $\omega|_{T^*\mathbb{R}^n\backslash\underline{0}}$ along the fibers of q. Let $a\colon \mathbb{P}_+(T^*\mathbb{R}^n) \to \mathbb{P}_+(T^*\mathbb{R}^n)$ be the canonical (antipodal) involution described in Subsection 1.3. Set $\eta := a^*\tilde{\omega}$. It is easy to see that

$$\int_{CC(P)} \omega = \int_{N(P)} \eta + \int_P \psi \text{ for any } P \in \mathcal{P}(\mathbb{R}^n);$$
$$\int_{CC(f)} \omega = \int_{N(f)} \eta + \int_{\mathbb{R}^n} f \cdot \psi \text{ for any } f \in \mathcal{F}_c(\mathbb{R}^n).$$

Thus by assumption we get

$$\int_{N(P)} \eta + \int_P \psi = 0 \tag{8.1.8}$$

for any $P \in \mathcal{P}(\mathbb{R}^n)$.

It was shown in [BB], Theorem 1, that a pair (η, ψ) with $\eta \in C^\infty(\mathbb{P}_+(T^*\mathbb{R}^n), \Omega^{n-1})$, $\psi \in C^\infty(\mathbb{R}^n, \Omega^n)$ satisfies the equality (8.1.8) for any compact *subanalytic* subset P if and only if it satisfies the following two conditions (where $\pi\colon \mathbb{P}_+(T^*\mathbb{R}^n) \to \mathbb{R}^n$ is the canonical projection):

$$\int_{\pi^{-1}(x)} \eta = 0 \text{ for any } x \in \mathbb{R}^n, \tag{8.1.9}$$
$$D\eta + \pi^*\psi = 0 \tag{8.1.10}$$

where $D\colon C^\infty(\mathbb{P}_+(T^*\mathbb{R}^n), \Omega^{n-1}) \to C^\infty(\mathbb{P}_+(T^*\mathbb{R}^n), \Omega^n)$ is an explicitly written differential operator of second order (introduced by Rumin in [R]).

However in the proof of the "if" part of Theorem 1 in [BB] the authors used equality (8.1.8) not for the whole class of compact subanalytic sets, but for the subclass of compact subanalytic submanifolds with boundary. Hence if (8.1.8) is satisfied for all $P \in \mathcal{P}(\mathbb{R}^n)$ then the conditions (8.1.9), (8.1.10) are satisfied, and hence (8.1.8) is satisfied for an *arbitrary* compact subanalytic subset $P \subset \mathbb{R}^n$ (again by Theorem 1 of [BB]).

In order to prove our lemma it is enough to show that (8.1.8) is satisfied for any bounded subanalytic subset P. Then we have

$$\int_{CC(\mathbf{1}_P)} \omega = \int_{CC(\mathbf{1}_{\bar{P}})} \omega - \int_{CC(\mathbf{1}_{\bar{P}\backslash P})} \omega = -\int_{CC(\mathbf{1}_{\bar{P}\backslash P})} \omega.$$

Since $\dim(\bar{P}\backslash P) < \dim P$ by Proposition 1.2.5(ii) we can use the induction on the dimension of P. Lemma is proved. □

Remark 8.1.2. The differential operator D was introduced by Rumin [R] for an arbitrary contact manifold, and it depends only on the contact structure. In our case for any smooth manifold X the space $\mathbb{P}_+(T^*X)$ has a canonical contact structure, and the operator D used in the proof of Lemma 8.1.1 corresponds to it.

8.2 Comparison of Filtrations

Let us define on $\mathcal{F}(X)$ a filtration by codimension of support:

$$W_i\big(\mathcal{F}(X)\big) := \big\{f \in \mathcal{F}(X) |\, \mathrm{codim}(\mathrm{supp}\ f) \geq i\big\}. \tag{8.2.1}$$

We have

$$\mathcal{F}(X) = W_0\big(\mathcal{F}(X)\big) \supset W_1(\mathcal{F}(X)) \supset \cdots \supset W_n\big(\mathcal{F}(X)\big) \supset W_{n+1}\big(\mathcal{F}(X)\big) = 0.$$

Proposition 8.2.1. *The canonical map*

$$\Xi \colon \mathcal{F}(X) \to V^{-\infty}(X)$$

is injective. Moreover for any $i = 0, 1, \ldots, n$, *and any* $f \in W_i(\mathcal{F}(X))\setminus W_{i+1}(\mathcal{F}(X))$ *there exists* $\phi \in W_{n-i}(V_c^\infty(X))$ *such that*

$$< \Xi(f), \phi > \neq 0.$$

Proof. Clearly it is enough to prove the second statement. Let us fix a constructible function $f \in W_i(\mathcal{F}(X))\setminus W_{i+1}(\mathcal{F}(X))$. Thus supp f is a subanalytic set and codim(supp f) $= i$.

One can choose a regular point $x \in$ supp f, a neighborhood U, a real analytic diffeomorphism $g \colon U \tilde{\to} \mathbb{R}^n$ such that $f|_U \circ g^{-1} = c \cdot 1\!\!1_{\mathbb{R}^{n-k}}$ where $\mathbb{R}^{n-k} \subset \mathbb{R}^n$ is the coordinate subspace, and $c \neq 0$ is a constant. Thus we may assume that $X = \mathbb{R}^n$, $f = 1\!\!1_{\mathbb{R}^{n-k}}$. Let us choose $\omega \in C_c^\infty(T^*\mathbb{R}^n, \Omega^n \otimes p^*o)$ as follows. Let $\{(q_1, \ldots, q_n)\}$ be coordinates on $\mathbb{R}^n$. Let $\{p_1, \ldots, p_n\}$ be dual coordinates on $\mathbb{R}^{n*}$. Let us fix a C^∞-smooth non-negative compactly supported function $\tau \colon \mathbb{R}^{n*} \to \mathbb{R}_{\geq 0}$ such that $\tau(0) > 0$. Let us take

$$\omega := \tau \cdot dx_1 \wedge \cdots \wedge dx_{n-k} \wedge dy_{n-k+1} \wedge \cdots \wedge dy_n.$$

Then clearly $\int_{CC(\mathbb{R}^{n-k})} \omega \neq 0$ and $\omega \in C_c^\infty(T^*\mathbb{R}^n, W_{n-i}(\Omega^n \otimes p^*o))$. □

From now on we will identify $\mathcal{F}(X)$ with the subspace of $V^{-\infty}(X)$ via the imbedding Ξ.

Proposition 8.2.2. (i) $\mathcal{F}(X)$ *is dense in* $V^{-\infty}(X)$ *in the weak topology.*
(ii) *For any* $i = 0, 1, \ldots, n$

$$\mathcal{F}(X) \cap W_i\big(V^{-\infty}(X)\big) = W_i\big(\mathcal{F}(X)\big).$$

Proof. (i) By the Hahn–Banach theorem it is enough to prove that for any $\phi \in V_c^\infty(X)\backslash\{0\}$ there exists $f \in \mathcal{F}(X)$ such that $< f, \phi > \neq 0$. Let us fix $\phi \in V_c^\infty(X)\backslash\{0\}$. One may find an open subset $U \subset X$ and a real analytic diffeomorphism $g\colon U \tilde{\to} \mathbb{R}^n$ such that $\phi|_U \not\equiv 0$. The smooth valuation $g_*\phi|_U \in V^\infty(\mathbb{R}^n)$ does not vanish identically. By Proposition 2.4.10 from [A5] there exists a convex compact set $K \in \mathcal{K}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that $(g_*\phi)(K) \neq 0$. Since every compact set can be approximated in the Hausdorff metric by convex compact polytopes, we may assume that K is a convex compact polytope, and hence a subanalytic set. Hence $g^{-1}(K)$ is a compact subanalytic subset of X. Take $f := 1\!\!1_{g^{-1}(K)}$. Then

$$< f, \phi > \neq 0.$$

Part (i) is proved.

(ii) First let us show the inclusion

$$\mathcal{F}(X) \cap W_i\big(V^{-\infty}(X)\big) \subset W_i\big(\mathcal{F}(X)\big). \tag{8.2.2}$$

Let $f \in \mathcal{F}(X)$ be such that $f \notin W_i(\mathcal{F}(X))$. Let us choose $l < i$ such that $f \in W_l(\mathcal{F}(X))\backslash W_{l+1}(\mathcal{F}(X))$. By Proposition 8.2.1 there exists $\phi \in W_{n-l}(V_c^\infty(X))$ such that $< f, \phi > \neq 0$. Hence $f \notin W_{l+1}(V^{-\infty}(X))$. Since $l + 1 \le i$ we have $f \notin W_i(V^{-\infty}(X))$. This proves the inclusion (8.2.2).

Let us prove the opposite inclusion

$$W_i\big(\mathcal{F}(X)\big) \subset \mathcal{F}(X) \cap W_i\big(V^{-\infty}(X)\big). \tag{8.2.3}$$

By Proposition 7.3.2 it is enough to show that for any $f \in W_i(\mathcal{F}(X))$, $\phi \in W_{n-i+1}(V_c^\infty(X))$

$$< f, \phi >= 0.$$

By Lemma 2.1.1 and 8.1.1 there exists a *compactly supported* form $\omega \in C_c^\infty(T^*X, W_{n-i+1}(\Omega^n \otimes p^*o))$ such that for any $h \in \mathcal{F}_c(X)$

$$< h, \phi >= \int_{CC(h)} \omega. \tag{8.2.4}$$

Since the form ω is compactly supported the equality (8.2.4) holds for any $h \in \mathcal{F}(X)$.

Let us assume now that $f = 1\!\!1_Q$ where Q is a subanalytic subset with $\operatorname{codim} Q \ge i$. We may assume that Q relatively compact. We have to show that $\int_{CC(Q)} \omega = 0$. It is enough to show that the restriction of ω to $\operatorname{supp}(CC(Q))$ vanishes. By Lemma 1.3.2 one can find a finite covering $\bar{Q} = \cup_\alpha Q_\alpha$ such that $CC(Q) \subset \cup_\alpha T^*_{Q_\alpha} X$. But since $\operatorname{codim} Q_\alpha \ge i$ it is obvious that the restriction of ω to $T^*_{Q_\alpha} X$ vanishes. The proposition is proved. $\square$

8.3 The Integration Functional vs. the Integration with Respect to the Euler Characteristic

On the space $\mathcal{F}_c(X)$ we have the linear functional $\mathcal{F}_c(X) \to \mathbb{C}$ of integration with respect to the Euler characteristic which is uniquely characterized by the property $1\!\!1_Q \mapsto \chi(Q)$ for any compact subanalytic subset Q (see [KS], §9.7). For a function $f \in \mathcal{F}_c(X)$ we will denote the integral of f with respect to the Euler characteristic by $\int f d\chi$.

Thus we have the canonical imbedding

$$\mathcal{F}_c(X) \hookrightarrow V_c^{-\infty}(X).$$

Proposition 8.3.1. *The restriction of the integration functional $\int : V_c^{-\infty} \to \mathbb{C}$ to $\mathcal{F}_c(X)$ is equal to the integration with respect to the Euler characteristic.*

Proof. Since the integration functional $\int : V_c^{-\infty}(X) \to \mathbb{C}$ to $\mathcal{F}_c(X)$ is continuous in the weak topology, Proposition 7.3.10 implies that there exists unique $\xi \in V^\infty(X)$ such that for any $\psi \in V_c^{-\infty}(X)$

$$\int \psi = < \psi, \xi > .$$

It is clear that if $\psi \in V_c^\infty(X)$ then

$$\int \psi = < \psi, \chi > .$$

Since $V_c^\infty(X)$ is dense in $V_c^{-\infty}(X)$ by Proposition 7.3.4, it follows that $\xi = \chi$.

Let us fix a Riemannian metric on X. By Theorems 1.5, 1.8 of [F2] there exists a form $\omega \in \tilde{C}^\infty(T^*X, \Omega^n \otimes p^*o)$ (which is a little modification of the Chern-Gauss-Bonnet form [C]) such that for any compact subset $P \subset X$ which is either subanalytic or belongs to $\mathcal{P}(X)$ one has

$$\chi(P) = \int_{CC(P)} \omega.$$

Then by the construction of the imbedding $\mathcal{F}(X) \hookrightarrow V^{-\infty}(X)$ and by Proposition 1.2.7(ii) we have for any $f \in \mathcal{F}_c(X)$

$$\int f = < f, \chi > = \int_{CC(f)} \omega.$$

The proposition is proved. □

8.4 The Euler–Verdier Involution vs. the Verdier Duality

The space of constructible functions $\mathcal{F}(X)$ has a canonical operator

$$\mathbb{D} \colon \mathcal{F}(X) \to \mathcal{F}(X)$$

called the Verdier duality (see [KS], §9.7). It satisfies $\mathbb{D}^2 = Id$, and for any function $f \in \mathcal{F}(X)$

$$CC(\mathbb{D}f) = a^* CC(f) \tag{8.4.1}$$

where $a \colon T^*X \to T^*X$ is the antipodal involution (Proposition 9.4.4 of [KS]). The main result of this subsection is the following proposition.

Proposition 8.4.1. *The restriction of the Euler–Verdier involution* $\sigma \colon V^{-\infty}(X) \to V^{-\infty}(X)$ *to* $\mathcal{F}(X)$ *is equal to* $(-1)^n \mathbb{D}$*.*

Proof. Let $f \in \mathcal{F}(X)$. We have to show that for any $\phi \in V_c^\infty(X)$ one has

$$< \sigma f, \phi >= (-1)^n < \mathbb{D}f, \phi > .$$

By Lemma 2.1.1 there exists $\omega \in C_c^\infty(T^*X, \Omega^n \otimes p^*o)$ such that for any $h \in \mathcal{F}(X)$ one has

$$< h, \phi >= \int_{CC(h)} \omega.$$

Then by the definition of σ we get

$$< \sigma f, \phi >=< f, \sigma\phi >= (-1)^n \int_{CC(f)} a^*\omega = s \int_{CC(\mathbb{D}f)} \omega =< \mathbb{D}f, \phi > .$$

The proposition is proved. □

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Geometric Applications of Chernoff-Type Estimates

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1 Introduction

In this paper we present a probabilistic approach to some geometric problems in asymptotic convex geometry. The aim of this paper is to demonstrate that the well known Chernoff bounds from probability theory can be used in a geometric context for a very broad spectrum of problems, and lead to new and improved results. We begin by briefly describing Chernoff bounds, and the way we will use them.

The following Proposition, which is a version of Chernoff bounds, gives estimates for the probability that at least βN trials out of N succeed, when the probability of success in one trial is p (the proof is standard, see e.g. [HR]).

Proposition 1 (Chernoff). *Let Z_i be independent Bernoulli random variables with mean $0 < p < 1$, that is, Z_i takes value 1 with probability p and value 0 with probability $(1-p)$. Then we have*

1) for every $\beta < p$

$$\mathbb{P}[Z_1 + \cdots + Z_N \geq \beta N] \geq 1 - e^{-NI(\beta,p)},$$

2) for every $\beta > p$

$$\mathbb{P}[Z_1 + \cdots + Z_N > \beta N] \leq e^{-NI(\beta,p)},$$

where $I(\beta,p) = \beta \ln(\beta/p) + (1-\beta)\ln((1-\beta)/(1-p))$.

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Assume that X_i is a sequence of independent non-negative random variables. For simplicity assume to begin with, that they are also identically distributed, and even bounded. A good example to consider is $X_i = \|U_i x\|$ where $\|\cdot\|$ is some norm on n-dimensional space $\mathbb{R}^n$, U_i a random orthogonal matrix (with respect to the normalized Haar measure on $O(n)$) and x some fixed point in the space. Define the sequence of partial sums $S_N = \sum_{i=1}^N X_i$. The law of large numbers says that $\frac{1}{N} S_N$ converges to the expectation $\mathbb{E}X_i$ as N tends to infinity. In our example the expectation is $|x|M$ where $M = \int_{S^{n-1}} \|u\| d\sigma(u)$ and σ is the rotation invariant probability measure on the sphere S^{n-1}.

To estimate the rate of convergence, one usually turns to large deviation theorems, for example the following well known Bernstein's inequality (see e.g. [BLM]). We say that a centered random variable X is a ψ_2 random variable if there exists some constant A such that $\mathbb{E}e^{X^2/A^2} = 2$, and the minimal A for which this inequality holds we call the ψ_2 norm of X. Below when we say the ψ_2 norm of X we mean the ψ_2 norm of the centered variable $(X - \mathbb{E}X)$.

Proposition 2 (Bernstein). *Let X_i be i.i.d. copies of the random variable X, and assume the X has ψ_2 norm A. Then for any $t > 0$*

$$\mathbb{P}\left[\left|\frac{1}{N} S_N - \mathbb{E}X\right| > t\right] \leq 2e^{-cNt^2}, \tag{1}$$

where $c = 1/8A^2$.

Sometimes it is important to get the probability in (1) to be very small. This is the case in the example of $X_i = \|U_i x\|$, if one wants to have an estimate for all points x in some large net on the sphere (we study this example in more detail in Section 4).

The obvious way to make the probability in (1) smaller is to increase t. However, once t is greater than $\mathbb{E}X$, the estimate in (1) makes sense only as an upper bound for S_N and provides no effective lower bound, since the trivial estimate $0 \leq S_N$ is always true.

Thus, we see that for positive random variables, an estimate of the type (1) does not fully answer our needs, and we actually want an estimate of the type

$$\mathbb{P}\left[\varepsilon\mathbb{E} \leq \frac{1}{N} S_N \leq t\mathbb{E}\right] \leq 1 - f(\varepsilon, t, N, X),$$

with f decaying exponentially fast to 0 with N, and moreover, such that the rate of decay will substantially improve as t tends to ∞ and ε tends to 0. This is the aim of our probabilistic method and the subject of the next discussion.

For $\frac{1}{N} S_N$ not to be very small, it is not obligatory that *all* X_is be large, it is enough if some fixed proportion of them are not small. This is the main idea behind our use of Chernoff bounds. The first time this method was applied in our field was in the paper of Milman and Pajor [MP], where in particular a global form of the low M^*-estimate was obtained.

Applying this in our scheme we let $Z_i = 1$ if $X_i \geq \varepsilon$ and $Z_i = 0$ if $X_i < \varepsilon$. Since all X_i are positive, having $\sum_{i=1}^N Z_i \geq \beta N$ means in particular that $\frac{1}{N} S_N \geq \beta\varepsilon$, and this happens with the probability written in Proposition 1, where $p = \mathbb{P}[X_i \geq \varepsilon]$, and β is any number smaller than this p.

Before we proceed let us analyze the estimate. We have

$$I(\beta, p) = u(\beta) - \beta \ln p - (1-\beta) \ln(1-p),$$

where we denoted $u(\beta) = [\beta \ln \beta + (1-\beta) \ln(1-\beta)]$. The term $u(\beta)$ is a negative, convex function which approaches 0 as $\beta \to 0$ and as $\beta \to 1$, and is symmetric about $1/2$ where it has a minima equal to $-\ln 2$. Thus the whole exponent is of the form

$$e^{-NI(\beta,p)} = p^{\beta N}(1-p)^{(1-\beta)N} e^{-Nu(\beta)} \leq (1-p)^{(1-\beta)N} 2^N. \tag{2}$$

We will usually use the latter, though sometimes we will need the better estimate including $u(\beta)$.

To use the full strength of (2), we will need to have the probability p of success increase rapidly as the parameters in question change. In our example, we will need $\mathbb{P}[X_i \geq \varepsilon]$ to approach 1 fast when $\varepsilon \to 0$. This is not always the case, and additional work is sometimes needed. This will best be demonstrated in Section 3.

In the remainder of this section we outline the main theorems to be proved in this paper and explain the notation to be used throughout.

In Section 2 we give an application to a problem from learning theory, improving a result of Cheang and Barron [CB]. The problem regards the approximation of the n-dimensional euclidean ball by a simpler body, which resembles a polytope but need not be convex, and is described by the set of points satisfying a certain amount of linear inequalities out of a given list of length N. In their paper [CB] Cheang and Barron showed that to ε-approximate the ball one can do with $N = C(n/\varepsilon)^2$ linear inequalities, and we improve this estimate (for fixed ε and $n \to \infty$) to $N = Cn \ln(1/\varepsilon)/\varepsilon^2$ (where C is a universal constant). We formulate our theorem (for the proof see [AFM]) and in the remainder of the section we show stability results.

In Section 3 we show three different applications to Khinchine-type inequalities. We reprove, with slightly worse constants, a theorem of Litvak, Pajor, Rudelson, Tomczak-Jaegermann and Vershynin, which is an isomorphic version of Khinchine inequality in the L_1 case, where instead of taking the average of the 2^n terms $|\langle x, \varepsilon \rangle|$ for $\varepsilon \in \{-1,1\}^n$, one averages only over $(1+\delta)n$ of them (for some fixed $\delta > 0$), and the constants in the corresponding inequality depend on δ. Another way to view this result is realizing an n-dimensional euclidean section of $\ell_1^{(1+\delta)n}$ by a matrix of random signs. Schechtman was the first who proved the existence of such an isomorphism for some universal (and large) δ_0, and also together with Johnson proved a non-random version of this fact, see [LPRTV] [S2] [JS]. We remark that an improvement of this result, with a much better dependence on δ will soon be published in [AFMS].

The next application answers a similar question, where instead of random sign vectors, the vectors are random with respect to the volume distribution in an isotropic convex body. We show that when the rows of an $(n \times (1+\delta)n)$ matrix are chosen randomly inside an isotropic convex body, again its image is an n-dimensional euclidean section of $\ell_1^{(1+\delta)n}$. There is a conceptual difference between this result and the preceding one, since now only the *rows* of the matrix are independent, and not *all* entries.

In another application, we reduce the level of randomness, substituting most of it by an explicit sign-matrix. We prove that a Hadamard $(n \times n)$ matrix with extra δn rows of random signs also realizes a euclidean section of $\ell_1^{(1+\delta)n}$, and moreover, the isomorphism constants are polynomially dependent on δ. This is an extension of a result by Schechtman [S1] where he used an $(n \times 2n)$ matrix whose upper half consisted of (a scalar multiple of) the identity matrix and all lower half entries were random signs.

In Section 4 we give a different type of application, proving a Dvoretzky-type theorem in global form. We show that the average of $C(a/M^*)^2$ random rotations of a convex body K (with half-mean-width M^* and half-diameter a, see definitions below) is isomorphic to the euclidean ball. This is well known, e.g. [MS]. In the proof we show how the probabilistic method can be adapted to give a new proof of the *upper* bound in this problem. As will be explained below, the main use of the Chernoff method is to provide lower bounds, while upper bounds can usually be obtained straightforwardly with the use of deviation inequalities. However, in the standard proof of the global Dvoretzky Theorem, the upper bound is obtained by using a deep geometric result about concentration on the product of spheres, which itself uses Ricci curvature (see [GrM]). We will show how standard concentration on the sphere, together with our method, provides an alternative proof for the bound. We then show how a reformulation of a conjecture by Vershynin, given by Latała and Oleszkiewicz [LO] about small ball probabilities will imply that the above is true for $(1+\delta)(a/M^*)^2$ random rotations, for any δ, with constants of isomorphism depending on δ, a result which will be optimal. In addition we give an alternative parameter that can be used to study these averages, similar to the one introduced by Klartag and Vershynin [KV], which in special cases gives improved results.

The paper includes both new proofs of known result and some new results, and our main goal is to show how the probabilistic method we describe here is applicable in many different situations, and in some sense can be considered as another systematic approach to obtaining lower and upper bounds. In many cases this unifies what were before individual proofs for specific problems.

Notation. We work in $\mathbb{R}^n$ which is equipped with the euclidean structure $\langle \cdot, \cdot \rangle$ and write $|\cdot|$ for the euclidean norm. The euclidean unit ball and sphere are denoted by D_n and S^{n-1} respectively. We write σ_n for the rotation invariant probability measure on S^{n-1}, and omit the index n when the dimension is clear from the context. Every symmetric (with respect to 0) convex body K in $\mathbb{R}^n$ induces the norm $\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$. The polar body of K is $K^\circ = \{y \in \mathbb{R}^n : \max_{x\in K} |\langle y, x\rangle| \le 1\}$ and it induces the dual norm $\|x\|_K^* = \|x\|_{K^\circ} = \max_{y\in K} |\langle y, x\rangle|$. We define $M(K) = \int_{S^{n-1}} \|u\|_K d\sigma_n(u)$ and $M^*(K) = \int_{S^{n-1}} \max_{y\in K} |\langle y, u\rangle| d\sigma_n(u)$. So, $M = M(K)$ is the average of the norm associated to K on the sphere and $M^* = M^*(K) = M(K^\circ)$ is half the mean width of K. We also denote by a and b the least positive constants for which $\frac{1}{a}|x| \le \|x\|_K \le b|x|$ holds true for every $x \in \mathbb{R}^n$. Thus, a is half of the diameter of K and $\frac{1}{b}$ is the in-radius of K (so, $\frac{1}{b}D \subseteq K \subseteq aD$). As usual in asymptotic geometric analysis, we will be dealing with finite dimensional normed spaces or convex bodies, and study behavior of some geometric parameters as the dimension grows to infinity. Thus, the dimension n is always assumed large, and the universal constants appearing throughout the paper, denoted usually by c, c_0, c_1, C, do not depend on the dimension and are just numerical constants which can be computed. In addition, throughout, we omit the notation $[\cdot]$ of integer values, and assume the numbers we deal with are integers when needed, to avoid notational inconvenience.

2 A ZigZag Approximation for Balls

2.1 The ZigZag Construction and the Main Theorem

We address the question of approximating the euclidean ball by a simpler set. In many contexts, polytopes are considered to be the simplest sets available, being the intersection of some number of half-spaces, or in other words the set of all points satisfying some list of N linear inequalities. However, it is well known and easy to check that to construct a polytope which is ε-close to the euclidean ball $D_n \subset \mathbb{R}^n$ in the Hausdorff metric one needs to use exponentially many half-spaces, $N \ge e^{Cn\ln(1/\varepsilon)}$ (this can be seen by assuming the polytope is inscribed in D_n, and estimating from above the volume of the cap that each half-space cuts off the sphere S^{n-1}). This is a huge number, and so a different kind of approximation was suggested, first used by Cybenko [C], and by Hornik, Stinchcombe and White [HSW].

The first good bounds in such an approximation result (we describe the approximating set below) were given by Barron [B]. These sets are implemented by what is called single hidden layer neural nets or perception nets, and we will use the simplest version of such sets, for which we suggested the name “ZigZag approximation”.

The approximating set is the following, it is no longer convex, but is still described by a list of linear inequalities. Given a set of N inequalities, and a number $k \le N$, the set consists of all points satisfying no less than k of the N inequalities. We learned of this approximation from a paper by Cheang and Barron [CB], where they showed that there exists a universal constant C

such that for any dimension n, one can find $N = C(n/\varepsilon)^2$ linear inequalities, such that the set of points satisfying at least k of the N inequalities is ε-close, in the Hausdorff metric, to D_n (where k is some proportion of N). This is already a huge improvement, from a set described by an exponential number of inequalities to a polynomial number.

Using our approach we improve (in the case of $n \to \infty$) their estimate to $N = Cn\ln(1/\varepsilon)/\varepsilon^2$ linear inequalities, and we use $k = N/2$. The formulation of our result is given in the following Theorem (see [AFM] for its proof).

Theorem 3. *There exists universal constants c, C such that for every dimension n, and every $0 < \varepsilon < 1$, letting $N = [Cn\ln(1/\varepsilon)/\varepsilon^2]$, if $z_1, \dots, z_N$ are random points with respect to Lebesgue measure σ on the sphere S^{n-1}, then with probability greater than $1 - e^{-cn}$, the set*

$$\mathcal{K} = \left\{ x \in \mathbb{R}^n : \exists i_1, \dots, i_{[N/2]} \text{ with } |\langle x, z_{i_j} \rangle| < \frac{c_0}{\sqrt{n}} \right\}$$

satisfies

$$(1 - \varepsilon) D_n \subset \mathcal{K} \subset (1 + \varepsilon) D_n,$$

where c_0 denotes the constant (depending on n, but converging to a universal constant as $n \to \infty$) for which $\sigma(u \in S^{n-1} : |\langle \theta, u \rangle| \le c_0/\sqrt{n}) = 1/2$ for some $\theta \in S^{n-1}$.

2.2 Stability Results

Theorem 3 above is stable, in the following sense, define the body

$$\mathcal{K}(\beta) = \left\{ x \in \mathbb{R}^n : \exists i_1, \dots, i_{[\beta N]} \text{ with } |\langle x, z_{i_j} \rangle| < \frac{c_0}{\sqrt{n}} \right\}$$

where we have changed the parameter $1/2$ into β. By stability we mean that for N large enough the two bodies $\mathcal{K}_1 = \mathcal{K}(\beta + \delta)$ and $\mathcal{K}_2 = \mathcal{K}(\beta - \delta)$ are close, in the Hausdorff distance, as long as $0 < \delta < \delta_0$, where δ_0 depends only on β. This will readily follow from the fact that both bodies will be close to the euclidean ball of the appropriate radius, depending on β.

We first remark that changing the constant c_0 in the definition of $\mathcal{K}(\beta)$ into c_1 results in multiplication of the body $\mathcal{K}(\beta)$ by the factor c_1/c_0. Thus if we denote by c_β the constant so that $\sigma(u \in S^{n-1} : |\langle \theta, u \rangle| \le c_\beta/\sqrt{n}) = \beta$ and define

$$\mathcal{K}'(\beta) = \left\{ x \in \mathbb{R}^n : \exists i_1, \dots, i_{[\beta N]} \text{ with } |\langle x, z_{i_j} \rangle| < \frac{c_\beta}{\sqrt{n}} \right\}$$

we will have $\mathcal{K}'(\beta) = (c_\beta/c_0)\mathcal{K}(\beta)$. Notice that the way we defined c_0 at the beginning it actually corresponds in the current notation to $c_{1/2}$.

Now, the fact that these bodies, $\mathcal{K}'(\beta)$, are equivalent to euclidean balls of radius 1 when N is sufficiently large follows in the same way as in Theorem

3. We give the sketch of the proof for $N = C(\beta,\varepsilon)n\log n$ and $\varepsilon > c/\sqrt{n}$. For the proof of the linear dependence on n see complete details in [AFM]. We pick a $1/n$ net of the sphere $(1-\varepsilon)S^{n-1}$. For a point x_0 in the net we check not only $x_0 \in \mathcal{K}'(\beta)$, but more, namely that there exist $i_1,\ldots,i_{[\beta N]}$ with $|\langle x_0, z_{i_j}\rangle| < c_\beta/\sqrt{n} - 1/n$.

Since the probability of a single event is

$$\sigma\left(u \in S^{n-1} : |\langle u,\theta\rangle| < \left(\frac{c_\beta}{\sqrt{n}} - \frac{1}{n}\right)/(1-\varepsilon)\right) = \beta + p_{\varepsilon,\beta}$$

for some $p_{\varepsilon,\beta} > 0$ (and as long as ε is not too small), we have by Chernoff bounds an exponential probability $1 - e^{-NI(\beta,\beta+p_{\varepsilon,\beta})}$ that x_0 satisfies βN of these inequalities. When N is large enough, greater than $C(\beta,\varepsilon)n\log n$, this probability suffices to take care of the whole net. Then for a point x in $(1-\varepsilon)S^{n-1}$ which is $1/n$-close to a point x_0 in the net, we have that for exactly the same indices, the inequalities $|\langle x, z_{i_j}\rangle| < c_\beta/\sqrt{n}$ are satisfied, which means that $x \in \mathcal{K}'(\beta)$. So we attained $(1-\varepsilon)D_n \subset \mathcal{K}'(\beta)$. The other inclusion is proved similarly.

This implies in particular that if N is large enough

$$(1-\varepsilon)\left(\frac{c_0}{c_{\beta+\delta}}\right)D_n \subset \mathcal{K}(\beta+\delta) \subset \mathcal{K}(\beta-\delta) \subset (1+\varepsilon)\left(\frac{c_0}{c_{\beta-\delta}}\right)D_n,$$

as long as $\delta < \delta_0(\beta)$.

The stability is reflected in the rate of change of c_β for β bounded away from 1, which one can estimate by standard volume estimates on the sphere. Thus, $c_{\beta+\delta} < c_{\beta-\delta}(1+C\delta)$. This is what we consider a stability result. We remark that it is not difficult to check that for, say, $\beta > 1/2$ and bounded away from 1, we have $c_0 c\beta < c_\beta < c_0 C\beta$ and thus

$$\left(\frac{c}{\beta}\right)D_n \subset \mathcal{K}(\beta) \subset \left(\frac{C}{\beta}\right)D_n.$$

(We mean here, that the constants c and C are universal for, say $1/2 < \beta < 3/4$, and in general depend only on δ_0 when we assume $1/2 < \beta < 1-\delta_0$.) The same is true for $\beta < 1/2$ and bounded away from 0.

The reason that stability results can be important is that sometimes one cannot check *exactly* if a proportion 1/2 of the inequalities is fulfilled, but *can* do the following weaker thing: to have a set so that each point in the set satisfies at least $1/2-\delta$ of the inequalities, and each point outside the set has at least $1/2-\delta$ inequalities which it violates. The stability result implies that we can be sure this set is $C\delta$-isomorphic to the euclidean ball (provided δ is in some bounded range).

Remark 1. The same type of results hold for the following body, where we omit the absolute value,

$$\mathcal{K}(\beta) = \left\{x \in \mathbb{R}^n : \exists i_1,\ldots,i_{[\beta N]} \text{ with } \langle x, z_{i_j}\rangle < \frac{c_0}{\sqrt{n}}\right\}.$$

Remark 2. The above discussion implies in particular a probabilistic approach to deciding whether a point is in the ball or not. Indeed, once we have a description of the ball as points satisfying at least 1/2 of the inequalities from a list of N inequalities, we can now for a given point pick randomly say 100 of the inequalities and check what proportion of them is satisfied. Again using Chernoff bounds, we can show that if it satisfies more than 1/2 of them there is a large probability that it is inside $(1+\varepsilon)D_n$ and if it violates more than 1/2 of the inequalities there is a large probability that it is outside $(1-\varepsilon)D_n$. The word "large" here is relative to the choice 100.

3 Khinchine-Type Inequalities, or Isomorphic Embeddings of ℓ_2^n into ℓ_1^N

3.1 Isomorphic Khinchine-Type Inequality

The classical Khinchine inequality states that for any $1 \le p < \infty$ there exist two constants $0 < A_p$ and $B_p < \infty$ such that

$$A_p \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \le \left(\mathrm{Ave}_{\varepsilon_1,\dots,\varepsilon_n = \pm 1} \left| \sum_{i=1}^n \varepsilon_i x_i \right|^p \right)^{1/p} \le B_p \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \tag{3}$$

holds true for every n and arbitrary choice of $x_1, \dots, x_n \in \mathbb{R}$.

In this section we show how Khinchine inequality can be realized without having to go through all 2^n summands in (3). We will insist that instead of going through all 2^n n-vectors of signs we use only N sign-vectors, where $N = (1+\delta)n$ and $0 < \delta < 1$ is any small positive number, and show that we can get inequalities like (3) loosing only in the constants. We know that one cannot do with less than n such vectors since ℓ_p and ℓ_2 are not isomorphic, and this means that the constants of isomorphism will depend on δ and explode as $\delta \to 0$.

Let us rewrite the inequality once again to make this clearer. For simplicity we only deal with the case $p = 1$; the same method works for all other $1 \le p \le 2$ (it is easy to see that $p = 1$ is the hardest case, because of monotonicity). We denote by $\varepsilon(j)$ an n-vector of ± 1, $\varepsilon(j) = (\varepsilon_{i,j})_{i=1}^n$. The average in (3) means summing over *all* possible vectors $\varepsilon(j)$, and there are 2^n of them. We wish to find vectors $\varepsilon(1), \dots, \varepsilon(N)$ such that

$$\frac{1}{N} \sum_{j=1}^N |\langle \varepsilon(j), x \rangle| \simeq |x|. \tag{4}$$

Notice that, obviously, this cannot be achieved by $\le n$ vectors since this would give an embedding of ℓ_2^n into ℓ_1^n. However, as we know that $\ell_1^{(1+\delta)n}$ *does* have isomorphic euclidean sections of dimension n (see [K]), it is conceivable that such an embedding can be constructed with a matrix of random signs.

This problem has a history. It was first shown by Schechtman in [S2] that the above is possible with a random selection of $N = Cn$ vectors, where C is a universal constant, and then repeated in [BLM] in a more general context including Kahane-type generalization. Schechtman showed that for this quantity of vectors, if chosen randomly, (4) holds with universal constants, with exponentially large probability. The question then remained whether the constant C can be reduced to be close to 1. This was resolved by Johnson and Schechtman, and follows from their paper [JS]. However, they showed the *existence* of such vectors, and not that it is satisfied for *random* $N = (1 + \delta)n$ sign-vectors. Very recently in a paper by Litvak, Pajor, Rudelson, Tomczak-Jaegermann and Vershynin [LPRTV] this was demonstrated. We reprove this result, using our method, getting slightly weaker dependence of the constants on δ. In a recent paper, joint with S. Sodin, we were able to significantly improve the dependence, from exponential in $(1/\delta)$ to polynomial, loosing only a little in the probability, see [AFMS].

One final remark is that even if we take an L_2 average instead of an L_1 average in formula (4) above, it is not correct that we can do with n random vectors alone. This is because, although the norm defined in (4) would be euclidean, it will correspond to some ellipsoid rather than to the standard ball D_n. This leads to the question of finding the smallest eigenvalue of an $(n \times n)$ matrix of random signs, which is itself an interesting question. Even the fact that with probability going to 1 exponentially fast such a matrix is *invertible* is a non trivial theorem due to Kahn, Komlós, and Szemerédy [KKS] (for a new improvement by Tao and Vu see [TV], see also [R]). The same question remains when one asks for smallest singular values of an $((1+\delta)n \times n)$ matrix of signs (where now the expectation of the smallest singular value is a constant depending on δ). This is also addressed in [LPRT], and follows also from our methods in the same way replacing $p = 1$ by $p = 2$. See also [AFMS] for better dependence on δ.

Our goal is to prove that with large probability on the choice of $N = (1 + \delta)n$ vectors $\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(N)$, where $\varepsilon(j) = (\varepsilon_{i,j})_{i=1}^n \in \{-1, 1\}^n$, we have for every x the estimate (4) where the isomorphism constants depend only on $\delta > 0$. Throughout this section we demonstrate our method by proving the following Theorem.

Theorem 4. *For any $0 < \delta < 1$ there exists a constant $0 < c(\delta)$, depending only on δ and universal constants $0 < c', C < \infty$, such that for large enough n, for $N = (1 + \delta)n$ random sign vectors $\varepsilon(1), \ldots, \varepsilon(N) \in \{-1, 1\}^n$, with probability greater than $1 - e^{-c'n}$, one has for every $x \in \mathbb{R}^n$*

$$c(\delta)|x| \le \frac{1}{N} \sum_{j=1}^{N} |\langle \varepsilon(j), x \rangle| \le C|x|.$$

Remark 1. The constant $c(\delta)$ which our proof provides is $c(\delta) = (c_1\delta)^{1+2/\delta}$, where c_1 is an absolute constant. The constant in [LPRTV] is better: $c_1^{1/\delta}$.

In [AFMS] we get a polynomial dependence on δ, but with a slightly worse exponent in the probability: $1 - e^{-c'\delta n^{1/6}}$.

Remark 2. It is easy to see that once you learn the theorem for small δ, it holds for large δ as well. This applies also to Theorem 7 and Theorem 11. Thus we may always assume that $\delta < \delta_0$ for some universal δ_0.

Before beginning the proof we want to remark on one more point. The technique we show below works for the $\ell_2^n \to \ell_2^N$ case as well, that is, to estimating the smallest singular number of an almost-square matrix. We present the proof for the $\ell_2^n \to \ell_1^N$ case, which is, even formally, more difficult. It is important to emphasize however that in the proof we do not *use* any known fact about the smallest singular number of the matrix (differently from what we do in [AFMS]). Thus, in fact, although proving $\ell_2^n \to \ell_1^N$ is formally more difficult, the main difficulty, and the reason for the exponentially bad bound that we get, lies primarily in the euclidean case. This section gives in particular another way to get lower bounds on smallest singular value of a random sign matrix using Chernoff bounds.

Proof of Theorem 4. We will denote $|||x||| = \frac{1}{N}\sum_{j=1}^{N} |\langle \varepsilon(j), x\rangle|$. This is a *random* norm depending on the choice of N sign vectors.

We need to estimate $\mathbb{P}[\varepsilon(1), \ldots, \varepsilon(N) : \forall x \in S^{n-1}\ c \leq |||x||| \leq C]$. The following step is standard: this probability is greater than

$$1 - \mathbb{P}\big[\exists x, |||x||| > C|x|\big] - \mathbb{P}\big[(\forall y, |||y||| \leq C|y|) \text{ and } (\exists x, |||x||| < c|x|)\big]. \quad (5)$$

We begin by estimating $\mathbb{P}[\exists x \in S^{n-1}, |||x||| > C]$. This is relatively easy, and does not require a new method; we do it in a similar way to the one in [BLM]: Let $\mathcal{N} = \{x_i\}_{i=1}^m$ be a $\frac{1}{2}$-net of S^{n-1}, with $m \leq 5^n$. For each $i = 1, \ldots, m$ define the random variables $\{X_{i,j}\}_{j=1}^N$ by

$$X_{i,j} = |\langle \varepsilon(j), x_i\rangle|,$$

and denote $r = \mathbb{E}|\langle \varepsilon, x\rangle|$. It is obvious that $r \leq |x| = 1$.

We use Proposition 2 from Section 1. It is well known that $X_{i,j}$ are ψ_2 random variables and $\|X_{i,j}\|_{\psi_2} \leq c_3$ for some absolute constant $c_3 > 0$ (it follows from Khinchine inequality and the basic facts about ψ_2 random variables). Proposition 2 then implies that for every $t > 0$, and a fixed i, we have

$$\mathbb{P}\left[\varepsilon(1), \ldots, \varepsilon(N) : \frac{1}{N}\sum_{j=1}^{N} X_{i,j} > r + t\right] \leq 2e^{-t^2 N/8c_3^2},$$

which in turn implies that (using that $r \leq 1$) for a fixed point $x_i \in \mathcal{N}$ and any $t > 1$ we have

$$\mathbb{P}\left[\varepsilon(1), \ldots, \varepsilon(N) : \frac{1}{N}\sum_{j=1}^{N} |\langle \varepsilon(j), x_i\rangle| > t\right] \leq 2e^{-(t-1)^2 N/8c_3^2}. \quad (6)$$

We choose t so that $2e^{(-(t-1)^2 N/8c_3^2)}5^n \le e^{-n}$, for example $t = 6c_3 + 1$. Then, with probability at least $1 - e^{-n}$, for every $i = 1, \ldots, m$,

$$\frac{1}{N}\sum_{j=1}^{N} |\langle \varepsilon(j), x_i \rangle| \le t.$$

We thus have an upper estimate for a net on the sphere. It is standard to transform this to an upper estimate on *all* the sphere (an important difference between lower and upper estimates). One uses consecutive approximation of a point on the sphere by points from the net to get that $|||x||| \le 2t = 12c_3 + 2$ for every $x \in S^{n-1}$. This completes the proof of the upper bound, where $C = 12c_3 + 2$ is our universal constant.

We now turn to the second term to be estimated in (5). Notice that when estimating this term we know *in advance* that the (random) norm $|||\cdot|||$ is bounded from above on the sphere. This is crucial in order to transform a lower bound on a net on the sphere to a lower bound on the whole sphere. For the lower bound we use our method, as described in Section 1, to estimate the following probability

$$\mathbb{P}\left[\left(\forall y \in S^{n-1}, |||y||| \le C\right) \text{ and } \left(\exists x \in S^{n-1}, |||x||| < c\right)\right]. \tag{7}$$

Let us denote by $p_{x,\alpha}$ the probability that for a random $\varepsilon \in \{-1,1\}^n$ we have $|\langle \varepsilon, x\rangle| \ge \alpha$, where $\alpha > 0$ and x is some point on S^{n-1}:

$$p_{x,\alpha} := \mathbb{P}\big[|\langle \varepsilon, x\rangle| \ge \alpha\big]. \tag{8}$$

If "doing an experiment" means checking whether $|\langle \varepsilon, x\rangle| \ge \alpha$ (with ε a random sign vector) then for $|||x|||$ to be greater than some c it is enough that c/α of the experiments succeed.

Of course, we will eventually not want to do this on all points x on the sphere, but just some dense enough set. This set turns out to be slightly more complicated than usual nets, because of the estimates we get for $p_{x,\alpha}$, but the underlying idea is still the usual simple one.

We first estimate $p_{x,\alpha}$. In estimating this probability we will consider two cases. Notice that in the simple example of $x = (1/2, 1/2, 0, \ldots, 0)$, for every $0 < \alpha < 1$ we have $p_{x,\alpha} = 1/2$. This is not a very high probability, and if we look again at the estimate (2) we see that we cannot make use of the parameters (in this case, decreasing α) to increase the rate of decay. This is a bad situation, however this is the worst that can happen, as shown in Lemma 5. Moreover, for *most* points x (these will be points x with 'many' small coordinates), a much better estimate holds, which we present in Lemma 6. The proof of the following lemma is not difficult, and we include it for the convenience of the reader.

Lemma 5. *There exists a universal constant $\alpha_0 > 0$ such that for every $x \in S^{n-1}$ we have*

$$\mathbb{P}\big[|\langle \varepsilon, x\rangle| \geq \alpha_0\big] \geq 1/2 \tag{9}$$

where $\varepsilon \in \{-1,1\}^n$ *is chosen uniformly.*

Proof. We prove this Lemma in two stages. First, assume that one of the coordinates of x is greater than or equal to α_0 (we later choose α_0, and it will be universal). Without loss of generality we may assume $x_1 \geq \alpha_0$. Then, using conditional probability

$$\mathbb{P}\Big[\Big|\sum_{i=1}^n \varepsilon_i x_i\Big| \geq \alpha_0\Big] \geq \frac{1}{2}\mathbb{P}\Big[\sum_{i=2}^n \varepsilon_i x_i \geq 0\Big] + \frac{1}{2}\mathbb{P}\Big[\sum_{i=2}^n \varepsilon_i x_i \leq 0\Big] = \frac{1}{2}.$$

This proves the statement in the case where one of the coordinates is greater than α_0. In the case where *all* the coordinates of x are smaller than α_0 we use the Berry-Esséen Theorem (see [Ha]), which will promise us that the distribution of the sum is close to gaussian, for which we can estimate the probability exactly. The theorem of Berry-Esséen states that for $X_1, X_2, \ldots$ independent random variables with mean zero and finite third moments, setting $S_n = \sum_{j=1}^n X_j$ and $s_n^2 = \mathbb{E}(S_n^2)$ one has

$$\sup_t \big|\mathbb{P}[S_n \leq s_n t] - \Phi(t)\big| \leq C' s_n^{-3} \sum_{j=1}^n \mathbb{E}(|X_j|)^3 \tag{10}$$

for all $n \geq 1$, where C' is a universal constant and where $\Phi(t)$ is the gaussian distribution function, i.e., $\Phi(t) = 1/\sqrt{2\pi}\int_{-\infty}^t e^{-s^2/2}ds$.

In our case, we let $X_j = \varepsilon_j x_j$, where ε_j's are independent ± 1 valued Bernoulli random variables. We are assuming that $\sum_{j=1}^n x_j^2 = 1$, and thus $s_n = 1$. Also, $\sum_{j=1}^n \mathbb{E}(|X_j|)^3 = \sum_{j=1}^n x_j^3$. Since we are in the case that for all j, $x_j < \alpha_0$, we have that $\sum_{j=1}^n \mathbb{E}(|X_j|)^3 \leq \alpha_0$. Inequality (10) tells us that

$$\sup_t \big|\mathbb{P}[\langle \varepsilon, x\rangle \leq t] - \Phi(t)\big| \leq C'\alpha_0.$$

We choose once $t = \alpha_0$ and once $t = -\alpha_0$, and get

$$\begin{aligned}\mathbb{P}[|\langle \varepsilon, x\rangle| \leq \alpha_0] &= \mathbb{P}[\langle \varepsilon, x\rangle \leq \alpha_0] - \mathbb{P}[\langle \varepsilon, x\rangle < -\alpha_0] \\ &\leq \Phi(\alpha_0) - \Phi(-\alpha_0) + 2C'\alpha_0 \\ &\leq \frac{2\alpha_0}{\sqrt{2\pi}} + 2C'\alpha_0.\end{aligned}$$

We choose $\alpha_0 = 1/(4(1/\sqrt{2\pi} + C'))$, then the sum is less than or equal to $1/2$ and this completes the proof of Lemma 5. □

Looking above, one sees that in the case where the coordinates of x are small we can very much improve the estimate $1/2$ in the lemma, by decreasing α_0. In the next lemma we push further this point of view. We estimate (8) when not necessarily *all* the coordinates are small (smaller than a), but a significant "weight" of them, γ^2, is. We can interplay with these two parameters a and γ, where for a given x, the parameter a determines γ, however it is the ratio that enters the estimate.

This has recently been done independently by the group Litvak, Pajor, Rudelson and Tomczak-Jaegermann in [LPRT], and the reader can either adapt the proof above or refer to Proposition 3.2 in [LPRT] for the proof of the following Lemma.

Lemma 6. *Let $x \in S^{n-1}$ and assume that for $j = 1, \dots, j_0$ we have $|x_j| < a$, and that $\sum_{j=1}^{j_0} x_j^2 > \gamma^2$. Then for any $\alpha > 0$ one has*

$$\mathbb{P}\big[|\langle \varepsilon, x\rangle| > \alpha\big] \geq 1 - \left(\frac{2\alpha}{\sqrt{2\pi}} + 2C'a\right)/\gamma$$

where C' is the universal constant from (10).

We return now to the proof of Theorem 4; we need to estimate the probability in (7). Note that we can bound it in the following way for any choice of a and γ (both x and y below are assumed to be in S^{n-1}):

$$\begin{aligned}
&\mathbb{P}\big[\,(\forall y, |||y||| \leq C) \text{ and } (\exists x \text{ s.t. } |||x||| < c)\,\big] \\
&\leq \mathbb{P}\Bigg[(\forall y, |||y||| \leq C) \text{ and } \Big(\exists x \text{ s.t. } \sum_{\{i:|x_i|\leq a\}} x_i^2 > \gamma^2 \text{ and } |||x||| < c\Big)\Bigg] \\
&\quad + \mathbb{P}\Bigg[(\forall y, |||y||| \leq C) \text{ and } \Big(\exists x \text{ s.t. } \sum_{\{i:|x_i|\leq a\}} x_i^2 \leq \gamma^2 \text{ and } |||x||| < c\Big)\Bigg].
\end{aligned}$$

This type of decomposition is by now considered standard, we were introduced to it by Schechtman, who used a similar decomposition in his paper [S1]. It is also used in [LPRT]. We need to estimate these two probabilities, choosing a and γ in the right way. We start by estimating the easy part, which is the second probability (again, in (11) both x and y belong to S^{n-1}):

$$\mathbb{P}\Bigg[(\forall y\ |||y||| \leq C) \text{ and } \Big(\exists x \text{ s.t. } \sum_{\{i:|x_i|\leq a\}} x_i^2 \leq \gamma^2 \text{ and } |||x||| < c\Big)\Bigg]. \tag{11}$$

If there exists $x \in S^{n-1}$ with $|||x||| < c$ and $\sum_{\{i:|x_i|\leq a\}} x_i^2 \leq \gamma^2$, then it is close to a vector with small support, let us denote it by $y = y(x)$. The vector $y(x)$ is defined as $y_i = 0$ when $|x_i| \leq a$ and $y_i = x_i$ when $|x_i| > a$. Thus $|x - y| < \gamma$. Since $|y| \leq |x| = 1$, it is clear that the support of y, the number of coordinates where y is non zero, cannot be larger than $[1/a^2]$. We prefer to use a normalized version, namely $y' = y/|y|$, which also has support no larger than $[1/a^2]$, is on the sphere, and satisfies

$$|y' - x| \leq |y' - y| + |y - x| \leq 1 - (1-\gamma^2)^{1/2} + \gamma \leq 2\gamma.$$

In addition we know that $|||y'||| \le |||x|||+|||y'-x||| \le c+C|x-y'| \le c+2C\gamma$.

We let $\mathcal{N}$ be a subset of S^{n-1} such that for every y' with $|y'| = 1$ and which is supported on no more than $[1/a^2]$ coordinates, there is a vector $v \in \mathcal{N}$ with $|y' - v| \le \theta_1$. (The parameter θ_1 will be chosen later.) For this we take a θ_1-net on each $[1/a^2]$-dimensional coordinate sub-sphere of S^{n-1}, and let $\mathcal{N}$ be the union of all these nets. We thus have $|\mathcal{N}| \le \binom{n}{[1/a^2]}(\frac{3}{\theta_1})^{[1/a^2]}$. If there exists x as above, and correspondingly y and y', then there exists $v \in \mathcal{N}$ with $|||v||| \le |||y'|||+|||v-y'||| \le c + 2C\gamma+C\theta_1$. Hence we can estimate probability (11) by

$$\mathbb{P}\big[\exists v \in \mathcal{N} : |||v||| \le c + 2C\gamma + C\theta_1\big]. \tag{12}$$

By Lemma 5, for a given $v \in \mathcal{N}$ (for any unit vector, for that matter) $p_{v,\alpha_0} = \mathbb{P}[|\langle\varepsilon, v\rangle| \ge \alpha_0] \ge 1/2$. In order to estimate the probability in (12), we choose in our scheme $\beta = 1/4$ (so, it is smaller than p_{v,α_0}) to be the proportion of "trials" $\{|\langle\varepsilon, v\rangle| \ge \alpha_0\}$ we want to succeed. We want $\beta\alpha_0 \ge c + 2C\gamma + C\theta_1$, so we have to make sure that γ, θ_1 and c are small enough, each say less than $\alpha_0/20C$. At this point we *choose* both γ and θ_1 to be equal $\alpha_0/20C$. The choice of c is postponed to later on since in the second part of the proof we have some more conditions on it.

Proposition 1 gives that for a given v

$$\mathbb{P}\big[|||v||| \le c + 2C\gamma + C\theta_1\big] \le e^{-NI(\frac{1}{4},\frac{1}{2})}.$$

Combining this with the size of $\mathcal{N}$, and the trivial calculation for $I(\frac{1}{4}, \frac{1}{2})$, we get that

$$\mathbb{P}\big[\exists v \in \mathcal{N} : |||v||| \le c + 2C\gamma + C\theta_1\big] \le \binom{n}{[1/a^2]}\Big(\frac{3}{\theta_1}\Big)^{[1/a^2]} e^{-c''n} \tag{13}$$

for $c'' = \ln(3^{3/4}/2)$.

We want this probability to be very small, less than $\frac{1}{2}e^{-c'n}$. Thus we get a restriction on a which is very mild (θ_1 has already been chosen), which we keep in mind for the time when we choose the constants. (The parameter a will later be chosen to be a small constant depending only on δ, and since n is assumed to be large, this condition will automatically be satisfied.)

We turn now to the more difficult task of estimating (again, x and y are assumed to be in S^{n-1}):

$$\mathbb{P}\Bigg[(\forall y\ |||y||| \le C) \text{ and } \Big(\exists x \text{ s.t.} \sum_{\{i:|x_i|\le a\}} x_i^2 > \gamma^2 \text{ and } |||x||| < c\Big)\Bigg]. \tag{14}$$

Let $\mathcal{N}$ be this time a θ-net on S^{n-1}, θ is yet another parameter we will choose later on. We can find one with cardinality $\le (\frac{3}{\theta})^n$. We bound (14) by

$$\mathbb{P}\big[\exists v \in \mathcal{N}' \text{ s.t. } |||v||| < c + C\theta\big] \tag{15}$$

where $\mathcal{N}' = \{v \in \mathcal{N} : \sum_{\{i:|v_i|\le a+\theta\}} v_i^2 \ge (\gamma-\theta)^2\}$. Indeed, if there exists $x \in S^{n-1}$ such that $\sum_{\{i:|x_i|\le a\}} x_i^2 > \gamma^2$ and $|||x||| < c$ then there is a vector $v \in \mathcal{N}$ such that $|x-v| \le \theta$ and we have $|||v||| \le |||x||| + |||x-v||| < c + C\theta$. Also, all the coordinates i for which $|x_i| \le a$ satisfy of course $|v_i| \le a+\theta$, and the square root of the sum of squares of these coordinates for v cannot differ by more than θ from the square root of the sum of squares of these coordinates for x. Therefore when taking squares the difference is at most $(\gamma-\theta)^2$. Hence if for the norm $|||\cdot|||$ there exist an $x \in S^{n-1}$ for (14), then there exists also some $v \in \mathcal{N}'$ for (15). By Lemma 6, for a given $v \in \mathcal{N}'$ we have for any $\alpha > 0$ that

$$p_{v,\alpha} = \mathbb{P}\big[|\langle\varepsilon, v\rangle| \ge \alpha\big] \ge 1 - \bigg(\frac{2\alpha}{\sqrt{2\pi}} + 2C'(a+\theta)\bigg)/(\gamma-\theta).$$

We return to our scheme, in order to estimate the probability in (15). Assume $\beta\alpha \ge c + C\theta$ (where β will be the portion of good trials out of N according to our scheme, and α another constant we later choose); Proposition 1 together with the estimate (2) gives that for a given v

$$\mathbb{P}\big[|||v||| \le c + C\theta\big] \le 2^N(1-p_{v,\alpha})^{(1-\beta)N},$$

and so for a given $v \in \mathcal{N}'$ we can estimate

$$\mathbb{P}\big[|||v||| \le c + C\theta\big] \le \bigg(2\Big(\frac{(2\alpha/\sqrt{2\pi}) + 2C'(a+\theta)}{(\gamma-\theta)}\Big)^{(1-\beta)}\bigg)^N.$$

Combining this with the size of $\mathcal{N}'$ (which is at most the size of $\mathcal{N}$) we get that

$$\mathbb{P}\big[\exists v \in \mathcal{N}' : |||v||| \le c + C\theta\big] \le \Big(\frac{3}{\theta}\Big)^n\bigg(2\Big(\frac{(2\alpha/\sqrt{2\pi}) + 2C'(a+\theta)}{(\gamma-\theta)}\Big)^{(1-\beta)}\bigg)^N.$$

We choose β such that $(1-\beta)(1+\delta)n = (1+\delta/2)n$, (so, $\beta = \delta/(2(1+\delta))$) thus we have (remembering that $N = (1+\delta)n$) that

$$\begin{aligned}&\mathbb{P}[\exists v \in \mathcal{N}' \text{ s.t. } |||v||| \le c + C\theta]\\ &\le \bigg[\Big(\frac{3}{\theta}\Big)2^{(1+\delta)}\frac{(2\alpha/\sqrt{2\pi}) + 2C'(a+\theta)}{(\gamma-\theta)}\bigg]^n \cdot \bigg[\frac{(2\alpha/\sqrt{2\pi}) + 2C'(a+\theta)}{(\gamma-\theta)}\bigg]^{(\delta/2)n}.\end{aligned}$$

We are now in the place to choose all the various constants. We let $a = c = \theta$. As θ will soon be chosen very small, smaller than $\gamma/2$ (which was already specified in the first part) we have that $\gamma - \theta$ is bounded from below by a universal constant $\alpha_0/40C$. We need to make sure that $\beta\alpha \ge c + C\theta$, so we let $\alpha = 12C\theta/\delta$. What we get, so far, without choosing θ yet, is that

$$\mathbb{P}\big[\exists v \in \mathcal{N}' \text{ s.t. } |||v||| \le c + C\theta\big] \le \Big(\frac{C_1}{\delta}\Big)^n \cdot \Big(\frac{C_2\theta}{\delta}\Big)^{(\delta/2)n}$$

for universal constants C_1 and C_2. To make this probability less than $\frac{1}{2}e^{-c'n}$ we choose $\theta \le (1/2e^{-c'}\delta/C_1)^{2/\delta}(\delta/C_2)$ and the proof of the estimate for the probability (14), and of the whole of Theorem 4, is complete. □

3.2 Euclidean Sections of ℓ_1^N Generated by Isotropic Convex Bodies

The second application we present also deals with Khinchine-type inequalities, this time when the matrix elements are chosen differently. The conceptual difference is that they are not all independent anymore.

Instead of considering the norm of the form given in (4), with N random sign vectors, we do the same but with vectors distributed uniformly in some isotropic convex body K (just as in (4) they were distributed uniformly in the discrete cube). By isotropic we mean that K satisfies $Vol(K) = 1$, $\int_K x = 0$ and, most importantly, for every $\theta \in S^{n-1}$ the integral $\int_K \langle x, \theta\rangle^2$ is a constant independent of θ, depending only on K, which is called the (square of the) isotropic constant of K and denoted L_K^2. It is easy to check that every body has a linear image which is isotropic. In other words, saying that the body is in isotropic position only means that we identify the right euclidean structure with which to work.

We want to check, as in Section 3.1, how close the randomly defined norm $\frac{1}{N}\sum_{j=1}^N |\langle z_j, x\rangle|$ is to being euclidean, when the points z_j are chosen randomly with respect to the volume distribution in K. We prove the following theorem.

Theorem 7. *For any* $0 < \delta < 1$ *there exist a constants* $0 < c(\delta)$*, depending only on* δ *and universal constants* $0 < c', C < \infty$ *such that for large enough* n*, for any convex body* $K \subset \mathbb{R}^n$ *in isotropic position, with probability greater than* $1 - e^{-c'n}$ *we have that*

$$c(\delta) L_K |x| \le \frac{1}{N}\sum_{j=1}^N |\langle z_j, x\rangle| \le C L_K |x|,$$

where $N = (1+\delta)n$ *and* z_j *are chosen independently and uniformly inside the body* K*.*

Proof. We begin with the upper estimate. As explained before in this paper, upper bounds usually present less difficulties, and the use of Chernoff bounds is not needed. When a point z is chosen uniformly inside a convex body, the distribution of the random variable $\langle x, z\rangle$ (where x is some fixed point) is *not necessarily* a ψ_2 distribution. For example for the unit ball of ℓ_1^n and the point $x = (1, 0, 0, \dots, 0)$, the decay of the distribution function is only exponential and not gaussian. This is the worst that can happen though. We say that a random variable X has ψ_1 behavior if there exists a constant λ such that $\mathbb{E}e^{X/\lambda} \le 2$. The smallest λ for which this inequality holds is what we call the ψ_1 norm of X. The following Lemma (resulting from the work of C. Borell) shows that our random variables are always ψ_1 (for proof see [MS] Appendix III and [GiM2] Section 1.3 and Lemma 2.1)

Lemma 8. *There exists a universal constant* C' *such that for any isotropic convex body* K*, and any direction* $\theta \in S^{n-1}$ *the random variable* $X = |\langle \theta, z\rangle|$

where z is chosen uniformly in K has ψ_1 distribution and its ψ_1 norm is equivalent to L_K and to its expectation, namely

$$L_K \le \|X\|_{\psi_1} \le C'\mathbb{E}X \le {C'}^2 L_K.$$

We thus need a proposition of the like of Proposition 2 but for ψ_1 distributions and it is the following, the proof of which is standard, in the same lines of the inequality in Proposition 2.

Proposition 9. *Let $\{X_j\}_{j=1}^N$ be i.i.d. copies of the random variable X. Assume that X is ψ_1 and that the ψ_1 norm of X is smaller than some constant A. Then for any t,*

$$\mathbb{P}\Big[\Big|\frac{1}{N}\sum_{j=1}^N X_j - \mathbb{E}X\Big| > t\Big] \le 2e^{-Nt/(3A)}. \tag{16}$$

Thus, for $t = C''L_K$ with C'' large enough, this probability is enough to take care of a 1/2 net of the sphere, and then by successive approximation one has an upper bound for the whole sphere.

We turn to the lower bound, where we will use our method. We need, as usual, to estimate the probability $\mathbb{P}[z \in K : |\langle x, z\rangle| < L_K\alpha]$. This is done in the following Proposition:

Proposition 10. *There exists a universal constant C_1 such that for any $\alpha > 0$ and for any symmetric isotropic convex body K, for every direction $u \in S^{n-1}$*

$$\mathbb{P}\big[x \in K : |\langle x, u\rangle| < L_K\alpha\big] < C_1\alpha.$$

Proof. We use two well known facts from Asymptotic Geometric Analysis. First, all central sections of an isotropic convex body have volume $\approx 1/L_K$. Second, for a centrally symmetric convex body K and a direction u, of all sections of K by hyperplanes orthogonal to u, the one with the largest volume is the central section (for proofs see e.g. the survey [GiM1]). In particular the two facts imply that there exists some universal constant C_1 such that for any direction u, any section of K orthogonal to u has $(n-1)$-dimensional volume $\le C_1/2L_K$. Now use Fubini Theorem to get that $\mathbb{P}[x \in K : |\langle x, u\rangle| < L_K\alpha] < C_1\alpha$. □

Notice that, differently from what was going on in Section 3.1, here for *any* point x, we can make the probability as small as we want by reducing α. This allows us to use just one simple net: take a θ-net $\mathcal{N}$ in S^{n-1}, with less than $(3/\theta)^n$ points x_i. Define the random variables $X_{i,j} = |\langle z_j, x_i\rangle|$. We know that for $\beta < 1 - C_1\alpha$ (which is hardly a restriction, α will be very small and so will β) we have

$$\mathbb{P}\Big[\frac{1}{N}\sum_{j=1}^N X_{i,j} > \beta L_K\alpha\Big] \ge 1 - e^{-NI(\beta, 1-C_1\alpha)}.$$

We choose β so that $(1+\delta)(1-\beta) = (1+\delta/2)$, hence $\beta = \delta/(2(1+\delta))$. We choose $\theta = \beta\alpha/2C$, where C comes from the upper bound (which is CL_K). To make sure that the probability that the above holds for all points in the net we ask that

$$\left(\frac{3}{\theta}\right)^n 2^N (C_1\alpha)^{(1+\frac{\delta}{2})n} \le \frac{1}{2}e^{-c'n}.$$

For this we choose $\alpha = (C_2\delta)^{2/\delta}$ for some universal C_2, and get the lower bound for each point of the θ-net of S^{n-1}. Now using the upper bound, for every $x \in S^{n-1}$ we have for some i that (denoting $|||x||| = \frac{1}{N}\sum_1^N |\langle x, z_i\rangle|$)

$$|||x||| \ge |||x_i||| - |||x - x_i||| \ge \beta L_K \alpha - \theta C L_K.$$

Thus the proof of the lower bound, and of Theorem 7, is complete. □

3.3 Reducing the Level of Randomness

Another variant of the question answered in Section 3.1 which we discuss in this section is related to a more "explicit" construction of n-dimensional euclidean sections of $\ell_1^{(1+\delta)n}$. In Section 3.1 we described Schechtman's question about realizing such a euclidean section by the image of a random sign matrix. In a different paper, [S1], Schechtman showed that for $\delta = 1$, that is, a $2n \times n$ matrix, one can take the upper half to be the identity matrix, and the lower to be n random sign vectors, and this gives an isomorphic euclidean section of ℓ_1^{2n}. Using this method we can also take only the identity with only δn additional random sign vectors (so, get a section of $\ell_1^{(1+\delta)n}$), and the isomorphism constant will depend on δ. Below we present a similar construction, in which we use our method to show that when the upper half (that is, the first n vectors) is a Hadamard matrix, namely a matrix of signs whose rows are orthogonal, and add to it δn random sign vectors below, you also get an isomorphic euclidean section of $\ell_1^{(1+\delta)n}$.

Remark. While it is not known precisely for which n a Hadamard matrix exists (the Hadamard conjecture is that they exist for $n = 1, 2$ and all multiples of 4), it is known that the orders of Hadamard matrices are dense in the sense that for all ε if n is sufficiently large there will exist a Hadamard matrix of order between n and $n(1-\varepsilon)$. However, we only use the fact that the first n rows are an orthonormal basis of $\mathbb{R}^n$ and Theorem 11 below holds if we replace the Hadamard matrix by any other orthonormal matrix (normalized properly). For more information on Hadamard matrices we refer the reader to [H]. We chose Hadamard matrices since this way the section we get is generated by a sign matrix.

Denote the rows of the $n \times n$ Hadamard matrix W_n by $(1/\sqrt{n})\varepsilon(j)$ for $j = 1, \dots, n$. They form an orthonormal basis of $\mathbb{R}^n$. We prove below that by adding the random sign vectors $\varepsilon(n+1), \dots, \varepsilon(n+\delta n)$ we get a matrix which gives an isomorphic euclidean section of $\ell_1^{(1+\delta)n}$. We prove

Theorem 11. *Let $0 < \delta < 1$, and denote $N = (1+\delta)n$. There exists a constant $c(\delta)$ depending only on δ, and universal constants c', C, such that for large enough n, with probability $1 - e^{-c'\delta n}$, for δn random sign-vectors $\varepsilon(j) \in \{-1,1\}^n$, with $j = n+1, \dots, n+\delta n$, one has for every $x \in \mathbb{R}^n$*

$$c(\delta)|x| \le \frac{1}{N}\sum_{j=1}^{N} |\langle x, \varepsilon(j)\rangle| \le (1+\sqrt{\delta}C)|x|, \tag{17}$$

where one may take $c(\delta) = c_1\delta^{3/2}/(1+\ln(1/\delta))$ for a universal c_1.

Proof. Since $(1/\sqrt{n})\varepsilon(1), \dots, (1/\sqrt{n})\varepsilon(n)$ is an orthonormal basis of $\mathbb{R}^n$, every $x \in S^{n-1}$ can be uniquely written as $x = (1/\sqrt{n})\sum a_i\varepsilon(i)$, and $a_i = (1/\sqrt{n})\langle x, \varepsilon(i)\rangle$. So, $\sum a_i^2 = 1$, and $a = (a_i)_{i=1}^n \in S^{n-1}$ depends on x. Our aim is to show that inequality (17) holds. We can rewrite it as

$$c(\delta)|x| \le \frac{1}{(1+\delta)}\frac{1}{n}\sum_{j=1}^{n}|\langle x, \varepsilon(j)\rangle| + \frac{1}{N}\sum_{j=1}^{\delta n}|\langle x, \varepsilon(n+j)\rangle| \le \big(1+\sqrt{\delta}C\big)|x|. \tag{18}$$

Fix $x \in S^{n-1}$. To prove the upper bound, first notice that the first summand satisfies

$$\frac{1}{(1+\delta)}\frac{1}{\sqrt{n}}\sum |a_i| \le \frac{1}{(1+\delta)}\sqrt{\sum a_i^2} = \frac{1}{(1+\delta)}.$$

As for the second one, we can use a standard upper bound approach as in Section 3.1. Notice that the second term is in fact

$$\Big[\frac{\delta}{1+\delta}\Big]\frac{1}{\delta n}\sum_{j=1}^{\delta n}\big|\langle x, \varepsilon(n+j)\rangle\big|,$$

so the upper bound we would expect for this part is $\simeq \delta C$. However, this is not true, since if we would go ahead trying to prove this, the probability we would get for an individual x to satisfy this would be $1 - e^{-c\delta n}$ and this is not enough to take care of say a 1/2-net of the sphere. Thus, we need to take a larger deviation in order to increase the probability. Take a 1/2-net $\mathcal{N}$ of S^{n-1}, then taking $t = C/(2\sqrt{\delta})$ in inequality (6) with $N = \delta n$ we get that for a fixed $x \in \mathcal{N}$

$$\mathbb{P}\Big[\frac{1}{\delta n}\sum_{j=1}^{\delta n}|\langle x, \varepsilon(n+j)\rangle| \le C/\big(2\sqrt{\delta}\big)|x|\Big] \ge 1 - e^{-\delta n (C/(2\sqrt{\delta})-1)^2/c_4}$$

for some universal c_4. For large enough C, this probability is enough to take care of the whole 1/2-net, and by successive approximation we get that with high probability $1 - e^{-cn}$ we have for every x

$$\left[\frac{\delta}{1+\delta}\right]\frac{1}{\delta n}\sum_{j=1}^{\delta n}|\langle x,\varepsilon(n+j)\rangle| \le \sqrt{\delta}C|x|. \tag{19}$$

The bound for the whole expression is thus as wanted, and in fact we will later use the bound $\sqrt{\delta}C$ for the second term separately.

For the lower bound, denote

$$A_\gamma = \Big\{x \in S^{n-1} : \frac{1}{\sqrt{n}}\sum_{i=1}^{n}|a_i| \le \gamma\Big\}.$$

If $x \notin A_\gamma$ then in inequality (17) we have

$$\frac{1}{N}\sum_{j=1}^{N}|\langle x,\varepsilon(j)\rangle| \ge \frac{1}{(1+\delta)}\frac{1}{n}\sum_{j=1}^{n}|\langle x,\varepsilon(j)\rangle| \ge \frac{\gamma}{(1+\delta)}$$

and so a lower bound of the order $\gamma/(1+\delta)$ holds. We want to choose γ so that all $x \in A_\gamma$ are taken care of by the δn random sign vectors, that is, by the right hand side term in equation (18).

We need the following observation: Let $\alpha < 1$ be some proportion. If $1/\sqrt{n}\sum_{i=1}^{n}|a_i| \le \gamma$, denote by a_{i_0} the term a_i which is in absolute value the (αn)'st largest one. Then

$$\gamma \ge \frac{1}{\sqrt{n}}\sum_{i=1}^{n}|a_i| \ge \frac{1}{\sqrt{n}}\sum_{\alpha n \text{ biggest}}|a_i| \ge \frac{\alpha n}{\sqrt{n}}\cdot|a_{i_0}| \ge \alpha\sqrt{n}\Big(\frac{1}{(1-\alpha)n}\sum_{i\in I}|a_i|^2\Big)^{\frac{1}{2}}$$

where I is the set of the $(1-\alpha)n$ coordinates a_i which are smallest in absolute value. Thus for some set I of coordinates, with $|I| = (1-\alpha)n$, we have $(\sum_{i\in I}|a_i|^2)^{1/2} \le \gamma\sqrt{1-\alpha}/\alpha$.

We let E stand for a subspace spanned by αn of the (normalized) Hadamard basis row vectors $\varepsilon(1),\dots,\varepsilon(n)$. The observation tells us that every $x \in A_\gamma$ can be written as $x = y + z$ with y in some such E, and $|z| < \gamma\sqrt{1-\alpha}/\alpha$. We will choose α so that the δn additional random vectors take care of *all* vectors in *all* the E's, with a lower bound c''. We will then choose γ such that $\gamma(\sqrt{1-\alpha}/\alpha)C\sqrt{\delta} \le c''/2$ (where C is from the upper bound in (19)) and by this we will finish, since then

$$\frac{1}{N}\sum_{j=1}^{\delta n}|\langle x,\varepsilon(n+j)\rangle| \ge \frac{1}{N}\sum_{j=1}^{\delta n}|\langle y,\varepsilon(n+j)\rangle| - C\sqrt{\delta}|z| \ge c''/2.$$

(So, we will have a lower bound $c(\delta) = \min(\gamma, c''/2)$.) We make sure that $\gamma\sqrt{1-\alpha}/\alpha < 1/2$, so that $|y| > 1/2$.

We thus have to find α and c'' such that for a set of $\binom{n}{\alpha n}$ subspaces E of dimension αn we have for all $y \in E \cap S^{n-1}$ that

$$\frac{1}{N}\sum_{j=1}^{\delta n} |\langle y, \varepsilon(n+j)\rangle| \geq c''.$$

We take a θ-net on this set (the value of θ will be chosen later). Its cardinality is less than $\binom{n}{\alpha n}(\frac{3}{\theta})^{\alpha n}$, this is $\leq (\frac{e}{\alpha})^{\alpha n}(\frac{3}{\theta})^{\alpha n}$. For a single y in the net we estimate the probability that

$$\frac{1}{N}\sum_{j=1}^{\delta n} |\langle y, \varepsilon(n+j)\rangle| \geq c''$$

by our usual method. The probability for a single experiment $|\langle y, \varepsilon\rangle| \geq \alpha_0 |y|$ is bounded below, for a suitably chosen α_0, by $1/2$, from Lemma 5. Choose, say $\beta = 1/4$, and just as in previous sections

$$\mathbb{P}\left[\frac{1}{N}\sum_{j=1}^{\delta n} |\langle y, \varepsilon(n+j)\rangle| > \frac{\delta}{1+\delta}\beta\alpha_0/2\right] \geq 1 - e^{-2c'\delta n}$$

(this is our *definition* of c'). We choose, say, $\theta = \alpha_0\sqrt{\delta}/(100C)$ (since the upper bound for the second part in (18) is $\sqrt{\delta}C$ and so we are able to transfer the bound from the net to the whole set) and then we choose α such that

$$\left(\frac{3e}{\alpha\theta}\right)^{\alpha n} e^{-2\delta n c'} \leq e^{-c'\delta n}.$$

This holds if $\alpha \leq c_2\delta/(1+\ln(1/\delta))$ for some universal c_2. This finally gives, say, $c'' = \delta\alpha_0/16$. We still have to return to γ, which we can choose to be $\gamma = \alpha c''/(4C\sqrt{\delta})$, and this is the order of the lower bound we achieve, $c(\delta) = c_1\delta^{3/2}/(1+\log(1/\delta))$. □

4 Dvoretzky-Type Theorems

In this Section we deal with a different question, namely with a global Dvoretzky-type theorem. We will first illustrate yet another application where our method works, reproving a well known version of the global Dvoretzky Theorem. First, we will show how the upper bound can be obtained using Chernoff's inequalities and standard concentration on the sphere. This is different from the standard way of proof for global Dvoretzky Theorem (which we also indicate below), where usually the upper bound is obtained by a deep geometric argument about concentration on the product of spheres, (inequality (20) below). The lower bound we then obtain by using our Chernoff scheme.

We will then state, as a conjecture, a natural strengthening of the global Dvoretzky Theorem (which would be optimal), the local analogue of which is known to hold. We show how this strengthened theorem would be implied by the validity of a small ball probability conjecture of Latała and Oleszkiewicz [LO].

In the last part of the section we discuss an alternative parameter that is of interest, and is similar to a parameter introduced by Klartag and Vershynin in [KV], and which clarifies some other cases where the global Dvoretzky-type theorem holds in an improved form.

4.1 About Global Dvoretzky's Theorem

The global analogue of Dvoretzky's Theorem first appeared in [BLM], in a non explicit form, and explicitly in [MS2], and is the following Theorem, which, by duality, means that the Minkowski average of $C'(a/M^*)^2$ random rotations of a convex body K with radius a and mean width M^* is isomorphic to a euclidean ball of radius M^*.

Theorem 12. *There exist universal constants c, c', C and C' such that for every symmetric convex body $K \subset \mathbb{R}^n$ satisfying $\frac{1}{b}D \subseteq K$, letting $M = M(K)$, we have with probability $1 - e^{-c'n}$, that the $N = C'(b/M)^2$ random orthogonal transformations $U_1, \dots, U_N \in O(n)$ satisfy for every $x \in \mathbb{R}^n$ that*

$$cM|x| \le \frac{1}{N}\sum_{i=1}^{N} \|U_i x\|_K \le CM|x|.$$

Remark. We later show that in fact the constant C' above can be chosen to be $(4+\delta)$ for any $\delta > 0$, and then all other constants depend on δ. We also later conjecture that in fact $(1+\delta)$ for any $\delta > 0$ should be the optimal constant.

It is clear that we are dealing with a lower and an upper bound for a sum of random variables. It is also clear what our experiments will be: for a random orthogonal transformation U_j, there is some fixed probability (for a given $x \in S^{n-1}$) that $\|U_j x\| \ge \alpha M$. We say that the experiment is a success if this happens. In fact, taking $\alpha = 1$ and taking M to be the median of the norm instead of its expectation (they are very close, see [MS]), this probability is exactly $1/2$. If at least $1/4$ of the trials succeed, we get the average above to be at least $M/4$. This can be thought of as the main idea, however, we need something stronger in order to get that $N \simeq (b/M)^2$ rotations are enough, and this naive approach will only give $N \simeq n$. which is typically much larger.

4.1.1 The Upper Bound, Using Concentration on the Product of Spheres

We start with the upper bound. The upper bound is usually handled with the estimate (see 6.5.2 in [MS]): Fix $x \in S^{n-1}$, then for random $U_j \in O(n)$, $j = 1, \dots, N$, and $t > 0$ we have

$$\mathbb{P}\Big[(U_1,\dots,U_N):\Big|\frac{1}{N}\sum_{j=1}^{N}\|U_jx_i\|-M\Big|>tM\Big]\le\sqrt{\frac{\pi}{2}}e^{-t^2N(M/b)^2(n-2)/2},\tag{20}$$

which is a concentration result on the product of N spheres.

Concentration on the product of n spheres is quite a strong tool, and at first glance this seems appropriate since we are searching for a strong result: not a sum of $N \simeq n$ variables, but much less (typically), $N \simeq (b/M)^2$. In what follows we will several times use the well known and easily provable fact that $b \le \sqrt{n}M$. To complete the upper bound using (20) we simply take a 1/2-net on the sphere, with at most 5^n points x_i. For each i we use (20) with, say, $t = 4$, and get that with probability at least $1 - 5^n\sqrt{\frac{\pi}{2}}e^{-8N(M/b)^2(n-2)}$ we have for every x_i in the net that

$$\frac{1}{N}\sum_{j=1}^{N}\|U_jx_i\| < 5M.$$

We clearly see that if $N \ge (b/M)^2$, the probability above is exponentially close to 1. Passing from a net on the sphere to the whole sphere, in an upper bound, is standard, and may be done by successive approximation, which gives us that for every $x \in \mathbb{R}^n$

$$\frac{1}{N}\sum_{j=1}^{N}\|U_jx_i\| < 10M.$$

Before moving to the lower bound, we would like to offer an alternative proof for the upper bound, which does not use (20) directly, but gives a proof of a slightly weaker estimate (which is sufficient for our needs) by using only Chernoff's bounds. We remark that (20), which is concentration on the product of n spheres, is a much deeper fact than the concentration estimate on the sphere, see [GrM].

4.1.2 The Upper Bound, Avoiding Concentration on the Product of Spheres

In this paper, up till now, we have mostly shown how the use of Chernoff bounds is useful in obtaining lower bounds, where the standard large deviation technology was not enough. Below we will show how standard concentration on the sphere, together with Chernoff bounds, provides an alternative proof for the upper bound. This approach was pursued further in the paper [Ar], and one of its merits is that it is quite robust.

We will use concentration on the sphere which states that

Lemma 13. *For* $t > 0$

$$\sigma\left(x \in S^{n-1} : |\|x\| - M| \ge tM\right) \le \sqrt{\frac{\pi}{2}}e^{-t^2(M/b)^2(n-2)/2},\tag{21}$$

and is simply the case $N = 1$ of inequality (20).

Fix $x \in S^{n-1}$, and denote

$$A_j = \{U \in O(n) : 2^j M < \|Ux\| \le 2^{j+1} M\},$$

where $j = t, t+1, \ldots, \log(b/M)$, for an integer $t \ge 2$. By Lemma 13 we thus have $\mathbb{P}(U \in A_j) \le \sqrt{\frac{\pi}{2}} e^{-(2^j-1)^2 (M/b)^2 (n-2)/2}$. We also denote $m_j = N 2^{-j}/j^2$. If out of the N transformations $U_1, \ldots, U_N$, for every $j \ge t$, less than m_j of them belong to A_j then

$$\frac{1}{N}\sum_{i=1}^{N} \|U_i x\| \le \left[2^t + \sum_{j=t}^{\log(b/M)} 2^{j+1}\frac{m_j}{N}\right] M \le (2^t+2)M.$$

Fix some $j \ge t$. We use Chernoff's Proposition 1 to bound from above the probability that more than m_j of the N transformations are in A_j. For us now $p = \sqrt{\frac{\pi}{2}} e^{-(2^j-1)^2 (M/b)^2 (n-2)/2}$ and $\beta = 2^{-j}/j^2$, and in particular $\beta > p$ since $j \ge 2$. Our scheme implies that this probability is bounded by $(2\sqrt{\frac{\pi}{2}})^N e^{-N(M/b)^2 (n-2)(2^j-2)/2j^2}$.

Adding these expressions up for $j = t, t+1, \ldots$ we get that

$$\mathbb{P}\left[(U_1, \ldots, U_N) : \frac{1}{N}\sum_{i=1}^{N} \|U_i x\| > (2^t+1)M\right] \le e^{-c' N (M/b)^2 (n-2)}$$

for some c' depending only on t, as long as t is bigger than some universal constant. (Where we use, as usual, that always $(b/M) \le \sqrt{n}$.) Thus by taking N above to be say $2\ln 5 (M/b)^{-2}/c'$ we get the upper bound $(2^t+2)M$ for a whole (1/2)-net. Successive approximation then gives the upper bound CM with $C = 2^{t+1} + 4$. Recall that t can be chosen to be anything above some universal constant C_0. Enlarging t will make the probability better, which means we can take N as any constant proportion, even < 1, of $(M/b)^{-2}$, and have an upper bound depending on this proportion. We have thus proved the upper bound, using only standard concentration, and Chernoff.

4.1.3 The Lower Bound

One crucial point in the lower bound's proof is estimating the probability of a success in a specific experiment, that is, $\mathbb{P}[U \in O(n) : \|Ux\| \ge \alpha M]$. What we usually need, is to be able to decrease this probability significantly by sending α to 0. The standard concentration argument on the sphere, such as Lemma 13, gives that for $\alpha < 1$ and a fixed $x \in S^{n-1}$

$$\mathbb{P}[U \in O(n) : \|Ux\| \ge \alpha M] \ge 1 - \sqrt{\frac{\pi}{8}} e^{-(1-\alpha)^2 (M/b)^2 (n-2)/2}. \qquad (22)$$

This is enough for proving Theorem 12 with a universal C', but sending α to 0 does not help to change the rate of decrease of the probability in (22), and this is the reason for not getting the conjectured (below) optimal constant.

To complete the proof of the lower bound we take again a net on the sphere, this time an $(1/4C)$-net where CM is the upper bound which we already know from either one of the two previous subsections. We use (22) and our scheme with $\alpha = 1/2$ and $\beta < \alpha$ (small) to be specified later, and have that for a given x in the net, with high probability, more than βN of the operators U_j satisfy $\|U_j x\| \geq M/2$, more precisely

$$\mathbb{P}\Big[U_1 \ldots, U_N \in O(n) : \frac{1}{N}\sum_{j=1}^{N} \|U_j x\| \geq \alpha\beta M\Big]$$
$$\geq 1 - 2^{-u(\beta)N}\left(\sqrt{\frac{\pi}{8}} e^{-\frac{1}{8}(M/b)^2(n-2)}\right)^{(1-\beta)N}.$$

We see that if $N = C'(\frac{b}{M})^2$ this probability is greater than

$$1 - 2^{-u(\beta)C'(b/M)^2} e^{-(1-\beta)(n-2)/8},$$

and so for $\beta = \beta_0$ for some universal β_0, and for large enough C' (which, notice, depends on the bound C we have achieved before), we can have this probability so big that it happens simultaneously for the whole net (and even this we can make sure happens with exponentially high probability). Now with use of the upper bound and the inverse triangle inequality we transfer the estimate to the whole sphere, and the proof is complete. □

4.2 With Conjectured Small-Ball Probability Estimate

In this section we discuss the following conjecture which is different from Theorem 12 by specifying the constant C' in that theorem.

Conjecture 14. For every δ there exists a constant $c(\delta)$ depending only on δ, and universal constants c' and C, such that for every symmetric convex body $K \subset \mathbb{R}^n$ satisfying $(1/b)D \subseteq K$, and with $M = M(K)$ there exist $N = (1+\delta)(b/M)^2$ orthogonal transformations $U_1, \ldots, U_N \in O(n)$ such that

$$c(\delta)M|x| \leq \frac{1}{N}\sum_{i=1}^{N} \|U_i x\|_K \leq CM|x|. \tag{23}$$

Remark. For large $\delta > 0$, this is Theorem 12. Also, we may assume $\delta < \delta_0$ for some universal δ_0 since if we prove Conjecture 14 for such δ it will then follow from standard arguments that the same holds for all $\delta > 0$. Finally, this is the best we can hope for in the general case, in the sense that using less than $N = (b/M)^2$ transformations U_i, we cannot expect the average always to be isomorphic to euclidean, as is implied by the example of a cylinder with basis of dimension $(M/b)^2 n$ (see a parallel local version in [GMT]; the local version also follows from an earlier result by Gordon [Go]).

To achieve such an estimate for N we need a stronger estimate than (22). Such an estimate was conjectured (for different applications) by Vershynin, and reformulated by Latała and Oleszkiewicz with an extra non-degeneration condition, as follows (see [LO], Conjecture 1 and its Corollaries). Below we formulate a variant of their conjecture, which was originally formulated in the Gaussian context, but the translation is straightforward. Notice that we formulate a variant with M being the mean of the norm whereas in [LO] the median is used;

Conjecture 15. For every constant $\kappa < 1$ there exists universal constants $C' = C'(\kappa)$, $c_0 = C_0(\kappa)$ and $w_0 = w_0(\kappa) > c_0(\kappa)$ such that if for some norm we have $(b/M)^2 \le n/w_0$ then for any $\alpha < 1$,

$$\sigma(x \in S^{n-1} : \|x\| < \alpha M) < (C'\alpha)^{\kappa((M/b)\sqrt{n}-c_0)_+^2}. \tag{24}$$

Notice that this estimates precisely the same quantity as in (22). Here we see that as $\alpha \to 0$, the estimates improve significantly.

Proof of the implication Conjecture 15 → Conjecture 14. We will first prove the implication in the non-degenerate case. Assume that Conjecture 15 is true. We start with a given $\delta > 0$, and first prove that the statement of Conjecture 14 must hold for bodies K with $b(K)/M(K) \le \sqrt{n/w_0}$, where $w_0 = \min(w_0(1-\delta/10), c_0/(10\delta))$ comes from the constants in Conjecture 15. We then show why knowing the conjecture in these cases implies all other cases.

So, let $\delta > 0$ and define $\kappa = (1 - \delta/10)$. By Conjecture 15, for every body satisfying $b(K)/M(K) \le \sqrt{n}/w_0$ we have that

$$\sigma(x \in S^{n-1} : \|x\| < \alpha M) < (C\alpha)^{(1-\delta/10)((M/b)\sqrt{n}-c_0)^2}$$

(where C, c_0 and κ, now depend δ). We choose β small enough so that $(1-\beta)N(M/b)^2(1-\delta/10)^2 \ge (1+\delta/2)$, so for example $\beta = \delta/10$ is small enough.

We now repeat the proof from Section 4.1, using the new estimate on the probability. Since β is very small (having assumed $\delta < \delta_0$), the term $u(\beta)$ from (2) has hardly any effect. We thus have an estimate $2^{u(\beta)N}(C'\alpha)^{(1+\delta/2)n}$ for the probability that for a single x a lower estimate in (23) of order $\beta\alpha M$ does not hold. We work the same way as in Section 3.1, having some choice of $\alpha = \alpha(\delta)$ for which this probability is small enough to take care of a whole $\alpha\beta/2C$-net of the sphere, where C is from the upper bound CM which we have already shown. This α will, as usual, be exponentially bad in δ, even before taking into account that $C' = C'(1-\delta/10)$ can itself have a bad dependency on δ. Notice that in this case we get not only the existence of operators U_i satisfying the inequality (23), but that (23) is true with high probability on the choice of operators (the probability coming from Chernoff, so, at least $1 - e^{-cn}$).

We turn to the case where, after specifying δ, we have a body for which $b(K)/M(K) > \sqrt{n}/w_0$ where w_0 was indicated above and depends only on

δ. This means in effect that K is very degenerate. We then do a preliminary "regulating" procedure. We pick randomly k operators U_i, $i = 1, \ldots, k$, for $k = 2(b/M)^2 w_0/n$. This is a small number depending on δ, since $k < 2w_0^2$. We now define K' to be the unit ball of the norm

$$\|x\|_{K'} = \frac{1}{k} \sum_{i=1}^{k} \|U_i x\|_K .$$

Of course, since M is simply the average of the norm on the sphere, we have that $M(K) = M(K')$. However, it is well known that the diameter of the average of k random rotates of a body is smaller by a factor about $1/\sqrt{k}$ than the diameter of the body. Since we are speaking about norms, this means that $b(K') \simeq b(K)/\sqrt{k}$. We will need the more precise result, namely that the diameter decreases almost isometrically by $1/\sqrt{k}$, provided k is not large compared to b/M, which is our case since K is degenerate. We formulate the lemma we need in its more familiar, dual form:

Lemma 16. *For any $0 < \varepsilon < 1$ there exist constants c_ε and $c(\varepsilon)$ such that for a symmetric body T, if $k < (c(\varepsilon)(\mathrm{diam}(T)/M^*(T))^{1/2}$ and $k \ln k \le n\varepsilon^2/8$ then for random $U_1, \ldots, U_k \in O(n)$ we have with probability greater than $1 - e^{-c_\varepsilon n/k^2}$ that*

$$\mathrm{diam}\left(\frac{1}{k} \sum_{i=1}^{k} U_i T\right) \le \frac{(1+\varepsilon)}{\sqrt{k}} \mathrm{diam}(T).$$

The proof of this fact follows from standard considerations, see [AM] for the case $k = 2$ which generalizes directly. In fact $c(\varepsilon)$ can be taken linear in ε and c_ε to be linear in ε^2.

Applying Lemma 16 to K° we get that for $k < \min(C(\varepsilon)(b/M)^2, n\varepsilon^2/8 \ln n)$ we have $b(K') \le (1+\varepsilon) b(K)/\sqrt{k}$. For our choice of k (and for a fixed $\delta > 0$) clearly the condition holds for n large enough, since w_0 doesn't depend on n. This means that $(b(K')/M(K')) \le (1+\varepsilon)\sqrt{n/2w_0}$, so as long as $\varepsilon \le \sqrt{2} - 1$ we may apply the proof of the first part to get that there exist rotations $V_1, \ldots, V_{N'}$, for $N' = (b(K')/M(K'))^2(1+\delta)$ so that

$$\frac{1}{N'} \sum_{j=1}^{N'} \|V_j x\|_{K'} \simeq |x|,$$

with constants of isomorphism depending only on δ. Taking the $N = N'k$ rotations $U_i V_j$ we have that

$$\frac{1}{N'k} \sum_{j=1}^{N'} \sum_{i=1}^{k} \|V_j U_i x\|_K \simeq |x|.$$

From our choice of k and the estimate on N' we see that

$$N = N'k = k\big(b(K')/M(K')\big)^2(1+\delta) \le (1+\varepsilon)(1+\delta)\big(b(K)/M(K)\big)^2.$$

For ε of the same order as δ, we have the desired result.

We remark that although we have proved the existence of a set of rotations, we provided rotations with a certain structure and did not show that for random N rotations (23) is satisfied.

Remark. A weakening of Conjecture 15 was proved by Latała and Oleszkiewicz, see Theorem 3 in [LO]. It states that for every symmetric K one has

$$\sigma(x \in S^{n-1} : \|x\| < \alpha M') < (12\alpha)^{((M'/b)\sqrt{n}-6)_+^2/4}, \tag{25}$$

where M' is the median of the norm (which, in the non degenerate case, is known to be close to the mean of the norm, M). This estimate can be used instead of (22), in the same way that (24) was used in the proof of the implication above, to prove Conjecture 14 with instead of constant $(1+\delta)$, constant $(4+\delta)$ for any $\delta > 0$. We omit the details.

4.3 Improvements in Some Special Cases

From our method of proof it is obvious that the parameter which plays the leading role in the lower bound is not $(M/b)^2$ but rather

$$\frac{1}{n}\log\sigma(x \in S^{n-1} : \|x\| > \alpha M).$$

(This parameter, for $\alpha = 1/2$, is very similar to the one introduced in [KV] to study local Dvoretzky type theorems.) To be precise, let us denote

$$f(\alpha) = \frac{1}{n}\log\Big(\frac{1}{\sigma}(x \in S^{n-1} : \|x\| > \alpha M)\Big).$$

Then for any proportion $\beta < \sigma(x \in S^{n-1} : \|x\| > \alpha M)$, that is, $\beta < 1 - e^{-nf(\alpha)}$) we have by Chernoff (2) that for a single x

$$\mathbb{P}\Big[U_1 \dots, U_N \in O(n) : \frac{1}{N}\sum_{i=1}^{N}\|U_i x\| \ge \beta\alpha M\Big] \ge 1 - e^{-nf(\alpha)N(1-\beta)}e^{-Nu(\beta)}.$$

If we want this probability to suffice for a $\beta\alpha/2C$-net of the sphere (where C is from the upper bound), we need to have

$$e^{-Nu(\beta)}e^{-nf(\alpha)N(1-\beta)}e^{n\log(1+4C/(\alpha\beta))} < 1.$$

This gives us the bound on N, namely that for every $\alpha < 1$ and $0 < \beta < 1 - e^{-nf(\alpha)}$, we may choose

$$N = 2\frac{\log(1+4C/(\alpha\beta))}{f(\alpha)(1-\beta) + u(\beta)/n}$$

and have a lower bound $(\alpha\beta/2)M$ on the norm defined in (23), where CM is the *upper* bound we have on this norm defined in (23). In other words, we can take N as close as we want to

$$\inf_{\alpha<1,0<\beta<1-e^{-nf(\alpha)}} \frac{\log(1+4C/(\alpha\beta))}{f(\alpha)(1-\beta)+u(\beta)/n},$$

getting that for the average of this number of rotations, assuming an upper bound CM, is isomorphic to euclidean, paying only with the isomorphism constants.

In many special cases the estimates for $f(\alpha)$ are better than what is given above, see examples in [KV]. The question remains whether one can give a general condition under which there are estimates for $f(\alpha)$ significantly better than (22) and (25).

Notice, however, that this is just the N for the lower bound, *assuming* an upper bound. It is well known that one always need to take at least $N = \lambda(b/M)^2$ for some constant $\lambda > 0$, to get the right order upper bound in (23). In particular, we need the upper bound so that we can transform the bounds on the net to bounds on the whole sphere. Thus, the improvement in the special cases where one computes $f(\alpha)$ and sees that it is larger than expected, i.e., that the infimum above is $o((b/M)^2)$, will be that averaging over $N = \lambda(b/M)^2$ rotations for a proportion $0 < \lambda$ is enough to get a norm isomorphic to euclidean.

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A Remark on the Surface Brunn–Minkowski-Type Inequality*

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In this note we would like to attract reader's attention to the following functional form for the surface Brunn–Minkowski-type inequality.

Theorem 1. *Let $0 < t < 1$ and let u, v, w be non-negative, quasi-concave, smooth functions on $\mathbf{R}^n$, such that $w(x) \to 0$, as $|x| \to \infty$, and*

$$w\big(tx + (1-t)y\big) \geq u(x)^t v(y)^{1-t}, \tag{1}$$

for all $x, y \in \mathbf{R}^n$. Then

$$\int |\nabla w(z)|\, dz \geq \left(\int |\nabla u(x)|\, dx \right)^t \left(\int |\nabla v(y)|\, dy \right)^{1-t}. \tag{2}$$

A function w is called quasi-concave, if $w(tx+(1-t)y) \geq \min\{w(x), w(y)\}$, whenever $x, y \in \mathbf{R}^n$ and $0 < t < 1$ (cf. e.g. [C-F] for an account on equivalent definitions and basic properties of such functions.) In particular, all log-concave functions are quasi-concave. In this case, the assumption on smoothness may be removed from the hypotheses of Theorem 1.

Let A and B be convex bodies in $\mathbf{R}^n$. Approximating these sets by smooth log-concave functions u and v, inequality (2) yields

$$S\big(tA + (1-t)B\big) \ \geq \ S(A)^t S(B)^{1-t}, \tag{3}$$

and by homogeneity, for $n \geq 2$,

$$S\big(tA + (1-t)B\big) \ \geq \ \Big[t\, S(A)^{1/(n-1)} + (1-t)\, S(B)^{1/(n-1)} \Big]^{n-1}, \tag{4}$$

where we use $S(\cdot)$ to denote the area size of the surface of a corresponding convex body. This is a Brunn–Minkowski-type inequality for the functional S, cf. [S]. The bound (4) is optimal in the sense that its right hand side

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provides minimum of $S(tA+(1-t)B)$ in terms of $S(A)$ and $S(B)$; however, the advantage of the "log-concave" form (3) is that it remains to formally hold when one of the sets is empty.

Thus, inequality (2) under the hypothesis (1) may be viewed as a functional form for (4). The class of quasi-concave functions is natural in Theorem 1, since only for such functions the sets of the form $A_u(\lambda) = \{x \in \mathbf{R}^n : u(x) \geq \lambda\}$ are convex, and since (4) is stated for convex sets.

Recall that, for all non-negative measurable functions u, v, and w, satisfying the condition (1), we have the Prékopa–Leindler inequality ([Pr1-2], [L])

$$\int w(z)\,dz \geq \left(\int u(x)\,dx\right)^t \left(\int v(y)\,dy\right)^{1-t}, \tag{5}$$

which represents a natural functional form for the volume Brunn–Minkowski inequality

$$\mathrm{vol}_n\big(tA+(1-t)B\big) \ \geq \ \Big[t\,\mathrm{vol}_n(A)^{1/n} + (1-t)\,\mathrm{vol}_n(B)^{1/n}\Big]^n. \tag{6}$$

Other functional forms of (6), earlier references and discussion of history may be found in S. Das Gupta [DG] and R. Gardner [G]. Prékopa–Leindler's theorem has found a number of interesting applications in Convex Geometry and Analysis; let us only mention the works by C. Borell [Bo1-2], K. Ball [Ba] and B. Maurey [M]. In fact, (5) being combined with (3) may also be used to derive inequality (2).

Indeed, assume the functions u and v are not identically zero, so that both vanish as $|x| \to \infty$, since w does. Hence, the sets $A_u(\lambda) = \{u \geq \lambda\}$, $\lambda > 0$, are convex, bounded (and perhaps empty), and similarly for v and w.

Now, by the hypothesis (1),

$$tA_u(\lambda_1) + (1-t)A_v(\lambda_2) \subset A_w(\lambda_1^t \lambda_2^{1-t}), \qquad \lambda_1, \lambda_2 > 0,\ t \in (0,1),$$

as long as both $A_u(\lambda_1)$ and $A_v(\lambda_2)$ are non-empty. Anyhow, by (3) and by monotonicity of S, the functions

$$f(\lambda) = S\big(A_u(\lambda)\big), \quad g(\lambda) = S\big(A_v(\lambda)\big), \quad h(\lambda) = S\big(A_w(\lambda)\big)$$

satisfy $h(\lambda_1^t \lambda_2^{1-t}) \geq f(\lambda_1)^t g(\lambda_2)^{1-t}$, for all $\lambda_1, \lambda_2 > 0$. This property is a multiplicative version of (1) in dimension one, and it also implies (5), ([Ba], Lemma 3), i.e.,

$$\int_0^{+\infty} h(\lambda)\,d\lambda \geq \left(\int_0^{+\infty} f(\lambda)\,d\lambda\right)^t \left(\int_0^{+\infty} g(\lambda)\,d\lambda\right)^{1-t}.$$

Finally, applying the coarea formula $\int_{\mathbf{R}^n} |\nabla u(x)|\,dx = \int_0^{+\infty} f(\lambda)\,d\lambda$ to u, as well as to the functions v and w, we arrive at the desired conclusion (2).

More generally, with a similar argument one may consider Choquet's integrals $\int \varphi\,d\mu \equiv \int_0^{+\infty} \mu\{\varphi \geq \lambda\}\,d\lambda$ for $\varphi \geq 0$ with an arbitrary monotone set function $\mu \geq 0$ on $\mathbf{R}^n$, such that $\mu(tA + (1-t)B) \geq \mu(A)^t \mu(B)^{1-t}$ in the class of all convex bodies in the n-space. Under the assumptions of Theorem 1, one then gets that

$$\int w \, d\mu \geq \left(\int u \, d\mu \right)^t \left(\int v \, d\mu \right)^{1-t}. \tag{7}$$

For example, the p-capacity $\mu(A) = \inf\{\int |\nabla g(x)|^p \, dx : g \geq 1_A, \, g \in C_0^\infty(\mathbf{R}^n)\}$ is included in (7) whenever $1 \leq p < n$. In that case, the log-concavity of μ was proved by C. Borell [Bo2] for $p = 2$, $n \geq 3$ (the case of Newton capacity) and by A. Colesanti and P. Salani for all $p < n$. When $p = 1$, (7) coincides with (2). On the other hand, when μ is Lebesgue measure, we return to (5).

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On Isoperimetric Constants for Log-Concave Probability Distributions*

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Summary. Lower bounds on the isoperimetric constant for logarithmically concave probability measures are considered in terms of the distribution of the Euclidean norm. A refined form of Kannan–Lovász–Simonovits' inequality is obtained.

Given a Borel probability measure μ on $\mathbf{R}^n$, its isoperimetric constant or, isoperimetric coefficient, is defined as the optimal value $h = h(\mu)$ satisfying an isoperimetric-type inequality

$$\mu^+(A) \geq h \min\big\{\mu(A), 1 - \mu(A)\big\}. \tag{1}$$

Here, A is an arbitrary Borel subset of $\mathbf{R}^n$ of measure $\mu(A)$ with μ-perimeter $\mu^+(A) = \lim_{\varepsilon\downarrow 0} \frac{\mu(A_\varepsilon)-\mu(A)}{\varepsilon}$, where $A_\varepsilon = \{x \in \mathbf{R}^n : |x - a| < \varepsilon,$ for some $a \in A\}$ denotes an open ε-neighbourhood of A with respect to the Euclidean distance.

The quantity $h(\mu)$ represents an important geometric characteristic of the measure and is deeply related to a number of interesting analytic inequalities. As an example, one may consider a Poincaré-type inequality

$$\int |\nabla f|^2 \, d\mu \ \geq\ \lambda_1 \int |f|^2 \, d\mu$$

in the class of all smooth functions f on $\mathbf{R}^n$ such that $\int f \, d\mu = 0$. The optimal value λ_1, the so-called spectral gap, satisfies $\lambda_1 \geq h^2/4$. This relation goes back to the work by J. Cheeger in the framework of Riemannian manifolds [C] and – in a more general form – to earlier works by V.G. Maz'ya (cf. [M1-2], [G]). The problem on bounding these two quantities from below has a long story. In this note we specialize to the class of log-concave probability measures, in which case, as was recently shown by M. Ledoux [L], λ_1 and h are equivalent ($\lambda_1 \leq 36\, h^2$).

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Following A. Prékopa [P], μ is called logarithmically concave (or, log-concave), if for all non-empty convex sets A, B in $\mathbf{R}^n$, and $t \in (0,1)$,

$$\mu\big((1-t)A + tB\big) \geq \mu(A)^{1-t}\mu(B)^t,$$

where $(1-t)A + tB = \{(1-t)a + tb : a \in A,\, b \in B\}$ denotes the Minkowski average. The definition reduces to the statement that μ is concentrated on some affine subspace L of $\mathbf{R}^n$, where it is absolutely continuous with respect to Lebesgue measure and has a density p, satisfying

$$p\big((1-t)x + ty\big) \geq p(x)^{1-t}p(y)^t, \quad \text{for all } \ x, y \in L,\ t \in (0,1). \tag{2}$$

For example, a uniform distribution over an arbitrary convex body K in $\mathbf{R}^n$ is log-concave (the convex body case). For a full description, including more general classes of convex measures, see C. Borell [Bor1-2]. In the sequel, by saying that μ is k-dimensional, we mean that the supporting subspace L has dimension k.

For any log-concave probability measure μ, its isoperimetric constant is positive and may be bounded from below, up to some universal constant $c > 0$, as

$$h(\mu) \geq \frac{c}{\int |x|\, d\mu(x)}. \tag{3}$$

This inequality was obtained by R. Kannan, L. Lovász and M. Simonovits for the convex body case as part of the study of randomized volume algorithms ([K-L-S], Main Theorem). Actually, their proof based on a localization lemma of Kannan and Lovász [K-L] may easily be extended to the general log-concave case. A different approach, using the Prékopa–Leindler functional form for the Brunn–Minkowski inequality, was later proposed in [B1].

Our aim is to get the following sharpening of the bound (3) involving the distribution of the Euclidean norm. Let $X = (X_1, \dots, X_n)$ be a random vector in $\mathbf{R}^n$ with distribution μ, and $|X| = (X_1^2 + \cdots + X_n^2)^{1/2}$ be its Euclidean length.

Theorem 1. *If μ is log-concave, then*

$$h(\mu) \geq \frac{c}{\mathrm{Var}(|X|^2)^{1/4}}, \tag{4}$$

where c is a universal constant.

Here, $\mathrm{Var}(|X|^2) = \mathbf{E}\,|X|^4 - (\mathbf{E}\,|X|^2)^2 = \int |x|^4\, d\mu(x) - (\int |x|^2\, d\mu(x))^2$ is the variance of $|X|^2$.

By Borell's lemma ([Bor1], Lemma 3.1), L^p-norms of $|X|$ are equivalent, so $\mathbf{E}|X|^4 \leq C^4\, (\mathbf{E}\,|X|)^4$, for some positive numerical constant C. Therefore,

$$\mathrm{Var}\big(|X|^2\big)^{1/4} \leq \big(\mathbf{E}|X|^4\big)^{1/4} \leq C\,\mathbf{E}\,|X|,$$

and thus (4) implies the K-L-S bound (3). To see that there can be an essential difference between (3) and (4), take the unit ball B in $\mathbf{R}^n$ with center at the origin and equip it with the normalized Lebesgue measure μ. Then, $|X|$ has the distribution function $F_n(t) = \mathbf{P}\{|X| \leq t\} = t^n$, $0 \leq t \leq 1$, so $\mathbf{E}\,|X| = \int_0^1 t\,dF_n(t) = \frac{n}{n+1}$. Hence, the right hand side of (3) is of order 1. On the other hand,

$$\mathrm{Var}(|X|^2) = \int_0^1 t^4\,dF_n(t) - \left(\int_0^1 t^2\,dF_n(t)\right)^2 = \frac{4n}{(n+2)^2(n+4)},$$

so, the right hand side of (4) is of order $\sqrt{n}$. Hence, in this case (4) provides a correct estimate for $h(\mu)$ with respect to the dimension n. Equivalently, if B is a ball of volume radius of order 1, then $h(\mu)$ is of order 1, as well.

More generally, suppose the measure μ is normalized to be in isotropy position in the sense that

$$\mathbf{E}\,\langle X, \theta\rangle^2 = \int \langle x, \theta\rangle^2\, d\mu(x) = |\theta|^2, \qquad \theta \in \mathbf{R}^n. \tag{5}$$

Then, $\mathbf{E}\,|X| \leq (\mathbf{E}|X|^2)^{1/2} = \sqrt{n}$, and (3) leads to $h(\mu) \geq c/\sqrt{n}$. It is unknown whether this bound can be improved in general. Nevertheless, by virtue of Theorem 1, one may reach an improvement for some classes of measures (or bodies). For example, one interesting class is described by the condition

$$\mathbf{E}\,X_i^2 X_j^2 \leq \mathbf{E}\,X_i^2\,\mathbf{E}\,X_j^2, \qquad i \neq j. \tag{6}$$

That is, a log-concave probability measure μ on $\mathbf{R}^n$ belongs to this class, if the squares of the coordinates have non-positive covariances $\mathrm{cov}(X_i^2, X_j^2)$. In this case, if $\mathbf{E}X_i^2 = 1$ for all $i \leq n$ (which holds under the isotropy assumption), we have that

$$\mathrm{Var}(|X|^2) = \sum_{i=1}^n \mathrm{Var}(X_i^2) + 2\sum_{i<j} \mathrm{cov}(X_i^2, X_j^2) \leq \sum_{i=1}^n \mathbf{E}\,X_i^4 \leq Cn,$$

for some universal constant C. Therefore, Theorem 1 yields:

Corollary 1. *If a log-concave isotropic measure μ on $\mathbf{R}^n$ satisfies* (6)*, then*

$$h(\mu) \geq \frac{c}{n^{1/4}}, \tag{7}$$

where c is a universal constant.

As a more specific case, consider the uniform distribution μ on the dilated ℓ_p^n-ball

$$K = \left\{x \in \mathbf{R}^n : |x_1|^p + \cdots + |x_n|^p \leq c^p\right\}$$

with parameter $1 \leq p \leq +\infty$ and with $c = c(p,n)$ chosen to satisfy the isotropy condition (5) (c is of order $n^{1/p}$). That the covariance property (6) is fulfilled for such a family of convex bodies was observed by K. Ball and I. Perissinaki [B-P]. As we mentioned, in the case $p = 2$, $h(\mu)$ is of order 1. The same is true for $p = +\infty$ (H. Hadwiger) and for the whole range $2 \leq p \leq +\infty$, since then μ can be obtained from the canonical Gaussian measure as Lipschitz transform. When $1 \leq p < 2$, the correct asymptotic with respect to the dimension seems to be unknown, and we can only state that $h(\mu) \geq c\, n^{-1/4}$. In this case, the constant is also believed to be of order 1; at least, this is inspired by concentration results, obtained by G. Schechtman and J. Zinn [S-Z]. More generally, Kannan, Lovász and Simonovits conjectured that $h(\mu)$ is of order 1 for arbitrary isotropic convex bodies.

Now, let us turn to the proof of Theorem 1. We use the localization argument of [K-L-S], but choose a somewhat different hypothesis in applying the localization lemma. The argument goes back to the bisection method of L. E. Payne and H. F. Weinberger [P-W]; similar ideas were also developed by M. Gromov and V. D. Milman in [G-M]; cf. also [A] and [F-G1,2]. Below we state as a lemma a slightly modified variant of Corollary 2.2 appearing in [K-L-S].

Lemma 1. *Let $\alpha, \beta > 0$, and suppose u_i, $i = 1,2,3,4$, are non-negative continuous functions on $\mathbf{R}^n$ such that for any segment $\Delta \subset \mathbf{R}^n$ and any affine function ℓ on Δ,*

$$\left(\int_\Delta u_1\, e^\ell\right)^\alpha \left(\int_\Delta u_2\, e^\ell\right)^\beta \leq \left(\int_\Delta u_3\, e^\ell\right)^\alpha \left(\int_\Delta u_4\, e^\ell\right)^\beta. \tag{8}$$

Then,

$$\left(\int_{\mathbf{R}^n} u_1\right)^\alpha \left(\int_{\mathbf{R}^n} u_2\right)^\beta \leq \left(\int_{\mathbf{R}^n} u_3\right)^\alpha \left(\int_{\mathbf{R}^n} u_4\right)^\beta. \tag{9}$$

The one-dimensional integrals in (8) are taken with respect to Lebesgue measure on Δ, while the integrals in (9) are n-dimensional.

It should be clear that Lemma 1 remains to hold for many discontinuous functions u_i, as well, like the indicator functions of open or closed sets in the space. For the uniform distribution μ on a convex body K in $\mathbf{R}^n$, the approach of [K-L-S] is to apply the lemma with $\alpha = \beta = 1$ to the functions of the form

$$u_1 = 1_A, \quad u_2 = 1_B, \quad u_3 = 1_C, \quad u_4(x) = \frac{\mathrm{const}\,|x|}{\varepsilon}\, 1_K(x),$$

where A and B are arbitrary "regular" disjoint subsets of $\mathbf{R}^n$ at the distance $\varepsilon = \mathrm{dist}(A,B)$ and where $C = \mathbf{R}^n \setminus (A \cup B)$. Then (9) turns into

$$\mu(A)\mu(B) \leq \mu(C)\, \frac{\mathrm{const}}{\varepsilon} \int |x|\, d\mu(x), \tag{10}$$

and letting $\varepsilon \to 0$, we arrive at the desired isoperimetric inequality (1) with $\frac{1}{h} = 2\,\mathrm{const} \int |x|\, d\mu(x)$. On the other hand, (8) turns into a one-dimensional inequality which is similar to (10). The only difference is that μ should be replaced by a specific probability measure μ_ℓ concentrated on Δ and having, up to a normalizing constant, the density e^ℓ with respect to Lebesgue measure on Δ. That is how, the bound (3) reduces to the one-dimensional inequality (10) in the body case.

More generally, if μ is absolutely continuous and has a density p satisfying (2), then in (8) we are dealing with a probability measure μ_ℓ, concentrated on Δ and having, up to a normalizing constant, the density pe^ℓ. It satisfies (2), so the bound (3), being stated for the class of all absolutely continuous log-concave probability measures on $\mathbf{R}^n$, may also be reduced to the inequality (10) about arbitrary log-concave measures on Δ. Therefore, we obtain the following corollary from Lemma 1:

Corollary 2. *Let g be a non-negative continuous function on $\mathbf{R}^n$. Let A, B be open disjoint subsets of $\mathbf{R}^n$ at distance $\varepsilon = \mathrm{dist}(A,B)$, and put $C = \mathbf{R}^n \setminus (A \cup B)$. If the inequality*

$$\mu(A)\mu(B) \leq \frac{\mu(C)}{\varepsilon} \int g\, d\mu \tag{11}$$

holds for any one-dimensional log-concave probability measure, then it holds for any n-dimensional log-concave probability measure on $\mathbf{R}^n$.

In the conclusion, the dimension is irrelevant and can be ignored.

Also, as we already mentioned, letting $\varepsilon \to 0$, (11) takes the form of an isoperimetric inequality

$$\mu(A)\mu(B) \leq \mu^+(C) \int g\, d\mu. \tag{12}$$

Actually, it is easy to show that (12) is equivalent to (11) when these inequalities are required to hold for all admissible partitions A, B, C (see e.g. [B-Z], Proposition 10.1). Recalling the definition (1) and using $2\mu(A)\mu(B) \geq \max\{\mu(A), \mu(B)\}$, one may reformulate Corollary 2 equivalently up to a factor as:

Corollary 3. *Given a non-negative continuous function g on $\mathbf{R}^n$, if the inequality $\frac{1}{h(\mu)} \leq \int g\, d\mu$ is fulfilled for any one-dimensional log-concave probability measure μ, then for any log-concave probability measure μ on $\mathbf{R}^n$, we have $\frac{1}{h(\mu)} \leq 2 \int g\, d\mu$.*

Proof of Theorem 1. If ξ is a random variable with a log-concave distribution μ on the real line, then

$$c_1 \sqrt{\mathrm{Var}(\xi)} \leq \frac{1}{h(\mu)} \leq c_2 \sqrt{\mathrm{Var}(\xi)}. \tag{13}$$

The optimal constants, which are not important for us, are $c_1 = 1/\sqrt{2}$, $c_2 = \sqrt{3}$ (cf. [B1], Proposition 4.1). Any one-dimensional log-concave probability measure μ on $\mathbf{R}^n$ may be viewed as the distribution of a random vector $a+\xi\theta$, where a, θ are orthogonal vectors, $|\theta|=1$, and ξ is a random variable with a log-concave distribution. Clearly, μ also satisfies (13). Hence, by Corollary 3, if the inequality

$$\sqrt{\mathrm{Var}(\xi)} \ \le\ \mathbf{E}g(a+\xi\theta) \tag{14}$$

holds for all ξ as above and for all vectors a, θ in $\mathbf{R}^n$, such that $\langle a,\theta\rangle = 0$, then

$$\frac{1}{h(\mu)} \ \le\ 2\,c_2 \int g\,d\mu \tag{15}$$

in the class of all log-concave probability measures μ on $\mathbf{R}^n$. We choose $g(x) = C\,|\,|x|^2-\alpha|^{1/2}$ with an arbitrary number α, but with a constant C to be specified. In this case, the quantity $\mathbf{E}g(a+\xi\theta) = C\mathbf{E}|\,|a|^2+\xi^2-\alpha|^{1/2}$ satisfies (14) in view of the equivalence of L^p-norms of polynomials with respect to log-concave distributions. To be more precise, if Q is a polynomial on $\mathbf{R}^n$ of degree d, and μ is a log-concave probability measure, then for $\|Q\|_p = (\int |Q|^p\,d\mu)^{1/p}$ there is the relation

$$\|Q\|_p \ \le\ c(d,p)\,\|Q\|_0, \quad p \ge 0, \tag{16}$$

with constants $c(d,p)$ depending on d and p, only (cf. [Bou], [B2], [B-G]). In particular, $\|Q\|_2 \le c\,\|Q\|_{1/2}$ for any quadratic function Q with $c = c(2,2)$. Therefore,

$$c\big(\mathbf{E}\,|\,|a|^2+\xi^2-\alpha|^{1/2}\big)^2 \ge \big(\mathbf{E}\,|\,|a|^2+\xi^2-\alpha|^2\big)^{1/2} \ge \mathrm{Var}(\xi^2)^{1/2}. \tag{17}$$

Also, if $L^2 = \mathbf{E}\xi^2$, we have

$$\begin{aligned}\mathrm{Var}(\xi^2)^{1/2} &\ge \|\xi^2-L^2\|_0 \ =\ \|\xi-L\|_0\,\|\xi+L\|_0 \\ &\ge \frac{1}{c^2}\|\xi-L\|_2\,\|\xi+L\|_2 \ \ge\ \frac{1}{c^2}\mathrm{Var}(\xi),\end{aligned}$$

where we applied (16) once more on the last step. Together with (17) this yields

$$C\,\mathbf{E}\,|\,|a|^2+\xi^2-\alpha|^{1/2} \ge \sqrt{\mathrm{Var}(\xi)}$$

with $C = c^{3/2}$, so the hypothesis (14) is fulfilled.

Now, let's look at the conclusion (15). If X is a random vector with distribution μ, by Jensen's inequality,

$$\int g\,d\mu = C\,\mathbf{E}\,|\,|X|^2-\alpha|^{1/2} \le C\,\big(\mathbf{E}\,|\,|X|^2-\alpha|^2\big)^{1/4}.$$

The right hand side is minimized for $\alpha = \mathbf{E}|X|^2$ and becomes $C\,\mathrm{Var}(|X|^2)^{1/4}$. Hence, $\frac{1}{h(\mu)} \le 2\,c_2 C\,\mathrm{Var}(|X|^2)^{1/4}$ which is the claim.

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A Remark on Quantum Ergodicity for CAT Maps

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1 Introduction

The purpose of this Note is to give an affirmative answer to a question raised in the paper of P. Kurlberg and Z. Rudnick [K-R]. We first briefly recall the background (see [K-R2]). Given $A \in SL_2(\mathbb{Z})$, consider the automorphism of the torus $\mathbb{T}^2 : x \mapsto Ax$.

Given $f \in \mathcal{C}^\infty(\mathbb{T}^2)$, the classical evolution defined by A is $f \mapsto f \circ A$. The quantization is obtained as follows. Let $N \in \mathbb{Z}_+$ be a large integer and consider the Hilbert space $\mathcal{H}_N = L^2(\mathbb{Z}_N)$, $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ with inner product

$$\langle \phi, \psi \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}_N} \phi(x) \overline{\psi(x)}.$$

The basic observables are given by the operators $T_N(n), n = (n_1, n_2) \in \mathbb{Z}^2$ defined as follows

$$\big(T_N(n)\phi\big)(x) = e^{i\pi \frac{n_1 n_2}{N}} e^{2\pi i \frac{n_2 x}{N}} \phi(x + n_1). \tag{1.1}$$

Writing $f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) e^{2\pi i n x}$, $f \in C^\infty(\mathbb{T}^2)$, its quantization is then defined by

$$Op_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) T_N(n). \tag{1.2}$$

Assume further that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfies

$$ab \equiv cd \equiv 0 \pmod 2.$$

One may then assign to A a unitary operator $U_N(A)$ called quantum propagator or quantized cat map, which satisfies the 'exact' Egorov theorem

$$U_N(A)^* Op_N(f) U_N(A) = Op_N(f \circ A). \tag{1.3}$$

We are concerned with the eigenfunctions of $U_N(A)$ which play the role of energy eigenstates.

It is shown in [K-R] that for N taken in a subsequence $\mathcal{N} \subset \mathbb{Z}_+$ of asymptotic density one, we have for all $f \in \mathcal{C}^\infty(\mathbb{T}^2)$

$$\max_{\psi} \left| \langle Op_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f \right| \underset{\substack{N\to\infty \\ N\in\mathcal{N}}}{\longrightarrow} 0 \tag{1.4}$$

where the maximum is taken over all normalized eigenfunctions ψ of $U_N(A)$.

The quantization of the cat map described above was proposed by Hannay and Berry [H-B]. A few comments at this point. In the context of cat maps, Schnirelman's general theorem when the classical dynamics is ergodic (which is the case when $A \in SL_2(\mathbb{Z})$ is hyperbolic) takes the following form. Let $f \in C^\infty(\mathbb{T}^2)$. If $\{\psi_j\}$ is an arbitrary orthonormal basis of $\mathcal{H}_N$ consisting of eigenfunctions of $U_N(A)$, there is a subset $J(N) \subset \{1, \dots, N\}$ such that $\frac{\# J(N)}{N} \to 1$ and for $j \in J(N)$

$$\langle Op_N(f)\psi_j, \psi_j\rangle \to \int_{\mathbb{T}^2} f \text{ when } N \to \infty. \tag{1.5}$$

Hence the [K-R] result (1.4) goes beyond (1.5), since they obtain a statement valid for all eigenfunctions of $U_N(A)$.

Previously, the only result providing an infinite set $\mathcal{N}$ of integers N (primes) satisfying (1.4) was due to Degli-Esposti, Graffi and Isola [D-G-I], conditional to GRH. The precise form of the [K-R] result is as follows (using previous notations)

$$\sum_{j=1}^{N} \left| \langle Op_N(f)\psi_j, \psi_j\rangle - \int_{\mathbb{T}^2} f \right|^4 \ll \frac{N(\log N)^{14}}{o(A,N)^2} \tag{1.6}$$

where $o(A,N)$ denotes the order of A mod N. (See [K-R], Theorem 2.) In order to derive (1.4) from (1.6), one needs to ensure that $o(A,N) \gg N^{1/2}$ for $N \in \mathcal{N}$. Verifying this property for sequence $\mathcal{N}$ of asymptotic density 1 is in fact a significant part of the [K-R] paper (the issue is related to the classical Gauss–Artin problem.) It is shown in [K-R] one may ensure for $N \in \mathcal{N}$ of asymptotic density 1, that

$$o(A,N) \gg N^{1/2} \exp\left((\log N)^\delta\right) \tag{1.7}$$

for some $\delta > 0$.

The authors raise the question how to get results when $o(A,N)$ is smaller than $N^{1/2}$. We will show here how to settle this problem using the new exponential sum bounds obtained in [BGK], [B], [B-C] for multiplicative subgroups G of finite fields and their products. These results provide nontrivial estimates even when G is very small.

They will allow us to deal with the case when $o(A,N) \gg N^\varepsilon$ (say for N prime) for an arbitrary small given $\varepsilon > 0$. Unlike a stronger statement such as (1.7), the generic validity of this last condition is essentially obvious to verify. Our results are stated in Proposition 2 (prime modulus) and Theorem 3 (arbitrary modulus). Note that in (3.1) below the discrepancy is estimated as $N^{-\delta}$, which is better than the bound obtained in [K-R].

The results of importance for what follows are the following

Theorem 1 (see [BGK] if $f = 1$ and [B-C] if $f > 1$). *Let $G < \mathbb{F}_{p^f}^*$ be of order t such that*

$$t > p^{\varepsilon f} \tag{1.8}$$

and

$$\max_{\substack{r|f \\ r<f}} (t, p^r - 1) < t^{1-\varepsilon} \tag{1.9}$$

where $\varepsilon > 0$ is an arbitrarily small given constant.

Then

$$\max_{\mathcal{X} \neq \mathcal{X}_0} \left| \sum_{x \in G} \mathcal{X}(x) \right| < Ct^{1-\delta} \tag{1.10}$$

where $\mathcal{X}$ runs over the nontrivial additive characters of $\mathbb{F}_{p^f}$, thus $\mathcal{X}(x) = e\big(\frac{1}{p} Tr(ax)\big), a \in \mathbb{F}_{p^f}^*$, and $\delta = \delta(\varepsilon) > 0$.

In the application below, $f = 2$.

Also needed is the following exponential sum bound in $\mathbb{F}_p \times \mathbb{F}_p$, obtained in [B].

Theorem 2 ([B]). *Let $G < \mathbb{F}_p^* \times \mathbb{F}_p^*$ be generated by $(\theta_1, \theta_2) \in \mathbb{F}_p^* \times \mathbb{F}_p^*$ satisfying*

$$O(\theta_1) > p^{\varepsilon} \tag{1.11}$$

$$O(\theta_2) > p^{\varepsilon} \tag{1.12}$$

$$O(\theta_1 \theta_2^{-1}) > p^{\varepsilon} \tag{1.13}$$

with $\varepsilon > 0$ a given arbitrary constant. We denote here $O(\theta)$ the multiplicative order of $\theta \in \mathbb{F}_p^*$.

There is $\delta = \delta(\varepsilon) > 0$ such that

$$\max_{(a_1,a_2) \neq (0,0)} \left| \sum_{x \in G} e_p(a_1 x_1 + a_2 x_2) \right| < C|G|^{1-\delta}. \tag{1.14}$$

Acknowledgement. The author is grateful to Z. Rudnick for his comments on an earlier version of this account.

2 The Prime Case

Considering first the case with $N = p$ prime, we show the following

Proposition 1. *For all* $\varepsilon > 0$*, there is* $\delta > 0$ *such that if* $o(A, N) > N^\varepsilon$*, then, assuming* n *and* nA *linearly independent mod* N*, we have*

$$\max_\psi \big|\langle T_N(n)\psi, \psi\rangle\big| < 2N^{-\delta} \tag{2.1}$$

with the maximum taken over the normalized eigenfunctions ψ *of* $U_N(A)$*.*

Proof. Denote $t = o(A, N)$. Since $U_N(A)$ is unitary, write for $j = 1, \dots, t$

$$\begin{aligned}\langle T_N(n)\psi, \psi\rangle &= \langle T_N(n)U_N(A)^j\psi, U_N(A)^j\psi\rangle \\ &= \frac{1}{t}\sum_{j=1}^{t}\langle U_N(A)^{-j}T_N(n)U_N(A)^j\psi, \psi\rangle. \end{aligned}\tag{2.2}$$

By Egorov's theorem (1.3), we have

$$U_N(A)^{-1}T_N(n)U_N(A) = T_N(nA) \tag{2.3}$$

and iterating

$$U_N(A)^{-j}T_N(n)U_N(A)^j = T_N(nA^j).$$

Hence from (2.2)

$$|\langle T_N(n)\psi, \psi\rangle| \le \|D(n)\| \tag{2.4}$$

where $D = D(n)$ is following operator on $\mathcal{H}_N$

$$D = \frac{1}{t}\sum_{j=1}^{t} T_N(nA^j) \tag{2.5}$$

and $\|\ \|$ stands for the operator norm.

Take a (sufficiently large) positive integer ℓ (to be specified) and estimate

$$\|D\|^{4\ell} \le \text{ trace } (DD^*)^{2\ell}. \tag{2.6}$$

Recall the following properties (see [K-R])

$$T_N(m)^* = T_N(-m) \tag{2.7}$$

and

$$T_N(m)T_N(n) = e_N\left(\frac{\omega(m,n)}{2}\right)T_N(m+n) \tag{2.8}$$

with

$$\omega(m,n) = m_1n_2 - m_2n_1.$$

Expanding (2.6) using (2.7)–(2.8) gives

$$(DD^*)^{2\ell} = \frac{1}{t^{4\ell}}\sum_{j_1,\dots,j_{4\ell}=1}^{t}\gamma_{j_1\dots j_{4\ell}}T_N\big(n(A^{j_1} - A^{j_2}\dots - A^{j_{4\ell}})\big) \tag{2.9}$$

where $|\gamma_{j_1} \dots j_{4\ell}| = 1$.

Next

$$\text{trace } T_N(n) = \begin{cases} N \text{ if } n = (0,0) \bmod N \\ 0 \text{ otherwise.} \end{cases} \tag{2.10}$$

It follows now from (2.9), (2.10) that

$$(2.6) \le t^{-4\ell} N.\#\big\{(j_1, \dots, j_{4\ell}) \in \{1, \dots, t\}^{4\ell} \,\big|\, n(A^{j_1} - \dots - A^{j_{4\ell}}) \equiv 0 \mod N\big\}. \tag{2.11}$$

The issue becomes now to estimate (2.11).

Recall that $N = p$ (prime).

Following [K-R], let K be the real quadratic field containing the eigenvalues of A (which are units) and O its maximal order. Let $\mathcal{P}$ be a prime of K lying above p and consider the residue class field $= O/\mathcal{P}$. If p splits, $K_p \simeq \mathbb{F}_p$ and if p is inert, $K_p \simeq \mathbb{F}_{p^2}$. Diagonalizing A over K_p, we obtain $A' = \left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$ and $n' = (n_1', n_2')$ in the eigenvector basis. Also $n_1' \neq 0, n_2' \neq 0$ in K_p as a consequence of the linear independence assumption for n and nA mod p. Our problem is therefore reduced to estimating the number (†) of solutions in $(j_1, \dots, j_{4\ell}) \in \{1, \dots, t\}^{4\ell}$ of the system of equations

$$\begin{cases} \sum_{s=1}^{4\ell} (-1)^s \varepsilon^{j_s} = 0 & (2.12) \\ \sum_{s=1}^{4\ell} (-1)^s \varepsilon^{-j_s} = 0 & (2.12') \end{cases}$$

in K_p. Here $\varepsilon \in K_p^*$ is of order t.

Case 1: The Split Case. Thus $K_p = \mathbb{F}_p$. Apply Theorem 2 with $\theta_1 = \varepsilon, \theta_2 = \varepsilon^{-1}$ for which $0(\theta_1) = 0(\theta_2) = t > p^\varepsilon$ and $0(\theta_1\theta_2^{-1}) = 0(\varepsilon^2) > \frac{t}{2} > \frac{1}{2}p^\varepsilon$. Hence (1.11) holds for some $\delta_1 = \delta_1(\varepsilon) > 0$.

Estimate by the circle method

$$\begin{aligned} (\dagger) &= \frac{1}{p^2} \sum_{0 \le a_1, a_2 < p} \left| \sum_{j=1}^{t} e_p(a_1\varepsilon^j + a_2\varepsilon^{-j}) \right|^{4\ell} \\ &< \frac{1}{p^2} t^{4\ell} + \max_{(a_1,a_2) \neq (0,0)} \left| \sum_{j=1}^{t} e_p(a_1\varepsilon^j + a_2\varepsilon^{-j}) \right|^{4\ell} \\ &< \frac{1}{p^2} t^{4\ell} + C^\ell t^{(1-\delta_1)4\ell} \\ &< t^{4\ell}(p^{-2} + C^\ell p^{-4\varepsilon\delta_1\ell}). \end{aligned} \tag{2.13}$$

Taking

$$\ell > \frac{1}{\varepsilon\delta_1} \tag{2.14}$$

it follows that (for p large enough)

$$(\dagger) < 2t^{4\ell}p^{-2}. \tag{2.15}$$

Case 2: The Inert Case. Then $K_p \approx \mathbb{F}_{p^2}$. Let $G = \{\varepsilon^j | 0 \leq j < t\} < K_p^*$. We have to distinguish 2 further subcases.

Assume first that $t = |G|$ satisfies

$$(t, p-1) < t^{1-\frac{\varepsilon}{2}} \tag{2.16}$$

so that condition (1.6) of Theorem 1 is fulfilled.

Then (1.7) holds with $\delta = \delta_1 = \delta_1(\varepsilon)$. By the circle method, we obtain again

$$\begin{aligned}(\dagger) &= \frac{1}{p^2}\sum_{\mathcal{X}}\Big|\sum_{x\in G}\mathcal{X}(x)\Big|^{4\ell}\\ &< \frac{t^{4\ell}}{p^2} + \max_{\mathcal{X}\neq\mathcal{X}_0}\Big|\sum_{x\in G}\mathcal{X}(x)\Big|^{4\ell}\\ &< t^{4\ell}\big(p^{-2} + Cp^{-4\ell\varepsilon\delta_1}\big)\\ &< 2t^{4\ell}p^{-2}\end{aligned} \tag{2.16$'$}$$

for a choice of ℓ as in (2.14).

Next, suppose (2.16) violated. Then $t = t_1 t_2$ where

$$t_1 | p-1 \text{ and } t_2 < t^{\varepsilon/2}.$$

Replace G by $G_1 = G^{t_2} < \mathbb{F}_p^*$ generated by $\varepsilon_1 = \varepsilon^{t_2}$ of order t_1 in $\mathbb{F}_p^*$, $t_1 > p^{\varepsilon/2}$.

Write $j \in \{0, 1, \dots, t-1\}$ in the form $j = j_1 t_2 + j_2$ with $j_1 \in \{0, 1, \dots, t_1 - 1\}$ and $j_2 \in \{0, 1, \dots, t_2 - 1\}$. Estimate

$$(\dagger) = \frac{1}{p^4}\sum_{a_1, a_2 \in \mathbb{F}_{p^2}}\Big|\sum_{j=0}^{t-1} e_p\big(Tr(a_1\varepsilon^j) + Tr(a_2\varepsilon^{-j})\big)\Big|^{4\ell}$$

and by Hölder's inequality

$$p^{-4}t_2^{4\ell-1}\sum_{a_1,a_2\in\mathbb{F}_{p^2}}\sum_{j_2=0}^{t_2-1}\Big|\sum_{j_1=0}^{t_1-1} e_p\big(Tr(a_1\varepsilon^{j_2})\varepsilon_1^{j_1} + Tr(a_2\varepsilon^{-j_2})\varepsilon_1^{-j_1}\big)\Big|^{4\ell} \tag{2.17}$$

the inner sum in (2.17) is again estimated by Theorem 2. Thus for some $\delta_1 = \delta\big(\frac{\varepsilon}{2}\big) > 0$

$$\Big|\sum_{j_1=0}^{t_1-1} e_p(b_1\varepsilon_1^{j_1} + b_2\varepsilon_1^{-j_1})\Big| < Ct_1^{1-\delta_1} \tag{2.18}$$

for $(b_1, b_2) \in \mathbb{F}_p \times \mathbb{F}_p$, $(b_1, b_2) \neq (0, 0)$.

Therefore clearly

$$
\begin{aligned}
(2.17) &\le p^{-4} t_1^{4\ell} t_2^{4\ell-1} \\
&\cdot \Big| \big\{ (a_1, a_2, j) \in \mathbb{F}_{p^2} \times \mathbb{F}_{p^2} \times \{0, 1, \dots, t_2 - 1\} \mid Tr(a_1 \varepsilon^j) = Tr(a_2 \varepsilon^{-j}) = 0 \big\} \Big| \\
&\quad + C t_2^{4\ell} t_1^{4\ell(1-\delta_1)} \\
&\le p^{-4} t_1^{4\ell} t_2^{4\ell-1} t_2 p^2 + C t_2^{4\ell} t_1^{4\ell(1-\delta_1)} \\
&\le t^{4\ell}(p^{-2} + C p^{-2\varepsilon\delta_1\ell}). \qquad (2.19)
\end{aligned}
$$

Taking $\ell > \frac{1}{\varepsilon\delta_1}$, we obtain again that

$$(\dagger) < 2p^{-2} t^{4\ell}. \qquad (2.20)$$

Thus (2.20) holds provided we take $\ell = \ell(\varepsilon)$ large enough, and gives the bound on the number of solutions of (2.12), (2.12′).

Returning to (2.11), we conclude that

$$(2.6) < \frac{2}{N}$$

hence

$$\|D\| < 2N^{-1/4\ell}. \qquad (2.21)$$

This proves (2.1).

Remark. As observed in [K-R], the condition of linear independence mod N of n and nA ($n \in \mathbb{Z}^2$ being fixed, $n \neq (0,0)$) is automatically satisfied for N a sufficiently large prime. Indeed, since A does not have rational eigenvectors, $\det(n, nA) \in \mathbb{Z}\backslash\{0\}$ for all $n \in \mathbb{Z}^2\backslash\{0\}$.

If $o(A,p) = t$, necessarily $p \mid \det(A^t - 1)$, where $\det(A^t - 1) \in \mathbb{Z}\backslash\{0\}$. Therefore a prime $p < T$ for which $o(A,p) < T^\varepsilon$ necessarily divides

$$B = \prod_{1 < t < T^\varepsilon} \det(A^t - 1). \qquad (2.22)$$

The number of these primes is at most $\log|B| < C.T^{2\varepsilon}$.

In view of Proposition 1, this shows the following

Proposition 2. *For all $\varepsilon > 0$, there is $\delta > 0$ and a sequence $\mathcal{S} = \mathcal{S}_\varepsilon$ of primes such that*

$$\#\{N \in \mathcal{S} \mid N < T\} < CT^\varepsilon \qquad (2.23)$$

and for all $n \in \mathbb{Z}^2\backslash\{(0,0)\}$

$$\max_\psi |\langle T_N(n)\psi, \psi\rangle| < N^{-\delta} \qquad (2.24)$$

if N is a sufficiently large prime, $N \notin \mathcal{S}$.
(The maximum taken over all normalized eigenfunctions ψ of $U_N(A)$.)

Hence, for $f \in \mathcal{C}^\infty(\mathbb{T}^2)$

$$\max_\psi \left| \langle Op_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f \right| < N^{-\delta} \qquad (2.25)$$

for N a sufficiently large prime outside $\mathcal{S}$.

3 The Case of General Modulus

We may now establish the following

Theorem 3. *There is a density 1 sequence $\mathcal{N}$ of integers N and $\delta > 0$ such that for all observables $f \in \mathcal{C}^\infty(\mathbb{T}^2)$, we have*

$$\max_\psi \left| \langle Op_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f \right| \ll C_f N^{-\delta} \text{ for } N \in \mathcal{N} \tag{3.1}$$

where the maximum is taken over all normalized eigenfunctions ψ of U_A.

Remark. Compared with [K-R], see in particular the combination of Corollary 9 and Theorem 17 in [K-R], what we get more is an $N^{-\delta}$ estimate rather than $1/\exp(\log N)^\delta$ for some $\delta > 0$.

The main ingredient is the improvement for N prime obtained in previous section.

Proof of Theorem 3. Fix a small positive number $\tau > 0$ (to be specified). Given a positive integer N, write $N = N_1^2 N_2$ with N_2 square-free. Since

$$\big|\{T < N < 2T \mid N_1 > T^\tau\}\big| < \sum_{T^\tau < N_1 \le T^{\frac{1}{2}}} \frac{T}{N_1^2} < T^{1-\tau} \tag{3.2}$$

we may restrict ourselves to integers N with square-free part $N_2 > N^{1-2\tau}$.

Next, we require that for any prime divisor p of N, $p > \sqrt{\log N}$, we have

$$o(A,p) > p^{\frac{1}{3}}. \tag{3.3}$$

As pointed out in the previous section, this property is satisfied for all primes $2^k \le p < 2^{k+1}$ except $2^{\frac{2}{3}k}$ of them. Our requirement (3.3) will therefore exclude from $[T, 2T]$ at most

$$\sum_{2T \gg 2^k > \sqrt{\log T}} 2^{\frac{2}{3}k} \frac{T}{2^k} \ll T(\log T)^{-1/6} \tag{3.4}$$

integers, which again leads to a density zero sequence. Given N as above, write $N = N_1^2 N_0 N'$ where $N_1 < N^\tau, N_0 < [\sqrt{\log N}]! < N^\tau$ and N' is a simple product of primes $p > \sqrt{\log N}$ for which (3.3) holds. Returning to the proof of Proposition 1, we estimate (2.11)

$$t^{-4\ell} N \big|\{(j_1, \dots, j_{4\ell}) \in \{1, \dots, t\}^{4\ell} \mid n(A^{j_1} - \dots - A^{j_{4\ell}}) \equiv 0 (\operatorname{mod} N)\}\big| \tag{3.5}$$

(up to this point no primality of N was involved).

For $M \in \mathbb{Z}_+$, denote $Mat_2(M)$ the 2×2 matrices over $\mathbb{Z}/M\mathbb{Z}$ and G_M its multiplicative subgroup $\{A^j \mid 0 \le j < o(A, M)\}$.

With previous decomposition of N, the map

$$G_N \to G_{N_1^2} \times G_{N_0} \times \prod_{p|N'} G_p$$

is injective. Defining

$$Q_M = \big|\big\{(\alpha_1, \ldots, \alpha_{4\ell}) \in G_M^{4\ell} \mid n(\alpha_1 - \cdots - \alpha_{4\ell}) \equiv 0 (\operatorname{mod} M)\big\}\big| \tag{3.6}$$

the last factor in (3.5) equals Q_N. Obviously

$$Q_N \leq Q_{N_1^2} \cdot Q_{N_0} \cdot \prod_{p|N'} Q_p. \tag{3.7}$$

Take $p|N'$ not dividing $\nu_n = \det(n, nA)$, so that n and nA are independent $\operatorname{mod} p$. Since (3.3) holds, the estimate (2.20) on (†) in the proof of Proposition 1 gives

$$Q_p < 2p^{-2}|G_p|^{4\ell} \tag{3.8}$$

where $\ell = \ell(\frac{1}{3})$ is some integer in particular independent of the choice of τ.

From (3.7), (3.8)

$$\begin{aligned} Q_M &< (N_1^2 N_0 \nu_n)^{16\ell} \prod_{\substack{p|N' \\ (p,\nu_n)=1}} \frac{2o(A,p)^{4\ell}}{p^2} \\ &< \frac{(N_1^2 N_0 \nu_n)^{16\ell+2}}{N^2} \left(\exp \frac{\log N}{\log\log N}\right) \left[\prod_{p|N_2} o(A,p)\right]^{4\ell} \\ &< C_A |n|^{40\ell} N^{60\tau\ell - 2} \left[\prod_{p|N_2} o(A,p)\right]^{4\ell} \end{aligned} \tag{3.9}$$

(N_2 = square free part of N).

At this point, recall Proposition 11 of [K-R]. It asserts that we may minorate

$$o(A, N) > c_A \frac{\prod_{p|N_2} o(A,p)}{\exp(3(\log\log N)^4)} \tag{3.10}$$

by further exclusion of N outside a density zero sequence

Substituting (3.10) in (3.9) gives,

$$\begin{aligned} Q_N &< C_A |n|^{40\ell} N^{60\tau\ell-2} \exp\big(13\ell(\log\log N)^4\big) o(A,N)^{4\ell} \\ &< C_A |n|^{40\ell} N^{61\tau\ell-2} o(A,N)^{4\ell}. \end{aligned} \tag{3.11}$$

Hence, from the argument in the initial part of the proof of Proposition 1

$$|\langle T_N(n)\psi, \psi\rangle| < C_A |n|^{10} N^{61\tau - \frac{1}{4\ell}}. \tag{3.12}$$

Choosing τ small enough, the claim easily follows.

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Some Arithmetical Applications of the Sum-Product Theorems in Finite Fields

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1 Introduction

The aim of this Note is to present a few more consequences of the combinatorial sum-product approach in prime fields to questions of obtaining nontrivial bounds on certain exponential sums and Waring type problems. Our analysis will depend heavily on results from [BGK] and [B]. Let us point out that although the statements are formulated in a non quantitative way, in principle more precise versions may be obtainable since all constants and exponents can be made explicit in [BGK], [B]. But their interest would surely not justify a rather tedious bookkeeping.

We first recall two theorems that characterize in a satisfactory way which subsets $A \subset \mathbb{F}_p$ and $A \subset \mathbb{F}_p \times \mathbb{F}_p$ have a 'small' sumset and product-set
$A + A = \{x + y | x, y \in A\}$
$A.A = \{x.y | x, y \in A\}$
(where $\mathbb{F}_p \times \mathbb{F}_p$ is endowed with its natural product structure).

Theorem 1 ([BKT] and [BGK]). *For all* $\varepsilon > 0$*, there is* $\delta > 0$ *such that if* $A \subset \mathbb{F}_p$ *and* $1 < |A| < p^{1-\varepsilon}$*, then*

$$|A + A| + |A.A| > |A|^{1+\delta}$$

and

Theorem 2 ([B]). *Fix* $\varepsilon > 0$*. There is* $\delta'(\delta) \overset{\delta \to 0}{\longrightarrow} 0$ *such that if* $A \subset \mathbb{F}_p \times \mathbb{F}_p$ $(p^\varepsilon < |A| < p^{2-\varepsilon})$ *satisfies*

$$|A + A| + |A.A| < p^\delta |A|$$

then

$$p^{1-\delta'} < |A| < p^{1+\delta'}$$

and there is a line $L \subset \mathbb{F}_p \times \mathbb{F}_p$ *of the form*

$$L = \{a\} \times \mathbb{F}_p, L = \mathbb{F}_p \times \{a\}$$

or

$$L = \big\{(x, ax) \mid x \in \mathbb{F}_p\big\}$$

such that

$$|A \cap L| > p^{1-\delta'}.$$

In [BKT], Theorem 1 is proven under the additional assumption that $p^\varepsilon < |A| < p^{1-\varepsilon}$, which is no restriction for our purpose. The proof of this result and of Theorem 2 is 'elementary', assuming the Pluck–Ruzsa sumset theory (see [Na], [T-V]) and T. Gowers' quantitative version of the Balog–Szemeredi theorem (see [BKT]).

2 Application to Exponential Sums

Both Theorems 1 and 2 have very significant applications to the theory of exponential sums since they provide estimates in situations which seem out of reach of classical methods such as Stepanov's method (cf. [K-S], [I-K]).

In [BGK], the following result for Gauss sums is proven (it is not formulated here in the sharpest form available).

Theorem 3. *For all $\delta > 0$, there is $\delta' > 0$ such that if $(k, p-1) < p^{1-\delta}$, then*

$$\max_{(a,p)=1} \left| \sum_{x=1}^{p} e_p(ax^k) \right| < Cp^{1-\delta'} \ . \tag{2.1}$$

Recall the classical bound due to Gauss

$$\max_{(a,p)=1} \left| \sum_{x=1}^{p} e_p(ax^k) \right| < (k, p-1)\sqrt{p} \tag{2.2}$$

which becomes trivial for $(k, p-1) \geq \sqrt{p}$. Results due to Garcia–Voloch, Shparlinski, Heath–Brown, Konyagin based on Stepanov's method (see [K-S], [K] for details) allowed eventually to establish (2.1) provided $(k, p-1) < p^{\frac{3}{4}-\varepsilon}$. The main ingredient in the proof of Theorem 3 is Theorem 1.

In a similar vein, Theorem 2 permits us to obtain an estimate for 'sparse' polynomials (originally established by Mordell [Mo] with much more restrictive conditions) under assumptions on the exponents that are essentially optimal.

Theorem 4 ([B]). *Let*

$$f(x) = \sum_{i=1}^{r} a_i x^{k_i} \in \mathbb{Z}[X] \ \textit{with} \ (a_i, p) = 1 \tag{2.3}$$

and exponents $k_1 < k_2 < \cdots < k_r$ *satisfying*

$$(k_i, p-1) < p^{1-\delta} \qquad (1 \le i \le r) \tag{2.4}$$

$$(k_i - k_j, p-1) < p^{1-\delta} \qquad (1 \le i \ne j < r) \tag{2.5}$$

where $\delta > 0$ *is arbitrary and fixed.*

Then

$$\Big|\sum_{x=1}^{p} e_p\big(f(x)\big)\Big| < Cp^{1-\delta'} \tag{2.6}$$

where $\delta' = \delta'(r, \delta) > 0$.

This estimate should be compared with the more precise (but also more restrictive) result of Cochrane and Pinner ([C-P]) improving upon Mordell's ([Mo]).

Theorem 5 ([C-P]). *Let* $f(x) \in \mathbb{Z}[X]$ *be as in* (2.3). *Then*

$$\Big|\sum_{x=1}^{p-1} e_p\big(f(x)\big)\Big| < \sqrt{2}^{r}(k_1 \dots k_r)^{\frac{1}{r^2}} p^{1-\frac{1}{2r}}. \tag{2.7}$$

For this bound to be indeed non-trivial, one needs in particular the geometric mean of the exponents $k_1, \dots, k_r$ to be small compared with $\sqrt{p}$, while Theorem 4 applies as soon as $k_1, \dots, k_r < p^{1-\delta}$, $\delta > 0$ arbitrary.

Recall also the example from [C-P] showing the relevance of condition (2.5) on differences of exponents. Take $r = 2, k_1 = 1, k_2 = \frac{p-1}{2} + 1$. Then for $f(x) = x - x^{\frac{p-1}{2}+1}$

$$\sum_{x=1}^{p-1} e_p\big(f(x)\big) = \frac{p-1}{2} + \sum_{(\frac{x}{p})=-1} e_p(2x) = \frac{p-1}{2} + 0(\sqrt{p})\ . \tag{2.8}$$

Of course, Theorems 4 and 5 only apply in a useful way to a restricted class of polynomials involving few monomials. This should be contrasted to A. Weil's result (see [I-K])

Theorem 6. *Let* $f(x) \in \mathbb{Z}[X]$ *be a polynomial of degree d, nontrivial over* $\mathbb{F}_p$. *Then*

$$\Big|\sum_{x=1}^{p} e_p\big(f(x)\big)\Big| \le (d-1)\sqrt{p} \tag{2.9}$$

providing non-trivial estimates for $d < \sqrt{p}$.

Our first result merges Theorem 4 with Theorem 6, providing an estimate for polynomials involving monomials x^k with all $k < p^{\frac{1}{2}-\varepsilon}$ except a few of them. This is presumably the largest class of polynomials for which presently we may establish a bound of the form $p^{1-\delta}$ (putting aside a different type of estimates of the more modest form $(1 - \frac{1}{p^{\gamma(f)}})p$, as obtained in [C-P-R] for a large class of polynomials $f(x)$).

Theorem 7. *For all $r \in \mathbb{Z}_+, \delta > 0$, there is $\delta' = \delta(r,\delta) > 0$ such that the following holds. Let $\{k_0, k_1, \dots, k_r\}$ be distinct exponents with*

$$k_0 < p^{\frac{1}{2}-\delta} \tag{2.10}$$

$$(k_i - k_0, p-1) < p^{1-\delta} \ . \tag{2.11}$$

Let

$$f(x) = a_1 x^{k_1} + \cdots + a_r x^{k_r} + a_0 x^{k_0} + g(x) \in \mathbb{Z}[X] \tag{2.12}$$

with $g(x)$ of degree $d < p^{\frac{1}{2}-\delta}$, not involving the monomial x^{k_0}, and $(a_0, p) = 1$.

Then

$$\Big| \sum_{x=1}^{p} e_p\big(f(x)\big) \Big| < p^{1-\delta'} \ . \tag{2.13}$$

Proof. We proceed by induction on r. If none of the frequencies $k_1, \dots, k_r$ are present, $f(x)$ is of degree at most $p^{\frac{1}{2}-\delta}$ and Weil's estimate (2.9) applies. Let now $r \geq 1$. Assume first

$$(k_r, p-1) < p^{1-\delta/4} \ . \tag{2.14}$$

Estimate (with $\ell \in \mathbb{Z}_+$ to be specified later)

$$\Big| \sum_{x=1}^{p} e_p\big(f(x)\big) \Big|^{\ell} \leq \sum_{y=1}^{p} \Big| \sum_{x_1^{k_r} + \cdots + x_\ell^{k_r} = y} e_p\big(\phi(x_1) + \cdots + \phi(x_\ell)\big) \Big| \tag{2.15}$$

where $\phi(x) \in \mathbb{Z}[X]$ is the polynomial

$$\phi(x) = a_1 x^{k_1} + \cdots + a_{r-1} x^{k_{r-1}} + a_0 x^{k_0} + g(x)$$

and the inner sum (2.14) extends over all $(x_1, \dots, x_\ell) \in \mathbb{F}_p^\ell$ such that

$$x_1^{k_r} + \cdots + x_\ell^{k_r} \equiv y \ (\mathrm{mod}\ p) \ .$$

Estimate further by the Cauchy–Schwartz inequality

$$(2.15) \leq$$

$$\sqrt{p} \Big\{ \sum_{x_1^{k_r} + \cdots + x_\ell^{k_r} = x_{\ell+1}^{k_r} + \cdots + x_{2\ell}^{k_r}} e_p\big(\phi(x_1) + \cdots + \phi(x_\ell) - \phi(x_{\ell+1}) \cdots - \phi(x_{2\ell})\big) \Big\}^{1/2} \ . \tag{2.16}$$

In order to apply the induction hypothesis, we introduce a new variable, observing that the condition

$$x_1^{k_r} + \cdots + x_\ell^{k_r} \equiv x_{\ell+1}^{k_r} + \cdots + x_{2\ell}^{k_r} \pmod p \tag{2.17}$$

is invariant under multiplication of $x_1, \dots, x_{2\ell}$ by an element $y \in \mathbb{F}_p^*$. Hence for the inner sum in (2.16)

$$\sum_{\dots} e_p\big(\phi(x_1)\dots - \phi(x_{2\ell})\big) = \frac{1}{p-1}\sum_{y=1}^{p-1}\sum_{\dots} e_p\big(\phi(yx_1)\cdots - \phi(yx_{2\ell})\big)$$
$$\leq \frac{1}{p-1}\sum_{\dots}\left|\sum_{y=1}^{p-1} e_p\big(\phi(yx_1)\cdots - \phi(yx_{2\ell})\big)\right| . \quad (2.18)$$

For $x_1, \dots, x_{2\ell}$ fixed, rewrite

$$\begin{aligned}\psi(y) &= \phi(yx_1)\cdots - \phi(yx_{2\ell})\\ &= b_1 y^{k_1} + \cdots + b_{r-1}y^{k_{r-1}} + b_0 y^{k_0} + h(y) \in \mathbb{Z}[Y]\end{aligned}$$

with

$$b_i = a(x_1^{k_i} + \cdots - x_{2\ell}^{k_i}) \quad (0 \leq i < r)$$

and

$$\text{degree } h \leq \text{degree } g.$$

The induction hypothesis applies provided

$$x_1^{k_0} + \cdots - x_{2\ell}^{k_0} \not\equiv 0 \pmod{p}$$

and we get

$$(2.18) \leq \frac{p^{1-\delta'_{r-1}}}{p-1}\left(\#\Big\{(x_1,\dots x_{2\ell}) \in \mathbb{F}_p^{2\ell} \;\Big|\; x_1^{k_r} + \cdots - x_{2\ell}^{k_r} = 0\Big\}\right) \quad (2.19)$$
$$+\left(\#\Big\{(x_1,\dots,x_{2\ell}) \in \mathbb{F}_p^{2\ell} \;\Big|\; x_1^{k_i} + \cdots - x_{2\ell}^{k_i} = 0 \text{ for } i = 0, r\Big\}\right) . \quad (2.20)$$

We claim that taking $\ell = \ell(\delta) \in \mathbb{Z}_+$ large enough, we can ensure that

$$\#\big\{(x_1,\dots,x_{2\ell}) \mid x_1^{k_r} + \cdots - x_{2\ell}^{k_r} = 0\big\} < 2p^{2\ell-1} \quad (2.21)$$

and

$$\#\big\{(x_1,\dots,x_{2\ell}) \mid x_1^{k_i} + \cdots - x_{2\ell}^{k_i} = 0 \text{ for } i = 0, r\big\} < 2p^{2(\ell-1)} . \quad (2.22)$$

From (2.19)–(2.22), it follows then that

$$(2.18) \lesssim p^{2\ell-1-\delta'_{r-1}} \quad (2.23)$$

and recalling (2.15), (2.16)

$$\left|\sum_{x=1}^{p} e_p\big(f(x)\big)\right| \lesssim p^{1-\frac{\delta'_{r-1}}{2\ell}} = p^{1-\delta'_r} \quad (2.24)$$

completing the argument.

Returning to (2.21), (2.22), we deduce (2.21) from Theorem 3 and (2.22) from Theorem 4 applied in the binomial case $r = 2$ with exponents k_0, k_r. We verify (2.22) ((2.21) is easier). Thus clearly from the circle method

$$(2.22) = \frac{1}{p^2} \sum_{x_1,\dots,x_{2\ell}=1}^{p} \sum_{\xi=1}^{p} \sum_{\xi'=1}^{p} e_p\big((x_1^{k_0} + \cdots - x_{2\ell}^{k_0})\xi + (x_1^{k_r} + \cdots - x_{2\ell}^{k_r})\xi'\big)$$

$$\leq p^{2\ell-2} + \frac{1}{p^2} \sum_{(\xi,\xi')\neq(0,0)} \Big| \sum_{x=1}^{p} e_p(\xi x^{k_0} + \xi' x^{k_r}) \Big|^{2\ell} . \tag{2.25}$$

In view of (2.10), (2.14) and (2.11) with $i = r$, application of Theorem 4 gives for $\xi \neq 0, \xi' \neq 0$

$$\Big| \sum_{x=1}^{p} e_p(\xi x^{k_0} + \xi' x^{k_r}) \Big| < p^{1-\gamma} \tag{2.26}$$

$\gamma = \gamma(\delta) > 0$. If $\xi = 0$ or $\xi' = 0$, apply of course Theorem 3. Therefore, assuming (2.26),

$$(2.25) \leq p^{2\ell-2} + p^{-2} p^2 p^{2\ell(1-\delta)} < 2p^{2(\ell-1)} \tag{2.27}$$

for appropriate choice of $\ell = \ell(\delta)$.

It remains to deal with the case $(k_r, p-1) > p^{1-\delta/4}$. Thus $p-1 | t.k_r$ where $t \in \mathbb{Z}_+, t < p^{\delta/4}$. Replace in (2.13) the variable x by $x.y^t$ and estimate by

$$\frac{1}{p-1} \sum_{x=1}^{p} \Big| \sum_{y=1}^{p-1} e_p\big(f(x.y^t)\big) \Big| \tag{2.28}$$

where

$$\mathbb{Z}[Y] \ni F(y) = f(x.y^t) = b_1 y^{tk_1} + \cdots + b_{r-1} y^{tk_{r-1}} + b_r + b_0 y^{tk_0} + h(y) \tag{2.29}$$

with

$$b_i = a_i x^{k_i} \qquad (0 \leq i \leq r)$$

and

$$\deg h = t.\deg g < p^{\frac{1}{2}-\delta+\frac{\delta}{4}} = p^{\frac{1}{2}-\frac{3\delta}{4}} .$$

The exponents $tk_0, tk_1, \dots, tk_{r-1}$ in (2.29) verify (2.10), (2.11) if we replace δ by $\frac{3\delta}{4}$. Application of the induction hypotheses gives

$$(2.28) < 1 + p^{1-\delta'(r-1,\frac{3\delta}{4})} . \tag{2.30}$$

This completes the proof.

Remarks.

(1). Again the [C-P] examples show the relevance of conditions (2.11). For instance, letting

$$f(x) = x^{\frac{p-1}{2}+1} + \cdots + x^{\frac{p-1}{2}+r} - x - \ldots - x^r \tag{2.31}$$

we get

$$\begin{aligned}\sum_{x=1}^{p-1} e_p\big(f(x)\big) &= \frac{p-1}{2} + \sum_{(\frac{x}{p})=-1} e_p\big(2(x+\cdots+x^r)\big)\\ &= \frac{p-1}{2} + 0(\sqrt{p})\end{aligned} \tag{2.32}$$

by Weil.

Avoiding conditions such as (2.5), (2.11) on differences of exponents is only possible by imposing also conditions on the coefficients of f.

(2). As observed by the referee, the proof of Theorem 7 permits also to establish the following bound on certain exponential sums involving a multiplicative character.

Theorem 7′. *Let $f(x) \in \mathbb{Z}[X]$ be as in Theorem 7 and $h(x) \in \mathbb{Z}[X]$ a polynomial of degree at most $p^{\frac{1}{2}-\delta}$. Then*

$$\Big|\sum_{x=1}^{p} \mathcal{X}\big(h(x)\big)e_p\big(f(x)\big)\Big| < p^{1-\delta'} \tag{2.33}$$

where $\mathcal{X}$ is a multiplicative character (mod p).

We are now using the Weil bound (Theorem 6) with the additional factor $\mathcal{X}\big(h(x)\big)$ in the sum.

3 Simultaneous Waring Problem (mod p) for Reciprocals

Our purpose is to show the following property.

Theorem 8. *Given $J \in \mathbb{Z}_+$ and $\kappa > 0$ arbitrary, there is a number $r = r(\kappa, J)$ such that for all $(a_1, \ldots, a_J) \in \mathbb{F}_p^J$, the congruences*

$$\sum_{s=1}^{r} (\overline{x}_s)^j \equiv a_j (\mathrm{mod}\, p) \qquad (1 \le j \le J) \tag{3.1}$$

have a solution $(x_1, \ldots, x_r) \in ([0, p^\kappa] \cap \mathbb{Z})^r$.

We denote here $\overline{x}$ the reciprocal of x (mod p), *i.e. $x\bar{x} \equiv 1$* (mod p).

When $J = 1$, the problem was solved by I. Shparlinski ([Shp]) answering a question of Erdös and Graham. His method is based on the use of Karatsuba's estimate for certain bilinear Klosterman sums. The general problem for one equation

$$\sum_{s=1}^{r} (\overline{x}_s)^j \equiv a \pmod{p} \tag{3.2}$$

($j \in \mathbb{Z}_+$ arbitrary fixed) was settled by E. Croot ([Cr]) and involves the sum-product theorem in $\mathbb{F}_p$ from [BKT]. Our approach is in the same spirit but we also bring the results and methods from [B] into play.

Instead of taking in (3.1) the variable $x_s \in [0, p^\kappa]$, we can impose the conditions $x_s \in [M_s, 2M_s]$ $(1 \le s \le r)$, assuming $M_s > p^\kappa, \kappa > 0$ arbitrary fixed. But we need to emphasize that if we were to consider *arbitrary* intervals $x_s \in I_s, I_s \subset [1, p]$ an interval of size p^κ, already for $J = 1$ we would need to assume $\kappa > \frac{1}{2}$. The argument described below is indeed not applicable for more general intervals. In this case the only available bound seems to be Weil's (after completing the sum), requiring $\kappa > \frac{1}{2}$ so far. Of relevance here is of course Hooley's 'R^*-conjecture' for incomplete Kloosterman sums

$$\sum_{x \in I} e_p(ax + b\overline{x}) \ . \tag{3.3}$$

Already establishing nontrivial bounds on (3.3) when $|I| < \sqrt{p}$ would be a breakthrough (cf. [H-B]).

Remark. An estimate $o(|I|)$ on (3.3) is obtainable for $I = [0, p^\kappa], \kappa > 0$ arbitrary, roughly speaking by retaining only integers x with sufficient many divisors and reducing the problem to multilinear estimates as described below. See also the remark at the end of this section.

Denote $\mathcal{P}$ the set of prime numbers.

Returning to Theorem 8, we produce $x_s \in [0, p^\kappa]$ as a product $x_{s,1} \ldots x_{s,\ell}$ where $x_{s,1}, \ldots, x_{s,\ell} \in [0, p^{\kappa/\ell}] \cap \mathcal{P}$ and where the integer ℓ is chosen sufficiently large (depending on r). In this form, we solve the representation problem (3.1) with the standard circle method and the following incomplete multilinear Kloosterman estimate.

Proposition 9. *Let $k_1, \ldots, k_J$ be fixed, distinct positive integers and $\kappa > 0$. Then there is*

$$\ell = \ell(k_1, \ldots, k_J) \in \mathbb{Z}_+$$

and

$$\delta = \delta(k_1, \ldots, k_J; \kappa) > 0$$

such that

$$\max_{(a_1, \ldots, a_J, p) = 1} \left| \sum_{\substack{0 < x_1, \ldots, x_\ell < p^{\kappa/\ell} \\ x_1, \ldots, x_\ell \in \mathcal{P}}} e_p\left(\sum_{j=1}^{J} a_j (\overline{x}_1 \ldots \overline{x}_\ell)^{k_j} \right) \right| < p^{\kappa - \delta} \ . \tag{3.4}$$

The main point is to establish Proposition 9. With Proposition 9 at hand, the existence of a solution to the representation problem

$$\sum_{s=1}^{r}(\overline{x}_{s,1}\cdots\overline{x}_{s,\ell})^{k_j}=a_j \pmod p \quad (1\le j\le J) \tag{3.5}$$

with $x_{s,1},\dots,x_{s,\ell}\in\mathcal{P}\cap[0,p^{\kappa/\ell}]$ and $r>r(k_1,\dots,k_J,\kappa)$ follows from a standard application of the circle method, providing moreover the expected asymptotic formula

$$\big(1+o(1)\big)p^{\kappa r-J}(\log p^{\kappa/\ell})^{-\ell r} \tag{3.6}$$

for the number of solutions of (3.5).

The proof of Proposition 9 is by induction on J and follows essentially the procedure as applied in [B] for Mordell polynomials (cf. Theorem 4 above).

Thus we establish the result for $J=1, J=2$ and derive then the general case $J>2$ rather easily.

Case $J=1$. Already here the special choice of the intervals will appear. Denote $\mathcal{M}=\mathcal{P}\cap[0,p^{\kappa/\ell}]$ and consider the sum

$$S=\sum_{x_1,\dots,x_\ell\in\mathcal{M}}e_p(a\overline{x}_1^k\cdots\overline{x}_\ell^k)\qquad (a,p)=1\ . \tag{3.7}$$

Let

$$M=|\mathcal{M}|\sim p^{\kappa/\ell}/\log p^{\kappa/\ell} \tag{3.8}$$

and assume

$$|S|>M^{\ell-\delta}. \tag{3.9}$$

Fix an integer $u\in\mathbb{Z}_+$ (to be specified) and write by Hölder's inequality

$$\begin{aligned}|S|&\le\sum_{x_2,\dots,x_\ell\in\mathcal{M}}\Big|\sum_{x_1\in\mathcal{M}}e_p(a\overline{x}_1^k\cdots\overline{x}_\ell^k)\Big|\\&\le M^{(\ell-1)(1-\frac{1}{2u})}\bigg(\sum_{x_2,\dots,x_\ell}\Big|\sum_{x_1}e_p(a\overline{x}_1^k\cdots\overline{x}_\ell^k)\Big|^{2u}\bigg)^{\frac{1}{2u}}\ .\end{aligned} \tag{3.10}$$

From (3.8), (3.9) it follows that

$$\Bigg|\sum_{\substack{x_{1,1},\,\dots,\,x_{1,2u}\,\in\,\mathcal{M}\\ x_2,\,\dots,\,x_\ell\,\in\,\mathcal{M}}}e_p\Big(a(\overline{x}_{1,1}^k-\overline{x}_{1,2}^k\cdots-\overline{x}_{1,2u}^k)\overline{x}_2^k\cdots\overline{x}_\ell^k\Big)\Bigg|>$$
$$M^{\ell+2u-1-2u\delta}\ . \tag{3.11}$$

For $y\in\mathbb{F}_p$, denote

$$\mu(y)=\big|\big\{(x_1,\dots,x_{2u})\in\mathcal{M}^{2u}\mid\overline{x}_1^k-\overline{x}_2^k\cdots-\overline{x}_{2u}^k=y\big\}\big|\ .$$

Thus

$$\sum_{y=1}^{p} \mu(y) = M^{2u} \tag{3.12}$$

and

$$\sum \mu(y)^2 = \left|\left\{(x_1, \dots, x_{4u}) \in \mathcal{M}^{4u} \mid \overline{x}_1^k + \cdots + \overline{x}_{2u}^k = \overline{x}_{2u+1}^k + \cdots + \overline{x}_{4u}^k\right\}\right| .$$

This is the place where the key arithmetical ingredient (the same as in Karatsuba's and Croot's argument). Thus the condition

$$\overline{x}_1^k + \cdots + \overline{x}_{2u}^k = \overline{x}_{2u+1}^k + \cdots + \overline{x}_{4u}^k$$

amounts to

$$\sum_{\alpha=1}^{4u} \pm \prod_{\beta \neq \alpha} x_\beta^k = 0 \pmod p \tag{3.13}$$

and all terms in (3.13) are at most $p^{\frac{\kappa}{\ell} k(4u-1)}$. Take

$$u = \left[\frac{\ell}{4\kappa k}\right] > 1 \tag{3.14}$$

(taking ℓ sufficiently large).

Then necessarily

$$\sum_{\alpha=1}^{4u} \pm \prod_{\beta \neq \alpha} x_\beta^k = 0 \quad (\text{in } \mathbb{Z})$$

and since $x_1, \dots, x_{4u}$ were assumed prime, we must have that $\{x_1, \dots, x_{2u}\} = \{x_{2u+1}, \dots, x_{4u}\}$ (dismissing small values of x). It follows that

$$\sum_{y=1}^{p} \mu(y)^2 < C_u M^{2u} . \tag{3.15}$$

Rewrite (3.11) as

$$\left| \sum_{y \in \mathbb{F}_p; x_2, \dots, x_\ell \in \mathcal{M}} \mu(y) e_p(ay\overline{x}_2^k \cdots \overline{x}_\ell^k) \right| > M^{\ell+2u-1-2u\delta} . \tag{3.16}$$

Repeat estimates (3.10), (3.11) replacing the variable x_1 by x_2. We obtain

$$M^{(\ell-2)(1-\frac{1}{2u})} \left[\sum \mu(y)^{\frac{2u}{2u-1}}\right]^{(1-\frac{1}{2u})} \cdot \left| \sum_{\substack{y_1, y_2 \in \mathbb{F}_p \\ x_3, \dots, x_\ell \in \mathcal{M}}} \mu(y_1)\mu(y_2) e_p(ay_1 y_2 \overline{x}_3^k \cdots \overline{x}_\ell^k) \right|^{\frac{1}{2u}} > M^{\ell+2u-1-2u\delta}$$

and invoking (3.11), (3.12)

$$\left| \sum_{y_1, y_2 \in \mathbb{F}_p; x_3, \ldots, x_\ell \in \mathcal{M}} \mu(y_1)\mu(y_2) e_p(a y_1 y_2 \overline{x}_2^k \cdots \overline{x}_\ell^k) \right| \gtrsim M^{\ell - 2 + 4u - 4u^2\delta} \ . \tag{3.17}$$

Continuing the process, we conclude that

$$\left| \sum \mu(y_1) \cdots \mu(y_\ell) e_p(a y_1 y_2 \cdots y_\ell) \right| \gtrsim M^{2\ell u - (2u)^\ell \delta} \ . \tag{3.18}$$

Recall that μ satisfies (3.12), (3.15) where by (3.14)

$$M^{2u} = p^{\frac{2\kappa}{\ell}[\frac{\ell}{4\kappa k}]} > p^{\frac{1}{3k}} \ . \tag{3.19}$$

Thus μ considered as a measure on $\mathbb{F}_p$ has entropy ratio at least $\frac{1}{3k}$ and the left side of (3.18) may be estimated by an application of Theorem 5 in [BGK], provided the number of factors $\ell > \ell(k)$.

Remark. Actually Theorem 5 in [BGK] is formulated for multilinear sums of the form

$$\sum_{y_1 \in A_1 \ldots y_\ell \in A_\ell} e_p(a y_1 \cdots y_\ell) \tag{3.20}$$

where $A_1, \ldots, A_\ell \subset \mathbb{F}_p$ and $|A_1|, \ldots, |A_\ell| > p^\sigma; \ell > \ell(\sigma)$. The passage to our setting where $\mathcal{X}_A$ is replaced by μ satisfying conditions (3.12), (3.15) is straightforward.

Returning to (3.18), application of the [BGK] result implies that

$$\left[\sum \mu(y) \right]^\ell p^{-\gamma} > c(\kappa, k) M^{2\ell u - (2u)^\ell \delta} \tag{3.21}$$

where $\gamma = \gamma(k) > 0$. Therefore

$$\delta > \frac{\gamma}{2(2u)^\ell} = \delta(k; \kappa) \ . \tag{3.22}$$

This takes care of the case $J = 1$.

Case $J = 2$. This is technically the most interesting one requiring some additional considerations involving the sum-product theorem (Theorem 2 above) in $\mathbb{F}_p \times \mathbb{F}_p$.

Thus we have to estimate the sum

$$S = \sum_{x_1, \ldots, x_\ell \in \mathcal{M}} e_p(a \overline{x}_1^{k_1} \cdots \overline{x}_\ell^{k_1} + b \overline{x}_1^{k_2} \cdots \overline{x}_\ell^{k_2}) \tag{3.23}$$

where $(a, p) = 1 = (b, p)$ and $\mathcal{M}$ as above.

Assume

$$|S| > M^{\ell-\delta}\ . \tag{3.23'}$$

Proceeding exactly as above, we obtain the analogue of (3.16)

$$\Bigg|\sum_{\substack{(y,z)\in\mathbb{F}_p^2\\ x_2,\dots,x_\ell\in\mathcal{M}}} \nu(y,z)e_p(ay\overline{x}_2^{k_1}\cdots\overline{x}_\ell^{k_1}+bz\overline{x}_2^{k_2}\cdots\overline{x}_\ell^{k_2})\Bigg| > M^{2u+\ell-1-2u\delta} \tag{3.24}$$

where now for $y,z\in\mathbb{F}_p$ we define

$$\nu(y,z) = \left|\left\{(x_1,\dots,x_{2u})\in\mathcal{M}_0^{2u}\ \Big|\ \begin{matrix}\overline{x}_1^{k_1}-\cdots-\overline{x}_{2u}^{k_1}=y\\ \overline{x}_1^{k_2}-\cdots-\overline{x}_{2u}^{k_2}=z\end{matrix}\right\}\right|. \tag{3.25}$$

Thus

$$\sum_{y,z\in\mathbb{F}_p}\nu(y,z) = M^{2u}$$

and

$$\sum_{y,z}\nu(y,z)^2 =$$
$$\left|\left\{(x_1,\dots,x_{4u})\in\mathcal{M}^{4u}\ \Big|\ \begin{matrix}\overline{x}_1^{k_1}+\cdots+\overline{x}_{2u}^{k_1}=\overline{x}_{2u+1}^{k_1}+\cdots+\overline{x}_{4u}^{k_1}\\ \overline{x}_1^{k_2}+\cdots+\overline{x}_{2u}^{k_2}=\overline{x}_{2u+1}^{k_2}+\cdots+\overline{x}_{4u}^{k_2}\end{matrix}\right\}\right|. \tag{3.26}$$

Take

$$u = \left[\frac{\ell}{20\kappa(k_1+k_2)}\right] > 1.$$

Already considering only the first condition

$$\overline{x}_1^{k_1}+\cdots+\overline{x}_{2u}^{k_1}=\overline{x}_{2u+1}^{k_1}+\cdots+\overline{x}_{4u}^{k_1}$$

in (3.26), we get (3.26) $< C_u M^{2u}$, hence

$$\sum_{y,z}\nu(y,z)^2 \lesssim M^{2u}.$$

We obtain again that

$$\Big|\sum \nu(y_1,z_1)\cdots\nu(y_\ell,z_\ell)e_p(ay_1\cdots y_\ell+bz_1\dots z_\ell)\Big| > M^{2\ell u-(2u)^\ell\delta}\ . \tag{3.27}$$

In order to obtain an upper bound on the left of (3.27), we apply a generalization of Theorem 5 in [BGK] to rings $R=\prod\mathbb{Z}_{q_j}$, which is given by Theorem 3.1 in [B-C]. For our application, $R=\mathbb{F}_p\times\mathbb{F}_p$. The result of [B-C] is formulated for sums of the form

$$\sum_{\substack{(y_i,z_i)\in A\\ 1\le i\le\ell}} e_p(ay_1\dots y_\ell+bz_1\dots z_\ell) \tag{3.28}$$

where $A \subset \mathbb{F}_p \times \mathbb{F}_p$ is subject to certain conditions. We assume

$$p^\gamma < |A| < p^{2-\gamma} \tag{3.29}$$

($\gamma > 0$ arbitrary fixed) and also that A does not have to large an intersection with the set $R\backslash R^*$ of non-invertible elements or with a set $S \subset R$ such that

$$|S| < p^{2-\gamma/10} \tag{3.30}$$

and S has a 'small' sum and product set in R. More precisely, assume

$$|A \cap (R\backslash R^*)| < p^{-\kappa_0}|A| \tag{3.31}$$

and also, for all $\xi \in R$

$$|(\xi + A) \cap S| < p^{-\kappa_0}|A| \tag{3.32}$$

whenever $S \subset \mathbb{F}_p \times \mathbb{F}_p$ satisfies (3.30) and

$$|S + S| + |S.S| < p^{\kappa_0}|S| \tag{3.33}$$

(letting $0 < \kappa_0 < \gamma$ be arbitrary and fixed).

Taking then $\ell > \ell(\kappa_0)$ in (3.28), Theorem 3.1 of [B-C] states a bound

$$|(3.28)| < p^{-\varepsilon}|A|^\ell \tag{3.34}$$

with $\varepsilon = \varepsilon(\kappa_0) > 0$.

In view of the first and second moment information on ν, we get in particular

$$p^{\frac{1}{11(k_1+k_2)}} < c_\ell \mathcal{M}^{2u} < |\operatorname{supp} \nu| < M^{2u} < p^{\frac{1}{10(k_1+k_2)}} \tag{3.35}$$

and we may take $\gamma = \frac{1}{20(k_1+k_2)}$ in (3.29).

It clearly suffices to show that for some $\kappa_0 = \kappa_0(k_1, k_2) > 0$

$$\nu(R\backslash R^*) < p^{-\kappa_0} M^{2u} \tag{3.36}$$

and for all $\xi \in \mathbb{F}_p \times \mathbb{F}_p$

$$\nu(\xi + S) < p^{-\kappa_0} M^{2u} \tag{3.37}$$

whenever $S \subset \mathbb{F}_p \times \mathbb{F}_p$ satisfies (3.30), (3.33).

Regarding (2.36), we need a bound for $k = k_1$ or $k = k_2$ on

$$\left|\left\{(x_1, \ldots, x_{2u}) \in \mathcal{M}^{2u} \mid \overline{x}_1^k - \cdots \overline{x}_{2u}^k = 0\right\}\right| . \tag{3.38}$$

The same argument as implying (3.15) shows that the left side of (3.38) is at most $C_u M^u$, so that by (3.35),

$$(3.38) < M^{2u} p^{-\frac{1}{30(k_1+k_2)}} . \tag{3.39}$$

Next, consider a subset $S \subset \mathbb{F}_p \times \mathbb{F}_p$ satisfying (3.30) with $\gamma = \frac{1}{20(k_1+k_2)}$ and (3.33) for sufficiently small κ_0 (to be specified).

Apply Theorem 2 above to $A = S$ with $\varepsilon = \gamma$ and $\delta = \kappa_0$.

Hence for some line $L \subset \mathbb{F}_p \times \mathbb{F}_p$ as described in Theorem 2, we have

$$p^{1-\kappa_1} < |S \cap L| < |S| < p^{1+\kappa_1} \tag{3.40}$$

with $\kappa_1 = \kappa_1(\kappa_0) \overset{\kappa_0 \to 0}{\longrightarrow} 0$.

Write $S_1 = S \cap L$ and

$$\mathcal{X}_S \leq \frac{1}{|S_1|} \sum_{\zeta \in S - S_1} \mathcal{X}_{\zeta + S_1}$$

hence

$$\begin{aligned} \nu(\xi + S) &\leq \frac{1}{|S_1|} \sum_{\zeta \in S - S_1} \nu(S_1 + \zeta + \xi) \\ &\overset{(3.40)}{<} p^{-1+\kappa_1} |S - S| \big(\max_\zeta \nu(L + \zeta) \big) \\ &< p^{2(\kappa_0 + \kappa_1)} \max_\zeta \nu(L + \zeta) \end{aligned} \tag{3.41}$$

since, by (3.33), (3.40) and Ruzsa's inequality

$$|S - S| < p^{2\kappa_0} |S| < p^{2\kappa_0 + \kappa_1 + 1}.$$

The case where L is a horizontal or vertical line amounts to an estimate on the size of the set

$$\big| \big\{ (x_1, \ldots, x_{2u}) \in \mathcal{M}^{2u} \mid \overline{x}_1^k - \cdots - \overline{x}_{2u}^k = a \big\} \big| \tag{3.42}$$

with $k = k_1$ or $k = k_2$ and $a \in \mathbb{F}_p$. Clearly (3.42) $\leq$ (3.38) $<$ (3.39).

Let now L be of the form

$$L = \{ (x, ax) \mid x \in \mathbb{F}_p \}$$

for some $a \in \mathbb{F}_p^*$.

We need to bound

$$\big| \big\{ (x_1, \ldots, x_{2u}) \in \mathcal{M}^{2u} \mid \overline{x}_1^{k_2} - \cdots - \overline{x}_{2u}^{k_2} = a(\overline{x}_1^{k_1} - \cdots - \overline{x}_{2u}^{k_1}) + b \big\} \big| \,. \tag{3.43}$$

Clearly

$$(3.45)^4 = \left| \left\{ (x_1, \ldots, x_{8u}) \in \mathcal{M}^{8u} \;\middle|\; \begin{matrix} (\overline{x}_1^{k_2} - \cdots - \overline{x}_{4u}^{k_2})(\overline{x}_{4u+1}^{k_1} - \cdots - \overline{x}_{8u}^{k_1}) \\ (\overline{x}_{4u+1}^{k_2} - \cdots - \overline{x}_{8u}^{k_2})(\overline{x}_1^{k_1} - \cdots - \overline{x}_{4u}^{k_1}) \end{matrix} = \right\} \right| . \tag{3.44}$$

Assume $k_1 > k_2$.

Rewrite the condition in (3.44) as

$$\left(\prod_{i=1}^{4u} x_i\right)^{k_1-k_2}\left(\sum_{i=1}^{4u} \pm \prod_{\substack{1\le j\le 4u\\ j\neq i}} x_j^{k_2}\right)\left(\sum_{i=4u+1}^{8u} \pm \prod_{\substack{4u<j\le 8u\\ j\neq i}} x_j^{k_1}\right) =$$

$$\left(\prod_{i=4u+1}^{8u} x_i\right)^{k_1-k_2}\left(\sum_{i=4u+1}^{8u} \pm \prod_{\substack{4u<j\le 8u\\ j\neq i}} x_j^{k_2}\right)\left(\sum_{i=1}^{4u} \pm \prod_{\substack{1\le j\le 4u\\ j\neq i}} x_j^{k_1}\right) . \qquad (3.45)$$

This equality is $(\bmod\, p)$. The terms in (3.45) are at most $p^{\frac{\kappa}{\ell}k_1(8u-1)} < p^{\frac{2}{5}}$ by definition of u. Thus (3.45) holds in $\mathbb{Z}$. Recall that the x_i are prime. Hence, if $1 \le i_1 \le 4u$, (3.45) implies that

$$x_{i_1} \Big| \prod_{\substack{1\le j\le 8u\\ j\neq i_1}} x_j \left(\sum_{i=4u+1}^{8u} \pm \prod_{\substack{4u<j\le 8u\\ j\neq i}} x_j^{k_2}\right) \qquad (3.46)$$

where we may assume the last factor non-vanishing.

Fix $x_{4u+1}, \dots, x_{8u} \in \mathcal{M}$. If an element in the set $\{x_1, \dots, x_{4u}\}$ only occurs once, it has to divide $\prod_{4u<j\le 8u} x_j\big(\sum_i \pm \prod_{\substack{4u<j\le 8u\\ j\neq i}} x_j^{k_2}\big)$ depending on $x_{4u+1}, \cdots, x_{8u}$.

Hence $\{x_1, \dots, x_{4u}\}$ is restricted to $p^{2u\frac{\kappa}{\ell}+}$ possibilities. Therefore (3.44) $<$ $p^{6u\frac{\kappa}{\ell}+}$ and by (3.35)

$$(3.43) < p^{\frac{3}{2}u\frac{\kappa}{\ell}+} < p^{-\frac{1}{90(k_1+k_2)}} M^{2u} . \qquad (3.47)$$

Recall also (3.39), (3.41) implies that

$$\nu(\xi + S) < p^{2(\kappa_0+\kappa_1)-\frac{1}{90(k_1+k_2)}} M^{2u} \qquad (3.48)$$

whenever S satisfies (3.30), (3.33).

Take $\kappa_0 = \kappa_0(k_1, k_2)$ small enough to ensure that

$$\kappa_0 < \kappa_1 < \frac{10^{-3}}{k_1 + k_2} . \qquad (3.49)$$

Then (3.48) certainly implies (3.37).

From Theorem 3.1 of [B-C], we conclude that for $\ell > \ell(\kappa_0) = \ell(k_1, k_2)$, the left side of (3.27) is at most $p^{-\varepsilon} M^{2\ell u}$, where $\varepsilon = \varepsilon(k_1, k_2) > 0$ (cf. (3.34)). The required lower bound

$$\delta > \frac{\varepsilon}{2(2u)^\ell} = \delta(k_1, k_2; \kappa)$$

on δ in (3.23′) follows.

This proves Proposition 9 for $J = 2$.

Inductive Step

For notational simplicity, take $J = 3$ (the argument is analogous in general). Let (replacing $\ell = 2\ell$)

$$S = \sum_{x_1,\dots,x_{2\ell}\in\mathcal{M}} e_p\bigg(\sum_{j=1}^{3} a_j \overline{x}_1^{k_j}\cdots\overline{x}_{2\ell}^{k_j}\bigg) \tag{3.50}$$

with $(a_j, p) = 1$ for $j = 1, 2, 3$ and where

$$\mathcal{M} = \mathcal{P} \cap [0, p^{\kappa/2\ell}]\ . \tag{3.51}$$

Define for $(y_1, y_2, y_3) \in \mathbb{F}_p^3$

$$\eta(y_1, y_2, y_3) = \big|\big\{(x_1, \dots, x_\ell) \in \mathcal{M}^\ell \mid \overline{x}_1^{k_j}\cdots\overline{x}_\ell^{k_j} = y_j \text{ for } j = 1, 2, 3\big\}\big|\ . \tag{3.52}$$

We may then rewrite (3.50) as

$$(3.50) = \sum_{\substack{y_1,\,y_2,\,y_3\\ z_1,\,z_2,\,z_3}} \eta(y)\eta(z)e_p\bigg(\sum_{j=1}^{3} a_j y_j z_j\bigg)\ . \tag{3.53}$$

Let $r \in \mathbb{Z}_+$ large enough (as will be specified) and write

$$\begin{aligned}(3.50) &\le M^{\ell(1-\frac{1}{2r})}\bigg[\sum_{y\in\mathbb{F}_p^3}\eta(y)\bigg|\sum_{z\in\mathbb{F}_p^3}\eta(z)e_p\bigg(\sum_{j=1}^{3} a_j y_j z_j\bigg)\bigg|^{2r}\bigg]^{\frac{1}{2r}}\\ &= M^{\ell(1-\frac{1}{2r})}\bigg[\sum_{y,z}\eta(y)\eta^{(2r)}(z)e_p\bigg(\sum_{j=1}^{3} a_j y_j z_j\bigg)\bigg]^{\frac{1}{2r}}\end{aligned} \tag{3.54}$$

where $\eta^{(2r)}$ denotes the $2r$-fold convolution $\eta * \cdots * \eta$.

Repeating the estimate w.r.t. the variable y, we get

$$(3.50) < M^{2\ell(1-\frac{1}{2r})}\bigg[\sum_{y,z}\eta^{(2r)}(y)\eta^{(2r)}(z)e_p\bigg(\sum_{j=1}^{3} a_j y_j z_j\bigg)\bigg]^{\frac{1}{4r^2}}\ . \tag{3.55}$$

The inner sum in (3.55) is bounded by

$$\begin{aligned}&\bigg[\sum_y \eta^{(2r)}(y)^2\bigg]^{1/2}\bigg[\sum_y\bigg|\sum_z \eta^{(2r)}(z)e_p\bigg(\sum_{j=1}^{3} a_j y_j z_j\bigg)\bigg|^2\bigg]^{1/2}\\ &= p^{3/2}\bigg[\sum_{y\in\mathbb{F}_p^3}\eta^{(2r)}(y)^2\bigg]\end{aligned} \tag{3.56}$$

by Parseval's identity.

Now

$$\begin{aligned}&\sum \eta^{(2r)}(y)^2\\ &= \left|\left\{(x_{i,s})_{\substack{1\le i\le \ell\\ 1\le s\le 4r}} \in \mathcal{M}^{4r\ell} \;\Big|\; \sum_{s=1}^{2r}\prod_{i=1}^{\ell} \overline{x}_{i,s}^{k_j} = \sum_{s=2r+1}^{4r}\prod_{i=1}^{\ell}\overline{x}_{i,s}^{k_j} \text{ for } j=1,2,3\right\}\right|\\ &\le \left|\left\{(x_{i,s})\in \mathcal{M}^{4r\ell} \mid \cdots \text{ for } j=1,2\right\}\right| . \qquad (3.57)\end{aligned}$$

We estimate (3.57) using the circle method and the bound (3.4) when $J=2$. Thus

$$\begin{aligned}(3.57) &= \frac{1}{p^2}\sum_{\xi\in\mathbb{F}_p^2}\left|\sum_{x_1,\dots,x_\ell\in\mathcal{M}} e_p\Big(\xi_1\prod_{i=1}^{\ell}\overline{x}_i^{k_1} + \xi_2\prod_{i=1}^{\ell}\overline{x}_i^{k_2}\Big)\right|^{4r}\\ &= \frac{1}{p^2}M^{4r\ell} + \frac{1}{p^2}\sum_{\xi\in\mathbb{F}_p^2\setminus\{(0,0)\}} |\cdots|^{4r} . \qquad (3.58)\end{aligned}$$

Assume $\ell \ge \ell(k_1,k_2)$ so that, recalling (3.51), (3.4) holds with κ replaced by $\frac{\kappa}{2}$ and exponents k_1,k_2, with $\delta' = \delta(k_1,k_2;\frac{\kappa}{2})$.

It follows that

$$\begin{aligned}(3.58) &< \frac{1}{p^2}\big(M^{4r\ell} + (p^2-1)p^{(\frac{\kappa}{2}-\delta')4r}\big)\\ &< \frac{1}{p^2}p^{2r\kappa}(1+p^{2-4r\delta'})\\ &< 2p^{2r\kappa-2} \qquad (3.59)\end{aligned}$$

taking

$$r = \left[\frac{1}{\delta'}\right].$$

Substitution of (3.59) in (3.56), (3.55) implies

$$\begin{aligned}(3.50) &< M^{2\ell(1-\frac{1}{2r})}p^{\frac{\kappa}{2r}-\frac{1}{8r^2}}\\ &< p^{\kappa-\frac{1}{8r^2}} . \qquad (3.60)\end{aligned}$$

This concludes the proof of Proposition 9 and hence Theorem 8.

Remark. Following the lines of the proof of Proposition 9, one may show

Proposition 10. *Given a positive integer d, there is a positive integer $\ell = \ell(d)$ such that if $f(x)\in\mathbb{Z}[X]$ is of degree at most d and $p > M_1,\dots,M_\ell > p^\delta$ ($\delta>0$ arbitrary and fixed), then*

$$\left|\sum_{x_i\in[M_i,2M_i](1\le i\le \ell)} e_p\big(f\overline{x}_1\cdots(\overline{x}_\ell)\big)\right| < p^{\delta'}\prod_{i=1}^{\ell} M_i \qquad (3.61)$$

where $\delta' = \delta(\ell,\delta)$.

Thus in Proposition 9, we establish (3.61) for the summation restricted to primes $x \in [M_i, 2M_i] \cap \mathcal{P}$. The advantage of using primes is that it simplifies the arithmetical issue of bounding the number of solutions of equations in $\mathbb{F}_p$

$$\overline{x}_1^k + \cdots + \overline{x}_u^k = \overline{x}_{u+1}^k + \cdots + \overline{x}_{2u}^k$$

or systems of such equations. If we don't assume the $x_i \in \mathcal{P}$ some extra difficulties of common divisors appear but they are only of technical nature.

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On the Maximal Number of Facets of 0/1 Polytopes*

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Summary. We show that there exist 0/1 polytopes in $\mathbb{R}^n$ whose number of facets exceeds $(cn/(\log n))^{n/2}$, where $c > 0$ is an absolute constant.

1 Introduction

Let P be a polytope with non-empty interior in $\mathbb{R}^n$. We write $f_{n-1}(P)$ for the number of its $(n-1)$-dimensional faces. Consider the class of 0/1 polytopes in $\mathbb{R}^n$; these are the convex hulls of subsets of $\{0,1\}^n$. In this note we obtain a new lower bound for the quantity

$$g(n) := \max\left\{f_{n-1}(P_n) : P_n \text{ is a } 0/1 \text{ polytope in } \mathbb{R}^n\right\}. \tag{1.1}$$

The problem of determining the correct order of growth of $g(n)$ as $n \to \infty$ was posed by Fukuda and Ziegler (see [Fu], [Z]). It is currently known that $g(n) \leq 30(n-2)!$ if n is large enough (see [FKR]). In the other direction, Bárány and Pór in [BP] determined that $g(n)$ is superexponential in n: they obtained the lower bound

$$g(n) \geq \left(\frac{cn}{\log n}\right)^{n/4}, \tag{1.2}$$

where $c > 0$ is an absolute constant. In [GGM] we showed that

$$g(n) \geq \left(\frac{cn}{\log^2 n}\right)^{n/2}. \tag{1.3}$$

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A more recent observation allows us to remove one logarithmic factor from the estimate in (1.3).

Theorem 1.1. *There exists a constant $c > 0$ such that*

$$g(n) \geq \left(\frac{cn}{\log n}\right)^{n/2}. \tag{1.4}$$

The method of proof of Theorem 1.1 is probabilistic and has its origin in the work of Dyer, Füredi and McDiarmid [DFM]. The proof is essentially the same with the one in [GGM], which in turn is based on [BP], with the exception of a different approach to one estimate, summarized in Proposition 3.1 below. We consider random ± 1 polytopes (i.e., polytopes whose vertices are independent and uniformly distributed vertices $\mathbf{X_i}$ of the unit cube $C = [-1,1]^n$). We fix $n < N \leq 2^n$ and consider the random polytope

$$K_N = \mathrm{conv}\{\mathbf{X}_1, \ldots, \mathbf{X}_N\}. \tag{1.5}$$

Our main result is a lower bound on the expectation $\mathbb{E}[f_{n-1}(K_N)]$ of the number of facets of K_N.

Theorem 1.2. *There exist two positive constants a and b such that: for all sufficiently large n, and all N satisfying $n^a \leq N \leq \exp(bn)$, one has that*

$$\mathbb{E}\big[f_{n-1}(K_N)\big] \geq \left(\frac{\log N}{a \log n}\right)^{n/2}. \tag{1.6}$$

The same result was obtained in [GGM] under the restriction $N \leq \exp(bn/\log n)$. This had a direct influence on the final estimate obtained, leading to (1.3).

The note is organized as follows. In Section 2 we briefly describe the method (the presentation is not self-contained and the interested reader should consult [BP] and [GGM]). In Section 3 we present the new technical step (it is based on a more general lower estimate for the measure of the intersection of a symmetric polyhedron with the sphere, which might be useful in similar situations). In Section 4 we use the result of Section 3 to extend the range of N's for which Theorem 1.2 holds true. Theorem 1.1 easily follows.

We work in $\mathbb{R}^n$ which is equipped with the inner product $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the Euclidean norm and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume, surface area, and the cardinality of a finite set, are all denoted by $|\cdot|$. We write $\partial(F)$ for the boundary of F. All logarithms are natural. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. The letters c, c', c_1, c_2 etc. denote absolute positive constants, which may change from line to line.

2 The Method

The method makes essential use of two families (Q^β) and (F^β) $(0 < \beta < \log 2)$ of convex subsets of the cube $C = [-1, 1]^n$, which were introduced by Dyer, Füredi and McDiarmid in [DFM]. We briefly recall their definitions. For every $\mathbf{x} \in C$, set

$$q(\mathbf{x}) := \inf \bigl\{\mathrm{Prob}\bigl(\mathbf{X} \in H\bigr) \colon \mathbf{x} \in H,\ H \text{ is a closed halfspace}\bigr\}. \tag{2.1}$$

The β-center of C is the convex polytope

$$Q^\beta = \bigl\{\mathbf{x} \in C \colon q(\mathbf{x}) \geq \exp(-\beta n)\bigr\}. \tag{2.2}$$

Next, define $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \tfrac{1}{2}(1 + x)\log(1 + x) + \tfrac{1}{2}(1 - x)\log(1 - x) \tag{2.3}$$

if $x \in (-1, 1)$ and $f(\pm 1) = \log 2$, and for every $\mathbf{x} = (x_1, \dots, x_n) \in C$ set

$$F(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i). \tag{2.4}$$

Then, F^β is defined by

$$F^\beta = \bigl\{\mathbf{x} \in C : F(\mathbf{x}) \leq \beta\bigr\}. \tag{2.5}$$

Since f is a strictly convex function on $(-1, 1)$, F^β is convex.

When $\beta \to \log 2$ the convex bodies Q^β and F^β tend to C. The main tool for the proof of Theorem 1.2 is the fact that the two families (Q^β) and (F^β) are very close, in the following sense.

Theorem 2.1. (i) *$Q^\beta \cap (-1, 1)^n \subseteq F^\beta$ for every $\beta > 0$.*
(ii) *There exist $\gamma \in \bigl(0, \frac{1}{10}\bigr)$ and $n_0 = n_0(\gamma) \in \mathbb{N}$ with the following property: If $n \geq n_0$ and $4 \log n/n \leq \beta < \log 2$, then*

$$F^{\beta-\varepsilon} \cap \gamma C \subseteq Q^\beta \tag{2.6}$$

for some $\varepsilon \leq 3 \log n/n$.

Part (i) of Theorem 2.1 was proved in [DFM]. Part (ii) was proved in [GGM] and strengthens a previous estimate from [BP].

Fix $n^8 \leq N \leq 2^n$ and define $\alpha = (\log N)/n$. The family (Q^β) is related to the random polytope K_N through a lemma from [DFM] (the estimate for ε claimed below is checked in [GGM]): If n is sufficiently large, one has that

$$\mathrm{Prob}\bigl(K_N \supseteq Q^{\alpha-\varepsilon}\bigr) > 1 - 2^{-(n-1)} \tag{2.7}$$

for some $\varepsilon \leq 3 \log n/n$.

Combining (2.7) with Theorem 2.1, one gets the following.

Lemma 2.2. *Let $n^8 \le N \le 2^n$ and $n \ge n_0(\gamma)$. Then,*

$$\mathrm{Prob}\big(K_N \supseteq F^{\alpha-\varepsilon} \cap \gamma C\big) > 1 - 2^{-(n-1)} \tag{2.8}$$

for some $\varepsilon \le 6 \log n/n$.

Bárány and Pór proved that K_N is weakly sandwiched between $F^{\alpha-\varepsilon} \cap \gamma C$ and $F^{\alpha+\delta}$ in the sense that $K_N \supseteq F^{\alpha-\varepsilon} \cap \gamma C$ and most of the surface area of $F^{\alpha+\delta} \cap \gamma C$ is outside K_N for small positive values of δ (the estimate for δ given below is checked in [GGM]).

Lemma 2.3. *If $n \ge n_0$ and $\alpha < \log 2 - 12n^{-1}$, then*

$$\mathrm{Prob}\big(\, |\partial(F^{\alpha+\delta}) \cap \gamma C \cap K_N| \ge \tfrac{1}{2} |\partial(F^{\alpha+\delta}) \cap \gamma C| \,\big) \le \tfrac{1}{100}. \tag{2.9}$$

for some $\delta \le 6/n$.

We will also need the following geometric lemma from [BP].

Lemma 2.4. *Let $\gamma \in \big(0, \frac{1}{10}\big)$ and assume that $\beta + \zeta < \log 2$. Then,*

$$|\partial(F^{\beta+\zeta}) \cap \gamma C \cap H| \le (3\zeta n)^{(n-1)/2} |S^{n-1}| \tag{2.10}$$

for every closed halfspace H whose interior is disjoint from $F^\beta \cap \gamma C$.

The strategy of Bárány and Pór (which is also followed in [GGM] and in the present note) is that for a random K_N and for each halfspace H_A which is defined by a facet A of K_N and has interior disjoint from K_N, we also have that H_A has interior disjoint from $F^{\alpha-\varepsilon} \cap \gamma C$ (from Lemma 2.2) and hence cuts a small amount (independent from A) of the surface of $\partial(F^{\alpha+\delta}) \cap \gamma C$ (from Lemma 2.4). Since the surface area of $\partial(F^{\alpha+\delta}) \cap \gamma C$ is mostly outside K_N (from Lemma 2.3) we see that the number of facets of K_N must be large, depending on the total surface of $\partial(F^{\alpha+\delta}) \cap \gamma C$. We will describe these steps more carefully in the last Section. First, we give a new lower bound for $|\partial(F^\beta) \cap \gamma C|$.

3 An Additional Lemma

The new element in our argument is the next Proposition.

Proposition 3.1. *There exists $r > 0$ with the following property: for every $\gamma \in (0,1)$ and for all $n \ge n_0(\gamma)$ and $\beta < c(\gamma)/r$ one has that*

$$|\partial(F^\beta) \cap \gamma C| \ge c(\gamma)^{n-1} (2\beta n)^{(n-1)/2} |S^{n-1}|, \tag{3.1}$$

where $c(\gamma) > 0$ is a constant depending only on γ.

Proof. We first estimate the product curvature $\kappa(\mathbf{x})$ of the surface $F(\mathbf{x}) = \beta$: in [GGM] it is proved that if $\beta < \log 2$ and $\mathbf{x} \in \gamma C$ with $F(\mathbf{x}) = \beta$, then

$$\frac{1}{\kappa(\mathbf{x})} \geq \left(1 - \gamma^2\right)^{n-1} (2\beta n)^{(n-1)/2}. \tag{3.2}$$

Let $\boldsymbol{\theta} \in S^{n-1}$ and write $\mathbf{x}(\boldsymbol{\theta}, \beta)$ for the point on the boundary of F^β for which $n \nabla F\big(\mathbf{x}(\boldsymbol{\theta}, \beta)\big)$ is a positive multiple of $\boldsymbol{\theta}$. This point is well-defined and unique if $0 < \beta < \big|\operatorname{supp} \boldsymbol{\theta}\big| (\log 2)/n$ (see [BP, Lemma 6.2]).

Let $r > 0$ be an absolute constant (which will be suitably chosen) and set

$$M_r = \left\{\boldsymbol{\theta} \in S^{n-1} : \sqrt{n/r}\, \boldsymbol{\theta} \in C\right\}. \tag{3.3}$$

The argument given in [BP, Lemma 6.3] shows that if $\beta < c_1(\gamma)/r$, then for every $\boldsymbol{\theta} \in M_r$ we have $\mathbf{x}(\boldsymbol{\theta}, \beta) \in \gamma C$. Also, we easily check that for every $\boldsymbol{\theta} \in M_r$ the condition $\big|\operatorname{supp} \boldsymbol{\theta}\big| \geq n/r$ is satisfied, and hence, if $\beta < c_1(\gamma)/r$ then $\mathbf{x}(\boldsymbol{\theta}, \beta)$ is well-defined and unique. We will estimate the measure of M_r.

Lemma 3.2. *There exists $r > 0$ such that: if $n \geq 3$ then*

$$|M_r| \geq e^{-n/2} |S^{n-1}|. \tag{3.4}$$

Proof. Write γ_n for the standard Gaussian measure on $\mathbb{R}^n$ and σ_n for the rotationally invariant probability measure on S^{n-1}. We use the following fact.

Fact 3.3. *If K is a symmetric convex body in $\mathbb{R}^n$ then*

$$\tfrac{1}{2}\, \sigma_n\big(S^{n-1} \cap \tfrac{1}{2} K\big) \leq \gamma_n(\sqrt{n} K) \leq \sigma_n(S^{n-1} \cap eK) + e^{-n/2}. \tag{3.5}$$

Proof of Fact 3.3. A proof appears in [KV]. We sketch the proof of the right hand side inequality (which is the one we need). Observe that

$$\sqrt{n} K \subseteq \left(\tfrac{1}{e} \sqrt{n} B_2^n\right) \cup C\left(\tfrac{1}{e} \sqrt{n} S^{n-1} \cap \sqrt{n} K\right) \tag{3.6}$$

where, for $A \subseteq \frac{1}{e}\sqrt{n} S^{n-1}$, we write $C(A)$ for the positive cone generated by A. It follows that

$$\gamma_n(\sqrt{n} K) \leq \gamma_n\big(\tfrac{1}{e}\sqrt{n} B_2^n\big) + \sigma\big(\tfrac{1}{e}\sqrt{n} S^{n-1} \cap \sqrt{n} K\big) \tag{3.7}$$

where σ denotes the rotationally invariant probability measure on $\frac{1}{e}\sqrt{n} S^{n-1}$. Now

$$\sigma\big(\tfrac{1}{e}\sqrt{n} S^{n-1} \cap \sqrt{n} K\big) = \sigma_n(S^{n-1} \cap eK), \tag{3.8}$$

and a direct computation shows that

$$\gamma_n\big(\rho\sqrt{n} B_2^n\big) \leq (\rho\sqrt{e})^n e^{-\rho^2 n/2} \tag{3.9}$$

for all $0 < \rho \leq 1$. It follows that

$$\gamma_n\left(\tfrac{1}{e}\sqrt{n}B_2^n\right) \leq \exp(-n/2). \tag{3.10}$$

From (3.7)–(3.10) we get the Fact. □

Proof of Lemma 3.2. Observe that

$$M_r = S^{n-1} \cap e\left(\sqrt{r/(e^2n)}\, C\right). \tag{3.11}$$

Hence

$$\begin{aligned} \frac{|M_r|}{|S^{n-1}|} &= \sigma_n(M_r) = \sigma_n\Big(S^{n-1} \cap e\big(\sqrt{r/(e^2n)}\, C\big)\Big) \\ &\geq \gamma_n\big((\sqrt{r}/e)\, C\big) - e^{-n/2} \\ &= d\big(\sqrt{r}/e\big)^n - e^{-n/2}, \end{aligned}$$

where

$$d(s) := \frac{1}{\sqrt{2\pi}} \int_{-s}^{s} e^{-t^2/2} dt. \tag{3.12}$$

Observe that $2e^{-n/2} < e^{-n/4}$ for $n \geq 3$. Choose $r > 0$ so that

$$d\big(\sqrt{r}/e\big) > e^{-1/4}; \tag{3.13}$$

this is possible, since $\lim_{s\to+\infty} d(s) = 1$. Then,

$$d\big(\sqrt{r}/e\big)^n > 2e^{-n/2} \tag{3.14}$$

for $n \geq 3$, which completes the proof. □

We can now finish the proof of Proposition 3.1. Writing $\mathbf{x}$ for $\mathbf{x}\big(\boldsymbol{\theta}, \beta\big)$ and expressing surface area in terms of product curvature (cf. [S, Theorem 4.2.4]), we can write

$$|\partial(F^\beta) \cap \gamma C| \geq \int_{M_r} \frac{1}{\kappa(\mathbf{x})}\, d\boldsymbol{\theta} \geq e^{-n/2}\big(1-\gamma^2\big)^{n-1}(2\beta n)^{(n-1)/2}|S^{n-1}|, \tag{3.15}$$

and the result follows. □

A General Version of Lemma 3.2. The method of proof of Lemma 3.2 provides a general lower estimate for the measure of the intersection of an arbitrary symmetric polyhedron with the sphere. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be non-zero vectors in $\mathbb{R}^n$ and consider the symmetric polyhedron

$$T = \bigcap_{j=1}^{m} \big\{x : |\langle x, \mathbf{u}_i\rangle| \leq 1\big\}. \tag{3.16}$$

The following theorem of Sidák (see [Si]) gives an estimate for $\gamma_n(T)$.

Fact 3.4 (Sidák's lemma). *If T is the symmetric polyhedron defined by* (3.16) *then*

$$\gamma_n(T) \geq \prod_{i=1}^{m} \gamma_n(\{x : |\langle x, \mathbf{u}_i\rangle| \leq 1\}) = \prod_{i=1}^{m} d\left(\frac{1}{\|\mathbf{u}_i\|_2}\right). \tag{3.17}$$

We will also use an estimate which appears in [Gi].

Fact 3.5. *There exists an absolute constant $\lambda > 0$ such that,* for any $t_1, \ldots, t_m > 0$,

$$\prod_{i=1}^{m} d\left(\frac{1}{t_i}\right) \geq \exp\left(-\lambda \sum_{i=1}^{m} t_i^2\right). \tag{3.18}$$

Consider the parameter $R = R(T)$ defined by

$$R^2(T) = \sum_{i=1}^{m} \|\mathbf{u}_i\|_2^2. \tag{3.19}$$

Let $s > 0$. Fact 3.4 shows that

$$\gamma_n(sT) \geq \prod_{i=1}^{m} d\left(\frac{s}{\|\mathbf{u}_i\|_2}\right). \tag{3.20}$$

Then, Fact 3.5 shows that

$$\gamma_n(sT) \geq \exp\bigl(-\lambda R^2(T)/s^2\bigr) \geq e^{-n/4} \geq 2e^{-n/2}, \tag{3.21}$$

provided that $n \geq 3$ and

$$s \geq \frac{2\sqrt{\lambda}R(T)}{\sqrt{n}}. \tag{3.22}$$

We then apply Fact 3.3 for the polyhedron $K = (s/\sqrt{n})T$ to get

$$\sigma_n\left(S^{n-1} \cap \frac{es}{\sqrt{n}}T\right) \geq \exp\bigl(-\lambda R^2(T)/s^2\bigr) - \exp(-n/2) \geq \tfrac{1}{2}\exp\bigl(-\lambda R^2(T)/s^2\bigr). \tag{3.23}$$

In other words, we have proved the following.

Proposition 3.3. *Let $n \geq 3$ and let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be non-zero vectors in $\mathbb{R}^n$. Consider the symmetric polyhedron*

$$T = \bigcap_{j=1}^{m} \bigl\{x : |\langle x, \mathbf{u}_i\rangle| \leq 1\bigr\},$$

and define

$$R^2(T) = \sum_{i=1}^{m} \|\mathbf{u}_i\|_2^2.$$

Then, for all $t \geq cR(T)/\sqrt{n}$ we have that

$$\sigma_n\bigl(S^{n-1} \cap (t/\sqrt{n})\, T\bigr) \geq \tfrac{1}{2}\exp\bigl(-cR^2(T)/t^2\bigr), \tag{3.24}$$

where $c > 0$ is an absolute constant.

4 Proof of the Theorems

Proof of Theorem 1.2. Let $\gamma \in (0,1)$ be the constant in Theorem 2.1. Assume that n is large enough and set $b = c(\gamma)/(2r)$, where $c(\gamma) > 0$ is the constant in Proposition 3.1.

Given N with $n^8 \leq N \leq \exp(bn)$, let $\alpha = (\log N)/n$. From Lemma 2.2 there exists $\varepsilon \leq 6 \log n/n$ such that

$$K_N \supseteq F^{\alpha-\varepsilon} \cap \gamma C \tag{4.1}$$

with probability greater than $1 - 2^{-n+1}$, and from Lemma 2.3 there exists $\delta \leq 6/n$ such that

$$|(\partial(F^{\alpha+\delta}) \cap \gamma C) \setminus K_N| \geq \tfrac{1}{2}|\partial(F^{\alpha+\delta}) \cap \gamma C| \tag{4.2}$$

with probability greater than $1 - 10^{-2}$. We assume that K_N satisfies both (4.1) and (4.2) (this holds with probability greater than $\frac{1}{2}$).

We apply Lemma 2.4 with $\beta = \alpha - \varepsilon$ and $\zeta = \varepsilon + \delta$: If A is a facet of K_N and H_A is the corresponding halfspace which has interior disjoint from K_N, then

$$|\partial(F^{\alpha+\delta}) \cap \gamma C \cap H_A| \leq \big(3n(\varepsilon+\delta)\big)^{(n-1)/2} |S^{n-1}|. \tag{4.3}$$

It follows that

$$\begin{aligned} f_{n-1}(K_N)\big(3n(\varepsilon+\delta)\big)^{(n-1)/2}|S^{n-1}| &\geq \sum_A |\partial(F^{\alpha+\delta}) \cap \gamma C \cap H_A| \\ &\geq \big|\big(\partial(F^{\alpha+\delta}) \cap \gamma C\big) \setminus K_N\big| \\ &\geq \tfrac{1}{2}|\partial(F^{\alpha+\delta}) \cap \gamma C|. \end{aligned}$$

Since $\alpha \leq b = c(\gamma)/(2r)$ and $\delta \leq 6/n$, we have $\alpha + \delta \leq c(\gamma)/r$ if n is large enough. Applying Proposition 3.1 with $\beta = \alpha + \delta$, we get

$$f_{n-1}(K_N)\big(3n(\varepsilon+\delta)\big)^{(n-1)/2} \geq \Big(c(\gamma)\sqrt{2\alpha n}\Big)^{n-1}, \tag{4.4}$$

for sufficiently large n. Since $\alpha n = \log N$ and $(\varepsilon + \delta)n \leq 12 \log n$, this shows that

$$f_{n-1}(K_N) \geq \left(\frac{c_1(\gamma)\log N}{\log n}\right)^{n/2} \tag{4.5}$$

with probability greater than $\frac{1}{2}$. □

Proof of Theorem 1.1. We can apply Theorem 1.2 with $N \geq \exp(bn)$ where $b > 0$ is an absolute constant. This shows that there exist 0/1 polytopes P in $\mathbb{R}^n$ with

$$f_{n-1}(P) \geq \left(\frac{cn}{\log n}\right)^{n/2}, \tag{4.6}$$

as claimed. □

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A Note on an Observation of G. Schechtman

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Summary. Using the *Gaussian min-max theorem* we provide a simple proof of a theorem observed by G. Schechtman, and a new application.

1 Two Applications of the Gaussian Min-Max Theorem

Theorem 1 below is the *second main theorem* in Schechtman's ([S2]). It also appears as Theorem 6.5.1 in the recent book of Talagrand ([T2]). Schechtman's proof uses the method of [S1] and the *majorizing measure* theorem of Talagrand [T1]. Some people, including myself, noted that the Theorem is in fact an elementary application of the *Gaussian min-max theorem* below, proved originally in 1985 in [Go1] (it appeared also in [K]), and is an easy modification on the method of the proof of Dvoretzky's theorem on $1+\varepsilon$ embedding of ℓ_2^k in an n-dimensional Banach space X, as was done originally in [Go1], and applied, see for example, Corollary 1.2 in [Go2]. I thank G. Schechtman for asking me to show how the *Gaussian min-max theorem* proves Theorem 1.

Let $T \subset S^{m-1}$ be a closed subset, $X = (\mathbb{R}^n, \|.\|)$ be a normed space, $\{e_i\}_{i=1}^n$ be the standard unit basis of $\mathbb{R}^n$. Following the notation of [S2], we set $\mathbf{E}(X) := \mathbf{E}\|\sum_{j=1}^n h_j e_j\|$, and $\mathbf{E}^*(T) = \mathbf{E}\max\{|\sum_{i=1}^m g_i t_i|;\ t = (t_1, \ldots, t_m) \in T\}$ where, throughout $\{g_i\}_{i=1}^m$, $\{h_j\}_{j=1}^n$ denote i.i.d. sequences of standard Gaussian random variables. $c, C, c_0, c_1, \ldots,$ denote positive constants.

Theorem 1. *Let* $0 < \varepsilon < 1$*, and assume* $\|x\| \le \|x\|_2$ *for all* $x \in \mathbb{R}^n$*. If* $\mathbf{E}^*(T) < \varepsilon \mathbf{E}(X)$*, then there is a linear map* $A : \mathbb{R}^m \to \mathbb{R}^n$ *such that*

$$\frac{\max_{t\in T} \|A(t)\|}{\min_{t\in T} \|A(t)\|} \le \frac{1+\varepsilon}{1-\varepsilon}\,.$$

Recall the *Gaussian min-max theorem* ([Go1]), a version of which is published in [Go2]:

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Theorem 2 (Gaussian min-max theorem). *Let $X_{i,j}$ and $Y_{i,j}$, $i = 1, ..., m$, $j = 1, ..., n$, be two centered Gaussian processes which satisfy the following inequalities for all indices:*
(*A*) $\mathbf{E}|X_{i,j} - X_{i,k}|^2 \le \mathbf{E}|Y_{i,j} - Y_{i,k}|^2$,
and, also satisfy for all $\ell \ne i$
(*B*) $\mathbf{E}|X_{i,j} - X_{\ell,k}|^2 \ge \mathbf{E}|Y_{i,j} - Y_{\ell,k}|^2$.
Then, $\mathbf{E}\min_i \max_j X_{i,j} \le \mathbf{E}\min_i \max_j Y_{i,j}$.

Remark 1. Note that if $m = 1$, part (B) is void, and the conclusion of the theorem says, $\mathbf{E}\max_j X_{1,j} \le \mathbf{E}\max_j Y_{1,j}$, therefore, given n, m if the inequalities of part *(A)* are equalities for all the indices i, j, k, then

$$\mathbf{E}\min_i \max_j X_{i,j} \le \mathbf{E}\min_i \max_j Y_{i,j} \le \mathbf{E}\max_{i,j} Y_{i,j} \le \mathbf{E}\max_{i,j} X_{i,j}.$$

Proof of Theorem 1. Consider the natural Gaussian operator from R^m to X, $G := \sum_{i=1}^m \sum_{j=1}^n g_{i,j} e_i \otimes e_j$ equipped with the norm $\|G_{|T}\| = \max_{t\in T, x^*\in B_{X^*}} \langle G(t), x^*\rangle$, here $\{g_{i,j}\}$ denote i.i.d. standard Gaussian variables. By the *Gaussian min-max theorem* if we consider the two Gaussian processes

$$X(t,x^*) = \langle g, t\rangle + \langle h, x^*\rangle,\ Y(t,x^*) = \langle G(t), x^*\rangle,$$

where $x^* \in B_{X^*}$, $t \in T$, and $g = (g_1, \dots, g_m)$, $h = (h_1, \dots, h_n)$ denote i.i.d. standard Gaussian vectors in R^m, R^n resp., then the conditions of Theorem 1 are satisfied (where $t \in T$ replaces the index i, and $x^* \in B_{X^*}$ replaces the index j). Indeed, if $t \in T$ and $x^*, y^* \in B_{X^*}$, then all inequalities in (A) are actually equalities since we have:

$$\mathbf{E}|X(t,x^*) - X(t,y^*)|^2 = \mathbf{E}|Y(t,x^*) - Y(t,y^*)|^2 = \|x^* - y^*\|_2^2,$$

and since $\langle x^*, y^*\rangle \le \|x^*\|_2\|y^*\|_2 \le \|x^*\|\|y^*\| \le 1$ for all $x^*, y^* \in B_{X^*}$, we have that the inequalities in (B) are satisfied for all $s, t \in T, x^*, y^* \in B_{X^*}$, indeed:

$$\begin{aligned}&\mathbf{E}\,|X(t,x^*) - X(s,y^*)|^2 - \mathbf{E}\,|Y(t,x^*) - Y(s,y^*)|^2\\ &= 2\big(1 - \langle s,t\rangle\big)\big(1 - \langle x^*, y^*\rangle\big) \ge 2\big(1 - \langle s,t\rangle\big)\big(1 - \|x^*\|_2\|y^*\|_2\big)\\ &\ge 2\big(1 - \langle s,t\rangle\big)\big(1 - \|x^*\|.\|y^*\|\big) \ge 0.\end{aligned}$$

Hence by the *Gaussian min-max theorem*

$$\begin{aligned}\mathbf{E}\,(X) - \mathbf{E}^*(T) &= \mathbf{E}\min_{t\in T}\max_{x^*\in B_{X^*}} X(t,x^*) \le \mathbf{E}\min_{t\in T}\max_{x^*\in B_{X^*}} Y(t,x^*)\\ &\le \mathbf{E}\max_{t\in T, x^*\in B_{X^*}} Y(t,x^*) \le \mathbf{E}\max_{t\in T, x^*\in B_{X^*}} X(t,x^*)\\ &= \mathbf{E}\,(X) + \mathbf{E}^*(T).\end{aligned}$$

Therefore, if $0 < \varepsilon < 1$ and $\mathbf{E}^*(T) \le \varepsilon\mathbf{E}(X)$ there is an operator A in the set of operators $\{G\}$ mapping the subset T to X which has "almost" constant norm in X for all $t \in T$, $1 \le \frac{\max_{t\in T}\|A(t)\|}{\min_{t\in T}\|A(t)\|} \le \frac{1+\varepsilon}{1-\varepsilon}$. This concludes the proof. □

We shall denote by $\alpha_1^*, \alpha_2^*, \ldots$, a non-increasing rearrangement of the absolute values of a given sequence of real numbers $\alpha_1, \alpha_2, \ldots$.

Another related application of the *Gaussian min-max theorem* is the following:

Theorem 3. *Let $X = (\mathbb{R}^n, \|.\|)$ be a 1-symmetric normed space, and d_X be a number, such that $\|x\| \le \|x\|_2 \le d_X\|x\|$ for every $x \in \mathbb{R}^n$. Let $1 \le k \le n$ and put $m(k) = [c_0 k/\ln(\frac{2n}{k})]$. Assume $1 \le c_1\sqrt{k}/(\mathbf{E}\,\|\sum_{i=m(k)+1}^n g_i^* e_i\|) \le d_X$, then there exists a k-codimensional subspace $Y \subset X^*$ such that for all $y \in Y$, $c_1\sqrt{k}/(\mathbf{E}\,\|\sum_{i=m(k)+1}^n g_i^* e_i\|)\,\|y\|_2 \le \|y\| \le d_X\|y\|_2$. In particular,*

$$d(\ell_2^{n-k}, Y) \le c_2 \frac{d_X}{\sqrt{k}}\,\mathbf{E}\left\|\sum_{i=m(k)+1}^{n} g_i^* e_i\right\|.$$

Proof. Consider the Gaussian linear map $G := \sum_{i=1}^n \sum_{j=1}^k g_{i,j} e_i \otimes e_j$ mapping the set $T_\lambda := \lambda B_{X^*} \cap S^{n-1}$ to the Euclidean space ℓ_2^k , where $d_X \ge \lambda \ge 1$, by assumption we have $B_{X^*} \subset B_2^n \subset d_X B_{X^*}$. We apply the preceding notation with $T = T_\lambda \subset S^{n-1}$ and take the range space of G to be ℓ_2^k. As in the proof of Theorem 1

$$\mathbf{E}(\ell_2^k) - \mathbf{E}^*(T_\lambda) \le \mathbf{E}\min_{t\in T_\lambda}\max_{x^*\in S^{k-1}} < G(t), x^* >= \mathbf{E}\min_{t\in T_\lambda}\|G(t)\|_2$$

where $\mathbf{E}\,(\ell_2^k) = \sqrt{2}\Gamma(\frac{k+1}{2})/\Gamma(\frac{k}{2})$, $\frac{k}{\sqrt{k+1}} \le \mathbf{E}\,(\ell_2^k) \le \sqrt{k}$. We are interested to find a value of λ which will make the left hand side positive. Towards this end, we shall estimate from above the value $\mathbf{E}^*(T_\lambda)$. Let $1 \le m \le n$ be an integer to be determined later. Since T_λ is 1-symmetric with respect to sign changes and permutations of coordinates

$$\begin{aligned} E^*(T_\lambda) &= \mathbf{E}\max_{t\in T_\lambda}\left|\sum_{i=1}^n t_i g_i\right| = \mathbf{E}\max_{t\in T_\lambda}\left|\sum_{i=1}^n t_i g_i^*\right| \\ &\le \mathbf{E}\max_{t\in T_\lambda}\left|\sum_{i=1}^m t_i g_i^*\right| + \mathbf{E}\max_{t\in T_\lambda}\left|\sum_{i=m+1}^n t_i g_i^*\right| \\ &\le \mathbf{E}\left(\sum_{i=1}^m (g_i^*)^2\right)^{1/2} + \lambda\,\mathbf{E}\left\|\sum_{i=m+1}^n g_i^* e_i\right\| \\ &\le c\sqrt{m\,\ln\left(\frac{2n}{m}\right)} + \lambda\,\mathbf{E}\left\|\sum_{i=m+1}^n g_i^* e_i\right\|. \end{aligned}$$

Now we select $m = m(k) := [c_0 k/\ln(\frac{2n}{k})]$ which is the solution for m of the equation $\mathbf{E}\,(\ell_2^k) \cong \sqrt{k} \cong \sqrt{m\,\ln(\frac{2n}{m})}$, and with this value $m(k)$ we choose

$$\lambda \cong \frac{\sqrt{m(k)\ln\left(\frac{2n}{m(k)}\right)}}{\mathbf{E}\left\|\sum_{i=m(k)+1}^{n} g_i^* e_i\right\|} \cong \frac{\sqrt{k}}{\mathbf{E}\left\|\sum_{i=m(k)+1}^{n} g_i^* e_i\right\|}$$

provided the number $\lambda \leq d_X$. Then it follows that $\mathbf{E}\min_{t\in T_\lambda}\|G(t)\| > 0$, hence there exists an operator G_0 of rank k which satisfies $\min_{t\in T_\lambda}\|G_0(t)\|_2 > 0$. Let $Y = G_0^{-1}(0)$ be the k-codimensional subspace of X^*. Since $Y \cap T_\lambda = \emptyset$, this implies that for each $y \in T_\lambda \cap Y$, $y \notin \lambda B_{X^*}$, i.e. each $y \in Y$ satisfies $d_X\|y\|_2 \geq \|y\| \geq \lambda\|y\|_2$. □

Corollary 1. *Given $0 < \varepsilon < 1$, $1 \leq k \leq n$, let T be any subset consisting of N points in S^{n-1}. If $c\sqrt{\ln N} < \mathbf{E}(\ell_2^k) - \varepsilon\mathbf{E}(\ell_2^n)$, there is a k-codimensional subspace $E_k \subset \mathbb{R}^n$ which misses T by distance greater than ε.*

Proof. As in Theorem 3, let $G : \mathbb{R}^n \to \ell_2^k$ be a Gaussian operator. Associate with each $t \in T$ a cap on the sphere centered at t with radius ε, $C_\varepsilon(t) := B_2^n(t,\varepsilon)\bigcap S^{n-1}$. Each point $x(t) \in C_\varepsilon(t)$ has the form $x(t) = t + ru$, where $0 \leq r \leq \varepsilon$, $u \in S^{n-1}$, and $\|x(t)\|_2 = \|t + ru\|_2 = 1$, set $T_\varepsilon := \bigcup_{t\in T} C_\varepsilon(t)$. As in Theorem 3, $\mathbf{E}\min_{x\in T_\varepsilon}\|G(x)\|_2 \geq \mathbf{E}(\ell_2^k) - \mathbf{E}^*(T_\varepsilon)$. Now, since the set $\{\sum_{i=1}^n t_i g_i; t \in T\}$ consists of N normalized standard Gaussians, it is well known and easy to prove that $\mathbf{E}^*(T) \leq c\sqrt{\ln N}$, so we have

$$\begin{aligned}\mathbf{E}^*(T_\varepsilon) &= \mathbf{E}\max_{t\in T,\, x(t)=t+ru\in C_\varepsilon(t)} |\langle g, t+ru\rangle| \\ &\leq \mathbf{E}\max_{t\in T}|\langle g,t\rangle| + \epsilon\mathbf{E}(\ell_2^n) = \mathbf{E}^*(T) + \varepsilon\mathbf{E}(\ell_2^n) \\ &\leq c\sqrt{\ln N} + \varepsilon\mathbf{E}(\ell_2^n),\end{aligned}$$

it follows that $\mathbf{E}\min_{x\in T_\varepsilon}\|G(x)\|_2 > 0$, and we conclude as in Theorem 3, that there is a k-codimensional subspace E_k which misses the set T_ε. □

Remark 2. It is easy to see that $\mathbf{E}(\ell_2^n) = \sqrt{n} - \frac{1}{4\sqrt{n}} + o(\frac{1}{\sqrt{n}})$. Thus, given $n = 1, 2, \ldots$, if $1 > \varepsilon_n = 1 - \theta_n \to 1$, where $n\theta_n^2 \to \infty$, and N_n are integers satisfying $\ln(N_n) \cong n\theta_n^2$, and $T_n \subset S^{n-1}$ are subsets containing N_n points, then for any sequence of integers $\ell_n \to \infty$ such that $\ell_n = o(n\theta_n)$, there exist subspaces $F_n \subset \mathbb{R}^n$, $\dim(F_n) = \ell_n$, which miss the sets T_n by distances greater than ε_n $(\to 1)$.

Remark 3. In Theorem 3, the selection of ℓ_2^k to be the range space of G was done because of convenience. Of course ℓ_2^k may be replaced by any k-dimensional $(1 \leq k \leq n)$ normed space $Z = (\mathbb{R}^k, \|.\|_Z)$ which satisfies on $\mathbb{R}^k$, $\|x\|_Z \leq \|x\|_2$ and we then aim to find λ for which $\min_{t\in T_\lambda}\|G(t)\|_Z > 0$. The change in the proof is that $\mathbf{E}(\ell_2^k)$ is replaced by $\mathbf{E}(Z)$, thus the integer $m(k)$ is replaced by the solution $m = m(Z) \cong [(\mathbf{E}(Z))^2/\ln(2n/(\mathbf{E}(Z))^2)]$ of the equation $\sqrt{m\ln(2n/m)} \cong \mathbf{E}(Z)$, provided $1 \leq m(Z) \leq n$, in which case we choose $\lambda \cong \mathbf{E}(Z)/(\mathbf{E}\|\sum_{i=m(Z)+1}^n g_i^* e_i\|)$ provided $\lambda \leq d_X$. This may produce many examples. Usually the choice of the "best" Z which will provide the largest λ, will depend on the choice of X.

Example 1. Given $1 \leq \ell \leq k \leq n$, take $Z_\ell = (\mathbb{R}^k, \|.\|)$ with norm $\|z\|_{Z_\ell} = \sqrt{\sum_{i=1}^{\ell}(z_i^*)^2}$. Note that when $\ell = k$ we are in the situation of Theorem 3. Then $\mathbf{E}(Z_l) \cong \sqrt{\ell \ln(2k/\ell)}$, so we obtain that

$$\mathbf{E} \min_{t \in T_\lambda} \|G(t)\|_{Z_\ell} \geq c' \sqrt{\ell \ln\left(\frac{2k}{\ell}\right)} - \left(c\sqrt{m \ln\left(\frac{2n}{m}\right)} + \lambda \mathbf{E}\left\|\sum_{i=m+1}^n g_i^* e_i\right\|\right),$$

we want to make the R.H.S. positive, so we select $m = m(Z_\ell) \cong \ell \ln(\frac{2k}{\ell})/\ln(\frac{2n}{\ell \ln(2k/\ell)})$, which is the solution for m of the equation $\ell \ln(\frac{2k}{\ell}) \cong m \ln(\frac{2n}{m})$, then choose $\lambda = \lambda_\ell := C\sqrt{\ell \ln(2k/\ell)}/\mathbf{E}\|\sum_{i=m(Z_\ell)+1}^n g_i^* e_i\|$, for an appropriate choice of the constant C. Now we maximize λ_ℓ over all $1 \leq \ell \leq k$, to get the largest possible value of λ for a given k.

Example 2. Theorem 3 provides immediately the well-known result of Gluskin [G] on embedding ℓ_2^k in ℓ_1^n. Let $X = \ell_\infty^n$, $0 < \theta < 1$, $k = [(1-\theta)n]$. Then $d_X = \sqrt{n}$, $m(k) = [c_1 k/\ln(\frac{2n}{k})]$, then there exists a k-codimensional subspace $Y \subset \ell_1^n$ for which

$$d(\ell_2^{n-k}, Y) \leq \frac{c_0 d_X}{\sqrt{k}} \mathbf{E}\left\|\sum_{i=m(k)+1}^n g_i^* e_i\right\|_\infty = c_0 \sqrt{\frac{n}{k}}\, \mathbf{E}\left(g_{m(k)+1}^*\right)$$
$$\cong \sqrt{\frac{n}{k} \ln\left(\frac{2n}{k}\right)} \cong \sqrt{\frac{1}{1-\theta} \ln\left(\frac{2}{1-\theta}\right)}.$$

Remark 4. Given a sequence of real numbers $\{t_i\}_{i=1}^n$, we denote its non-decreasing rearrangement by $k - \min\{t_i\}_{i=1}^n$, thus $1 - \min\{t_i\}_{i=1}^n = \min\{t_i\}_{i=1}^n$, $2 - \min\{t_i\}_{i=1}^n$ is the next smallest, etc., finally, $n - \min\{t_i\}_{i=1}^n = \max\{t_i\}_{i=1}^n$. Paper [GoLSW2] provides upper and lower bounds for the values $\mathbf{E}(k - \min\{|x_i f_i(\omega)|^p\}_{i=1}^n)$ for $k = 1, 2, \ldots, n$ whenever $f_i(\omega), i = 1, \ldots, n$ are i.i.d. random variables, satisfying some general conditions (including the p-stable random variables and others), and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are real numbers, and $1 \leq p \leq \infty$. In [GoLSW1] the case of Gaussian random variables is developed. Thus, to compute the values $\mathbf{E}\|\sum_m^n g_i^* e_i\|$ for a given norm and an arbitrary interval of integers $[m, n] \subseteq [1, N]$, one may use the estimates provided in these papers, provided that the norm $\|.\|$ can somehow be evaluated, such as in the case of ℓ_p^n spaces.

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Marginals of Geometric Inequalities

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Summary. This note consists of three parts. In the first, we observe that a surprisingly rich family of functional inequalities may be proven from the Brunn–Minkowski inequality using a simple geometric technique. In the second part, we discuss consequences of a functional version of Santaló's inequality, and in the third part we consider functional counterparts of mixed volumes and of Alexandrov–Fenchel inequalities.

1 Introduction

In this note we review a simple, folklore, method for obtaining a functional inequality – an inequality about functions – from a geometric inequality, which here means an inequality about shapes and bodies. Given a compact set $K \subset \mathbb{R}^n$ and a k-dimensional subspace $E \subset \mathbb{R}^n$, the marginal of K on the subspace E is the function $f_{K,E} : E \to [0, \infty)$ defined as

$$f_{K,E}(x) = \mathrm{Vol}_{n-k}\big(K \cap [x + E^{\perp}]\big)$$

where $E^{\perp}$ is the orthogonal complement to E in $\mathbb{R}^n$, and Vol_{n-k} is the induced Lebesgue measure on the affine subspace $x + E^{\perp}$. A trivial observation is that an inequality of the form $\mathrm{Vol}_n(A) \geq \mathrm{Vol}_n(B)$ implies the inequality $\int_E f_{A,E} \geq \int_E f_{B,E}$. Thus geometric inequalities give rise to certain functional inequalities in a lower dimension.

The idea of recovering functional inequalities from different types of inequalities in higher dimension is not new, and neither is the use of marginals as explained above (see, e.g., [Bo, Er] or [KLS, page 548]). In this note we observe that this obvious method, when applied to some classical geometric inequalities, entails non-trivial functional inequalities. In particular, this method yields conceptually simple proofs of logarithmic Sobolev inequalities, Prékopa–Leindler and other inequalities: All follow as marginals of the Brunn–Minkowski inequality. Marginals of the Brunn–Minkowski inequality are the

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subject of the first part of this paper, that consists of Section 2 and Section 3. Although no new mathematical statements are presented in this part of the note, we hope that some readers will benefit from the clear geometric flavor added to the known proofs of these inequalities, in particular the approach of Bobkov and Ledoux to the gaussian log-Sobolev inequality [BoL]. We would also like to acknowledge the great influence of K. Ball's work [B2] and F. Barthe's work [Ba1] on our understanding of the interplay between log-concave functions and convex sets.

An application of the "marginals of geometric inequalities" approach to Santaló's inequality was carried out in [ArtKM]. By appropriately taking marginals of both sides of Santaló's inequality, the following new inequality was established: For any integrable function $g : \mathbb{R}^n \to [0, \infty)$ with a positive integral, there exists $x_0 \in \mathbb{R}^n$ such that $\tilde{g}(x) = g(x - x_0)$ satisfies

$$\int_{\mathbb{R}^n} \tilde{g} \int_{\mathbb{R}^n} \tilde{g}^\circ \leq (2\pi)^n \tag{1}$$

where $f^\circ(x) = \inf_{y \in \mathbb{R}^n} \left[e^{-\langle x,y \rangle} / f(y) \right]$ for any $f : \mathbb{R}^n \to [0, \infty)$. In the case where g is assumed to be an even function, the inequality (1) was proven by K. Ball [B1]. If $\int x g^\circ(x) dx = 0$, then we can take $x_0 = 0$ in (1). In that case, equality in (1) holds if and only if g is a gaussian function. Additionally, the left hand side of (1) is always bounded from below by c^n, for a universal constant $c > 0$ (see [KlM]).

Santalò's inequality, once translated into its functional form (1), attains power of its own. For example, it was shown in [ArtKM] following ideas of Maurey [M], that the inequality (1) implies a sharp concentration inequality for Lipshitz functions of gaussian variables. The second part of this paper describes further applications of the functional Santaló inequality (1). For example, with the aid of the transportation of measure technique, we derive the following corollary:

Corollary 1.1. *Let $K, T \subset \mathbb{R}^n$ be centrally-symmetric, convex bodies, and denote by $D \subset \mathbb{R}^n$ the standard Euclidean unit ball in $\mathbb{R}^n$. Then,*

$$\mathrm{Vol}_n(K \cap_2 T)\mathrm{Vol}_n(K^\circ \cap_2 T) \leq \mathrm{Vol}_n(D \cap_2 T)^2 \tag{2}$$

where $K^\circ = \{x \in \mathbb{R}^n; \forall y \in K, \ \langle x, y \rangle \leq 1\}$ is the polar body, and $A \cap_2 B$ is defined as follows: If A is the unit ball of the norm $\| \cdot \|_A$ and B is the unit ball of the norm $\| \cdot \|_B$, then $A \cap_2 B$ is defined as the unit ball of the norm $\|x\|_{A \cap_2 B} = \sqrt{\|x\|_A^2 + \|x\|_B^2}$.

Here, a convex body is a compact, convex set with a non-empty interior. Note that $A \cap B \subset A \cap_2 B \subset \sqrt{2}(A \cap B)$ for any centrally-symmetric convex sets $A, B \subset \mathbb{R}^n$. Thus, Corollary 1.1 immediately implies that

$$\mathrm{Vol}_n(K \cap T)\mathrm{Vol}_n(K^\circ \cap T) \leq 2^n \mathrm{Vol}_n(D \cap T)^2 \tag{3}$$

for any centrally-symmetric convex bodies $K, T \subset \mathbb{R}^n$. Inequality (3) is probably not sharp; The constant 2^n on the right hand side seems unnecessary. The validity of (3), without the 2^n factor, was conjectured by Cordero–Erausquin in [C-E]. Cordero–Erausquin proved this conjecture for the case where $K, T \subset \mathbb{R}^{2n}$ are unit balls of complex Banach norms, and T is invariant under complex conjugation [C-E]. Another case in which a sharp version of (3) is known to hold, without the 2^n factor, is the case where T is an unconditional convex body. This follows from the methods in [C-EFM], and was also observed independently by Barthe and Cordero–Erausquin [Ba2]. Corollary 1.1 is derived from more general principles in Section 5, and so is the following corollary.

Corollary 1.2. *Let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be a convex, even function, and let $\alpha > 0$ be a parameter. Let μ be a measure on $\mathbb{R}^n$ whose density $F = \frac{d\mu}{dx}$ is*

$$F(x) = \int_0^\infty t^{n+1} e^{-\alpha t^2} e^{-\psi(tx)} dt. \tag{4}$$

Then, for any centrally-symmetric, convex body $K \subset \mathbb{R}^n$,

$$\mu(K)\mu(K^\circ) \leq \mu(D)^2. \tag{5}$$

What types of measures arise in Corollary 1.2? By plugging in (4), e.g., $\alpha = 1, \psi(x) = \|x\|^2$ for some norm $\|\cdot\|$ on $\mathbb{R}^n$, we deduce that a measure μ whose density is $1/(1 + \|x\|^2)^{n+2}$ satisfies (5). Observe that these measures are not log-concave (see Section 4 for definition).

The third part of this note focuses on the Alexandrov–Fenchel inequalities for mixed volumes. Let $f : \mathbb{R}^n \to [0, \infty)$ be a function that is concave on its support. We define the Legendre transform of f to be

$$\mathcal{L}' f(x) = \sup_{y; f(y) > 0} \left[f(y) - \langle x, y \rangle \right].$$

Note that $\mathcal{L}'f$ is convex. We use the notation $\mathcal{L}'$ and not $\mathcal{L}$, since our transform is slightly different from the standard Legendre transform $\mathcal{L}$ of convex functions (see, e.g., [Ar] or (42) below). The transforms $\mathcal{L}$ and $\mathcal{L}'$ differ mainly by a trivial minus sign.

Theorem 1.3. *Let $f_0, ..., f_n : \mathbb{R}^n \to [0, \infty)$ be compactly-supported, continuous functions, that are concave on their support. Assume also that $\mathcal{L}' f_0, ..., \mathcal{L}' f_n$ posses continuous second derivatives. Denote*

$$V(f_0, ..., f_n) = \int_{\mathbb{R}^n} [\mathcal{L}' f_0](x) D\big(\mathrm{Hess}[\mathcal{L}' f_1](x), ..., \mathrm{Hess}[\mathcal{L}' f_n](x)\big) dx \tag{6}$$

where D stands for mixed discriminant (see, e.g., the Appendix below) and Hess *stands for the Hessian of a function. Then:*

1. *The multilinear form $V(f_0, ..., f_n)$ may be extended to be defined for all compactly-supported non-negative functions that are concave on their support (without any smoothness or even continuity assumptions). The quantity $V(f_0, ..., f_n)$ is finite also in this extended domain of definition.*
2. *The multilinear form V is continuous with respect to pointwise convergence of functions, in the space of compactly-supported non-negative functions that are concave on their support.*
3. *The multilinear form V is fully symmetric, i.e. for any permutation $\sigma \in S_{n+1}$,*
$$V(f_0, ..., f_n) = V(f_{\sigma(0)}, ..., f_{\sigma(n)}),$$
whenever $f_0, ..., f_n : \mathbb{R}^n \to [0, \infty)$ are compactly-supported functions that are concave on their support.
4. *Let $f_0, ..., f_n, g_0, ..., g_n : \mathbb{R}^n \to [0, \infty)$ be compactly-supported functions that are concave on their support. If $f_0 \geq g_0, ..., f_n \geq g_n$, then*
$$V(f_0, ..., f_n) \geq V(g_0, ..., g_n) \geq 0.$$
5. *Let $f_0, ..., f_n : \mathbb{R}^n \to [0, \infty)$ be compactly-supported functions that are concave on their support. The following "hyperbolic-type" inequality holds:*
$$V(f_0, f_1, ..., f_n)^2 \geq V(f_0, f_0, f_2, ..., f_n)V(f_1, f_1, f_2, ..., f_n). \tag{7}$$

The analogy with mixed volumes of convex bodies is clear (see Section 5). Note that a function $g : \mathbb{R}^n \to \mathbb{R}$ is of the form $g = \mathcal{L}'f$ for some compactly-supported function $f : \mathbb{R}^n \to [0, \infty)$ that is concave on its support, if and only if g is convex and
$$\forall x \in \mathbb{R}^n, \quad 0 \leq g(x) - h_T(x) \leq C \tag{8}$$
for some $C > 0$ and a compact, convex set $T \subset \mathbb{R}^n$, where $h_T(x) = \sup_{y \in T}\langle x, y\rangle$ is the supporting functional of T. Thus, we could have reformulated Theorem 1.3 in terms of convex functions satisfying condition (8), rather than in terms of Legendre transform of concave functions.

Theorem 1.4. *Let $K \subset \mathbb{R}^n$ be a compact, convex set, and let $f_0, ..., f_n : K \to [0, \infty)$ be concave functions that vanish on ∂K. Assume further that the functions have continuous second derivatives in the interior of K, and that the second derivatives are bounded. Denote*
$$I(f_0, ..., f_n) = \int_K f_0(x) D\big(-\mathrm{Hess} f_1(x), ..., -\mathrm{Hess} f_n(x)\big) dx, \tag{9}$$
where, as before, D stands for the mixed discriminant and Hess *stands for the Hessian of a function. Then:*

1. *The multilinear form* $I(f_0, ..., f_n)$ *is finite, and continuous with respect to pointwise convergence of functions (yet, trying to extend the multilinear form* I *to non-smooth functions, we may encounter situations where* $I = \infty$. *We thus choose to formally confine the domain of definition of* I *to smooth functions).*
2. *The multilinear form* I *is fully symmetric, i.e. for any permutation* $\sigma \in S_{n+1}$,
$$I(f_0, ..., f_n) = I(f_{\sigma(0)}, ..., f_{\sigma(n)}),$$
whenever $f_0, ..., f_n : K \to [0, \infty)$ *are concave functions that vanish on* ∂K *and have continuous, bounded, second derivatives in the interior of* K.
3. *Let* $f_0, ..., f_n, g_0, ..., g_n : K \to [0, \infty)$ *be concave functions that vanish on* ∂K *and have continuous, bounded, second derivatives in the interior of* K. *If* $f_0 \geq g_0, ..., f_n \geq g_n$ *then*
$$I(f_0, ..., f_n) \geq I(g_0, ..., g_n) \geq 0.$$
4. *Let* $f_0, ..., f_n : K \to [0, \infty)$ *be concave functions that vanish on* ∂K *and have continuous, bounded, second derivatives in the interior of* K. *The following "elliptic-type" inequality holds:*
$$I(f_0, f_1, ..., f_n)^2 \leq I(f_0, f_0, f_2, ..., f_n) I(f_1, f_1, f_2, ..., f_n). \tag{10}$$

The only significant difference between V from Theorem 1.3 and I from Theorem 1.4, is the fact that the Legendre transform is applied to the functions in Theorem 1.3 (compare the definition (9) with the definition (6)). The "elliptic" inequality (10) is transformed into the "hyperbolic" inequality (7) after an application of the Legendre transform. It would be desirable to have a deeper understanding of this fact. In particular, our proofs of (7) and of (10) are completely different; We would like to see a unifying scheme for both inequalities. Such a unifying approach might possibly shed new light on the highly non-trivial Alexandrov–Fenchel inequalities. The proofs of Theorem 1.3 and Theorem 1.4 appear in Section 5.1 and Section 5.2, respectively. Section 5 constitutes the third part of this note.

For the convenience of the reader, we also include a short appendix regarding some standard properties of mixed discriminants. Here, the letter D denotes both the unit Euclidean ball and the mixed discriminant, but the context will always distinguish between the two meanings.

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2 The Basic Setting

We work in Euclidean spaces $\mathbb{R}^m$, for various $m > 0$, and we denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the usual norm and scalar product in $\mathbb{R}^m$. Let $n, s > 0$ be integers, and let $f : \mathbb{R}^n \to [0,\infty)$ be a function. The support of f, denoted by $\mathrm{Supp}(f)$, is the closure of $\{x \in \mathbb{R}^n; f(x) > 0\}$. We say that f is s-concave if $\mathrm{Supp}(f)$ is compact, convex and $f^{\frac{1}{s}}$ is concave on $\mathrm{Supp}(f)$. An s-concave function is continuous in the interior of its support (see e.g., [Ro]). With any function $f : \mathbb{R}^n \to [0,\infty)$ we associate a set

$$\mathcal{K}_f = \left\{(x,y) \in \mathbb{R}^{n+s} = \mathbb{R}^n \times \mathbb{R}^s; x \in \mathrm{Supp}(f), |y| \le f^{\frac{1}{s}}(x)\right\} \tag{11}$$

where, for given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$, (x,y) are coordinates in $\mathbb{R}^{n+s}$. If the function f is measurable, so is the set $\mathcal{K}_f$. Additionally, the set $\mathcal{K}_f$ is convex if and only if f is s-concave. We also note that

$$\mathrm{Vol}(\mathcal{K}_f) = \int_{\mathrm{Supp}(f)} \kappa_s \cdot \left(f^{\frac{1}{s}}(x)\right)^s dx = \kappa_s \int f \tag{12}$$

where $\kappa_s = \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$ is the volume of the s-dimensional Euclidean unit ball. For $\lambda > 0$ and $f : \mathbb{R}^n \to [0,\infty)$, we define the function $\lambda \times_s f : \mathbb{R}^n \to [0,\infty)$ to be

$$[\lambda \times_s f](x) = \lambda^s f\left(\frac{x}{\lambda}\right). \tag{13}$$

Note that $\mathcal{K}_{\lambda \times_s f} = \lambda \mathcal{K}_f = \{\lambda y; y \in \mathcal{K}_f\}$, and hence we view $\lambda \times_s f$ as a functional analog to homothety of bodies. If f is an s-concave function, so is $\lambda \times_s f$. Recall that for two sets $A, B \subset \mathbb{R}^n$, their Minkowski sum is defined by $A + B = \{a + b; a \in A, b \in B\}$. For two functions $f, g : \mathbb{R}^n \to [0,\infty)$, we define their "$s$-Minkowski sum" as

$$[f \oplus_s g](x) = \left(\sup_{\substack{y \in \mathrm{Supp}(f), z \in \mathrm{Supp}(g) \\ x = y+z}} f(y)^{\frac{1}{s}} + g(z)^{\frac{1}{s}}\right)^s \tag{14}$$

whenever $x \in \mathrm{Supp}(f) + \mathrm{Supp}(g)$. If $x \notin \mathrm{Supp}(f) + \mathrm{Supp}(g)$, we set $[f \oplus_s g](x) = 0$. Our definition is motivated by the fact that

$$\mathcal{K}_{f \oplus_s g} = \mathcal{K}_f + \mathcal{K}_g.$$

Note that whenever f, g are s-concave, the function $f \oplus_s g$ is also s-concave. The $\oplus_s$ and $\times_s$ operations induce a convex cone structure on the class of s-concave functions.

Arguably one of the most useful geometric inequalities in the theory of convex bodies is the Brunn–Minkowski inequality. This inequality states that for any non-empty compact sets $A, B \subset \mathbb{R}^m$,

$$\mathrm{Vol}(A+B)^{\frac{1}{m}} \ge \mathrm{Vol}(A)^{\frac{1}{m}} + \mathrm{Vol}(B)^{\frac{1}{m}}. \tag{15}$$

There are at least a handful of completely different proofs of (15), see, e.g., [BonF], [Bru1, Bru2], [GiM], [Gr], [GrM], [HO], [KnS], [Mc2]. For instance, along the lines of Blaschke's proof, we may use the easily verified fact that for any hyperplane $H \subset \mathbb{R}^m$,

$$S_H(A+B) \supset S_H(A) + S_H(B),$$

where S_H is the Steiner symmetrization with respect to the hyperplane H (see, e.g., [BonF]). We now derive (15) by applying a suitable sequence of Steiner symmetrizations, such that $S_{H_1}...S_{H_k}(A+B), S_{H_1}...S_{H_k}(A)$ and $S_{H_1}...S_{H_k}(B)$ converge to Euclidean balls when $k \to \infty$.

The Brunn–Minkowski inequality (15) for $(n+s)$-dimensional sets implies that for any $\lambda, \mu > 0$ and measurable functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\mathrm{Vol}^*_{n+s}\big(\mathcal{K}_{[\lambda \times_s f] \oplus_s [\mu \times_s g]}\big)^{\frac{1}{n+s}} \geq \lambda \mathrm{Vol}_{n+s}(\mathcal{K}_f)^{\frac{1}{n+s}} + \mu \mathrm{Vol}_{n+s}(\mathcal{K}_g)^{\frac{1}{n+s}} \tag{16}$$

where Vol^*_{n+s} stands for outer Lebesgue measure (the set $\mathcal{K}_{[\lambda \times_s f] \oplus_s [\mu \times_s g]}$ may be non-measurable). We immediately conclude that (16) translates, using (12), to the following inequality: For all $\lambda, \mu > 0$, an integer $s > 0$ and measurable functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\Big(\int^*_{\mathbb{R}^n} [\lambda \times_s f] \oplus_s [\mu \times_s g] \Big)^{\frac{1}{n+s}} \geq \lambda \Big(\int_{\mathbb{R}^n} f \Big)^{\frac{1}{n+s}} + \mu \Big(\int_{\mathbb{R}^n} g \Big)^{\frac{1}{n+s}} \tag{17}$$

where $\int^*$ is the outer integral. We summarize this discussion with the following theorem.

Theorem 2.1. *Let $f, g, h : \mathbb{R}^n \to [0, \infty)$ be three integrable functions, and $s, \lambda, \mu > 0$ be real numbers. Assume that for any $x, y \in \mathbb{R}^n$,*

$$h\big(\lambda x + \mu y\big) \geq \Big(\lambda f(x)^{\frac{1}{s}} + \mu g(y)^{\frac{1}{s}}\Big)^s. \tag{18}$$

Then,

$$\Big(\int h \Big)^{\frac{1}{n+s}} \geq \lambda \Big(\int f \Big)^{\frac{1}{n+s}} + \mu \Big(\int g \Big)^{\frac{1}{n+s}}.$$

Proof. Assume first that s is an integer. In this case, the theorem follows from (17), as $h \geq [\lambda \times_s f] \oplus_s [\mu \times_s g]$ pointwise, and $\int h = \int^* h$. The case of an integer s suffices for all the applications we present below. Next, assume that $s = p/q$ is a rational number, and $p, q > 0$ are integers. Note that by Hölder's inequality, for any $x_1, ..., x_q, y_1, ..., y_q \in \mathbb{R}^n$,

$$\begin{aligned} \lambda \prod_{i=1}^q f(x_i)^{\frac{1}{qs}} + \mu \prod_{i=1}^q g(y_i)^{\frac{1}{qs}} &\leq \Big(\prod_{i=1}^q \big(\lambda f(x_i)^{\frac{1}{s}} + \mu g(y_i)^{\frac{1}{s}}\big) \Big)^{\frac{1}{q}} \\ &\leq \prod_{i=1}^q h(\lambda x_i + \mu y_i)^{\frac{1}{qs}} \end{aligned} \tag{19}$$

where the second inequality follows from (18). Our derivation of (19) is inspired by [GrM]. For a function $r : \mathbb{R}^n \to [0,\infty)$ we define ad-hoc $\tilde{r} : \mathbb{R}^{nq} \to [0,\infty)$ by $\tilde{r}(x) = \tilde{r}(x_1, ..., x_q) = \prod_{i=1}^{q} r(x_i)$ where $x = (x_1, ..., x_q) \in (\mathbb{R}^n)^q$ are coordinates in $\mathbb{R}^{nq}$. Thus, (19) implies that for any $x, y \in \mathbb{R}^{nq}$,

$$\tilde{h}(\lambda x + \mu y) \geq \left(\lambda \tilde{f}(x)^{\frac{1}{qs}} + \mu \tilde{g}(y)^{\frac{1}{qs}}\right)^{qs}. \tag{20}$$

Note that $qs = p$ is an integer, which is the case we already dealt with. Hence

$$\begin{aligned}\Big(\int h\Big)^{\frac{1}{n+s}} &= \Big(\int \tilde{h}\Big)^{\frac{1}{q(n+s)}} \geq \lambda\Big(\int \tilde{f}\Big)^{\frac{1}{q(n+s)}} + \mu\Big(\int \tilde{g}\Big)^{\frac{1}{q(n+s)}} \\ &= \lambda\Big(\int f\Big)^{\frac{1}{n+s}} + \mu\Big(\int g\Big)^{\frac{1}{n+s}}\end{aligned}$$

and the theorem is proven for the case of a rational $s > 0$. The case of a real $s > 0$ follows by a standard approximation argument. □

Theorem 2.1 was first proven, for the case $n = 1$, by Henstock and Macbeath [HeM]. Later, it was proven for all $n \geq 1$ by Dinghas [D], by Borell [Bor] and by Brascamp–Lieb [BrL] independently. The notation in [Bor, BrL] is different from ours, and it covers only the case where $\lambda + \mu = 1$ in Theorem 2.1 (yet the general case follows easily). However, the framework in [Bor, BrL] also covers the case where $s \leq -n$, which does not seem to fit well into our discussion.

When $\lambda + \mu = 1$, letting s tend to infinity in Theorem 2.1, we recover the Prékopa–Leindler inequality [Le, Pr1, Pr2] as follows:

Corollary 2.2. *Let $f, g, h : \mathbb{R}^n \to [0,\infty)$ be three integrable functions and $0 < \lambda < 1$. Assume that for any $x, y \in \mathbb{R}^n$,*

$$h\left(\lambda x + (1-\lambda)y\right) \geq f(x)^{\lambda} g(y)^{1-\lambda}.$$

Then,

$$\int h \geq \Big(\int f\Big)^{\lambda}\Big(\int g\Big)^{1-\lambda}. \tag{21}$$

Proof. The argument is standard. Fix $M > 1$. The basic observation is that,

$$\left(\lambda x^{\frac{1}{s}} + (1-\lambda) y^{\frac{1}{s}}\right)^{s} \stackrel{s\to\infty}{\longrightarrow} x^{\lambda} y^{1-\lambda} \tag{22}$$

uniformly for $(x, y) \in \left(\frac{1}{M}, M\right) \times \left(\frac{1}{M}, M\right)$. Therefore for any $\varepsilon > 0$ there is $s_0(\varepsilon, M) > 0$, such that whenever $s > s_0(\varepsilon, M)$ and $\frac{1}{M} < f(x), g(y) < M$,

$$h\big(\lambda x + (1-\lambda)y\big) + \varepsilon \geq \left(\lambda f(x)^{\frac{1}{s}} + (1-\lambda) g(y)^{\frac{1}{s}}\right)^{s}.$$

Denote $K_f^M = \{x \in \mathbb{R}^n; \frac{1}{M} < f(x) < M\}$ and $K_g^M = \{x \in \mathbb{R}^n; \frac{1}{M} < g(x) < M\}$. Theorem 2.1 implies that for $\varepsilon > 0, s > s_0(\varepsilon, M)$,

$$\int_{\lambda K_f^M + (1-\lambda)K_g^M} \big[h(x) + \varepsilon\big] dx \geq \left[\lambda\Big(\int_{K_f^M} f\Big)^{\frac{1}{n+s}} + (1-\lambda)\Big(\int_{K_g^M} g\Big)^{\frac{1}{n+s}}\right]^{n+s}$$
$$\geq \Big(\int_{K_f^M} f\Big)^{\lambda}\Big(\int_{K_g^M} g\Big)^{1-\lambda}.$$

Since f, g are integrable, the sets $K_f^M, K_g^M \subset \mathbb{R}^n$ are bounded, and so is $\lambda K_f^M + (1-\lambda)K_g^M$. Letting first ε tend to zero, and then M tend to infinity, we conclude (21). □

Note that in the proof of Corollary 2.2, we could confine s to be an integer, and use the simpler inequality (17) rather than Theorem 2.1. The proof of Corollary 2.2 is a prototype for the results we will obtain in the next section. The idea is to consider a geometric inequality in dimension $n+s$, to use the marginal of both sides of the inequality, and then let the extra dimension s tend to infinity. Thus our inequalities are traces of higher dimensional geometric inequalities, when the dimension tends to infinity.

3 Minkowski's Inequality

Suppose $K \subset \mathbb{R}^n$ is a convex set with the origin in its interior. For $x \in \mathbb{R}^n$ we define

$$\|x\|_K = \inf\left\{\lambda > 0; \frac{x}{\lambda} \in K\right\}.$$

Then $\|\cdot\|_K$ is the (perhaps non-symmetric) norm whose unit ball is K. The dual norm, which again may be non-symmetric, is $\|x\|_* = \sup_{y \in K}\langle x, y\rangle$. In this section we will prove the following theorem:

Theorem 3.1. *Let $K \subset \mathbb{R}^n$ be a convex set with the origin in its interior. Let $\|\cdot\|$ be the norm that K is its unit ball (it may be a non-symmetric norm). Let $1 \leq p < \infty$, and let $F : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with $\int |F|^p, \int |\nabla F|^p < \infty$. Then,*

$$\int_{\mathbb{R}^n} F^p(x) \log \frac{cF^p(x)}{\int F^p(y)dy} dx \leq \int_{\mathbb{R}^n} \|\nabla F(x)\|^p dx \tag{23}$$

where $c = \mathrm{Vol}_n(K^\circ)e^n(\frac{q}{p})^{\frac{n}{q}}\Gamma(\frac{n}{q}+1)$, and $q \geq 1$ satisfies $\frac{1}{p}+\frac{1}{q} = 1$ (for $p = 1$ the value of c is $\mathrm{Vol}_n(K^\circ)e^n$, as interpreted by continuity). If $p > 1$, then equality in (23) holds for $F(x) = \alpha e^{-\|x\|_*^q/q}$, where $\|\cdot\|_*$ is the dual norm and $\alpha > 0$ is an arbitrary real number. The constant c is also optimal in the case $p = 1$.

Theorem 3.1 is equivalent, by a quick scaling argument produced below, to a family of inequalities which were explicitly stated and proven by Gentil [G] and independently by Agueh, Ghoussoub and Kang [AGK] (see also Remark (2) on Page 320 in [C-ENV]). The proof in [AGK] uses the mass-transportation method developed by Cordero–Erausquin, Nazareth and Villani [C-ENV] for the study of Sobolev and Gagliardo–Nirenberg inequalities. The proof in [G] relies on the Prékopa–Leindler inequality, and is related to the proof of the gaussian logarithmic Sobolev inequality by Bobkov and Ledoux [BoL]. Our approach is closer to that of Gentil, as we use Brunn–Minkowski, and our main contribution here is the clear geometric framework.

The case $p = 2$ and $\|\cdot\|$ being the Euclidean norm in (23) is particularly interesting; In this case (23) is simply equivalent to Stam's inequality from information theory [St]. Setting $F(x) = G(\sqrt{2}x)$ in (23) we may rewrite inequality (23) for $p = 2, \|\cdot\| = |\cdot|$ as follows:

$$\int_{\mathbb{R}^n} G^2(x) \log \frac{(e\sqrt{2\pi})^n G^2(x)}{\int G^2(y)dy} dx \leq 2 \int_{\mathbb{R}^n} |\nabla G(x)|^2 dx, \tag{24}$$

for any function G such that the right-hand side is finite. Furthermore, substituting $G(x) = \frac{e^{-\frac{|x|^2}{4}}}{(2\pi)^{\frac{n}{4}}} f(x)$ in (24), we obtain after integration by parts that

$$\int f^2(x) \log \frac{f^2(x)}{\int f^2(y) d\gamma_n(y)} d\gamma_n(x) \leq 2 \int |\nabla f(x)|^2 d\gamma_n(x) \tag{25}$$

where $\frac{d\gamma_n}{dx} = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$ is the density of the standard gaussian measure on $\mathbb{R}^n$. Inequality (25) is the logarithmic Sobolev inequality for the gaussian measure, first explicitly stated by Gross [Gro]. Inequality (25) is fundamental in the study of concentration inequalities in Gauss space, see [L]. We learned the fact that (25) and Stam's inequality are easily equivalent from [Be1, Be2]. In [Be2] it is also shown how (25) directly implies Nash's inequality.

For two sets $K, T \subset \mathbb{R}^m$ we denote the "T-surface area of K" by

$$\tilde{\mathcal{S}}(K;T) = \frac{1}{m} \lim_{\varepsilon \to 0^+} \frac{\mathrm{Vol}_m(K + \varepsilon T) - \mathrm{Vol}_m(K)}{\varepsilon}$$

if the limit exists. The Brunn–Minkowski inequality implies that

$$\begin{aligned}\mathrm{Vol}_m(K + \varepsilon T) &\geq \left(\mathrm{Vol}_m(K)^{\frac{1}{m}} + \varepsilon \mathrm{Vol}_m(T)^{\frac{1}{m}}\right)^m \\ &\geq \mathrm{Vol}_m(K) + m\varepsilon \mathrm{Vol}_m(K)^{\frac{m-1}{m}} \mathrm{Vol}_m(T)^{\frac{1}{m}}.\end{aligned}$$

Consequently, whenever $\tilde{\mathcal{S}}(K;T)$ exists,

$$\tilde{\mathcal{S}}(K;T) \geq \mathrm{Vol}_m(K)^{\frac{m-1}{m}} \mathrm{Vol}_m(T)^{\frac{1}{m}}. \tag{26}$$

Inequality (26) is known as the Minkowski inequality (see e.g. [S, Theorem 3.2.1]). Note that K, T might be non convex in (26). Following our interest in marginals of Minkowski's inequality (26), we define, for any functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\tilde{\mathcal{S}}_s(f; g) = \frac{1}{n+s} \lim_{\varepsilon \to 0^+} \frac{\int [f \oplus_s (\varepsilon \times_s g)] - \int f}{\varepsilon} \tag{27}$$

whenever the integrals are defined and the limit exists. We interpret the Minkowski inequality (26) as follows:

Proposition 3.2. *Fix $s > 0$. Let $f, g : \mathbb{R}^n \to [0, \infty)$ be integrable functions such that $\tilde{\mathcal{S}}_s(f; g)$ exists. Then,*

$$\tilde{\mathcal{S}}_s(f; g) \geq \Big(\int f\Big)^{1 - \frac{1}{n+s}} \Big(\int g\Big)^{\frac{1}{n+s}}. \tag{28}$$

If $f = \lambda \times_s g$ and g is s-concave, then equality holds in (28).

Proof. By Theorem 2.1, whenever the functions are integrable,

$$\begin{aligned} \int \big[f \oplus_s (\varepsilon \times_s g)\big] &\geq \bigg(\Big(\int f\Big)^{\frac{1}{n+s}} + \varepsilon\Big(\int g\Big)^{\frac{1}{n+s}}\bigg)^{n+s} \\ &\geq \Big(\int f\Big) + \varepsilon(n+s)\Big(\int f\Big)^{1-\frac{1}{n+s}} \Big(\int g\Big)^{\frac{1}{n+s}}. \end{aligned}$$

We assume that $\tilde{\mathcal{S}}(f; g)$ exists, hence the definition (27) implies the desired inequality. It is easy to verify that equality holds when $f = \lambda \times_s g$ is s-concave. □

Recall from Section 1 that for a 1-concave function $f : \mathbb{R}^n \to [0, \infty)$, its Legendre transform is

$$\mathcal{L}' f(x) = \sup_{y \in \mathrm{Supp}(f)} \big[-\langle x, y\rangle + f(y)\big]. \tag{29}$$

The function $\mathcal{L}' f : \mathbb{R}^n \to \mathbb{R}$ is always convex. Additionally, for any numbers $\lambda, \mu > 0$ and functions $f, g : \mathbb{R}^n \to [0, \infty)$,

$$\mathcal{L}' \left\{ \big[(\lambda \times_s f) \oplus_s (\mu \times_s g) \big]^{\frac{1}{s}} \right\} = \lambda \mathcal{L}'\big(f^{\frac{1}{s}}\big) + \mu \mathcal{L}'\big(g^{\frac{1}{s}}\big), \tag{30}$$

as the reader may easily verify. The inverse transform is

$$\mathcal{L}'^{-1} f(x) = \inf_{y \in \mathbb{R}^n} \big[\langle x, y\rangle + f(y)\big].$$

If f is 1-concave, then $\mathcal{L}'^{-1}\mathcal{L}' f = f$ on $\mathrm{Supp}(f)$. In this case, if $x \notin \mathrm{Supp}(f)$ then $\mathcal{L}'^{-1}\mathcal{L}' f(x) = -\infty$. Moreover, note that when f is concave, and is also differentiable and strictly concave in some neighborhood of a point x, then

$$y = \nabla f(x) \quad \Leftrightarrow \quad x = -\nabla \mathcal{L}' f(y).$$

Lemma 3.3. *Let $s > 0$ be an integer, and let $f, g : \mathbb{R}^n \to [0, \infty)$. Assume that f is continuous and that g is compactly-supported. Assume further that f is continuously differentiable in the interior of* $\mathrm{Supp}(f)$. *Then, for $x \in \mathbb{R}^n$ with $f(x) > 0$,*

$$\frac{d}{d\varepsilon}\big[f \oplus_s (\varepsilon \times_s g)\big](x)\Big|_{\varepsilon=0} = s f^{\frac{s-1}{s}}(x)\mathcal{L}'\big[g^{\frac{1}{s}}\big]\left(\nabla f^{\frac{1}{s}}(x)\right).$$

Moreover,

$$\frac{[f \oplus_s (\varepsilon \times_s g)](x) - f(x)}{\varepsilon} \overset{\varepsilon \to 0}{\longrightarrow} s f^{\frac{s-1}{s}}(x)\mathcal{L}'\left[g^{\frac{1}{s}}\right]\left(\nabla f^{\frac{1}{s}}(x)\right)$$

locally uniformly in x in the interior of $\mathrm{Supp}(f)$.

Proof. Begin with the definitions (13) and (14). For sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\big[f \oplus_s (\varepsilon \times_s g)\big](x) &= \sup_{\substack{y \in \mathrm{Supp}(f), z \in \mathrm{Supp}(g) \\ x = y + \varepsilon z}} \left(f^{\frac{1}{s}}(y) + \varepsilon g^{\frac{1}{s}}(z)\right)^s \\ &= \sup_{z \in \mathrm{Supp}(g)} \left(f^{\frac{1}{s}}(x - \varepsilon z) + \varepsilon g^{\frac{1}{s}}(z)\right)^s\end{aligned}$$

(all we need is that $x - \varepsilon z \in \mathrm{Supp}(f)$ for all $z \in \mathrm{Supp}(g)$; Recall that $\mathrm{Supp}(g)$ is a bounded set). Since f is smooth and $f(x) > 0$, then $f^{\frac{1}{s}}$ is also continuously differentiable in a neighborhood of x, and

$$f^{\frac{1}{s}}(x - \varepsilon z) = f^{\frac{1}{s}}(x) - \varepsilon\langle \nabla f^{\frac{1}{s}}(x), z\rangle + |\varepsilon z|\alpha_x(\varepsilon z),$$

where $\alpha_x(y) \to 0$ as $y \to 0$, locally uniformly in x. Therefore,

$$\big[f \oplus_s (\varepsilon \times_s g)\big]^{\frac{1}{s}}(x) = f^{\frac{1}{s}}(x) + \varepsilon \sup_{z \in \mathrm{Supp}(g)} \left[-\langle \nabla f^{\frac{1}{s}}(x), z\rangle + g^{\frac{1}{s}}(z) + |z|\alpha_x(\varepsilon z)\right].$$

Denote $\alpha'_x(\varepsilon) = \sup_{z \in \mathrm{Supp}(g)} |z||\alpha_x(\varepsilon z)|$. Since $\mathrm{Supp}(g)$ is compact, then $\alpha'_x(\varepsilon) \to 0$ as $\varepsilon \to 0$ locally uniformly in x, and

$$\left|\,[f \oplus_s (\varepsilon \times_s g)]^{\frac{1}{s}}(x) - f^{\frac{1}{s}}(x) - \varepsilon \sup_{z \in \mathrm{Supp}(g)} \left[-\langle \nabla f^{\frac{1}{s}}(x), z\rangle + g^{\frac{1}{s}}(z)\right]\,\right| \le \varepsilon \alpha'_x(\varepsilon).$$

By (29) we conclude that

$$\frac{d}{d\varepsilon}\left[f \oplus_s (\varepsilon \times_s g)\right]^{\frac{1}{s}}(x)\Big|_{\varepsilon=0} = \mathcal{L}'\left[g^{\frac{1}{s}}\right]\left(\nabla f^{\frac{1}{s}}(x)\right)$$

and that the ε-derivative converges locally uniformly in x. This in turn implies that

$$\frac{d}{d\varepsilon}\left[f \oplus_s (\varepsilon \times_s g)\right](x)\Big|_{\varepsilon=0} = s f^{\frac{s-1}{s}}(x)\mathcal{L}'\left[g^{\frac{1}{s}}\right]\left(\nabla f^{\frac{1}{s}}(x)\right)$$

where the derivative converges locally uniformly in x. □

Lemma 3.4. *Let $s > 1$, and let $f, g : \mathbb{R}^n \to [0, \infty)$ be compactly-supported bounded functions. Assume that the function f is of the form $f(x) = (A - G(x))_+^p$ for some $A, p > 0$ and for a continuous function $G : \mathbb{R}^n \to \mathbb{R}$, continuously differentiable in a neighborhood of $\{x; G(x) \leq A\}$. Assume that ∇G does not vanish on $\{x; G(x) = A\}$. Then,*

$$\tilde{\mathcal{S}}_s(f; g) = \frac{s}{n+s} \int_{\mathrm{Supp}(f)} f^{\frac{s-1}{s}}(x) \mathcal{L}' \left[g^{\frac{1}{s}} \right] \left(\nabla f^{\frac{1}{s}}(x) \right) dx < \infty. \tag{31}$$

Proof. Our task is basically to justify differentiation under the integral sign (see Lemma 5.2 for a less technical argument of the same spirit). For $\varepsilon > 0$, denote $F(\varepsilon, x) = \frac{[f \oplus_s (\varepsilon \times_s g)](x) - f(x)}{\varepsilon}$. According to (27),

$$\tilde{S}(f, g) = \frac{1}{n+s} \lim_{\varepsilon \to 0^+} \int F(\varepsilon, x) dx. \tag{32}$$

Let K be a compact set contained in the interior of $\mathrm{Supp}(f)$. By Lemma 3.3,

$$F(\varepsilon, x) \quad \overset{\varepsilon \to 0}{\longrightarrow} \quad s f^{\frac{s-1}{s}}(x) \mathcal{L}' \left[g^{\frac{1}{s}} \right] \left(\nabla f^{\frac{1}{s}}(x) \right)$$

uniformly on K. We conclude that

$$\int_K F(\varepsilon, x) dx \quad \overset{\varepsilon \to 0}{\longrightarrow} \quad s \int_K f^{\frac{s-1}{s}}(x) \mathcal{L}' \left[g^{\frac{1}{s}} \right] \left(\nabla f^{\frac{1}{s}}(x) \right) dx \tag{33}$$

for any compact set K contained in the interior of $\mathrm{Supp}(f)$. For $\delta > 0$, let K^δ be a compact set, contained in the interior of $\mathrm{Supp}(f)$, such that $\mathrm{Supp}(f) \setminus K^\delta$ is contained in a δ-neighborhood of $\partial \mathrm{Supp}(f)$. We will show that

$$\lim_{\delta \to 0^+} \limsup_{\varepsilon \to 0^+} \left| \int_{\mathbb{R}^n \setminus K^\delta} F(\varepsilon, x) dx \right| = 0. \tag{34}$$

It is straightforward to obtain (31) from (32), (33) and (34). Hence we focus our attention on proving (34). Denote $R = \max\{|x|; x \in \mathrm{Supp}(g)\}, m = \sup g^{1/s}$. Then for any $0 < \varepsilon < \frac{\delta}{R}$,

$$\int_{\mathbb{R}^n \setminus K^\delta} \left[f \oplus_s (\varepsilon \times_s g) \right](x) dx \leq \int_{(\partial \mathrm{Supp}(f))_\delta} \left(\sup_{|z| \leq R} \left[f(x - \varepsilon z)^{\frac{1}{s}} + \varepsilon m \right] \right)^s dx$$

where $T_\delta = \{x \in \mathbb{R}^n; \exists y \in T, |y - x| < \delta\}$ for any $T \subset \mathbb{R}^n$. Recall that $f(x) = (A - G(x))_+^p$ and denote $G_\varepsilon(x) = \inf_{|z-x|<\varepsilon} G(z)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n \setminus K^\delta} F(\varepsilon, x) &\leq \int_{(\partial \mathrm{Supp}(f))_\delta} \frac{\left[\left(A - G_{\varepsilon R}(x) \right)_+^{\frac{p}{s}} + \varepsilon m \right]^s - \left(A - G(x) \right)_+^p}{\varepsilon} dx \\ &\leq \int_{(\partial \mathrm{Supp}(f))_\delta} C + \frac{\left(A - G_{\varepsilon R}(x) \right)_+^p - \left(A - G(x) \right)_+^p}{\varepsilon} dx \end{aligned}$$

for a small enough $\delta, \varepsilon > 0$, where in this proof we denote by c, C, c' etc. positive numbers independent of ε and δ. Therefore,

$$\Big| \int_{\mathbb{R}^n \setminus K^\delta} F(\varepsilon, x) \Big| \leq C \mathrm{Vol}_n((\partial \mathrm{Supp}(f))_\delta)$$

$$+ \int_{\mathrm{Supp}(f) \cap (\partial \mathrm{Supp}(f)_\delta} \hat{C} \left[(A - G(x))^{p-1} + 1 \right] \frac{G(x) - G_{\varepsilon R}(x)}{\varepsilon} dx \qquad (35)$$

$$+ \int_{(\partial \mathrm{Supp}(f))_\delta \setminus \mathrm{Supp}(f)} \frac{(A - G_{\varepsilon R}(x))_+^p}{\varepsilon} dx. \qquad (36)$$

Clearly $\mathrm{Vol}_n((\partial \mathrm{Supp}(f))_\delta) \to 0$ as $\delta \to 0$. Next, we will bound (35). Since G is continuously differentiable, we have that $\frac{G(x) - G_{R\varepsilon}(x)}{\varepsilon} < C$ on $\mathrm{Supp}(f)_\delta$. As the gradient of G does not vanish on the compact set $\partial \mathrm{Supp}(f)$, and since the vector $\nabla g(x)$ is normal to $\partial \mathrm{Supp}(f)$ for $x \in \partial \mathrm{Supp}(f)$, we conclude that for $x \in \mathrm{Supp}(f)$,

$$G(x) < A - c \cdot d\big(x, \partial \mathrm{Supp}(f)\big)$$

whenever $d(x, \partial \mathrm{Supp}(f)) < \tilde{c}$, where $d(x, A)$ stands for the distance between x and A. Therefore, (35) is smaller than

$$\int_{\mathrm{Supp}(f) \cap (\partial \mathrm{Supp}(f))_\delta} \tilde{C} \left[d\big(x, \partial \mathrm{Supp}(f)\big)^{p-1} + 1 \right] dx.$$

The latter integral actually converges even when we replace the domain of integration with the entire $\mathrm{Supp}(f)$, because $p > 0$. Hence (35) tends to zero as $\delta \to 0$, regardless of ε. All that remains is to bound (36). The integrand of (36) is non-zero only on $(\partial \mathrm{Supp}(f))_{\varepsilon R} \setminus \mathrm{Supp}(f)$. The volume of this set is bounded by $\tilde{C}\varepsilon$, and thus (36) is smaller than

$$\tilde{C} \sup_{x \in (\partial \mathrm{Supp}(f))_{\varepsilon R} \setminus \mathrm{Supp}(f)} \big(A - G_{\varepsilon R}(x)\big)_+^p \overset{\varepsilon \to 0^+}{\longrightarrow} 0$$

independently of δ. This establishes (34) and the lemma is proven. □

Next, we will prove Theorem 3.1. Aside from some technicalities, Theorem 3.1 follows simply by letting s tend to ∞ in Minkowski's inequality, in the form of Proposition 3.2.

Proof of Theorem 3.1. First, assume that $p > 1$. Let $s > 1$ and denote

$$g^{\frac{1}{s}}(x) = (1 - \|x\|_*^q)_+^{\frac{1}{q}}.$$

Then g is concave and compactly-supported. Hölder's inequality implies that

$$\left[\mathcal{L}' g^{\frac{1}{s}} \right](x) = (1 + \|x\|^p)^{\frac{1}{p}}.$$

Next, let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that $h(x) \to \infty$ when $|x| \to \infty$. Assume that h is a continuously differentiable function whose gradient is non-zero for $x \neq 0$. Assume also that

$$\int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \left(h^{10}(x) + |\nabla h(x)|^p\right) dx < \infty. \tag{37}$$

Introduce

$$f^{\frac{1}{s}}(x) = \left(s - h(x)\right)_+^{\frac{1}{q}}.$$

Then f is compactly-supported, and by Lemma 3.4,

$$\tilde{\mathcal{S}}(f;g) = \frac{s}{n+s} \int_{\mathbb{R}^n} \left(s - h(x)\right)_+^{\frac{s-1}{q}} \left(1 + \frac{\|\nabla h(x)\|^p}{q^p (s - h(x))_+}\right)^{\frac{1}{p}} dx.$$

Set $t = \frac{1}{s}$. Proposition 3.2 along with some simple manipulations yields that for any $t > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(1 - th(x)\right)_+^{\frac{1}{q}\left(\frac{1}{t}-1\right)} \left(1 + \frac{t\|\nabla h(x)\|^p}{q^p \left(1 - th(x)\right)_+}\right)^{\frac{1}{p}} dx \\ &\geq (1+nt) \left(\int_{\mathbb{R}^n} (1 - th(x))_+^{\frac{1}{qt}} dx\right)^{1-\frac{t}{nt+1}} \left(\frac{1}{t^{\frac{n}{q}}} \int_{\mathbb{R}^n} (1 - \|x\|_*^q)_+^{\frac{1}{qt}} dx\right)^{\frac{t}{nt+1}}. \end{aligned} \tag{38}$$

Note that by Proposition 3.2, equality in (38) holds for $h(x) = \|x\|_*^q$. Denote by $A(t)$ and by $B(t)$ the left and right hand sides of inequality (38), respectively. Then $A(t), B(t) \to \int e^{-h(x)/q}$ as $t \to 0$, and hence we set $A(0) = B(0) = \int e^{-h(x)/q}$. Our integrability assumptions on h allow us to differentiate $A(t)$ under the integral sign (see, e.g. [AlB], Theorem 20.4). We obtain

$$A'(0) = \int e^{-\frac{h(x)}{q}} \left(-\frac{h^2(x)}{2q} + \frac{h(x)}{q} + \frac{\|\nabla h(x)\|^p}{pq^p}\right) dx. \tag{39}$$

Regarding differentiation of the right hand side, recall that K° is the unit ball of $\|\cdot\|_*$. Note that

$$\begin{aligned} \frac{1}{t^{\frac{n}{q}}} \int_{\mathbb{R}^n} (1 - \|x\|_*^q)_+^{\frac{1}{qt}} dx &= \mathrm{Vol}(K^\circ) \frac{1}{t^{\frac{n}{q}+1}} \int_0^1 (1 - s^q)^{\frac{1}{qt}-1} s^{q-1} s^n ds \\ &= \frac{\mathrm{Vol}(K^\circ)}{t^{\frac{n}{q}+1} q} \int_0^1 s^{\frac{n}{q}} (1-s)^{\frac{1}{qt}-1} ds \\ &= \frac{\mathrm{Vol}(K^\circ) q^{\frac{n}{q}} \Gamma\left(\frac{n}{q}+1\right)}{tn+1} \cdot \frac{\left(\frac{1}{qt}\right)^{\frac{n}{q}} \Gamma\left(\frac{1}{qt}\right)}{\Gamma\left(\frac{1}{qt} + \frac{n}{q}\right)} \end{aligned}$$

which tends to $\mathrm{Vol}(K^\circ) c'_{n,q} = \mathrm{Vol}(K^\circ) q^{\frac{n}{q}} \Gamma(n/q + 1)$ as $t \to 0$. Next, we will compute the derivative of $B(t)$ (again, using differentiation under the integral sign, justified by [AlB], Theorem 20.4). We derive

$$B'(0) = \int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \left[-\frac{h^2(x)}{2q} + n - \log \int e^{-\frac{h(y)}{q}} dy + \log\left(\mathrm{Vol}(K^\circ) c'_{n,q}\right)\right] dx.$$

Since $A(0) = B(0)$ and $A(t) \geq B(t)$ for all t, we conclude that $A'(0) \geq B'(0)$. Thus,

$$\int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \left(\frac{h(x)}{q} + \frac{\|\nabla h(x)\|^p}{pq^p} \right) dx \geq \int_{\mathbb{R}^n} e^{-\frac{h(x)}{q}} \log \frac{\tilde{c}_{n,q} \mathrm{Vol}(K^\circ)}{\int e^{-\frac{h(y)}{q}} dy} dx$$

where $\tilde{c}_{n,q} = e^n c'_{n,q} = (eq^{1/q})^n \Gamma(n/q+1)$, with equality for $h(x) = \|x\|_*^q$. Now, introduce $\tilde{F}(x) = e^{-\frac{h(x)}{pq}}$. Then we get

$$p^{p-1} \int \|\nabla \tilde{F}(x)\|^p dx \geq \int \tilde{F}^p(x) \log \frac{\tilde{c}_{n,q} \mathrm{Vol}(K^\circ) \tilde{F}^p(x)}{\int \tilde{F}^p(y) dy} dx, \qquad (40)$$

and equality holds for $\tilde{F}(x) = e^{-\frac{\|x\|_*^q}{pq}}$. Our final manipulation is setting $\tilde{F}(x) = F(x/p^{1/q})$. We obtain, after a simple change of variables,

$$\int \|\nabla F(x)\|^p dx \geq \int F^p(x) \log \frac{c_{n,q} \mathrm{Vol}(K^\circ) F^p(x)}{\int F^p(y) dy} dx$$

where $c_{n,q} = p^{-n/q} \tilde{c}_{n,q} = (e(q/p)^{1/q})^n \Gamma(1+n/q)$. Equality holds for $F(x) = e^{-\|x\|_*^q/q}$. Note that if F is smooth, decays fast enough at infinity, and the gradient of F does not vanish for $x \neq 0$, then the integrability assumption (37) on $h(x) = -c_1 \log F(c_2 x)$ automatically holds. This implies inequality (23) for a class of functions F that is dense in $W^{1,p}(\mathbb{R}^n)$. A standard approximation argument entails the conclusion of the theorem for any function F with $\int |F|^p, \int |\nabla F|^p < \infty$. This ends the case $p > 1$. The case $p = 1$ of inequality (23) is obtained by continuity, with the sharp constant $e^n = \lim_{q \to \infty} (e(\frac{q}{p})^{1/q})^n \Gamma(1 + \frac{n}{q})$. This concludes the proof. □

Next we present the equivalence of Theorem 3.1 and the inequalities proven by Gentil [G] and by Agueh, Ghoussoub and Kang [AGK]. Note that our formulation is indeed equivalent to that in [G, AGK], since a convex function that is homogenous of degree p, is necessarily $\|x\|^p$ for some norm $\|\cdot\|$, which is not necessarily a symmetric norm.

Corollary 3.5. *Let $K \subset \mathbb{R}^n$ be a convex set with the origin in its interior. Let $\|\cdot\|$ be the (possibly non-symmetric) norm for which K is its unit ball. Let $1 \leq p < \infty$, and let $F : \mathbb{R}^n \to [0, \infty)$ be a smooth function with $\int F^p(x) dx = 1$. Then,*

$$\int F^p(x) \log F^p(x) dx + \log \left[c_{n,p} \mathrm{Vol}(K^\circ) \right] \leq n \log \left(\int \|\nabla F(x)\|^p dx \right)^{\frac{1}{p}}$$

where $c_{n,p} = [(eq)^{\frac{n}{q}} n^{\frac{n}{p}} \Gamma(\frac{n}{q}+1)]/p^n$ and $\frac{1}{p} + \frac{1}{q} = 1$ (the constant $c_{n,1} = n^n$ is interpreted by continuity). If $p > 1$, equality holds for $F(x) = \alpha e^{-\beta \|x\|_*^q}$, where $\|\cdot\|_*$ is the dual norm, and $\alpha, \beta > 0$ are such that $\int F^p(x) dx = 1$. The constant is also optimal for $p = 1$.

Proof. The argument is standard. For any $t > 0$, let $G_t(x) = F(tx)$. Applying Theorem 3.1 for the function G_t, we obtain

$$t^p \int \|\nabla F(x)\|^p dx \geq \int F^p(x) \log \frac{cF^p(x)}{\int F^p(y)dy} dx + (n \log t) \int F^p(x)dx.$$

Optimizing over t, we set $t = (\frac{n \int F^p(x)dx}{p \int \|\nabla F(x)\|^p dx})^{\frac{1}{p}}$. Thus,

$$\frac{n}{p} \int F^p(x)dx \\ \geq \int F^p(x) \log \frac{cF^p(x)}{\int F^p(y)dy} dx + \frac{n}{p} \int F^p(x)dx \cdot \log \frac{n \int F^p(x)dx}{p \int \|\nabla F(x)\|^p dx}.$$

Recall that $\int F^p(x) = 1$. We conclude that

$$\frac{n}{p} \log \int \|\nabla F(x)\|^p dx \geq \int F^p(x) \log F^p(x)dx + \log c + \frac{n}{p}\Big(\log \frac{n}{p} - 1\Big). \quad \square$$

4 Santaló's Inequality

Let $K \subset \mathbb{R}^n$ be a compact set. Santaló's inequality (see, e.g. [MeP]) states that for some $x_0 \in \mathbb{R}^n$, and $\tilde{K} = K - x_0$ we have

$$\mathrm{Vol}_n(\tilde{K})\mathrm{Vol}_n(\tilde{K}^\circ) \leq \mathrm{Vol}_n(D)^2 \tag{41}$$

where, as before, $\tilde{K}^\circ = \{x \in \mathbb{R}^n; \forall y \in \tilde{K},\ \langle x, y\rangle \leq 1\}$ is the polar body and $D \subset \mathbb{R}^n$ is the Euclidean unit ball. Inequality (1), which is a functional version of Santaló's inequality, was proven in [ArtKM] by taking marginals of both sides in (41). See also [B1, FM]. Here, for simplicity, we focus attention on the case where the functions involved are even, as in [B1]. Recall that the standard Legendre transform of a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined by (e.g. [Ar])

$$\mathcal{L}\varphi(x) = \sup_{y \in \mathbb{R}^n} \big[\langle x, y\rangle - \varphi(y)\big]. \tag{42}$$

For a convex continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ we have $\mathcal{L}\mathcal{L}\varphi = \varphi$. Note that the only function for which $\mathcal{L}\varphi = \varphi$ is $\varphi(x) = |x|^2/2$. In the case of even functions, inequality (1) reads as follows:

Proposition 4.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an even, measurable function such that $0 < \int e^{-\varphi} < \infty$. Then,*

$$\int_{\mathbb{R}^n} e^{-\varphi} dx \int_{\mathbb{R}^n} e^{-\mathcal{L}\varphi} dx \leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} dx\right)^2$$

with equality iff φ is a.e. a positive definite quadratic form.

The inequality in Proposition 4.1 is due to K. Ball [B1], and the equality case was settled in [ArtKM]. One advantage of switching from geometric inequalities to analytic ones, is the availability of a new arsenal of analytic techniques. This will be demonstrated in this section, where we apply the results of Brenier, McCann and Caffarelli to Proposition 4.1.

We begin with standard definitions. A measure μ on $\mathbb{R}^n$ is a logarithmically concave measure (log-concave for short) if for any compact sets $A, B \subset \mathbb{R}^n$ and $0 < \lambda < 1$,

$$\mu\big(\lambda A + (1-\lambda)B\big) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}. \tag{43}$$

The Lebesgue measure on $\mathbb{R}^n$ is a log-concave measure, as follows from the Brunn–Minkowski inequality (15). Given a function $f : \mathbb{R}^n \to [0,\infty)$, we say that f is a log-concave function if

$$f\big(\lambda x + (1-\lambda)y\big) \geq f(x)^{\lambda} f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n, 0 < \lambda < 1$. The notions of a log-concave function and a log-concave measure are closely related. Borell showed in [Bor] that if μ is a measure on $\mathbb{R}^n$ whose support is not contained in any affine hyperplane, then μ is a log-concave measure if and only if μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$, and the density of μ is a log-concave function. In this section we will prove the following:

Theorem 4.2. *Let $f : \mathbb{R}^n \to (-\infty,\infty]$ be an even measurable function, let $\alpha > 0$, and assume that μ is an even log-concave measure on $\mathbb{R}^n$. Then,*

$$\int_{\mathbb{R}^n} e^{-\alpha f} d\mu \int_{\mathbb{R}^n} e^{-\alpha \mathcal{L} f} d\mu \leq \left(\int_{\mathbb{R}^n} e^{-\alpha \frac{|x|^2}{2}} d\mu \right)^2 \tag{44}$$

whenever at least one of the integrals on the left-hand side is both finite and non-zero.

We recently learned that Theorem 4.2 was also proven independently, using the same method as ours, by Barthe and Cordero–Erausquin [Ba2]. Given two Borel probability measures μ_1, μ_2 on $\mathbb{R}^n$ and a Borel map $T : \mathbb{R}^n \to \mathbb{R}^n$ we say that T transports μ_1 to μ_2 (or pushes forward μ_1 to μ_2) if for any Borel set $A \subset \mathbb{R}^n$,

$$\mu_2(A) = \mu_1\big(T^{-1}(A)\big).$$

Equivalently, for any compactly-supported, bounded, measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \varphi(x) d\mu_2(x) = \int_{\mathbb{R}^n} \varphi(Tx) d\mu_1(x).$$

Brenier's theorem [Bre], as refined by McCann [Mc1], is the following:

Theorem 4.3. *Let μ_1 and μ_2 be two probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to the standard Lebesgue measure. Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ such that $T = \nabla F$ exists μ_1-almost everywhere, and T transports μ_1 to μ_2. Moreover, the map T, called "Brenier map", is uniquely determined μ_1-almost everywhere.*

A corollary of the uniqueness part in Theorem 4.3 is that if both μ_1 and μ_2 are invariant under a linear map $L \in GL_n(\mathbb{R})$, then T is also invariant under L. Recall that we denote by γ_n the standard gaussian probability measure on $\mathbb{R}^n$, i.e. $\frac{d\gamma_n}{dx} = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$. For the case where $\mu_1 = \gamma_n$ and $\frac{d\mu_2}{d\gamma_n}$ is a log-concave function, the following useful result was proven by Caffarelli [C]:

Theorem 4.4. *Let μ be a probability measure on $\mathbb{R}^n$ such that $\psi = \frac{d\mu}{d\gamma_n}$ exists, and is a log-concave function. Let T be the Brenier map that transports γ_n to μ. Then T is a non-expansive map, i.e. $|Tx - Ty| \le |x - y|$ for all $x, y \in \mathbb{R}^n$.*

The following simple lemma demonstrates a certain relation between Legendre transform and non-expansive maps.

Lemma 4.5. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a function. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a non-expansive map. Denote*

$$a(x) = f(Tx) + \frac{|x|^2 - |Tx|^2}{2}, \quad b(x) = \mathcal{L}f(Tx) + \frac{|x|^2 - |Tx|^2}{2}.$$

Then, for any $x \in \mathbb{R}^n$,

$$b(x) \ge \mathcal{L}a(x).$$

Proof. By (42), for any $x, y \in \mathbb{R}^n$,

$$f(Tx) + \mathcal{L}f(Ty) \ge \langle Tx, Ty \rangle.$$

Hence, for all $x, y \in \mathbb{R}^n$,

$$f(Tx) - \frac{|Tx|^2}{2} + \mathcal{L}f(Ty) - \frac{|Ty|^2}{2} \ge -\frac{|Tx - Ty|^2}{2} \ge -\frac{|x - y|^2}{2}$$

as T is a non-expansive map. We conclude that

$$a(x) + b(y) = f(Tx) + \frac{|x|^2 - |Tx|^2}{2} + \mathcal{L}f(Ty) + \frac{|y|^2 - |Ty|^2}{2} \ge \langle x, y \rangle.$$

The definition (42) implies that $b \ge \mathcal{L}a$ (and also that $a \ge \mathcal{L}b$). □

Proof of Theorem 4.2. First consider the case $\alpha = 1$. We may clearly assume that the support of μ is n-dimensional (otherwise, we may pass to a subspace of a lower dimension). By Borell's theorem, $\psi := \frac{d\mu}{dx}$ exists and is a log-concave function. Let $d\nu(x) = \frac{1}{\kappa}\psi(x)e^{-|x|^2/2}dx$ where $\kappa = \int \psi(x)e^{-|x|^2/2}dx$. Then,

$$\int_{\mathbb{R}^n} e^{-f} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu = \kappa^2 \int_{\mathbb{R}^n} e^{\frac{|x|^2}{2} - f(x)} d\nu(x) \int_{\mathbb{R}^n} e^{\frac{|x|^2}{2} - \mathcal{L}f(x)} d\nu(x).$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the Brenier map that transports the probability measure γ_n to the probability measure ν. Then,

$$\int_{\mathbb{R}^n} e^{-f} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu = \kappa^2 \int_{\mathbb{R}^n} e^{\frac{|Tx|^2}{2} - f(Tx)} d\gamma_n(x) \int_{\mathbb{R}^n} e^{\frac{|Tx|^2}{2} - \mathcal{L}f(Tx)} d\gamma_n(x)$$
$$= \frac{\kappa^2}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\Big(\frac{|Tx|^2 - |x|^2}{2} - f(Tx) \Big) dx$$
$$\cdot \int_{\mathbb{R}^n} \exp\Big(\frac{|Tx|^2 - |x|^2}{2} - \mathcal{L}f(Tx) \Big) dx.$$

Denote $g(x) = f(Tx) + \frac{|x|^2 - |Tx|^2}{2}$ and $h(x) = \mathcal{L}f(Tx) + \frac{|x|^2 - |Tx|^2}{2}$. Note that from Theorem 4.4, we know that T is a non-expansive map. Lemma 4.5 implies that $h \geq \mathcal{L}g$. Furthermore, since ψ is even, by the uniqueness of the Brenier map (Theorem 4.3) we also know that T is an even map. Hence h and g are even functions. Assume that $0 < \int_{\mathbb{R}^n} e^{-f} d\mu = \frac{\kappa}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-g} < \infty$. Proposition 4.1 implies that

$$\frac{\kappa^2}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-g} \int_{\mathbb{R}^n} e^{-h} \leq \frac{\kappa^2}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-g} \int_{\mathbb{R}^n} e^{-\mathcal{L}g} \leq \kappa^2$$

and the theorem follows for $\alpha = 1$. If $0 < \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu = \frac{\kappa}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-h} < \infty$ we repeat the last argument with h in place of g (note that $g \geq \mathcal{L}h$). This ends the case $\alpha = 1$.

For the general case, let μ_α be the measure defined by $\mu_\alpha(A) = \mu(\alpha^{-\frac{1}{2}} A)$. Note that

$$\int \varphi(x) d\mu_\alpha(x) = \int \varphi(\sqrt{\alpha} x) d\mu(x) \tag{45}$$

for any test function φ. Let $g : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary even function, and set $f(x) = \alpha g(x/\sqrt{\alpha})$. It is readily verified that $\mathcal{L}f(x) = \alpha \mathcal{L}g(x/\sqrt{\alpha})$. The measure μ_α is log-concave and even. Since f is also an even function, we conclude, from the case treated above, that

$$\int_{\mathbb{R}^n} e^{-f} d\mu_\alpha \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu_\alpha \leq \Big(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} d\mu_\alpha \Big)^2$$

whenever the integrals converge. This translates, with the help of (45), into

$$\int_{\mathbb{R}^n} e^{-f(\sqrt{\alpha} x)} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}f(\sqrt{\alpha} x)} d\mu \leq \Big(\int_{\mathbb{R}^n} e^{-\frac{\alpha |x|^2}{2}} d\mu \Big)^2.$$

According to the definition of f we get that

$$\int_{\mathbb{R}^n} e^{-\alpha g} d\mu \int_{\mathbb{R}^n} e^{-\alpha \mathcal{L}g} d\mu \leq \Big(\int_{\mathbb{R}^n} e^{-\frac{\alpha |x|^2}{2}} d\mu(x) \Big)^2$$

whenever $0 < \int e^{-\alpha g} d\mu < \infty$ or $0 < \int e^{-\alpha \mathcal{L}g} d\mu < \infty$. □

Remark 4.6. For $n = 1$, the equality case in Theorem 4.2 is easily characterized: If μ is not a multiple of the Lebesgue measure on $\mathbb{R}$, then equality holds if and only if $f(x) = |x|^2/2$. If μ is a multiple of the Lebesgue measure on $\mathbb{R}$, then equality holds if and only if $f(x) = cx^2$ for some $c > 0$.

Theorem 4.2 has some interesting consequences, two of which were formulated in Section 1.

Proof of Corollary 1.1. For a centrally-symmetric convex set $A \subset \mathbb{R}^n$, we denote by $\|\cdot\|_A$ the norm whose unit ball is A. Let $d\mu = e^{-\|x\|_T^2/2}dx$, and consider the function $f(x) = \|x\|_K^2/2$. Then

$$\mathcal{L}f(x) = \frac{\|x\|_{K^\circ}^2}{2}.$$

Note that for any centrally-symmetric convex set $A \subset \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} e^{-\frac{\|x\|_A^2}{2}} dx = 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right) \mathrm{Vol}_n(A)$$

(see e.g., [P], page 11). In particular,

$$\int_{\mathbb{R}^n} e^{-\frac{\|x\|_K^2}{2}} d\mu = \int_{\mathbb{R}^n} e^{-\frac{\|x\|_K^2+\|x\|_T^2}{2}} dx = 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right) \mathrm{Vol}_n(K \cap_2 T)$$

and similar identities hold for $K^\circ \cap_2 T$ and $D \cap_2 T$. By Theorem 4.2,

$$\int e^{-\frac{\|x\|_K^2}{2}} d\mu \int e^{-\frac{\|x\|_{K^\circ}^2}{2}} d\mu \le \left(\int e^{-\frac{|x|^2}{2}} d\mu\right)^2.$$

We conclude that

$$\mathrm{Vol}_n(K \cap_2 T)\mathrm{Vol}_n(K^\circ \cap_2 T) \le \mathrm{Vol}_n(D \cap_2 T)^2$$

and (2) is proven. □

Proof of Corollary 1.2. Introduce $d\nu = e^{-\psi}dx$ and note that ν is even and log-concave. Then, for an arbitrary centrally-symmetric convex set $K \subset \mathbb{R}^n$,

$$\begin{aligned}\int e^{-\alpha\frac{\|x\|_K^2}{2}} d\nu &= \int_0^\infty \alpha t e^{-\frac{\alpha t^2}{2}} \nu(tK)dt \\ &= \alpha \int_0^\infty t e^{-\frac{\alpha t^2}{2}} \int_{tK} e^{-\psi(x)} dx dt \\ &= \alpha \int_0^\infty \int_K t^{n+1} e^{-\frac{\alpha t^2}{2}} e^{-\psi(tx)} dx dt = \alpha\mu(K)\end{aligned}$$

(everything is positive, so we may interchange the order of integration). Therefore, the inequality

$$\int_{\mathbb{R}^n} e^{-\frac{\alpha\|x\|_K^2}{2}} d\nu \int_{\mathbb{R}^n} e^{-\frac{\alpha\|x\|_{K^\circ}^2}{2}} d\nu \le \left(\int_{\mathbb{R}^n} e^{-\frac{\alpha|x|^2}{2}} d\nu\right)^2$$

of Theorem 4.2 translates to

$$\alpha^2\mu(K)\mu(K^\circ) \le \alpha^2\mu(D)^2.$$

This concludes the proof. □

Remarks.

1. Assume that μ is an even, log-concave measure, whose density is $F(x)$. Assume further that $F(x) = F(x_1, ..., x_n)$ actually depends only on $x_1, ..., x_{\lfloor \varepsilon n \rfloor}$, for some $0 < \varepsilon < 1$. By using techniques similar to those in [ArtKM], it is possible to show that for any centrally-symmetric convex set $K \subset \mathbb{R}^n$,
$$\mu(K)\mu(K^\circ) \le \big(1 + c(\varepsilon)\big)\mu(D)^2$$
for some function $c(\varepsilon)$ that tends to zero as $\varepsilon \to 0$. The important feature is that $c(\varepsilon)$ depends solely on ε (and not on the dimension n).
2. What is the class of measures μ that satisfy (44), for all even measurable functions f and $\alpha > 0$? This class contains all even, log-concave measures, according to Theorem 4.2. If F is a density of a measure satisfying (44) and $\beta > 0$, then also the measure whose density is the function
$$x \mapsto \int_0^\infty t^{n+1} e^{-\beta t^2} F(tx) dt \tag{46}$$
satisfies (44), for all even functions f and $\alpha > 0$. This follows by combining the one-dimensional Prékopa–Leindler inequality with the proof of Corollary 1.2, similarly to the argument in [B1] (see also [ArtKM, Theorem 2.1]). We omit the details. We conclude that the class of densities of measures μ that satisfy (44) is closed under the transform (46).

5 Mixed Volumes

5.1 The V Functional

As observed by Minkowski (see, e.g., [S]), for any compact, convex sets $K_1, ..., K_N \subset \mathbb{R}^n$, the function
$$(\lambda_1, ..., \lambda_N) \mapsto Vol_n\bigg(\sum_{i=1}^N \lambda_i K_i\bigg),$$
defined for $\lambda_1, ..., \lambda_N > 0$, is a homogeneous polynomial of degree $n+1$ in the variables $\lambda_1, ..., \lambda_N$. Minkowski concluded (see, e.g., the Appendix here) that there exists a unique symmetric multilinear n-form V defined on the space of compact, convex sets in $\mathbb{R}^n$ such that
$$\mathrm{Vol}(K) = V(K, ..., K)$$
for any compact, convex set $K \subset \mathbb{R}^n$. The symmetry and multilinearity mean that

1. For any compact, convex sets $A, B, K_2, ..., K_n \subset \mathbb{R}^n$ and $\lambda, \mu > 0$,
$$V(\lambda A + \mu B, K_2, ..., K_n) = \lambda V(A, K_2, ..., K_n) + \mu V(B, K_2, ...K_n).$$

2. For any compact, convex sets $K_1, ..., K_n \subset \mathbb{R}^n$ and a permutation $\sigma \in S_n$,

$$V(K_1, ...K_n) = V(K_{\sigma(1)}, ..., K_{\sigma(n)}).$$

We say that $V(K_1, ..., K_n)$ is the mixed volume of $K_1, ..., K_n$. The mixed volume $V(K_1, ..., K_n)$ depends continuously on the convex sets $K_1, ..., K_n$, with respect to the Hausdorff metric on the space of convex sets. Two fundamental properties of mixed volumes of convex bodies are:

1. $K_1 \subset T_1, ..., K_n \subset T_n$ imply that $0 \leq V(K_1, ..., K_n) \leq V(T_1, ..., T_n)$.
2. Alexandrov–Fenchel inequalities:

$$V(C, T, K_1, ..., K_{n-2})^2 \geq V(C, C, K_1, ..., K_{n-2})V(T, T, K_1, ..., K_{n-2})$$

for any compact, convex sets $C, T, K_1, ..., K_{n-2} \subset \mathbb{R}^n$.

Functional analogs of mixed volumes of convex bodies will be considered here. We will restrict ourselves to 1-concave functions, as the formulae are simpler in this case. Part of our discussion generalizes directly to the s-concave case, with an integer s. For any 1-concave functions $f_1, ..., f_N : \mathbb{R}^n \to \mathbb{R}$, the function

$$(\lambda_1, ..., \lambda_N) \mapsto \int \left[(\lambda_1 \times_1 f_1) \oplus_1 ... \oplus_1 (\lambda_N \times_1 f_N)\right] = \mathrm{Vol}_{n+1}\left(\sum_{i=1}^{N} \lambda_i \mathcal{K}_{f_i}\right),$$

defined for $\lambda_1, ..., \lambda_N > 0$, is a homogeneous polynomial of degree n in the variables $\lambda_1, ..., \lambda_N$. This follows from Minkowski's theorem (recall that 1-concave functions have compact support, hence the integral is always finite). Therefore there exists a unique symmetric multilinear $(n+1)$-form V defined on the space of 1-concave functions on $\mathbb{R}^n$ that satisfies the following:

1. For any 1-concave functions $f_0, ..., f_n : \mathbb{R}^n \to [0, \infty)$, and any permutation $\sigma \in S_{n+1}$,
$$V(f_0, ..., f_n) = V(f_{\sigma(0)}, ..., f_{\sigma(n)}).$$
2. For any 1-concave functions $f, g, h_1, ..., h_n : \mathbb{R}^n \to [0, \infty)$ and $\lambda, \mu > 0$,
$$V((\lambda \times_1 f) \oplus_1 (\mu \times_1 g), h_1, ..., h_n) = \lambda V(f, h_1, ..., h_n) + \mu V(g, h_1, ..., h_n).$$
3. For any 1-concave function $f : \mathbb{R}^n \to [0, \infty)$,
$$V(f, ..., f) = \int_{\mathbb{R}^n} f(x)dx.$$
4. If $f_0 \leq g_0, ..., f_n \leq g_n$ are all 1-concave functions, then
$$0 \leq V(f_0, ..., f_n) \leq V(g_0, ..., g_n).$$

5. For any 1-concave functions $f, g, h_2, ..., h_n$,

$$V(f, g, h_2, ..., h_n)^2 \geq V(f, f, h_2, ..., h_n) V(g, g, h_2, ..., h_n).$$

The proof of these five properties is a direct application of the known properties of Minkowski's mixed volumes and our definitions (11), (12), (13) and (14). We will see below that the multilinear form V satisfies the conclusions of Theorem 1.3 (and agrees with the definition given in the formulation of Theorem 1.3).

Mixed volumes of convex bodies are continuous with respect to the Hausdorff metric. We thus conclude that $V(f_0, ..., f_n)$ is continuous with respect to uniform convergence in the functions $f_0, ..., f_n$. Indeed, if $f, f^1, f^2, ... : \mathbb{R}^n \to [0, \infty)$ are 1-concave functions such that $f^m \to f$ uniformly in $\mathbb{R}^n$, then $\mathcal{K}_{f^m} \to \mathcal{K}_f$ in the Hausdorff metric. Actually, arguing as in Theorem 10.8 from [Ro], it is not very difficult to see that if $f^m \to f$ pointwise in $\mathbb{R}^n$, then $\mathcal{K}_{f^m} \to \mathcal{K}_f$ in the Hausdorff metric. We thus conclude that V satisfies property 1 from Theorem 1.3. The next lemma is standard in convex analysis, and follows e.g. from Theorem 1.1 in [CoH1]. We omit the details.

Lemma 5.1. *Let $f, f_1, f_2, ... : \mathbb{R}^n \to [0, \infty)$ be continuous, 1-concave functions. Assume that $f_k \to f$ uniformly in $\mathbb{R}^n$ when $k \to \infty$. Then, for any continuous, non-negative function $\varphi : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} \varphi\big(\nabla f_k(x)\big) dx \stackrel{k \to \infty}{\longrightarrow} \int_{\mathbb{R}^n} \varphi\big(\nabla f(x)\big) dx$$

(since the functions are concave on their support, the gradient exists a.e. and so the integrals are well-defined).

The next lemma is a minor modification of Lemma 3.4 (for the case $s = 1$).

Lemma 5.2. *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be continuous, 1-concave functions. Then,*

$$V(f, ..., f, g) = \frac{1}{n+1} \int_{\mathrm{Supp}(f)} \mathcal{L}' g\big(\nabla f(x)\big) dx. \tag{47}$$

Proof. Since $\mathcal{L}' g$ is a non-negative continuous function, both sides in (47) are continuous in f with respect to uniform convergence, according to Lemma 5.1. By approximation, we may assume that f equals, on its support, to the minimum of finitely many affine functionals; Indeed, the set of functions of this form is dense among continuous 1-concave functions, in the topology of uniform convergence. Thus, we may suppose that

$$\mathrm{Supp}(f) = \bigcup_{i=1}^N A_i, \quad i \neq j \ \Rightarrow \ A_i \cap A_j = \emptyset,$$

for some convex sets $A_1, ..., A_N$, and that for $x \in A_i$ we have $f(x) = \langle x, \theta_i \rangle + c_i$. Let $R = \max_{x \in \mathrm{Supp}(g)} |x|$. If $x \in A_i$ and $d(x, A_j) > R\varepsilon$ for all $j \neq i$, then

$$\begin{aligned}\big[f \oplus_1 (\varepsilon \times_1 g)\big](x) &= \sup_{\substack{y \in \mathrm{Supp}(f), z \in \mathrm{Supp}(g) \\ y + \varepsilon z = x}} \big[f(y) + \varepsilon g(z)\big] \\ &= \sup_{z \in \mathrm{Supp}(g)} \big[\langle x - \varepsilon z, \theta_i\rangle + c_i + \varepsilon g(z)\big]\end{aligned}$$

as $z \in \mathrm{Supp}(g)$ implies that $y = x - \varepsilon z \in A_i \subset \mathrm{Supp}(f)$. Hence,

$$\big[f \oplus_1 (\varepsilon \times_1 g)\big](x) = f(x) + \varepsilon \sup_{z \in \mathrm{Supp}(g)} \big[g(z) - \langle z, \theta_i\rangle\big] = f(x) + \varepsilon \mathcal{L}' g(\theta_i).$$

Denote $B_\varepsilon = \{x \in \mathbb{R}^n; \exists i = 1, ..., N; d(x, \partial A_i) < R\varepsilon\}$. Then $\mathrm{Vol}_n(B_\varepsilon) \le C\varepsilon$, for some $C > 0$ independent of ε, and

$$\int_{\mathrm{Supp}(f) \setminus B_\varepsilon} \big[f \oplus_1 (\varepsilon \times_1 g)\big](x) dx = \int_{\mathrm{Supp}(f) \setminus B_\varepsilon} f(x) + \varepsilon \mathcal{L}' g\big(\nabla f(x)\big) dx.$$

Let $\omega(\delta)$ be the modulus of continuity of f, and let $M = \sup g$. Then,

$$\begin{aligned}&\left| \int_{B_\varepsilon} \big[f \oplus_1 (\varepsilon \times_1 g)\big](x) dx - \int_{B_\varepsilon} f(x) dx \right| \\ &= \left| \int_{B_\varepsilon} \sup_{z \in \mathrm{Supp}(g)} \big[f(x - \varepsilon z) - f(x) + \varepsilon g(z)\big] dx \right| \\ &\le \mathrm{Vol}_n(B_\varepsilon)\big(\omega(R\varepsilon) + \varepsilon M\big).\end{aligned}$$

Note that $\mathrm{Supp}(f \oplus_1 (\varepsilon \times_1 g)) \subset \mathrm{Supp}(f) \cup B_\varepsilon$. Consequently,

$$\begin{aligned}&\left| \int_{\mathbb{R}^n} \big[f \oplus_1 (\varepsilon \times_1 g)\big](x) dx - \int_{\mathbb{R}^n} f(x) - \varepsilon \int_{\mathrm{Supp}(f) \setminus B_\varepsilon} \mathcal{L}' g\big(\nabla f(x)\big) dx \right| \\ &\qquad < \mathrm{Vol}_n(B_\varepsilon)\big(\omega(R\varepsilon) + \varepsilon M\big).\end{aligned}$$

Since $\omega(R\varepsilon) \to 0$ as $\varepsilon \to 0$, we conclude that

$$\frac{1}{\varepsilon}\left[\int_{\mathbb{R}^n} f \oplus_1 (\varepsilon \times_1 g) - \int_{\mathbb{R}^n} f\right] \overset{\varepsilon \to 0}{\longrightarrow} \int_{\mathrm{Supp}(f)} \mathcal{L}' g\big(\nabla f(x)\big). \tag{48}$$

By linearity and symmetry of V,

$$\begin{aligned}\int_{\mathbb{R}^n} f \oplus_1 (\varepsilon \times_1 g) &= V\big(f \oplus_1 (\varepsilon \times_1 g), ..., f \oplus_1 (\varepsilon \times_1 g)\big) \\ &= \left(\int_{\mathbb{R}^n} f\right) + (n+1)\varepsilon V(f, ..., f, g) + O(\varepsilon^2).\end{aligned} \tag{49}$$

The lemma follows from (48) and (49). □

We have proven almost all of the properties of V that were announced in Theorem 1.3. In fact, all that remains is to show that our definition of V agrees with the one given in Theorem 1.3.

Lemma 5.3. *Let $f_0, ..., f_n$ be continuous, 1-concave functions on $\mathbb{R}^n$. Assume that $\mathcal{L}' f_1, ..., \mathcal{L}' f_n$ have continuous second derivatives. Then,*

$$V(f_0, ..., f_n) = \frac{1}{n+1} \int_{\mathbb{R}^n} \mathcal{L}' f_0(y) D\big(\mathrm{Hess}\mathcal{L}' f_1(y), ..., \mathrm{Hess}\mathcal{L}' f_n(y)\big) dy$$

where D is the mixed discriminant.

Proof. By Lemma 5.2, for any continuous 1-concave functions f, g, we have

$$\begin{aligned} V(f, ..., f, g) &= \frac{1}{n+1} \int_{\mathrm{Supp}(f)} \mathcal{L}' g(\nabla f) \\ &= \frac{1}{n+1} \int_{Im(\nabla f)} \mathcal{L}' g(y) \det \mathrm{Hess}\big(\mathcal{L}' f(y)\big) dy \qquad (50) \end{aligned}$$

where we have used the following standard change of variables: We set $y = \nabla f(x)$ and so $x = -\nabla \mathcal{L}' f(y)$. Note that

$$y \notin Im(\nabla f) = \big\{\nabla f(z); z \in \mathrm{Supp}(f)\big\} \quad \Rightarrow \quad \mathcal{L}' f(y) = \sup_{x \in \mathrm{Supp}(f)} \langle y, -x \rangle.$$

Hence $\mathcal{L}' f$ equals the support function of the convex set $-\mathrm{Supp}(f)$ on the complement of $\overline{Im(\nabla f)}$. We conclude that if $y \notin \overline{Im(\nabla f)}$, then $\mathcal{L}' f(ty) = t\mathcal{L}' f(y)$ for t close to 1, and hence $\det(\mathrm{Hess}\mathcal{L} f(y)) = 0$. Hence we may extend the integral in (50) and write,

$$V(f, ..., f, g) = \frac{1}{n+1} \int_{\mathbb{R}^n} \mathcal{L}' g(y) \det \mathrm{Hess}\big(\mathcal{L}' f(y)\big) dy.$$

By polarizing, we obtain

$$V(f_0, ..., f_n) = \frac{1}{n+1} \int_{\mathbb{R}^n} \mathcal{L}' f_0(y) D(\mathrm{Hess}\mathcal{L}' f_1, ..., \mathrm{Hess}\mathcal{L}' f_n) dy. \qquad \square$$

The proof of Theorem 1.3 is complete. We transfer our attention to the functional I.

5.2 The *I* Functional

The I functional was considered, using different terminology, in [CoH2] and in [TW3]. In the latter work, applications to partial differential equations were discussed. Let $K \subset \mathbb{R}^n$ be a compact, convex set. Recall that for $f_0, ..., f_n : K \to [0, \infty)$ smooth, concave functions that vanish on ∂K and that have bounded derivatives in the interior of K, we set

$$I(f_0, ..., f_n) = \int_K f_0(x) D\big(-\mathrm{Hess} f_1(x), ..., -\mathrm{Hess} f_n(x)\big) dx. \qquad (51)$$

The multilinear form I is continuous with respect to pointwise convergence of the functions $f_0, ..., f_n$. This is essentially the content of Theorem 1.1 in [CoH1]. Unlike the multilinear form V from the previous section, the extension of I to general concave functions that vanish on ∂K, but that are not assumed to have bounded derivatives, may fail to be finite (e.g. $K = [-1, 1] \subset \mathbb{R}$ and $f_0(t) = f_1(t) = \sqrt{1 - t^2}$). We therefore choose not to extend the definition of I to the class of general concave functions. Detailed explanations regarding such "Hessian measures" appear in [CoH1, CoH2, TW1, TW2, TW3].

Let us begin with establishing the symmetry of I. This symmetry is based on a certain relation between mixed discriminants and Hessians. Some readers might prefer to formulate this relation in the language of exterior forms, which is more suitable for applications of Stokes theorem (see, e.g., [Gr]). We stick to the more elementary mixed discriminants. Following the notation of [R], we define the Kronecker symbol $\delta^{i_1,...,i_k}_{j_1,...,j_k}$ to be 1 if $i_1, ..., i_k$ are distinct and are an even permutation of $j_1, ..., j_k$, to be -1 if $i_1, ..., i_k$ are distinct and are an odd permutation of $j_1, ..., j_k$, and to be zero otherwise. $[A]^i_j$ denotes the (i, j)-element of the matrix A. Then if $A_1, ..., A_n$ are $n \times n$ symmetric matrices,

$$D(A_1, ..., A_n) = \frac{1}{n!} \sum \delta^{i_1,...,i_n}_{j_1,...,j_n} [A_1]^{i_1}_{j_1} ... [A_n]^{i_n}_{j_n}$$

where the sum is over all $i_1, ..., j_n, j_1, ..., j_n \in \{1, ..., n\}$. For matrices A, B we write $\langle A, B \rangle = Tr(A^t B)$, for A^t being the transpose of A, and $Tr(A)$ standing for the trace of the matrix A. This is indeed a scalar product. We define $T(A_1, ..., A_{n-1})$ to be the unique matrix such that $\langle T(A_1, ..., A_{n-1}), B \rangle = D(A_1, ..., A_{n-1}, B)$ for any matrix B. In coordinates,

$$\big[T(A_1, ..., A_{n-1})\big]^i_j = \frac{1}{n!} \sum \delta^{i_1,...,i_{n-1},i}_{j_1,...,j_{n-1},j} [A_1]^{i_1}_{j_1} ... [A_{n-1}]^{i_{n-1}}_{j_{n-1}}$$

where the sum is over all $i_1, ..., j_{n-1}, j_1, ..., j_{n-1} \in \{1, ..., n\}$.

Given a symmetric matrix A, we denote by $[A]_i$ the i^{th} row or column of A. The next lemma was essentially noted in [R].

Lemma 5.4. *Let $f^1, ..., f^{n-1} : \mathbb{R}^n \to \mathbb{R}$ be functions with continuous third derivatives. Then for any $1 \le i \le n$,*

$$\text{div}\, \big[T\big(\text{Hess}(f^1), ..., \text{Hess}(f^{n-1})\big)\big]_i = 0,$$

or equivalently, for any fixed $u \in \mathbb{R}^n$, $\text{div}\,\big(T(\text{Hess}(f^1), ..., \text{Hess}(f^{n-1}))u\big) = 0.$

Proof. We need to prove that for any $1 \le i \le n$,

$$\sum_{j=1}^{n} \frac{\partial}{\partial j} \sum \delta^{i_1,...,i_{n-1},i}_{j_1,...,j_{n-1},j} f^1_{i_1,j_1} ... f^{n-1}_{i_{n-1},j_{n-1}} = 0.$$

We write f_j for the derivative with respect to the j^{th} variable. It is sufficient to prove that for any $1 \le i, k \le n$,

$$\sum \delta^{i_1,\dots,i_{n-1},i}_{j_1,\dots,j_{n-1},j} f^1_{i_1,j_1} \dots f^{k-1}_{i_{k-1},j_{k-1}} f^k_{i_k,j_k,j} f^{k+1}_{i_{k+1},j_{k+1}} \dots f^{n-1}_{i_{n-1},j_{n-1}} = 0 \tag{52}$$

where the sum is over $i_1, \dots, i_{n-1}, j_1, \dots, j_{n-1}, j$. Since $f_{i_k,j_k,j} = f_{i_k,j,j_k}$, then the left-hand side of (52) is equal to

$$\sum \delta^{i_1,\dots,i_{n-1},i}_{j_1,\dots,j_{n-1},j} f^1_{i_1,j_1} \dots f^{k-1}_{i_{k-1},j_{k-1}} f^k_{i_k,j,j_k} f^{k+1}_{i_{k+1},j_{k+1}} \dots f^{n-1}_{i_{n-1},j_{n-1}} \tag{53}$$

(j_k and j were switched). But since δ is reversed when we switch j and j_k, then (52) also equals the negative of the left-hand side of (53). We conclude that the sum is zero. □

We would also like to use $I(f_0, \dots, f_n)$ for non-concave functions. For any bounded, sufficiently smooth functions $f_0, \dots, f_n : K \to [0, \infty)$ with bounded first and second derivatives, we use (51) as the definition of $I(f_0, \dots, f_n)$.

Lemma 5.5. *Let $K \subset \mathbb{R}^n$ be a convex set, and let $f_0, \dots, f_n : K \to [0, \infty)$ be bounded functions that vanish on ∂K. Assume that these functions have continuous third derivatives in the interior of K, and that the first and second derivatives are bounded in the interior of K. Then, for any permutation $\sigma \in S_{n+1}$*

$$I(f_0, \dots, f_n) = I(f_{\sigma(0)}, \dots, f_{\sigma(n)}).$$

Moreover,

$$I(f_0, \dots, f_n) = \int_K D\big(-\mathrm{Hess}(f_2), \dots, -\mathrm{Hess}(f_n), \nabla f_0 \otimes \nabla f_1\big).$$

Proof. Since mixed discriminant is symmetric, clearly

$$I(f_0, f_1, \dots, f_n) = I(f_0, f_{\sigma(1)}, \dots, f_{\sigma(n)})$$

for any permutation σ of $\{1, \dots, n\}$. Thus it suffices to show that

$$I(f, g, h_2, \dots, h_n) = I(g, f, h_2, \dots, h_n)$$

for any bounded functions $f, g, h_2, \dots, h_n : K \to [0, \infty)$, that vanish on ∂K, have continuous third derivatives in the interior of K, and whose first and second derivatives are bounded in the interior of K. Abbreviate $T = T(-\mathrm{Hess}(h_2), \dots, -\mathrm{Hess}(h_n))$. Fix $1 \le i \le n$. By Stokes Theorem,

$$0 = \int_{\partial K} g f_i \langle [T]_i, \nu_x \rangle dx = \int_K \mathrm{div}\big(f_i g [T]_i\big) \tag{54}$$

where ν_x is the outer unit normal to ∂K at x. The use of Stokes theorem here is legitimate: To see this, take a sequence of domains $K_\delta \subset K$ with $K_\delta \to K$. In K_δ we may clearly apply Stokes theorem. By our assumptions,

$[T]_i, f_i$ are bounded on K, and hence $(gf_i[T]_i)(x) \to 0$ uniformly as $x \to \partial K$. This justifies (54). We conclude that

$$0 = \int_K f_i g \operatorname{div}([T]_i) + f_i\langle \nabla g, [T]_i\rangle + g\langle \nabla f_i, [T]_i\rangle.$$

By Lemma 5.4, $\operatorname{div}([T]_i) = 0$, and summing for all $1 \le i \le n$,

$$\int_K \sum_{i=1}^n f_i\langle \nabla g, [T]_i\rangle + \int_K \sum_{i=1}^n g\langle \nabla f_i, [T]_i\rangle = 0.$$

By the definitions of I and T,

$$I(g, f, h_2, ..., h_n) = \int_K \sum_{i=1}^n \langle [T]_i, -\nabla f_i\rangle\, g(y)dy = \int_K \sum_{i=1}^n f_i\langle \nabla g, [T]_i\rangle.$$

We conclude that

$$\begin{aligned} &I(g, f, h_2, ..., h_n) \\ &= \int_K \langle T, \nabla f \otimes \nabla g\rangle = \int_K D\big(-\operatorname{Hess}(h_2), ..., -\operatorname{Hess}(h_n), \nabla f \otimes \nabla g\big). \end{aligned}$$

Since $\operatorname{Hess}(h_i)$ is a symmetric matrix for $i = 2, ..., n$ and $(\nabla f \otimes \nabla g)^t = \nabla g \otimes \nabla f$, by (56) from the Appendix, we conclude that

$$\begin{aligned} D\big(-\operatorname{Hess}(h_2), ..., -\operatorname{Hess}(h_n), \nabla f \otimes \nabla g\big) \\ = D\big(-\operatorname{Hess}(h_2), ..., -\operatorname{Hess}(h_n), \nabla g \otimes \nabla f\big) \end{aligned}$$

and hence I is symmetric in f and g. □

Proof of Theorem 1.4. The multilinear form I is finite, since it is the integral of a continuous function on a compact set. The continuity of I was discussed right after (51). Thus the first property in Theorem 1.4 is valid. According to Lemma 5.5, the functional $I(f_0, ..., f_n)$ is symmetric for functions $f_0, ..., f_n$ which are sufficiently smooth in the interior of K. By continuity, we obtain property 2 of Theorem 1.4. To obtain property 3, note that $-\operatorname{Hess}(f_0), ..., -\operatorname{Hess}(f_n)$ are non-negative definite matrices, and hence $D(-\operatorname{Hess}(f_0), ..., -\operatorname{Hess}(f_n)) \ge 0$. Therefore, if $f_0 \ge g_0, ..., f_n \ge g_n$, then

$$\begin{aligned} I(f_0, f_1, ..., f_n) &= \int_K f_0 D\big(-\operatorname{Hess}(f_1), ..., -\operatorname{Hess}(f_n)\big) \\ &\ge \int_K g_0 D\big(-\operatorname{Hess}(f_1), ..., -\operatorname{Hess}(f_n)\big) = I(g_0, f_1, f_2, ..., f_n) \\ &\ge I(g_0, g_1, f_2, ..., f_n) \ge ... \ge I(g_0, ..., g_n) \ge I(0, ..., 0) = 0. \end{aligned}$$

Property 3 is thus established. It remains to prove property 4. This proof is similar to the proof of the Cauchy-Schwartz inequality. It is enough to consider sufficiently smooth concave functions $f, g, h_2, ..., h_n : K \to [0, \infty)$. For $t \in \mathbb{R}$, the function $f + tg$ may fail to be concave. Nevertheless, we still have,

$$I(f+tg, f+tg, h_2, ..., h_n) = \int_K (f+tg) D\big(-\mathrm{Hess}(f+tg), -\mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n)\big).$$

According to Lemma 5.5, for any $t \in \mathbb{R}$,

$$I(f+tg, f+tg, h_2, ..., h_n) = \int_K D\big(-\mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n), \nabla(f+tg) \otimes \nabla(f+tg)\big).$$

Note that $\nabla(f+tg) \otimes \nabla(f+tg)$ is a non-negative definite matrix. Since $-\mathrm{Hess}(h_2), ..., -\mathrm{Hess}(h_n)$ are also non-negative definite, we conclude that

$$I(f+tg, f+tg, h_2, ..., h_n) = t^2 I(g, g, h_2, ..., h_n) + 2tI(f, g, h_2, ..., h_n) + I(f, f, h_2, ..., h_n) \geq 0$$

for all $t \in \mathbb{R}$. The fact that the quadratic function $I(f+tg, f+tg, h_2, ..., h_n)$ is always non-negative, entails that its discriminant is non-positive. This is exactly the content of Property 4. The proof is complete. □

6 Appendix: Mixed Discriminants

Given $p : \mathbb{R}^m \to \mathbb{R}$ a homogeneous polynomial of degree k, there exists a unique symmetric multilinear form $\tilde{p} : (\mathbb{R}^m)^k \to \mathbb{R}$ such that

$$p(x) = \tilde{p}(x, x, ..., x)$$

for any $x \in \mathbb{R}^m$. We say that $\tilde{p}$ is the polarization of p. This is proven e.g. in Appendix A in [H]. In particular, let A be an $n \times n$ matrix. Then $\det(A)$ is a homogeneous polynomial of degree n in the n^2 matrix elements. Hence, we may define the "mixed discriminant of the matrices $A_1, ..., A_n$" to be $D(A_1, ..., A_n)$, a multilinear symmetric form such that

$$\det(A) = D(A, ..., A)$$

for any matrix A. Note that by linearity,

$$\det\left(\sum_{i=1}^N \lambda_i A_i \right) = \sum_{i_1, \ldots, i_n \in \{1, \ldots, N\}} D(A_{i_1}, ..., A_{i_n}) \prod_{j=1}^n \lambda_{i_j}. \tag{55}$$

In fact, (55) is the essence of the proof of the existence of the polarization. Also, since $\det(A) = \det(A^t)$, then

$$D(A_1, ..., A_n) = D(A_1^t, ..., A_n^t). \tag{56}$$

The mixed discriminants satisfy various inequalities. We would like to mention only Alexandrov's inequality, from which it follows that the mixed discriminant of non-negative definite matrices is a non-negative number.

Lemma 6.1. *Let $A_1, ..., A_{n-2}, B, C$ be non-negative definite $n \times n$ symmetric matrices. Then,*

$$D(A_1, ..., A_{n-2}, B, C)^2 \geq D(A_1, ..., A_{n-2}, B, B) D(A_1, ..., A_{n-2}, C, C). \tag{57}$$

Sketch of Proof. (See [H], pages 63-65.) First, suppose that the matrices are positive definite. Let $p(A) = det(A) = D(A, ..., A)$. For any symmetric matrix A and a positive-definite matrix B, the polynomial in the variable t,

$$p(tB + A) = \det(B) \det\left(tId + \sqrt{B^{-1}} A \sqrt{B^{-1}}\right)$$

has only real roots, as $\sqrt{B^{-1}} A \sqrt{B^{-1}}$ is a real, symmetric matrix. By Rolle's Theorem,

$$\frac{d}{dt} p(tB + A) = nD(B, tB + A, ..., tB + A)$$

also has only real roots. The fact that $D(C, tB + A, ..., tB + A)$ has only real roots for any positive definite matrices B, C and any symmetric matrix A, follows from the general theory of hyperbolic polynomials (see e.g. Proposition 2.1.31 in [H]). We may now differentiate $D(C, tB+A, ..., tB+A)$ and so forth. By induction we conclude that

$$q(t) = D(A_1, ..., A_{n-2}, tB + C, tB + C) \tag{58}$$

has only real roots for any positive definite matrices $A_1, ..., A_{n-2}, B$ and any symmetric matrix C. Since q is a quadratic polynomial, its discriminant is non-negative, which is exactly the inequality (57). Thus (57) is proven, for the case of positive definite matrices. The inequality for non-negative definite matrices follows by continuity. □

Remark. The fact that $D(A_1, ..., A_{n-3}, tB + C, tB + C, tB + C)$ has only real roots implies the inequality

$$6a_0a_1a_2a_3 - 4a_2^3a_0 + 3a_2^2a_1^2 - 4a_3a_1^3 - a_3^2a_0^2 \geq 0$$

which holds for any non-negative definite matrices, where

$$a_i = D(A_1, ..., A_{n-3}; B, i; C, 3 - i),$$

i.e. B appears i times, C appears $3 - i$ times. See also [Ros].

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Deviation Inequalities on Largest Eigenvalues

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Summary. In these notes[1], we survey developments on the asymptotic behavior of the largest eigenvalues of random matrix and random growth models, and describe the corresponding known non-asymptotic exponential bounds. We then discuss some elementary and accessible tools from measure concentration and functional analysis to reach some of these quantitative inequalities at the correct small deviation rate of the fluctuation theorems. Results in this direction are rather fragmentary. For simplicity, we mostly restrict ourselves to Gaussian models.

Introduction

In the recent years, important developments took place in the analysis of the spectrum of large random matrices and of various random growth models. In particular, universality questions at the edge of the spectrum has been conjectured, and settled, for a number of apparently disconnected examples.

Let $X^N = (X_{ij}^N)_{1 \leq i,j \leq N}$ be a complex Hermitian matrix such that the entries on and above the diagonal are independent complex (real on the diagonal) centered Gaussian random variables with variance σ^2. Denote by $\lambda_1^N, \ldots, \lambda_N^N$ the real eigenvalues of X^N. Under the normalization $\sigma^2 = \frac{1}{4N}$ of the variance, the famous Wigner theorem indicates that the spectral measure $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i^N}$ converges to the semicircle law, supported on $(-1,+1)$. Furthermore, the largest eigenvalue $\lambda_{\max}^N$ converges almost surely to 1, the right-end point of the support of the semicircle law. As one main achievement in the recent developments of random matrix theory, it has been proved in the early nineties by P. Forrester [Fo1] and C. Tracy and H. Widom [T-W1] that the fluctuations of the largest eigenvalue are given by

$$N^{2/3}\big(\lambda_{\max}^N - 1\big) \to F$$

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where F is the so-called Tracy–Widom distribution. A similar conclusion holds for real Gaussian matrices, and the result has been extended by A. Soshnikov [So1] to classes of real or complex matrices with independent entries under suitable moment assumptions.

In the striking contribution [B-D-J], J. Baik, P. Deift and K. Johansson proved in 1999 that the Tracy–Widom distribution governs the fluctuation of an apparently completely disconnected model, namely the length of the longest increasing subsequence in a random partition. Denote indeed by L_n the length of the longest increasing subsequence in a random permutation chosen uniformly in the symmetric group over n elements. Then, as shown in [B-D-J],

$$\frac{1}{2n^{1/6}}\big(L_n - 2\sqrt{n}\,\big) \to F$$

weakly, with F the Tracy–Widom distribution. (Note that the normalization is given by the third power of the mean order $2\sqrt{n}$, as it would be the case if we replace $\lambda_{\max}^N$ by $N\lambda_{\max}^N$ in the random matrix model.)

Since then, universality of the Tracy–Widom distribution is conjectured for a number of models, and has been settled recently for some specific ones, including corner growth models, last-passage times in directed percolation, exclusion processes, Plancherel measure, random Young tableaux... For example, let $w(i,j)$, $i,j \in \mathbb{N}$, be independent exponential or geometric random variables. For $M \geq N \geq 1$, set

$$W = W(M,N) = \max_{\pi} \sum_{(i,j)\in\pi} w(i,j)$$

where the maximum runs over all up/right paths π in $\mathbb{N}^2$ from $(1,1)$ to (M,N). The random growth function W may be interpreted as a directed last-passage time in percolation. K. Johansson [Joha1] showed that, for every $c \geq 1$, up to some normalization factor,

$$\frac{1}{N^{1/3}}\Big(W\big([cN],N\big) - \omega N\Big) \to F$$

weakly, where again F is the Tracy–Widom distribution (and ω the mean parameter).

These attractive results, and the numerous recent developments around them (cf. the review papers [Baik2], [Joha4], [T-W4]...) emphasize the unusual rate $(\text{mean})^{1/3}$ and the central role of the new type of distribution F in the fluctuations of largest eigenvalues and random growth models. The analysis of these models is actually made possible by a common determinantal point process structure and asymptotics of orthogonal polynomials for which sophisticated tools from combinatorics, complex analysis, integrable systems and probability theory have been developed. This determinantal structure is also the key to the study of the spacings between the eigenvalues, a topic of major interest in the recent developments of random matrix theory which led in particular to striking conjectures in connection with the Riemann zeta function (cf. [De], [Fo2], [Fy], [Kö], [Meh]...).

In these notes, we will be concerned with the simple question of non-asymptotic exponential deviation inequalities at the correct fluctuation rate in some of the preceding limit theorems. We only concentrate on the order of growth and do not discuss the limiting distributions. For example, in the preceding setting of the largest eigenvalue $\lambda_{\max}^N$ of random matrices, we would be interested to find, for fixed $N \geq 1$ and $\varepsilon > 0$, (upper-) estimates on

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1+\varepsilon\}\big) \quad \text{and} \quad \mathbb{P}\big(\{\lambda_{\max}^N \leq 1-\varepsilon\}\big)$$

which fit the weak convergence rate towards the Tracy–Widom distribution. In a sense, this purpose is similar to the Gaussian tail inequalities for sums of independent random variables in the context of the classical central limit theorem. Several results, usually concerned with large and moderate deviation asymptotics and convergence of moments, deal with this question in the literature. However, not all of them are easily accessible, and usually require a rather heavy analysis, connected with stationary phase asymptotics of contour integrals or non-classical analytical schemes of the theory of integrable systems such as Riemann–Hilbert asymptotic methods. In any case, the conclusions so far only deal with rather restricted classes of models. For example, in the random matrix context, only (complex) Gaussian entries allow at this point for satisfactory deviation inequalities at the appropriate rate. Directed percolation models have been answered only for geometric or exponential weights.

The aim of these notes is to provide a few elementary tools, some of them of functional analytic flavour, to reach some of these deviation inequalities. (We will only be concerned with upper bounds.) A first attempt in this direction deals with the modern tools of measure concentration. Measure concentration typically produces Gaussian bounds of the type

$$\mathbb{P}\Big(\Big\{\big|\lambda_{\max}^N - \mathbb{E}(\lambda_{\max}^N)\big| \geq r\Big\}\Big) \leq C\,\mathrm{e}^{-Nr^2/C}, \quad r \geq 0,$$

for some $C > 0$ independent of N. These inequalities are rather robust and hold for large families of distributions. While they describe the correct large deviations, they however do not reflect the small deviations at the rate $(\text{mean})^{1/3}$ of the Tracy–Widom theorem. Further functional tools (if any) would thus be necessary, and such a program was actually advertised by S. Szarek in [Da-S]. We present here a few arguments of possible usefulness to this task, relying on Markov operator ideas such as hypercontractivity and integration by parts. In particular, we try to avoid saddle point analysis on Laplace integrals for orthogonal polynomials which are at the root of the asymptotic results. We however still rely on determinantal and orthogonal polynomial representations of the random matrix models. Certainly, suitable bounds on orthogonal polynomials might supply for most of what is necessary to our purpose. Our first wish was actually to try a few abstract and (hopefully) general arguments to tackle some of these questions in the hope of extending some conclusions to more general models. The various conclusions from this particular viewpoint are however far from complete, not always optimal, and do not really extend to new examples of interest. It is the hope of the future research that new tools may answer in a more satisfactory way some of these questions.

The first part describes, in the particular example of the Gaussian Unitary Ensemble, the fundamental determinantal structure of the eigenvalue distribution and the orthogonal polynomial method which allow for the fluctuation and large deviation asymptotics of the top eigenvalues of random matrix and random growth models. In the second part, we present the known exponential deviation inequalities which may be drawn from the asymptotic theory and technology. Part 3 addresses the measure concentration tools in this setting, and discusses both their usefulness and limitations. In Part 4, the tool of hypercontractivity of Markov operators is introduced to the task of deviation and variance inequalities at the Tracy–Widom rate. The last part presents some moment recurrence equations which may be obtained from integration by parts for Markov operators, and discusses their interest in deviation inequalities both above and below the limiting expected mean.

These notes are only educational and do not present any new result. They moreover focus on a very particular aspect of random matrix theory, ignoring some main developments and achievements. In particular, references are far from exhaustive. Instead, we try to refer to some general references where more complete expositions and pointers to the literature may be found. We apologize for all the omissions and inaccuracies in this respect. In connection with these notes, let us thus mention, among others, the book [Meh] by M. L. Mehta which is a classical reference on the main random matrix ensembles from the mathematical physics point of view. It contains in particular numerous formulas on the eigenvalue densities, their correlation functions etc. The very recent third edition presents in addition some of the latest developments on the asymptotic behaviors of eigenvalues of random matrices. The monograph [De] by P. Deift discusses orthogonal polynomial ensembles and presents an introduction to the Riemann–Hilbert asymptotic method. P. Forrester [Fo] extensively describes the various mathematical physics models of random matrices and their relations to integrable systems. The survey paper by Z. D. Bai [Bai] offers a complete account on the spectral analysis of large dimensional random matrices for general classes of Wigner matrices by the moment method and the Stieltjes transform. The short reviews [Joha4], [T-W4], [Baik2] provide concise presentations of some main recent achievements. The lectures [Fy] by Y. Fyodorov are an introduction to the statistical properties of eigenvalues of large random Hermitian matrices, and treat in particular the paradigmatic example of the Gaussian Unitary Ensemble (much in the spirit of these notes). Finally, the recent nice and complete survey on orthogonal polynomial ensembles by W. König [Kö] achieves an accessible and inspiring account to some of these important developments. More references may be downloaded from the preceding ones.

1 Asymptotic Behaviors

In this first part, we briefly present some basic facts about the asymptotic analysis of the largest eigenvalues of random matrix and random growth models. For simplicity, we restrict ourselves to some specific models (mostly the Hermite and Meixner Ensembles) for which complete descriptions are available. We follow the recent literature on the subject. In the particular example of the Gaussian Unitary Ensemble, we fully examine the basic determinantal structure of the correlation functions and the orthogonal polynomial method. We further discuss Coulomb gas and random growth functions, as well as large deviation asymptotics.

1.1 The Largest Eigenvalue of the Gaussian Unitary Ensemble

One main example of interest throughout these notes will be the so-called Gaussian Unitary Ensemble (GUE). This example is actually representative of a whole family of models. Consider, for each integer $N \geq 1$, $X = X^N = (X_{ij}^N)_{1 \leq i,j \leq N}$ a $N \times N$ selfadjoint centered Gaussian random matrix with variance σ^2. By this, we mean that X is a $N \times N$ Hermitian matrix such that the entries above the diagonal are independent complex (real on the diagonal) Gaussian random variables with mean zero and variance σ^2 (the real and imaginary parts are independent centered Gaussian variables with variance $\sigma^2/2$). Equivalently, the random matrix X is distributed according to the probability distribution

$$\mathbb{P}(dX) = \frac{1}{Z} \exp\big(-\operatorname{Tr}(X^2)/2\sigma^2\big)dX \tag{1.1}$$

on the space $\mathcal{H}_N \cong \mathbb{R}^{N^2}$ of $N \times N$ Hermitian matrices where

$$dX = \prod_{1 \leq i \leq N} dX_{ii} \prod_{1 \leq i < j \leq N} d\,\mathrm{Re}\,(X_{ij}) d\,\mathrm{Im}\,(X_{ij})$$

is Lebesgue measure on $\mathcal{H}_N$ and $Z = Z_N$ the normalizing constant. This probability measure is invariant under the action of the unitary group on $\mathcal{H}_N$ in the sense that UXU^* has the same law as X for each unitary element U of $\mathcal{H}_N$. The random matrix X is then said to be element of the Gaussian Unitary Ensemble (GUE) ("ensemble" for probability distribution).

The real case is known as the Gaussian Orthogonal Ensemble (GOE) defined by a real symmetric random matrix $X = X^N = (X_{ij}^N)_{1 \leq i,j \leq N}$ such that the entries X_{ij}^N, $1 \leq i \leq j \leq N$, are independent centered real-valued Gaussian random variables with variance σ^2 ($2\sigma^2$ on the diagonal). Equivalently, the distribution of X on the space $\mathcal{S}_N$ of $N \times N$ symmetric matrices is given by

$$\mathbb{P}(dX) = \frac{1}{Z} \exp\big(-\operatorname{Tr}(X^2)/4\sigma^2\big)dX \tag{1.2}$$

(where now dX is Lebesgue measure on $\mathcal{S}_N$). This distribution is invariant by the orthogonal group.

For such a symmetric or Hermitian random matrix $X = X^N$, denote by $\lambda_1^N, \ldots, \lambda_N^N$ its (real) eigenvalues.

It is a classical result due to E. Wigner [Wig] that, almost surely,

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^N} \to \nu \tag{1.3}$$

in distribution as $\sigma^2 \sim \frac{1}{4N}$, $N \to \infty$, where ν is the semicircle law with density $\frac{2}{\pi}(1-x^2)^{1/2}$ with respect to Lebesgue measure on $(-1,+1)$. This result has been extended, on the one hand, to large classes of both real (symmetric) and complex (Hermitian) random matrices with non-Gaussian independent (subject to the symmetry condition) entries, called Wigner matrices, under the variance normalization $\sigma^2 = \mathbb{E}(|X_{ij}|^2) \sim \frac{1}{4N}$, $i < j$. The basic techniques include moment methods, to show the convergence of

$$\frac{1}{N}\,\mathbb{E}\Big(\mathrm{Tr}\,\big((X^N)^p\big)\Big)$$

to the p-moment ($p \in \mathbb{N}$) of the semicircle law, or the Stieltjes transform (a kind of moment generating function) method. Another point of view on the Stieltjes transform is provided by the free probability calculus ([Vo], [V-D-N], [H-P], [Bi]...). In the particular example of the GUE, simple orthogonal polynomial properties may be used (see below). Actually, all these arguments first establish convergence of the mean spectral measure

$$\mu^N = \mathbb{E}\bigg(\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^N}\bigg). \tag{1.4}$$

This convergence has been improved to the almost sure statement (1.3) in [Ar]. We refer to the paper [Bai] by Z. D. Bai for a complete account on spectral distributions of large Wigner matrices not addressed here. On the other hand, Wigner's theorem has been extended to orthogonal or unitary invariant ensembles of the type (1.1) or (1.2) where X^2 is replaced (by the functional calculus) by $v(X)$ for some suitable function $v : \mathbb{R} \to \mathbb{R}$. The main tool in this case is the Stieltjes transform, and the limiting spectral distribution (or equilibrium measure, cf. [De], [H-P], [S-T]...) then depends on (the potential) v. The quadratic potential is the only one leading to independent entries X_{ij}, $1 \le i \le j \le N$, in the matrix X with law (1.1) or (1.2).

It is also well-known that in the GUE and GOE models, as well as in the more general setting of Wigner matrices (cf. [Bai]), under suitable moment hypotheses, the largest eigenvalue $\lambda_{\max}^N = \max_{1 \le i \le N} \lambda_i^N$ converges almost surely, as $\sigma^2 = \frac{1}{4N}$, to the right-end point of the support of the semicircle law, that is 1 in the normalization chosen here. (By symmetry, the smallest eigenvalue converges to -1. The result extends to the k-th extremal eigenvalues for every fixed k.) For the orthogonal and unitary invariant ensembles, the convergence is towards the right-end point of the compact support of the limiting spectral distribution (cf. [De]).

As one of the main recent achievements of the theory of random matrices, it has been shown by P. Forrester [Fo1] (in a mathematical physics language) and C. Tracy and H. Widom [T-W1] that the fluctuations of the largest eigenvalue $\lambda_{\max}^N$ of a GUE random matrix $X = X^N$ with $\sigma^2 = \frac{1}{4N}$ around its expected value 1 takes place at the rate $N^{2/3}$. More precisely,

$$N^{2/3}(\lambda_{\max}^N - 1) \to F_{\mathrm{GUE}} \tag{1.5}$$

weakly where F_{GUE} is the so-called (GUE) Tracy–Widom distribution. Note that the normalization $N^{2/3}$ may be somehow guessed from the Wigner theorem since, for $\varepsilon > 0$ small,

$$\mathrm{Card}\,\{1 \leq i \leq N; \lambda_i^N > 1 - \varepsilon\} \sim N\,\nu\big((1-\varepsilon, 1]\big) \sim N\varepsilon^{3/2}$$

so that for ε of the order of $N^{-2/3}$ the probability $\mathbb{P}(\{\lambda_{\max}^N \leq 1 - \varepsilon\})$ should be stabilized. The new distribution F_{GUE} occurs as a Fredholm determinant

$$F_{\mathrm{GUE}}(s) = \det\Big([\mathrm{Id} - K_{\mathrm{Ai}}]_{\mathrm{L}^2(s,\infty)}\Big), \quad s \in \mathbb{R}, \tag{1.6}$$

of the integral operator associated to the Airy kernel K_{Ai}, as a limit in this regime of the Hermite kernel using Plancherel–Rotach orthogonal polynomial asymptotics (see below). C. Tracy and H. Widom [T-W1] were actually able to provide an alternate description of this new distribution F_{GUE} in terms of some differential equation as

$$F_{\mathrm{GUE}}(s) = \exp\bigg(-\int_{2s}^{\infty}(x - 2s)u(x)^2 dx\bigg), \quad s \in \mathbb{R}, \tag{1.7}$$

where $u(x)$ is the solution of the Painlevé II equation $u'' = 2u^3 + xu$ with the asymptotics $u(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}}\,\mathrm{e}^{-\frac{2}{3}x^{3/2}}$ as $x \to \infty$. Similar conclusions hold for the Gaussian Orthogonal Ensemble (GOE) with a related limiting distribution F_{GOE} of the Tracy–Widom type [T-W2]. Random matrix theory is also concerned sometimes with quaternionic entries leading to the Gaussian Simplectic Ensemble (GSE), cf. [Meh], [T-W2]. A few characteristics of the distribution F_{GUE} are known. It is non-centered, with a mean around $-.879$, and its respective behaviors at $\pm\infty$ are given by

$$C^{-1}\,\mathrm{e}^{-Cs^3} \leq F_{\mathrm{GUE}}(-s) \leq C\,\mathrm{e}^{-s^3/C} \tag{1.8}$$

and

$$C^{-1}\,\mathrm{e}^{-Cs^{3/2}} \leq 1 - F_{\mathrm{GUE}}(s) \leq C\,\mathrm{e}^{-s^{3/2}/C} \tag{1.9}$$

for s large and C numerical (cf. e.g. [Au], [John], [L-M-R]...)

As already emphasized in the introduction, the Tracy–Widom distributions actually appeared recently in a number of apparently disconnected problems, from the length of the longest increasing subsequence in a random permutation, to corner growth models, last-passage times in oriented percolation, exclusion processes, Plancherel measure, random Young tableaux etc, cf. [Joha4], [T-W4], [Baik2], [Kö]... The Tracy–Widom distributions are conjectured to be the universal limiting laws for this type of models, with a common rate (mean)$^{1/3}$ (in contrast with the (mean)$^{1/2}$ rate of the classical central limit theorem).

The fluctuation result (1.5) has been extended by A. Soshnikov in the striking contribution [So1] to Wigner matrices $X = X^N$ with real or complex non-Gaussian independent entries with variance $\sigma^2 = \mathbb{E}(|X_{ij}|^2) = \frac{1}{4N}$ and a Gaussian control of the moments $\mathbb{E}(|X_{ij}|^{2p}) \leq (Cp)^p$, $p \in \mathbb{N}$, $1 \leq i < j \leq N$. In particular, the assumptions cover the case of matrices $X = (X_{ij}/2\sqrt{N})_{1 \leq i,j \leq N}$ where the X_{ij}'s, $i \leq j$, are independent symmetric Bernoulli variables. This is one extremely rare case so far for which universality of the Tracy–Widom distributions has been fully justified. Interestingly enough, one important aspect of Soshnikov's remarkable proof is that it is actually deduced from the GUE or GOE cases by a moment approximation argument (and not directly from the initial matrix distribution). In another direction, asymptotics of orthogonal polynomials have been deeply investigated to extend the GUE fluctuations to large classes of unitary invariant ensembles. Depending on the structure of the underlying orthogonal polynomials, the proofs can require rather deep arguments involving the steepest descent/stationary phase method for Riemann–Hilbert problems (cf. [De], [Ku], [Baik2], [B-K-ML-M]...). For a strategy based on $1/n$-expansion in unitary invariant random matrix ensembles avoiding the Riemann–Hilbert analysis, see [P-S]. Further developments are still in progress.

1.2 Determinantal Representations

The analysis of the GUE, and more general unitary invariant ensembles, is made possible by the determinantal representation of the eigenvalue distribution as a Coulomb gas and the use of orthogonal polynomials. This determinantal point process representation is the key towards the asymptotics results on eigenvalues of large random matrices, both inside the bulk (spacing between the eigenvalues) and at the edge of the spectrum. We follow below the classical literature on the subject [Meh], [De], [Fo2], [P-L]... to which we refer for further details.

Keeping the GUE example, by unitary invariance of the ensemble (1.1) and the Jacobian change of variables formula, the distribution of the eigenvalues $\lambda_1^N \leq \cdots \leq \lambda_N^N$ of $X = X^N$ on the Weyl chamber $E = \{x \in \mathbb{R}^N; x_1 < \cdots < x_N\}$ may be shown to be given by

$$\frac{1}{Z}\,\Delta_N(x)^2\prod_{i=1}^N d\mu(x_i/\sigma) \tag{1.10}$$

where

$$\Delta_N = \Delta_N(x) = \prod_{1\le i<j\le N}(x_j - x_i)$$

is the Vandermonde determinant, $d\mu(x) = \mathrm{e}^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ the standard normal distribution on $\mathbb{R}$ and $Z = Z_N$ the normalization factor. We actually extend the probability distribution (1.10) to the whole of $\mathbb{R}^N$ by symmetry under permutation of the coordinates, and thus speak, with some abuse, of the joint distribution of the eigenvalues $(\lambda_1^N, \dots, \lambda_N^N)$ as a random vector in $\mathbb{R}^N$.

It is on the basis of the representation (1.10) that the so-called orthogonal polynomial method may be developed. Denote by P_ℓ, $\ell \in \mathbb{N}$, the normalized Hermite polynomials with respect to μ, which form an orthonormal basis of $\mathrm{L}^2(\mu)$. Since, for each ℓ, P_ℓ is a polynomial function of degree ℓ, elementary manipulations on rows or columns show that the Vandermonde determinant $\Delta_N(x)$ is equal, up to a constant depending on N, to

$$D_N = D_N(x) = \det\big(P_{\ell-1}(x_k)\big)_{1\le k,\ell\le N}\,.$$

The following lemma is then a useful tool in the study of the correlation functions. It is a simple consequence of the definition of the determinant together with Fubini's theorem.

Lemma 1.1. *On some measure space $(S, \mathcal{S}, m)$, let φ_i, ψ_j, $i, j = 1, \dots, N$, be square integrable functions. Then*

$$\int_{S^N}\det\big(\varphi_i(x_j)\big)_{1\le i,j\le N}\det\big(\psi_i(x_j)\big)_{1\le i,j\le N}\prod_{k=1}^N dm(x_k) = N!\det\bigg(\int_S \varphi_i\psi_j\,dm\bigg)_{1\le i,j\le N}.$$

Replacing thus Δ_N by D_N in (1.10), a first consequence of Lemma 1.1 applied to $\varphi_i = \psi_j = P_{\ell-1}$ and $dm = \mathbf{1}_{(-\infty,t/\sigma]}d\mu$ is that, for every $t \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}\big(\{\lambda_{\max}^N \le t\}\big) &= \frac{1}{Z'}\int_{\mathbb{R}^N} D_N(x)^2\prod_{k=1}^N dm(x_k)\\ &= \det\Big(\langle P_{\ell-1}, P_{k-1}\rangle_{\mathrm{L}^2(]-\infty,t/\sigma],d\mu)}\Big)_{1\le k,\ell\le N}\\ &= \det\Big(\mathrm{Id} - \langle P_{\ell-1}, P_{k-1}\rangle_{\mathrm{L}^2((t/\sigma,\infty),d\mu)}\Big)_{1\le k,\ell\le N}\end{aligned} \tag{1.11}$$

where $Z' = \int_{\mathbb{R}^N} D_N(x)^2\prod_{k=1}^N dm(x_k)$ and $\langle\cdot\,,\cdot\rangle_{\mathrm{L}^2(A,d\mu)}$ is the scalar product in the Hilbert space $\mathrm{L}^2(A, d\mu)$, $A \subset \mathbb{R}$.

On the basis of Lemma 1.1 and the orthogonality properties of the polynomials P_ℓ, the eigenvalue vector $(\lambda_1^N, \dots, \lambda_N^N)$ may be shown to have determinantal correlation functions in terms of the (Hermite) kernel

$$K_N(x,y) = \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y), \quad x, y \in \mathbb{R}. \tag{1.12}$$

The following statement provides such a description.

Proposition 1.2. *For any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{E}\bigg(\prod_{i=1}^N \big[1 + f(\lambda_i^N)\big] \bigg) \\ = \sum_{r=0}^N \frac{1}{r!} \int_{\mathbb{R}^r} \prod_{i=1}^r f(\sigma x_i) \det \big(K_N(x_i, x_j)\big)_{1 \le i,j \le r} d\mu(x_1) \cdots d\mu(x_r).$$

Proof. Starting from the eigenvalue distribution (1.10), we have

$$\mathbb{E}\bigg(\prod_{i=1}^N \big[1 + f(\lambda_i^N)\big] \bigg) = \frac{1}{Z'} \int_{\mathbb{R}^N} \prod_{i=1}^N \big[1 + f(\sigma x_i)\big] D_N(x)^2 d\mu(x_1) \cdots d\mu(x_N)$$

where, as above, $Z' = \int_{\mathbb{R}^N} D_N(x)^2 d\mu(x_1) \cdots d\mu(x_N)$. By Lemma 1.1, $Z' = N!$ while similarly

$$\int_{\mathbb{R}^N} \prod_{i=1}^N \big[1 + f(\sigma x_i)\big] D_N(x)^2 d\mu(x_1) \cdots d\mu(x_N) \\ = N! \det \Big(\langle (1+g) P_{\ell-1}, P_{k-1} \rangle_{\mathrm{L}^2(\mu)} \Big)_{1 \le k, \ell \le N} \\ = N! \det \Big(\mathrm{Id} + \langle P_{\ell-1}, P_{k-1} \rangle_{\mathrm{L}^2(g d\mu)} \Big)_{1 \le k, \ell \le N}$$

where we set $g(x) = f(\sigma x)$, $x \in \mathbb{R}$. Hence,

$$\mathbb{E}\bigg(\prod_{i=1}^N \big[1 + f(\lambda_i^N)\big] \bigg) = \det \Big(\mathrm{Id} + \langle P_{\ell-1}, P_{k-1} \rangle_{\mathrm{L}^2(g d\mu)} \Big)_{1 \le k, \ell \le N}.$$

Now, the latter is equal to

$$\sum_{r=0}^N \frac{1}{r!} \sum_{\ell_1, \dots, \ell_r = 1}^N \det \Big(\langle P_{\ell_i - 1}, P_{\ell_j - 1} \rangle_{\mathrm{L}^2(g d\mu)} \Big)_{1 \le i,j \le r},$$

and thus, by Lemma 1.1 again, also to

$$\sum_{r=0}^{N}\frac{1}{r!}\sum_{\ell_1,\dots,\ell_r=1}^{N}\frac{1}{r!}\int_{\mathbb{R}^r}\det\big(f(\sigma x_j)P_{\ell_i-1}(x_j)\big)_{1\le i,j\le r}$$
$$\times\det\big(P_{\ell_i-1}(x_j)\big)_{1\le i,j\le r}d\mu(x_1)\cdots d\mu(x_r).$$

By the Cauchy–Binet formula, this amounts to

$$\sum_{r=0}^{N}\frac{1}{r!}\int_{\mathbb{R}^r}\prod_{i=1}^{r}f(\sigma x_i)\det\big(K_N(x_i,x_j)\big)_{1\le i,j\le r}d\mu(x_1)\cdots d\mu(x_r)$$

which is the announced claim. □

What Proposition 1.2 (more precisely its immediate extension to the computation of $\mathbb{E}\big(\prod_{i=1}^{N}[1+f_i(\lambda_i^N)]\big)$ for bounded measurable functions $f_i:\mathbb{R}\to\mathbb{R}$, $i=1,\dots,N$) puts forward is the fact that the distribution of the eigenvalues, and its marginals, are completely determined by the kernel K_N of (1.12) In particular, replacing f by εf in Proposition 1.2 and letting $\varepsilon\to 0$, the mean spectral measure μ^N of (1.4) is given, for every bounded measurable function f, by

$$\mathbb{E}\bigg(\frac{1}{N}\sum_{i=1}^{N}f(\lambda_i^N)\bigg)=\int_{\mathbb{R}}f(\sigma x)\,\frac{1}{N}\sum_{\ell=0}^{N-1}P_\ell^2d\mu. \tag{1.13}$$

Choosing $f=-\mathbf{1}_{(t,\infty)}$ in Proposition 1.2 shows at the other end that the distribution of the largest eigenvalue $\lambda_{\max}^N$ may be expressed by

$$\mathbb{P}\big(\{\lambda_{\max}^N\le t\}\big)$$
$$=\sum_{r=0}^{N}\frac{(-1)^r}{r!}\int_{(t/\sigma,\infty)^r}\det\big(K_N(x_i,x_j)\big)_{1\le i,j\le r}d\mu(x_1)\cdots d\mu(x_r),\quad t\in\mathbb{R}. \tag{1.14}$$

This identity emphasizes the distribution of the largest eigenvalue $\lambda_{\max}^N$ as the Fredholm determinant of the (finite rank) operator

$$\varphi\mapsto\int_{t/\sigma}^{\infty}\varphi(y)K_N(\cdot,y)d\mu(y)$$

with kernel K_N. That this expression be called a determinant is justified in particular by (1.11) (cf. e.g. [Du-S] or [G-G] for generalities on Fredholm determinants).

A classical formula due to Christoffel and Darboux (cf. [Sze]) indicates that

$$K_N(x,y)=\kappa_N\,\frac{P_N(x)P_{N-1}(y)-P_{N-1}(x)P_N(y)}{x-y},\quad x,y\in\mathbb{R}. \tag{1.15}$$

(Note, see Part 3, that $P'_N = \sqrt{N}\, P_{N-1}$.) In the regime given by (1.5), set then $t = 1 + sN^{-2/3}$, while as usual $\sigma^2 = \frac{1}{4N}$. After a change of variables in (1.14),

$$\mathbb{P}\big(\{\lambda^N_{\max} \le 1 + sN^{-2/3}\}\big) = \sum_{r=0}^{N} \frac{(-1)^r}{r!} \int_{(s,\infty)^r} \det\big(\widetilde{K}_N(x_i,x_j)\big)_{1\le i,j\le r} dx_1 \cdots dx_r$$

where

$$\widetilde{K}_N(x,y) = K_N\Big(2\sqrt{N} + 2xN^{-1/6}, 2\sqrt{N} + 2yN^{-1/6}\Big) \cdot \sqrt{\frac{2}{\pi}}\,\frac{1}{N^{1/6}}\, \mathrm{e}^{-[\sqrt{N}+xN^{-1/6}]^2} \mathrm{e}^{-[\sqrt{N}+yN^{-1/6}]^2}.$$

Now, in this regime, the kernel $\widetilde{K}_N(x,y)$ may be shown to converge to the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}\,(x)\mathrm{Ai}'\,(y) - \mathrm{Ai}'\,(x)\mathrm{Ai}\,(y)}{x-y}\,, \quad x,y \in \mathbb{R},$$

through the appropriate asymptotics on the Hermite polynomials known as Plancherel–Rotach asymptotics (cf. [Sze], [Fo1]...). Here Ai is the special Airy function solution of $\mathrm{Ai}'' = x\mathrm{Ai}$ with the asymptotics $\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}}\, \mathrm{e}^{-\frac{2}{3}x^{3/2}}$ as $x \to \infty$. By further functional arguments, the convergence may be extended at the level of Fredholm determinants to show that, for every $s \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}\big(\{\lambda^N_{\max} \le 1 + sN^{-2/3}\}\big) \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \int_{(s,\infty)^r} \det\big(K_{\mathrm{Ai}}(x_i,x_j)\big)_{1\le i,j\le r}\, dx_1 \cdots dx_r \\
&= \det\Big([\mathrm{Id} - K_{\mathrm{Ai}}]_{\mathrm{L}^2(s,\infty)}\Big) = F_{\mathrm{GUE}}(s),
\end{aligned}$$

justifying thus (1.5) (cf. [Fo1], [T-W1], [De]).

1.3 Coulomb Gas and Random Growth Functions

Probability measures on $\mathbb{R}^N$ of the type (1.10) may be considered in more generality. Given for example a (continuous or discrete) probability measure ρ on $\mathbb{R}^N$, and $\beta > 0$, let

$$dQ(x) = \frac{1}{Z}\, \big|\Delta_N(x)\big|^\beta d\rho(x) \tag{1.16}$$

where $Z = \int |\Delta_N|^\beta d\rho < \infty$ is the normalization constant. As we have seen, such probability distributions naturally occur as the joint law of the eigenvalues of matrix models. For example, in the GUE case (cf. (1.10)), $\beta = 2$ and

ρ is a product Gaussian measure. (In the GOE and GSE cases, $\beta = 1$ and 4 respectively, cf. [Meh].) For more general (orthogonal, unitary or simplectic) ensembles induced by the probability law $Z^{-1} \exp(-\mathrm{Tr}\, v(X))dX$ on matrices, ρ is the product measure of the density $\mathrm{e}^{-v(x)}$.

One general idea is that among reasonable families of distributions ρ, for example product measures of identical factors, the asymptotic behavior of the probability laws (1.16) is governed by the Vandermonde determinant, and thus exhibits common features. It would be of interest to describe a few general facts about these laws. Distributions of the type (1.16) are called Coulomb gas in mathematical physics. The largest eigenvalue of the matrix models thus appears here as the rightmost point or charge $\max_{1 \le i \le N} x_i$ under (1.16).

When $d\rho(x) = \prod_{i=1}^N d\mu(x_i)$ for some probability measure μ on $\mathbb{R}$ or $\mathbb{Z}$, and $\beta = 2$, the preceding Coulomb gas distributions may be analyzed through the orthogonal polynomials of the underlying probability measure μ (provided they exists) as in the example of the GUE discussed previously. In particular, the correlation functions admit determinantal representations. In this case, the probability measures (1.16) are thus sometimes called orthogonal polynomial ensembles. Accordingly, the joint law of the eigenvalues of the GUE is called the Hermite (orthogonal polynomial) Ensemble. In what follows, we only consider Coulomb gas of this sort given as orthogonal polynomial ensembles (cf. [De], [Kö]...).

Following the analysis of the GUE, fluctuations of the largest eigenvalue or rightmost charge of orthogonal polynomial ensembles toward the Tracy–Widom distribution may be developed on the basis of the common Airy asymptotics of orthogonal polynomials (at this regime). The principle of proof extends to kernels K_N properly convergent as $N \to \infty$ to the Airy kernel. When the orthogonal polynomials admit suitable integral representations, the asymptotic behaviors may generally be obtained from a saddle point analysis. For example, the ℓ-th Hermite polynomial may be described as

$$P_\ell(x) = \frac{(-1)^\ell}{\sqrt{\ell!}}\, \mathrm{e}^{x^2/2} \frac{d^\ell}{dx^\ell}\big(\mathrm{e}^{-x^2/2}\big),$$

and thus, after a standard Fourier identity,

$$P_\ell(x) = \frac{(-i)^\ell}{\sqrt{\ell!}}\, \mathrm{e}^{x^2/2} \int_{-\infty}^{+\infty} s^\ell \mathrm{e}^{isx - s^2/2}\, \frac{ds}{\sqrt{2\pi}}\,.$$

The asymptotic behavior as $\ell \to \infty$ may then be handled by the so-called saddle point method (or steepest descent, or stationary phase) of asymptotic evaluation of integrals of the form

$$\int_\Gamma \phi(z) \mathrm{e}^{t\psi(z)} dz$$

over a contour Γ in the complex plane as the parameter t is large (cf. [Fy] for a brief introduction). While these asymptotics are available for the classical orthogonal polynomials from suitable representation of their generating series, the study of more general weights can lead to rather delicate investigations. This might require deep arguments involving steepest descent methods of highly non-trivial Riemann–Hilbert analysis as developed by P. Deift and X. Zhou [D-Z]. We refer to the monograph [De] by P. Deift for an introduction to these methods and complete references up to 1999, including the important contribution [DKMcVZ]. A further introduction is the set of notes [Ku] including more recent developments and references. See also references in [Baik2]. Discrete orthogonal polynomial ensembles are deeply investigated in [B-K-ML-M]. When suitable contour integral representations of the kernels are available, the standard saddle point method is however enough to determine the expected asymptotics (see e.g. [Joha2] for an example of regularized Wigned matrices).

The orthogonal polynomial method cannot be developed however outside the (complex) case $\beta = 2$. Specific arguments have to be found. The real case for example uses Pfaffians and requires non-trivial modifications (cf. [Meh]). In particular, it is possible to relate the asymptotic behavior of the largest eigenvalues of the GOE to the one of the GUE through a generalized two-dimensional kernel, and thus to conclude to similar fluctuation results (cf. [T-W2], [Wid1]). In particular, the limiting GOE Tracy–Widom law takes the form

$$F_{\mathrm{GOE}}(s) = F_{\mathrm{GUE}}(s)^{1/2} \exp\bigg(-\frac{1}{2}\int_{2s}^{\infty} u(x)dx \bigg), \quad s \in \mathbb{R}.$$

Coulomb gas associated to the classical orthogonal polynomial ensembles are of particular interest. Among these ensembles, the Laguerre and (discrete) Meixner ensembles play a central role and exhibit some remarkable features. The Laguerre Ensemble represents the joint law of the eigenvalues of Wishart matrices. Let G be a complex $M \times N$, $M \geq N$, random matrix the entries of which are independent complex Gaussian random variables with mean zero and variance σ^2, and set $Y = Y^N = G^*G$. The law of Y defines a unitary invariant probability measure on $\mathcal{H}_N$, and the distribution of the eigenvalues is given by a Coulomb gas (1.16) with $\beta = 2$ and ρ (up to the scaling parameter σ) the product measure of the Gamma distribution $d\mu(x) = \Gamma(\gamma+1)^{-1}x^\gamma \mathrm{e}^{-x}dx$ on $(0,\infty)$ with $\gamma = M - N$. The Laguerre polynomials being the orthogonal polynomials for the Gamma law, the corresponding joint distribution of the eigenvalues is called the Laguerre Ensemble. Real Wishart matrices are defined similarly (with $\beta = 1$), and Wishart matrices with non-Gaussian entries may also be considered. The limiting spectral measure of Wishart matrices, as $\sigma^2 \sim \frac{1}{4N}$, and $M \sim cN$, $c \geq 1$, is described by the so-called Marchenko–Pastur distribution (or free Poisson law) [M-P] (cf. [Bai]).

The Meixner Ensemble is associated to a discrete weight. Let μ be the so-called negative binomial distribution on $\mathbb{N}$ with parameters $0 < q < 1$ and $\gamma > 0$ given by

$$\mu\big(\{x\}\big) = \frac{(\gamma)_x}{x!}\, q^x(1-q)^\gamma, \quad x \in \mathbb{N}, \tag{1.17}$$

where $(\gamma)_x = \gamma(\gamma+1)\cdots(\gamma+x-1)$, $x \geq 1$, $(\gamma)_0 = 1$. If $\gamma = 1$, μ is just the geometric distribution with parameter q. The orthogonal polynomials for μ are called the Meixner polynomials (cf. [Sze], [Ch], [K-S]).

As already mentioned above, asymptotics of the Laguerre and Meixner polynomials may then be used as for the GUE, but with increased technical difficulty, to show that the largest eigenvalue of (properly rescaled) Wishart matrices and the rightmost charge, that is the function $\max_{1\leq i\leq N} x_i$ ('largest eigenvalue'), of the Meixner orthogonal polynomial Ensemble, fluctuate around their limiting value at the Tracy–Widom regime. This has been established by K. Johansson [Joha1] for the Meixner Ensemble. The Laguerre Ensemble appears as a limit as $q \to 1$, and has been investigated independently by I. Johnstone [John] who also carefully analyzes the real case along the lines of [T-W2]. In [So2], A. Soshnikov extends these conclusions to Wishart matrices with non-Gaussian entries following his previous contribution [So1] for Wigner matrices.

Further classical orthogonal polynomial ensembles may be considered. For example, fluctuations of the Jacobi Ensemble constructed over Jacobi polynomials and associated to Beta matrices are addressed in [Co].

The Laguerre and Meixner Ensembles actually share some specific Markovian type properties which make them play a central role in connection with various probabilistic models. In the remarkable contribution [Joha1], K. Johansson indeed showed that the Meixner orthogonal polynomial Ensemble entails an extremely rich mathematical structure connected with many different interpretations. In particular, its rightmost charge may be interpreted in terms of shape functions and last-passage times. Let $w(i,j)$, $i,j \in \mathbb{N}$, be independent geometric random variables with parameter q, $0 < q < 1$. For $M \geq N \geq 1$, set

$$W = W(M,N) = \max_{\pi} \sum_{(i,j)\in\pi} w(i,j) \tag{1.18}$$

where the maximum runs over all up/right paths π in $\mathbb{N}^2$ from $(1,1)$ to (M,N). An up/right path π from $(1,1)$ to (M,N) is a collection of sites $\{(i_k,j_k)\}_{1\leq k\leq M+N-1}$ such that $(i_1,j_1) = (1,1)$, $(i_{M+N-1}, j_{M+N-1}) = (M,N)$ and $(i_{k+1},j_{k+1}) - (i_k,j_k)$ is either $(1,0)$ or $(0,1)$. The random growth function W may be interpreted as a directed last-passage time in percolation. Using the Robinson–Schensted–Knuth correspondence between permutations and Young tableaux (cf. [Fu]), K. Johansson [Joha1] proved that, that for every $t \geq 0$,

$$\mathbb{P}\big(\{W \leq t\}\big) = Q\Big(\big\{\max_{1\leq i\leq N} x_i \leq t + N - 1\big\}\Big) \tag{1.19}$$

where Q is the Meixner orthogonal polynomial Ensemble with parameters q and $\gamma = M - N + 1$. As described in [Joha1], this model is also closely related to the one-dimensional totally asymmetric exclusion process. It may also be interpreted as a randomly growing Young diagram or a zero-temperature directed polymer in a random environment (cf. also [Kö]).

Provided with this correspondence, the fluctuations of the rightmost charge of the Meixner Ensemble may be translated on the growth function $W(M,N)$. As indeed shown in [Joha1], for every $c \geq 1$, (some multiple of) the random variable

$$\frac{W\big([cN],N\big) - \omega N}{N^{1/3}},$$

where

$$\omega = \frac{(1+\sqrt{qc}\,)^2}{1-q} - 1,$$

converges weakly to the Tracy–Widom distribution F_{GUE}.

In the limit as $q \to 1$, the model covers the fluctuation of the largest eigenvalue of Wishart matrices, studied independently in [John]. Namely, if w is a geometric random variable with parameter $0 < q < 1$, as $q \to 1$, $(1-q)w$ converges in distribution to a exponential random variable with parameter 1. If $W = W(M,N)$ is then understood as a maximum over up/right paths of independent such exponential random variables, the identity (1.19) then translates into

$$\mathbb{P}\big(\{W \leq t\}\big) = Q\Big(\big\{\max_{1\leq i\leq N} x_i \leq t\big\}\Big), \quad t \geq 0, \tag{1.20}$$

where Q is now the Coulomb gas of the Laguerre Ensemble with parameter $\sigma = 1$. (It should be mentioned that no direct proof of (1.20) is so far available.) This example thus admits the double description as a largest eigenvalue of random matrices and a last-passage time.

The central role of the Meixner model covers further instances of interest. Among them are the Plancherel measure and the length of the longest increasing subsequence in a random permutation. (See [A-D] for a general presentation on the length of the longest increasing subsequence in a random permutation.) It was namely observed by K. Johansson [Joha3] (see also [B-O-O]) that, as $q = \frac{\theta}{N^2}$, $N \to \infty$, $\theta > 0$, the Meixner orthogonal polynomial Ensemble converges to the θ-Poissonization of the Plancherel measure on partitions. Since the Plancherel measure is the push-forward of the uniform distribution on the symmetric group S_n by the Robinson–Schensted–Knuth correspondence which maps a permutation $\sigma \in S_n$ to a pair of standard Young tableaux of the same shape, the length of the first row is equal to the length $L_n(\sigma)$ of the longest increasing subsequence in σ. As a consequence, in this regime,

$$\lim_{N\to\infty} \mathbb{P}\big(\big\{W(N,N) \leq t\big\}\big) = \mathbb{P}\big(\{L_{\mathcal{N}} \leq t\}\big), \quad t \geq 0,$$

where $\mathcal{N}$ is an independent Poisson random variable with parameter $\theta > 0$. The orthogonal polynomial approach may then be used to produce a new proof of the important Baik–Deift–Johansson theorem [B-D-J] on the fluctuations of L_n stating that

$$\frac{L_n - 2\sqrt{n}}{2n^{1/6}} \to F_{\mathrm{GUE}} \tag{1.21}$$

in distribution.

The Markovian properties of the specific geometric and exponential distributions make it thus possible to fully analyze the shape functions W and their asymptotic behaviors. (If the definition of up/right paths is modified, a few more isolated cases have been studied [Baik1], [Joha1], [Joha3], [Se2], [T-W3]...) It would be a challenging question to establish the same fluctuation results, with the same $(\mathrm{mean})^{1/3}$ rate, for random growth functions $W(M,N)$ (1.18) constructed on more general families of distributions of the $w(i,j)$'s, such as for example Bernoulli variables. While superadditivity arguments show that $W([cN],N)/N$ is convergent almost surely as $N \to \infty$ under rather mild conditions, fluctuations around the (usually unknown) limit are almost completely open so far. Even the variance growth (see Part 4) has not yet been determined.

1.4 Large Deviation Asymptotics

In addition to the preceding fluctuation results for the largest eigenvalues or rightmost charges of orthogonal polynomial ensembles, some further large deviation theorems have been investigated during the past years. The analysis again relies of the determinantal structure of Coulomb gas together with a careful examination of the equilibrium measure from the logarithmic potential point of view [S-T].

For example, translated into the framework of the preceding random growth function $W(M,N)$ defined from geometric random variables, K. Johansson also proved in the contribution [Joha1] a large deviation theorem in the form of

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\Big(\Big\{W\big([cN],N\big) \geq N(\omega+\varepsilon)\Big\}\Big) = -J(\varepsilon) \tag{1.22}$$

for each $\varepsilon > 0$, where J is an explicit function such that $J(x) > 0$ if $x > 0$ (see below). The result is actually due to T. Seppäläinen [Se3] in the simple exclusion process interpretation of the model. The large deviation principle on the left of the mean takes place at the speed N^2 and expresses

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\Big(\Big\{W\big([cN],N\big) \leq N(\omega-\varepsilon)\Big\}\Big) = -I(\varepsilon) \tag{1.23}$$

for each $\varepsilon > 0$, where $I(x) > 0$ for $x > 0$. As we will see it below, the rate functions J and I of (1.22) and (1.23) actually partly reflect the $N^{2/3}$ rate of the fluctuation results.

In contrast with the fluctuation theorems of the preceding section which rely on specific orthogonal polynomial asymptotics, such large deviation principles hold for large classes of (both continuous or discrete) Coulomb gas (1.16)

$$dQ(x) = \frac{1}{Z} \left|\Delta_N(x)\right|^\beta \prod_{i=1}^N d\mu(x_i),$$

for arbitrary $\beta > 0$ and under mild hypotheses on μ (cf. [Joha1], [BA-D-G], [Fe]). They are closely related to the large deviation principles at the level of the spectral measures emphasized by D. Voiculescu [Vo] (as a microstate description) and G. Ben Arous and A. Guionnet [BA-G] (cf. [H-P]) (as a Sanov type theorem). These results examine the large deviation principles for the empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ at the speed N^2 in the space of probability measures on $\mathbb{R}$. The rate function is minimized at the equilibrium measure, almost sure limit of the empirical measure (the semicircle law for example in case of the Hermite Ensemble).

The corresponding rate function of the large deviation principles for the right-most charges (largest eigenvalues) is then usually deduced from the one for the empirical measures. The speed of convergence is however different on the right and on the left of the mean. In the example of the largest eigenvalue $\lambda_{\max}^N$ of the GUE with $\sigma^2 = \frac{1}{4N}$, it is shown in [BA-D-G] that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\big(\{\lambda_{\max}^N \geq 1+\varepsilon\}\big) = -J_{\mathrm{GUE}}(\varepsilon) \tag{1.24}$$

where, for every $\varepsilon > 0$,

$$J_{\mathrm{GUE}}(\varepsilon) = 4\int_0^\varepsilon \sqrt{x(x+2)}\,dx. \tag{1.25}$$

Note that $J_{\mathrm{GUE}}(\varepsilon)$ is of the order of $\varepsilon^{3/2}$ for the small values of ε, in accordance with the Tracy–Widom theorem (1.5). Similarly, on the left of the mean,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\big(\{\lambda_{\max}^N \leq 1-\varepsilon\}\big) = -I_{\mathrm{GUE}}(\varepsilon) \tag{1.26}$$

for some function I_{GUE} such that $I_{\mathrm{GUE}}(x) > 0$ for every $x > 0$ (cf. also [Joha1], [Fe]). The speed N^2 partly indicates that the largest eigenvalues tend to accumulate below the right-end point of the support of the spectrum. The Laguerre and Meixner examples will be discussed in the next part. It is expected that for large classes of potentials v in the driving measure $d\mu = \mathrm{e}^{-v}dx$ of the Coulomb gas Q, the corresponding rate function J on the right of the mean is such that $J(\varepsilon) \sim \varepsilon^{3/2}$ for small ε. Large deviations for the length of the longest increasing subsequence have been described in [Se1], [D-Ze].

2 Known Results on Non-Asymptotic Bounds

The purpose of these notes is to describe some non-asymptotic exponential deviation inequalities on the largest eigenvalues or rightmost charges of random matrix and random growth models at the order (mean)$^{1/3}$ of the fluctuation results. It actually turns out that several results are already available in the literature, motivated by convergence of moments in Tracy–Widom type theorems or moderate deviation principles interpolating between fluctuations and large deviations. We thus survey here some results developed to this aim, which however, as we will see it, usually require a rather heavy analysis and only concern some rather specific models. In particular, Wigner matrices or random growth functions with arbitrary weights do not seem to have been accessed by any method so far. We treat upper deviation inequalities both above and below the mean, and analyze their consequences to variance inequalities.

2.1 Upper Tails on the Right of the Mean

As presented in the first part, the Tracy–Widom theorem on the behavior of the largest eigenvalue $\lambda_{\max}^N$ of the GUE with the scaling $\sigma^2 = \frac{1}{4N}$ expresses that

$$\lim_{N\to\infty} \mathbb{P}\big(\{\lambda_{\max}^N \leq 1 + sN^{-2/3}\}\big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R}. \tag{2.1}$$

In addition to this fluctuation result, the largest eigenvalue $\lambda_{\max}^N$ also satisfies the large deviation theorems of Section 1.4. In order to quantify these asymptotic results, one would be interested in finding (upper-) estimates, for each fixed $N \geq 1$ and $\varepsilon > 0$, on $\mathbb{P}(\{\lambda_{\max}^N \geq 1 + \varepsilon\})$ and $\mathbb{P}(\{\lambda_{\max}^N \leq 1 - \varepsilon\})$. Actually, from the discussion on the speed of convergence in the large deviation asymptotics (1.24) and (1.26) and the behaviors (1.8) and (1.9) of the Tracy–Widom distribution F_{GUE}, we typically expect that for some $C > 0$ and all $N \geq 1$,

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1 + \varepsilon\}\big) \leq C\,\mathrm{e}^{-N\varepsilon^{3/2}/C} \quad \text{and} \quad \mathbb{P}\big(\{\lambda_{\max}^N \leq 1 - \varepsilon\}\big) \leq C\,\mathrm{e}^{-N^2\varepsilon^3/C}$$

for $\varepsilon > 0$. The range of interest concerns particularly small $\varepsilon > 0$ to cover the values $\varepsilon = sN^{-2/3}$ in (2.1), justifying the terminology of small deviation inequalities. Bounds of this type may then be used towards convergence of moments and variance bounds, or moderation deviation results. (We do not address the question of lower estimates which does not seem to have been investigated in the literature.)

A first approach to such a project would be to carefully follow the proof of the Tracy–Widom theorem, and to control the various Fredholm determinants by appropriate (finite range) bounds on orthogonal polynomials. This is the route taken by G. Aubrun in [Au] which allowed him to state the following small deviation inequality for the largest eigenvalue $\lambda_{\max}^N$ of the GUE (with $\sigma^2 = \frac{1}{4N}$).

Proposition 2.1. *For some numerical constant $C > 0$, and all $N \geq 1$ and $\varepsilon > 0$,*

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1 + \varepsilon\}\big) \leq C\,\mathrm{e}^{-N\varepsilon^{3/2}/C}.$$

As announced, when $\varepsilon = sN^{-2/3}$, the deviation inequality of Proposition 2.1 fits the fluctuation result (2.1). The bound is also in accordance with the tail behavior (1.9) of the Tracy–Widom distribution F_{GUE} at $+\infty$.

This line of reasoning can certainly be pushed similarly for the orthogonal polynomial ensembles for which a Tracy–Widom theorem holds, and for which asymptotics of orthogonal polynomials together with the corresponding bounds are available. This issue is seemingly not clearly addressed in the literature. Results of this type seem to be discussed in particular in [G-T-W]. As already emphasized, this might however require a quite deep analysis, including steepest descent arguments of Riemann–Hilbert type (cf. [De]). An attempt relying on measure concentration and weak convergence to the equilibrium measure is undertaken in [Bl] to yield asymptotic deviation inequalities of the correct order for some families of unitary invariant ensembles.

Another direction to deviation inequalities on the right of the mean may be developed in the context of last-passage times, relying on superadditivity and large deviation asymptotics. Let us consider for example the random growth function $W(M,N)$ of Part 1, last-passage time in directed percolation for geometric random variables with parameter $0 < q < 1$. As we have seen it, up to some multiplicative factor,

$$\lim_{N\to\infty} \mathbb{P}\Big(\big\{W\big([cN],N\big) \le \omega N + sN^{1/3}\big\}\Big) = F_{\mathrm{GUE}}(s), \quad s \in \mathbb{R} \tag{2.2}$$

where we recall that

$$\omega = \frac{(1+\sqrt{qc}\,)^2}{1-q} - 1.$$

Fix $N \ge 1$ and $c \ge 1$, and set $W = W([cN],N)$. As for the largest eigenvalue $\lambda^N_{\max}$ of the GUE, to quantify (2.2), we may ask for exponential bounds on the probabilities

$$\mathbb{P}\big(\{W \ge N(\omega+\varepsilon)\}\big) \quad \text{and} \quad \mathbb{P}\big(\{W \le N(\omega-\varepsilon)\}\big)$$

for $\varepsilon > 0$. One may also ask for example for bounds on the variance of W, which are expected to be of the order of $N^{2/3}$ by (2.2).

As observed by K. Johansson in [Joha1], inequalities on $\mathbb{P}(\{W \ge N(\omega + \varepsilon)\})$ may be obtained from the large deviation asymptotics (1.22) together with a superadditivity argument (compare [Se1]). It is indeed immediate to see that $W(M,N)$ is superadditive in the sense that

$$W(M,N) + W\big([M+1,2M],[N+1,2N]\big) \le W(2M,2N)$$

where $W([M+1,2M],[N+1,2N])$ is understood as the supremum over all up/right paths from $(M+1,N+1)$ to $(2M,2N)$. Since $W([M+1,2M],[N+1,2N])$ is independent with the same distribution as $W(M,N)$, it follows that for every $t \ge 0$,

$$\mathbb{P}\big((\{W(M,N) \ge t\}\big)^2 \le \mathbb{P}\big((\{W(2M,2N) \ge 2t\}\big).$$

Iterating, for every integer $k \ge 1$,

$$\mathbb{P}\big(\big(\{W(M,N) \geq t\}\big)^{2^k} \leq \mathbb{P}\big(\big(\{W(2^k M, 2^k N) \geq 2^k t\}\big).$$

Together with the large deviation property (1.22), as $k \to \infty$, for every fixed $N \geq 1$ and $\varepsilon > 0$,

$$\mathbb{P}\big(\{W \geq N(\omega + \varepsilon)\}\big) \leq \mathrm{e}^{-NJ(\varepsilon)}. \tag{2.3}$$

Now the function $J(\varepsilon)$ is explicitly known. It has however a rather intricate description, based itself on the knowledge of the equilibrium measure of the Meixner Ensemble. Precisely, as shown in [Joha1],

$$J(\varepsilon) = J_{\mathrm{MEIX}}(\varepsilon) = \frac{1}{1-q}\int_1^x (x-y)\Big[\frac{c-q}{y+B} + \frac{1-qc}{y+D}\Big]\frac{dy}{\sqrt{y^2-1}}$$

where

$$x = 1 + \frac{(1-q)\varepsilon}{2\sqrt{qc}}, \quad B = \frac{c+q}{2\sqrt{qc}}, \quad D = \frac{1+qc}{2\sqrt{qc}}.$$

One may nevertheless check that

$$J(\varepsilon) \geq C^{-1}\min(\varepsilon, \varepsilon^{3/2}), \quad \varepsilon > 0,$$

where $C > 0$ only depends on c and q. As a consequence, we may state the following exponential deviation inequality. Note that this conclusion requires both the delicate large deviation theorem (1.22) for the Meixner Coulomb gas together with the deep combinatorial description (1.19) (in order to make use of superadditivity of the growth function $W(M,N)$).

Proposition 2.2. *For some constant $C > 0$ only depending on the parameter $0 < q < 1$ of the underlying geometric distribution and $c \geq 1$, and all $N \geq 1$ and $\varepsilon > 0$,*

$$\mathbb{P}\big(\{W \geq N(\omega + \varepsilon)\}\big) \leq C\,\mathrm{e}^{-N\min(\varepsilon,\varepsilon^{3/2})/C}.$$

Note that in addition to the small deviation inequality at the Tracy-Widom rate, Proposition 2.2 also emphasizes the order $\mathrm{e}^{-N\varepsilon/C}$ for the large values of ε due to the precise knowledge of the rate function J_{MEIX}. We will come back to this observation in the context of the GUE below.

The explicit knowledge of the rate function J_{MEIX} actually allows one to make use of the non-asymptotic inequality (2.3) for several related models. For example as we already saw it, if w is geometric with parameter $0 < q < 1$, then as $q \to 1$, $(1-q)w$ converges in distribution to an exponential random variable with parameter 1. In this limit, (2.3) turns into

$$\mathbb{P}\big(\{W \geq N(\omega + \varepsilon)\}\big) \leq \mathrm{e}^{-NJ_{\mathrm{LAG}}(\varepsilon)} \tag{2.4}$$

where now W is the supremum (1.18) over up/right paths of independent exponential random variables with parameter 1, $\omega = (1+\sqrt{c})^2$ and J_{LAG} is the Laguerre rate function

$$J_{\mathrm{LAG}}(\varepsilon) = \int_1^x (x-y)\,\frac{(1+c)y + 2\sqrt{c}}{(y+B)^2}\,\frac{dy}{\sqrt{y^2-1}}$$

with

$$x = 1 + \frac{\varepsilon}{2\sqrt{c}}\,, \quad B = \frac{1+c}{2\sqrt{c}}\,.$$

One may check similarly that $J_{\mathrm{LAG}}(\varepsilon) \geq C^{-1}\min(\varepsilon, \varepsilon^{3/2})$ so that the bound (2.4) thus provides an analogue of Propositions 2.1 and 2.2 for the Laguerre Ensemble for both the interpretation in terms of the last-passage time W or the largest eigenvalue of Wishart matrices.

Now one may further go from Wishart matrices to random matrices from the GUE, and actually recover in this way Proposition 2.1. Namely, recalling the Wishart matrix $Y = Y^N = GG^*$ with variance σ^2, as $M \to \infty$,

$$\sigma^{-1}\sqrt{M}\Big(\frac{Y}{M} - \sigma^2\,\mathrm{Id}\Big) \to X$$

in distribution where X follows the GUE law (with variance σ^2). In particular,

$$\sigma^{-1}\sqrt{M}\Big(\frac{1}{M}\,\lambda^N_{\max}(Y) - \sigma^2\,\mathrm{Id}\Big) \to \lambda^N_{\max}(X). \tag{2.5}$$

Now, after the scaling $\sigma^2 = \frac{1}{4N}$, (2.4) indicates that for $M = [cN]$, $c \geq 1$,

$$\mathbb{P}\big(\{\lambda^N_{\max}(Y) \geq \tfrac{\omega+\varepsilon}{4}\}\big) \leq \mathrm{e}^{-NJ_{\mathrm{LAG}}(\varepsilon)}$$

for every $N \geq 1$ and $\varepsilon > 0$. Change then ε into $2\sqrt{c}\,\varepsilon$ and take the limit (2.5) as $c \to \infty$. Denoting as usual by $\lambda^N_{\max}$ the largest eigenvalue of the GUE with variance $\sigma^2 = \frac{1}{4N}$, it follows that

$$\mathbb{P}\big(\{\lambda^N_{\max} \geq 1+\varepsilon\}\big) \leq \mathrm{e}^{-NJ_{\mathrm{GUE}}(\varepsilon)} \tag{2.6}$$

where $J_{\mathrm{GUE}}(\varepsilon)$, $\varepsilon > 0$, was given in (1.25) as

$$J_{\mathrm{GUE}}(\varepsilon) = 4\int_0^\varepsilon \sqrt{x(x+2)}\,dx.$$

Since $J_{\mathrm{GUE}}(\varepsilon) \geq C^{-1}\max(\varepsilon^2, \varepsilon^{3/2})$, $\varepsilon > 0$, we thus recover in this way Proposition 2.1, and actually more precisely

$$\mathbb{P}\big(\{\lambda^N_{\max} \geq 1+\varepsilon\}\big) \leq C\,\mathrm{e}^{-N\max(\varepsilon^2,\varepsilon^{3/2})/C} \tag{2.7}$$

for every $\varepsilon > 0$. As in Proposition 2.2, this exponential deviation inequality emphasizes both the small deviations of order $\varepsilon^{3/2}$ in accordance with the Tracy–Widom theorem and the large deviations of the order ε^2.

It may actually be shown directly that (2.6) follows from the large deviation principle (1.24) as a consequence of superadditivity, however on some related representation. It has been proved namely, via various arguments, that the largest eigenvalue $\lambda_{\max}^N$ from the GUE with $\sigma^2 = 1$ has the same distribution as

$$\sup \sum_{i=1}^{N} (B_{t_i}^i - B_{t_{i-1}}^i) \tag{2.8}$$

where $B^1, \ldots, B^N$ are independent standard Brownian motions and the supremum runs over all $0 = t_0 \le t_1 \le \cdots \le t_{N-1} \le 1$. Proofs in [Bar], [G-T-W] are based on the Robinson–Schensted–Knuth correspondence, while in [OC-Y] advantage is taken from non-colliding Brownian motions and queuing theory, and include generalizations related to the classical Pitman theorem (see [OC]). The representation (2.8) may be thought of as a kind of continuous version of directed last-passage percolation for Brownian paths. On the basis of this identification, it is not difficult to adapt the superadditivity argument developed for the random growth function (1.18) to deduce (2.6) from the large deviation bound (1.24). In any case, the price to pay to reach Propositions 2.1 and 2.2 is rather expensive.

It should be pointed out that outside these specific models, non-asymptotic small deviation inequalities at the Tracy–Widom rate are so far open. Universality conjectures would expect similar deviation inequalities for general Wigner matrices or directed last passage times W with general independent weights. Soshnikov's proof [So1] only allows for asymptotic inequalities (cf. Section 5.2). Similarly, only the choice of geometric and exponential random variables w_{ij} gives rise so far to statements such as Proposition 2.2.

The central role of the Meixner model shows, by appropriate scalings and the explicit expression of the rate function J_{MEIX}, that the tail (2.3) actually covers further instances of interest. As discussed in Section 1.3, one such instance is the length of the longest increasing subsequence in a random permutation and the Baik–Deift–Johansson theorem (1.21). Namely, in the regime $q = \frac{\theta}{N^2}$, $N \to \infty$, the deviation inequality (2.3) may indeed be used to show that, for every $n \ge 1$ and every $\varepsilon > 0$,

$$\mathbb{P}\big(\big\{L_n \ge 2\sqrt{n}\,(1+\varepsilon)\big\}\big) \le C \exp\Big(-\frac{1}{C}\sqrt{n}\,\min\big(\varepsilon^{3/2}, \varepsilon\big)\Big) \tag{2.9}$$

where $C > 0$ is numerical, in accordance thus with (1.21) and the large deviation theorem of [D-Z]. The previous bound also matches the upper tail moderate deviation theorem of [L-M].

2.2 Upper Tails on the Left of the Mean

We next turn to the probability that the largest eigenvalue or rightmost charge is less than or equal to the right-end point of the spectral measure. As already mentioned, the intuition, together with the large deviation asymptotics (1.23) and (1.26), suggests that it is much smaller than the probability that the largest eigenvalue exceeds the right-end point. Let us consider again the Meixner model in terms of the directed last-passage time function W of

(1.18) with geometric random variables. We thus look for the probability that $W = W([cN], N)$ is less than or equal to $N(\omega - \varepsilon)$ for each $\varepsilon > 0$ and fixed $N \geq 1$. Things are here much more delicate. Seemingly, only a few results are available, relying furthermore on delicate and quite difficult to access methods and arguments. The following result has been put forward in [BDMcMZ] by refined Riemann–Hilbert steepest descent methods in order to investigate convergence of moments and moderate deviations. Some related estimates are developed in [B-D-R] in the context of random Young tableaux, and in [L-M-R] for the length of the longest increasing subsequence.

Proposition 2.3. *For some constant $C > 0$ only depending on the parameter $0 < q < 1$ of the underlying geometric distribution and $c \geq 1$, and all $N \geq 1$ and $0 < \varepsilon \leq \omega$,*

$$\mathbb{P}\big(\{W \leq N(\omega - \varepsilon)\}\big) \leq C\,\mathrm{e}^{-N^2\varepsilon^3/C}.$$

(Actually, the statement in [BDMcMZ] seems to concern only large values of N.)

As for Proposition 2.1, the preceding inequality matches the behavior at $-\infty$ of the Tracy–Widom distribution F_{GUE} given by (1.8).

After [B-D-R] and [BDMcMZ], H. Widom [Wid2] noticed a somewhat less precise estimate, replacing $N^2\varepsilon^3$ by its square root, using a more simple trace bound, however still requiring steepest descent. We will come back to this observation in Part 5.

It is plausible that the behavior of the constant C in Proposition 2.3 allows for limits to the Laguerre and Hermite Ensembles as in the preceding section. This is however not completely obvious from the analysis in [BDMcMZ]. On the other hand, there is no doubt that a similar Riemann–Hilbert analysis may be performed analogously for these examples, and that the statements corresponding to Proposition 2.3 hold true. We may for example guess the following for the largest eigenvalue $\lambda_{\max}^N$ of the GUE with $\sigma^2 = \frac{1}{4N}$.

Proposition 2.4. *For some numerical constant $C > 0$, and all $N \geq 1$ and $0 < \varepsilon \leq 1$,*

$$\mathbb{P}\big(\{\lambda_{\max}^N \leq 1 - \varepsilon\}\big) \leq C\,\mathrm{e}^{-N^2\varepsilon^3/C}.$$

As already mentioned, similar estimates have been obtained in [L-M-R] in the proof of the lower tail moderate deviations for longest increasing subsequences, where, based on the investigation [B-D-J], the following speed of convergence is established: there exists a numerical constant $C > 0$ such that for every $n \geq 1$ and every $0 < \varepsilon \leq 1$,

$$\mathbb{P}\big(\big\{L_n \leq 2\sqrt{n}\,(1 - \varepsilon)\big\}\big) \leq C\,\mathrm{e}^{-n\,\varepsilon^3/C}. \tag{2.10}$$

2.3 Variance Inequalities

The non-asymptotic deviation inequalities of Sections 2.1 and 2.2 allow for convergence of moments towards the Tracy–Widom distribution [BDMcMZ],

[Wid2]. In particular, they may easily be combined to reach variance bounds. For example, the next statement on the growth function $W = W([cN], N)$ follows from Propositions 2.2 and 2.3.

Corollary 2.5. *For some constant $C > 0$ (only depending on q and c), and every $N \geq 1$,*

$$\mathrm{var}\,(W) = \mathbb{E}\Big(\big[W - \mathbb{E}(W)\big]^2\Big) \leq CN^{2/3}.$$

Proof. Fix $N \geq 1$. We may write

$$\begin{aligned} N^{-2}\,\mathrm{var}\,(W) &\leq N^{-2}\,\mathbb{E}\big([W-\omega]^2\big) \\ &\leq \int_0^\infty \mathbb{P}\big(\{W \geq N(\omega+t)\}\big)dt^2 \\ &\quad + \int_0^\omega \mathbb{P}\big(\{W \leq N(\omega-t)\}\big)dt^2. \end{aligned}$$

By Proposition 2.2,

$$\int_0^\infty \mathbb{P}\big(\{W \geq N(\omega+t)\}\big)dt^2 \leq C\int_0^\infty \mathrm{e}^{-N\min(t,t^{3/2})/C}dt^2 \leq CN^{-4/3}$$

where C, here and below, may vary from line to line. On the other hand, by Proposition 2.3,

$$\int_0^\omega \mathbb{P}\big(\{W \leq N(\omega-t)\}\big)d(t^2) \leq C\int_0^\omega \mathrm{e}^{-N^2t^3/C}dt^2 \leq CN^{-4/3}.$$

The proposition is established. □

It is worthwhile mentioning that a weaker bound in Proposition 2.3, with $N^2\varepsilon^3$ replaced by $N\varepsilon^{3/2}$ as proved in [Wid2], is sufficient for the proof of Corollary 2.5.

Taking Proposition 2.4 for granted, we get similarly for the largest eigenvalue $\lambda_{\max}^N$ of the GUE with $\sigma^2 = \frac{1}{4N}$ the following variance bound.

Corollary 2.6. *For some numerical constant $C > 0$, and all $N \geq 1$,*

$$\mathrm{var}\,\big(\lambda_{\max}^N\big) \leq CN^{-4/3}.$$

As the proof of Corollary 2.5 shows, we actually have that for some $C > 0$ and all $N \geq 1$,

$$\mathbb{E}\big(|\lambda_{\max}^N - 1|^2\big) \leq CN^{-4/3}.$$

The same arguments furthermore leads to

$$\sup_N \mathbb{E}\Big(\big|N^{2/3}\big(\lambda_{\max}^N - 1\big)\big|^p\Big) < \infty$$

for any $p > 0$ which allow for convergence of moments in the Tracy–Widom theorem. In particular,

$$\big|\mathbb{E}\big(\lambda_{\max}^N\big) - 1\big| \le CN^{-2/3}. \tag{2.11}$$

We will observe in Part 5 below that moment recurrence equations may be used to see that actually

$$\mathbb{E}\big(\lambda_{\max}^N\big) \le 1 - \frac{1}{CN^{2/3}} \tag{2.12}$$

for some $C > 0$ and all $N \ge 1$. In particular thus, (2.11) means that

$$1 - \frac{1}{C'N^{2/3}} \le \mathbb{E}\big(\lambda_{\max}^N\big) \le 1 - \frac{1}{CN^{2/3}}$$

for some $C, C' > 0$ and all $N \ge 1$.

3 Concentration Inequalities

We present here a few classical measure concentration tools that may be used in the investigation of exponential deviation inequalities on the largest eigenvalues and growth functions. These tools are of general interest, apply in rather large settings and provide useful information at the level of large deviation bounds for both extremal eigenvalues and spectral distributions. While measure concentration yields the appropriate (Gaussian) large deviation bounds, it however does not produce the correct small deviation rate $(\text{mean})^{1/3}$ of the asymptotic theorems presented in Part 1. For simplicity, we mostly detail below the relevant inequalities in the Gaussian case. We successively present concentration inequalities for largest eigenvalues and random growth functions, spectral measures, as well as Coulomb gas.

3.1 Concentration Inequalities for Largest Eigenvalues

Let μ denote the standard Gaussian measure on $\mathbb{R}^n$ with density $(2\pi)^{-\frac{n}{2}}\,\mathrm{e}^{-\frac{|x|^2}{2}}$ with respect to Lebesgue measure. One basic concentration property (cf. [Le1]) indicates that for every Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$ with $\|F\|_{\mathrm{Lip}} \le 1$, and every $r \ge 0$,

$$\mu\big(\big\{F \ge \textstyle\int F d\mu + r\big\}\big) \le \mathrm{e}^{-r^2/2}. \tag{3.1}$$

Together with the same inequality for $-F$, for every $r \ge 0$,

$$\mu\big(\big\{|F - \textstyle\int F d\mu| \ge r\big\}\big) \le 2\,\mathrm{e}^{-r^2/2}. \tag{3.2}$$

The same inequalities hold for a median of F instead of the mean. Independence upon the dimension of the underlying state space is one crucial aspect of these properties.

We may apply for example these inequalities to the largest eigenvalue $\lambda_{\max}^N$ of the GUE. Namely, by the variational characterization,

$$\lambda_{\max}^N = \sup_{|u|=1} uX^N u^*, \tag{3.3}$$

so that $\lambda_{\max}^N$ is easily seen to be a 1-Lipschitz map of the N^2 independent real and imaginary entries X_{ii}, $1 \leq i \leq N$, $\mathrm{Re}\,(X_{ij})/\sqrt{2}$, $\mathrm{Im}\,(X_{ij})/\sqrt{2}$, $1 \leq i < j \leq N$, of X^N. Together with the scaling of the variance $\sigma^2 = \frac{1}{4N}$ we thus get the following concentration inequality on $\lambda_{\max}^N$.

Proposition 3.1. *For all $N \geq 1$ and $r \geq 0$,*

$$\mathbb{P}\Big(\Big\{\big|\lambda_{\max}^N - \mathbb{E}\big(\lambda_{\max}^N\big)\big| \geq r\Big\}\Big) \leq 2\,\mathrm{e}^{-2Nr^2}, \quad r \geq 0.$$

As a consequence, note that $\mathrm{var}\,(\lambda_{\max}^N) \leq C\,N^{-1}$ that should be compared with Corollary 2.6. Actually, while Proposition 3.1 describes the Gaussian decay of $\lambda_{\max}^N$ for the large values of r, it does not catch the $r^{3/2}$ rate of the small deviation inequality (2.7). It actually seems that viewing the largest eigenvalue as one particular example of Lipschitz function of the entries of the matrix does not reflect enough the structure of the model. This comment more or less applies to all the results presented here deduced from the concentration principle.

A similar inequality holds for the GOE, and actually for more general families of Gaussian matrices. Before however going on with further applications of the general principle of measure concentration, a few words are necessary at the level of the centerings. The inequalities emphasized in Part 2 indeed discuss exponential deviation inequalities from the limiting expected value (for example 1 for the scaled largest eigenvalue $\lambda_{\max}^N$ of the GUE) while the concentration principle typically produces tail inequalities around some mean (or median) value of the given functional (such as $\mathbb{E}(\lambda_{\max}^N)$). A comparison thus requires proper control over $\mathbb{E}(\lambda_{\max}^N)$ or similar average values. In the example of the GUE, (2.11) is of course enough to this task, but to make the concentration inequalities relevant by themselves, one needs independent estimates. A few remarks in this regard may be developed.

Keep again the GUE example. We may ask whether $\mathbb{E}(\lambda_{\max}^N)$, or a median of $\lambda_{\max}^N$, are smaller than 1, or at least suitably controlled. As emphasized in [Da-S], Gaussian comparison principles are of some help to this task. Consider the real-valued Gaussian process

$$G_u = uX^N u^* = \sum_{i,j=1}^N X_{ij} u_i \overline{u_j}, \quad |u| = 1,$$

where $u = (u_1, \ldots, u_N) \in \mathbb{C}^N$. It is immediate to check that for every $u, v \in \mathbb{C}^N$,

$$\mathbb{E}\big(|G_u - G_v|^2\big) = \sigma^2 \sum_{i,j=1}^N |u_i\overline{u_j} - v_i\overline{v_j}|^2.$$

Hence, if we define the Gaussian process indexed by $u \in \mathbb{C}^N$, $|u| = 1$,

$$H_u = \sum_{i=1}^N g_i \,\mathrm{Re}\,(u_i) + \sum_{j=1}^N h_j \,\mathrm{Im}\,(u_j)$$

where $g_1, \dots, g_N,\ h_1, \dots, h_N$ are independent standard Gaussian variables, then, for every u, v such that $|u| = |v| = 1$,

$$\mathbb{E}\big(|G_u - G_v|^2\big) \le 2\sigma^2 \mathbb{E}\big(|H_u - H_v|^2\big).$$

By the Slepian–Fernique lemma (cf. [L-T]),

$$\mathbb{E}\Big(\sup_{|u|=1} G_u\Big) \le \sqrt{2}\,\sigma\,\mathbb{E}\Big(\sup_{|u|=1} H_u\Big) \le 2\sqrt{2}\,\sigma\,\mathbb{E}\bigg(\bigg[\sum_{i=1}^N g_i^2\bigg]^{1/2}\bigg).$$

When $\sigma^2 = \frac{1}{4N}$, we thus get that

$$\mathbb{E}\big(\lambda_{\max}^N\big) \le \sqrt{2}. \tag{3.4}$$

Together with the one-sided version of the inequality of Proposition 3.1, for every $r \ge 0$,

$$\mathbb{P}\big(\{\lambda_{\max}^N \ge \sqrt{2} + r\}\big) \le \mathrm{e}^{-2Nr^2}$$

that thus agrees with (2.7) for r large.

It is worthwhile mentioning that in the real GOE case, the comparison theorem may be sharpened into

$$\mathbb{E}\big(\lambda_{\max}^N\big) \le 4\sigma^2 \mathbb{E}\bigg(\bigg[\sum_{i=1}^N g_i^2\bigg]^{1/2}\bigg) < 1 \tag{3.5}$$

(cf. [Da-S]). In particular therefore

$$\mathbb{P}\big(\{\lambda_{\max}^N \ge 1 + r\}\big) \le \mathrm{e}^{-Nr^2}$$

for every $r \ge 0$, which is more directly comparable to the Tracy–Widom theorem. However (3.5) is not sharp enough to reach (2.12).

Bounds such as (3.4) or (3.5) extend to the class of sub-Gaussian distributions (cf. [L-T], [Ta3]) including thus random matrices with symmetric Bernoulli entries. They may then be combined as above with Proposition 3.3 below.

On the basis of the supremum representation (3.3) of the largest eigenvalue or the very definition (1.18) of last passage time in oriented percolation, one may actually wonder whether bounds on the supremum of Gaussian or more general processes $(Z_t)_{t\in T}$ may be useful in this type of investigation. Numerous developments took place in the last decades (cf. [L-T] and the recent monograph [Ta3]) in the analysis of bounds on $\mathbb{E}(\sup_{t\in T} Z_t)$ and $\mathbb{P}(\{\sup_{t\in T} Z_t \geq r\})$, $r \geq 0$, with rather sophisticated chaining arguments involving metric entropy or majorizing measures. For real symmetric matrices X, the task would be for example to investigate processes given by

$$Z_u = uX^N u^* = \sum_{i,j=1}^{N} X_{ij} u_i u_j, \quad |u| = 1,$$

where $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$ and X_{ij}, $1 \leq i \leq j \leq N$, are independent centered either Gaussian or Bernoulli variables, and to study the size of the unit sphere $|u| = 1$ under the L^2-metric

$$\mathbb{E}\big(|Z_u - Z_v|^2\big) = \sum_{i,j=1}^{N} |u_i u_j - v_i v_j|^2, \quad |u| = |v| = 1.$$

While these tools provide general, typically Gaussian, bounds, the unusual and more refined rates from random matrix theory do not seem to have been accessed so far from this point of view. It might be a worthwhile project to investigate this question in more detail.

We now come back to the application of measure concentration to general families of random matrices and random growth functions. This is actually the main interest in the theory. The concentration inequality of Proposition 3.1 indeed applies to large families of both real and complex random matrices, the entries of which form a random vector with a dimension free concentration property. For notational simplicity, we only deal below with real matrices but up to numerical factors, all the results hold similarly in the complex case. That is, we are looking for measures μ on $\mathbb{R}^n$, representing the joint law of the entries of a given matrix, which satisfy, as (3.1) or (3.2) for Gaussian measures, the dimension free concentration inequality

$$\mu\big(\big\{|F - \textstyle\int F d\mu| \geq r\big\}\big) \leq C\, \mathrm{e}^{-r^2/C}, \quad r \geq 0 \tag{3.6}$$

for some $C > 0$ independent of n and every 1-Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$. The mean may be replaced by a median of F. Actually, other tails than Gaussian may be considered, and we refer to [Le1] for a general account on the concentration of measure phenomenon and examples satisfying it. Now, it is immediate (cf. e.g. (3.3)) that the singular values (resp. eigenvalues) of a $N \times N$ matrix X (resp. symmetric matrix) are Lipschitz functions of the vector of the N^2 (resp. $N(N+1)/2$) entries of X. One thus immediately concludes to concentration inequalities of the type of Proposition 3.1 for singular values or eigenvalues of matrices the joint law of the entries satisfying a concentration inequality (3.6). This observation already yields various concentration inequalities for singular values and eigenvalues of families of Gaussian matrices. Another simple example of interest consists of matrices with independent uniform entries (which may be realized as a contraction of Gaussian variables). The following proposition summarizes this conclusion. Note that if $X = (X_{ij})_{1 \le i,j \le N}$ is a real symmetric $N \times N$ random matrix, then its eigenvalues are 1-Lipschitz functions of the entries X_{ii}, $1 \le i \le N$, $\sqrt{2}\, X_{ij}$, $1 \le i < j \le N$ (justifying in particular the normalization of the variances in the GOE). For simplicity, we do not distinguish below between the diagonal and non-diagonal entries, and simply use that the eigenvalues are Lipschitz with a Lipschitz coefficient less than or equal to $\sqrt{2}$ with respect to the vector X_{ij}, $1 \le i \le j \le N$.

Proposition 3.2. *Let $X = (X_{ij})_{1 \le i,j \le N}$ be a real symmetric $N \times N$ random matrix and $Y = (Y_{ij})_{1 \le i,j \le N}$ be a real $N \times N$ random matrix. Assume that the distributions of the random vectors X_{ij}, $1 \le i \le j \le N$, and Y_{ij}, $1 \le i, j \le N$, in respectively $\mathbb{R}^{N(N+1)/2}$ and $\mathbb{R}^{N^2}$ satisfy the dimension free concentration property (3.6). Then, if τ is any eigenvalue of X, respectively singular value of Y, for every $r \ge 0$,*

$$\mathbb{P}\big(\big\{|\tau - \mathbb{E}(\tau)| \ge r\big\}\big) \le C\,\mathrm{e}^{-r^2/2C}, \quad \textit{resp.} \quad C\,\mathrm{e}^{-r^2/C}.$$

We next discuss two examples of distributions satisfying concentration inequalities of the type (3.6) and illustrate there application to matrix models.

A first class of interest consists of measures satisfying a logarithmic Sobolev inequality which form a natural extension of the Gaussian example. A probability measure μ on $\mathbb{R}$ or $\mathbb{R}^n$ is said to satisfy a logarithmic Sobolev inequality if for some constant $C > 0$

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \le 2C \int_{\mathbb{R}^n} |\nabla f|^2 d\mu \tag{3.7}$$

for every smooth enough function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\int f^2 d\mu = 1$. The prototype example is the standard Gaussian measure on $\mathbb{R}^n$ which satifies (3.7) with $C = 1$. Another example consists of probability measures on $\mathbb{R}^n$ of the type $d\mu(x) = \mathrm{e}^{-V(x)} dx$ where $V - c\,(|x|^2/2)$ is convex for some $c > 0$ which satisfy (3.7) for $C = 1/c$. An important aspect of the logarithmic Sobolev inequality is its stability by product that yields dimension free constants. That is, if $\mu_1, \ldots, \mu_n$ are probability measures on $\mathbb{R}$ satisfying the logarithmic Sobolev inequality (3.7) with the same constant C, then the product measure $\mu_1 \otimes \cdots \otimes \mu_n$ also satisfies it (on $\mathbb{R}^n$) with the same constant. The application of logarithmic Sobolev inequalities to measure concentration is developed by the so-called Herbst argument that indicates that if μ satisfies (3.7), then for any 1-Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$ and any $\lambda \in \mathbb{R}$,

$$\int \mathrm{e}^{\lambda F} d\mu \le \mathrm{e}^{\lambda \int F d\mu + C\lambda^2/2}.$$

In particular, by a simple use of Markov's exponential inequality (for both F and $-F$), for any $r \ge 0$,

$$\mu\big(\{|F - \textstyle\int F d\mu| \geq r\}\big) \leq 2\,\mathrm{e}^{-r^2/2C},$$

so that the dimension free concentration property (3.6) holds. We refer to [Le1] for a complete discussion on logarithmic Sobolev inequalities and measure concentration. Related Poincaré inequalities, in connection with variance bounds and exponential concentration, may be considered similarly in this context and in the applications below.

As a consequence of this discussion, if $X = (X_{ij})_{1\leq i,j\leq N}$ is a real symmetric $N \times N$ random matrix and $Y = (Y_{ij})_{1\leq i,j\leq N}$ a real $N \times N$ random matrix such that the entries X_{ij}, $1 \leq i \leq j \leq N$ and Y_{ij}, $1 \leq i, j \leq N$ define random vectors in respectively $\mathbb{R}^{N(N+1)/2}$ and $\mathbb{R}^{N^2}$ the law of which satisfy the logarithmic Sobolev inequality (3.7), then the conclusion of Proposition 3.2 holds. By the product property of logarithmic Sobolev inequalities, this is in particular the case if the variables X_{ij} and Y_{ij} are independent and satisfy (3.7) with a common constant C (for example, they have a common distribution $\mathrm{e}^{-v}dx$ where $v'' \geq c = \frac{1}{C} > 0$). In particular thus, if $\lambda^N_{\max}$ denotes the largest eigenvalue of X,

$$\mathbb{P}\Big(\Big\{\big|\lambda^N_{\max} - \mathbb{E}(\lambda^N_{\max})\big| \geq r\Big\}\Big) \leq 2\,\mathrm{e}^{-r^2/4C}, \quad r \geq 0$$

for every $r \geq 0$.

Another family of interest are product measures. The application of measure concentration to this class however requires an additional convexity assumption on the functionals. Indeed, if μ is a product measure on $\mathbb{R}^n$ with compactly supported factors, a fundamental result of M. Talagrand [Ta2] shows that (3.2) holds for every Lipschitz convex function. More precisely, assume that $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ where each μ_i is supported on $[a, b]$. Then, for every 1-Lipschitz convex function $F : \mathbb{R}^n \to \mathbb{R}$,

$$\mu\big(\{|F - m| \geq r\}\big) \leq 4\,\mathrm{e}^{-r^2/4(b-a)^2} \tag{3.8}$$

where m is a median of F for μ. (Classical arguments, cf. [Le1], allow for the replacement of m by the mean of F up to numerical constants.) Since, by the variational characterization, the largest eigenvalue $\lambda^N_{\max}$ of symmetric (or Hermitian) matrices is clearly a convex function of the entries, such a statement may immediately be applied to yield concentration inequalities similar to Propositions 3.1 and 3.2.

Proposition 3.3. *Let X be a real symmetric $N \times N$ matrix such that the entries X_{ij}, $1 \leq i \leq j \leq N$, are independent random variables with $|X_{ij}| \leq 1$. Denote by $\lambda^N_{\max}$ the largest eigenvalue of X. Then, for any $r \geq 0$,*

$$\mathbb{P}\Big(\big\{|\lambda^N_{\max} - M| \geq r\big\}\Big) \leq 4\,\mathrm{e}^{-r^2/32}$$

where M is a median of $\lambda^N_{\max}$.

Up to some numerical constants, the median may be replaced by the mean. A similar result is expected for all the eigenvalues. A partial result in [A-K-V] yields a bound of the order of $4\,\mathrm{e}^{-r^2/32\min(k,N-k+1)^2}$ on the k-th largest eigenvalue which, for k far from 1 or N is much bigger than the corresponding one in the Gaussian case for example. The analogous question for singular values (in particular the smallest one) in this context seems also to be open. Further inequalities on eigenvalues and norms following this principle, together with additional material, are discussed in [Mec] and [G-P].

We refer to [Ta2], [Le1] for further examples of distributions with the concentration property.

Similar measure concentration tools may be developed at the level of random growth functions. Consider for example an array $(w_{ij})_{1\le i\le M, 1\le j\le N}$ of real-valued random variables and let, as in the preceding sections,

$$W = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}$$

where the sup runs over all up/right paths from $(1,1)$ to (M,N). It is clear that $F(x) = \sup \sum_{(i,j)\in\pi} x_{ij}$ is a Lipschitz map of the MN coordinates $(x_{ij})_{1\le i\le M, 1\le j\le N}$ with Lipschitz constant $\sqrt{M+N-1}$. The following statement is thus an immediate consequence of the basic concentration principle. It applies thus in particular to independent Gaussian variables, or more general distributions satisfying a logarithmic Sobolev inequality. Since F is clearly a convex function of the coordinates, the result also applies to independent random variables with compact supports (such as for example Bernoulli variables).

Proposition 3.4. *Let $(w_{ij})_{1\le i\le M, 1\le j\le N}$ be a set of real-valued random variables such that the distribution on $\mathbb{R}^{MN}$ satisfies the concentration property (3.6) for all Lipschitz convex functions. Then, for any $r\ge 0$,*

$$\mathbb{P}\Big(\big\{\big|W - \mathbb{E}(W)\big| \ge r\big\}\Big) \le C\,\mathrm{e}^{-r^2/C(M+N-1)}.$$

While again of interest, and of rather wide applicability, this exponential bound however does not describe the expected rate drawn from the Meixner model as examined in the previous sections. In particular, the variance growth drawn from Proposition 3.4 with $M = N$ only yields $\mathrm{var}\,(W) \le C'\,N$ (where $C' > 0$ only depends on C) while it is expected to be of the order of $N^{2/3}$ (cf. Corollary 2.5). Similar comments apply to the concentration inequalities for the length of the longest increasing subsequence investigated in [Ta2] which do not match the Baik–Deift–Johansson theorem (1.21). Indeed, building on the general principle underlying (3.8), M. Talagrand got for example that

$$\mathbb{P}\big(\big\{|L_n - m| \ge r\big\}\big) \le 4\,\mathrm{e}^{-r^2/8m}$$

for $0 \le r \le m$.

3.2 Concentration Inequalities for Spectral Distributions

The general concentration principles do not yield the correct small deviation rate at the level of the largest eigenvalues. They however apply to large classes of Lipschitz functions. In particular, as investigated by A. Guionnet and O. Zeitouni [G-Z], applications to functionals of the spectral measure yield sharp exponential bounds in accordance with the large deviation asymptotics for empirical measures (cf. Section 1.4). For example, if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, it is not difficult to check that $F = 1/\sqrt{N} \sum_{i=1}^{N} f(\lambda_i^N)$ is a Lipschitz function of the (real and imaginary) entries of X^N. Moreover, if f is convex on the real line, then F is convex on the space of matrices (Klein's lemma). Therefore, the general concentration principle may be applied to functions of the spectral measure. For example, if X is a GUE random matrix with variance $\sigma^2 = \frac{1}{4N}$, and if $f : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz, as a consequence of (3.2), for any $r \geq 0$,

$$\mathbb{P}\bigg(\bigg\{\bigg|\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i^N) - \int f d\mu^N\bigg| \geq r\bigg\}\bigg) \leq 2\,\mathrm{e}^{-2N^2r^2} \tag{3.9}$$

(where we recall that μ^N is the mean spectral measure (1.4)). Inequality (3.9) is in accordance with the N^2 speed of the large deviation principles for spectral measures. With the additional assumption of convexity on f, similar inequalities hold for real or complex matrices the entries of which are independent with bounded support. The various examples of distributions with the measure concentration property discussed for example in the previous sections may thus be developed similarly at the level of the spectral measures, and Propositions 3.2 and 3.3 have immediate counterparts for the Lipschitz functions F as above. We may for example state the following.

Proposition 3.5. *Let $X = (X_{ij})_{1 \leq i,j \leq N}$ be a real symmetric $N \times N$ random matrix. Assume that the distribution of the random vector X_{ij}, $1 \leq i \leq j \leq N$, in $\mathbb{R}^{N(N+1)/2}$ satisfy the dimension free concentration property (3.6) for all Lipschitz (resp. Lipschitz and convex) functions Then, for any 1-Lipschitz (resp. 1-Lipschitz and convex) function $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathbb{P}\bigg(\bigg\{\bigg|\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i^N) - \int f d\mu^N\bigg| \geq r\bigg\}\bigg) \leq C\,\mathrm{e}^{-Nr^2/2C}$$

for all $r \geq 0$.

Extended inequalities have been investigated along these lines in [G-Z] to which we refer for further applications to various families of random matrices.

Interestingly enough, these concentration inequalities may be used to improve the Wigner theorem from the statement on the mean spectral measure to the almost sure conclusion. For example, in the context of the GUE, as a consequence of (3.9),

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i^N) - \int f d\mu^N \to 0$$

almost surely for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$. Assuming that $\mu^N \to \nu$, the semicircle law, it easily follows after a density argument that, almost surely,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i^N} \to \nu$$

weakly as probability measures on $\mathbb{R}$.

3.3 Concentration Inequalities for Coulomb Gas

The GUE model shares both the structure of a Wigner matrix with independent entries and the one of a unitary invariant ensemble. As a unitary ensemble, we have seen in Part 1 how the joint eigenvalue distribution may be represented as a Coulomb gas (1.16). Under suitable convexity assumption on the underlying potential, Coulomb gas actually also share concentration properties which follow from general convexity principles.

Let indeed, as in (1.16),

$$dQ(x) = \frac{1}{Z}\,\big|\Delta_N(x)\big|^\beta d\rho(x)$$

where ρ is a probability measure on $\mathbb{R}^N$ and $Z = Z_N = \int |\Delta_N|^\beta d\rho < \infty$ the normalization constant. For particular values of $\beta > 0$ and suitable distributions ρ, Q thus represents the eigenvalue distribution of some random matrix model. We consider probability measures ρ given by $d\rho = \mathrm{e}^{-V}dx$ for some symmetric (invariant by permutation of the coordinates) potential $V : \mathbb{R}^N \to \mathbb{R}$. Typically, in the context of eigenvalues of random matrix models, $V(x) = \sum_{i=1}^N v(x_i)$, $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, where $v : \mathbb{R} \to \mathbb{R}$ is the underlying potential of the matrix distribution $\exp(-\mathrm{Tr}\, v(X))dX$. Assume now that $V(x) - c\,\frac{|x|^2}{2}$ is convex for some $c > 0$. For example, if

$$V(x) = \frac{|x|^2}{2} = \frac{1}{2}\sum_{i=1}^{N} x_i^2,$$

we would deal with the joint eigenvalue distribution of the GUE. By exchangeability, we may describe equivalently the measure Q by

$$dQ(x) = \frac{N!}{Z}\,\Delta_N(x)^\beta\, \mathbf{1}_E\, d\rho(x) \tag{3.10}$$

where $E = \{x \in \mathbb{R}^N; x_1 < \cdots < x_N\}$. Now, $\log \Delta_N^\beta$ is concave on the convex set E, so that the probability measure Q of (3.10) enters the general setting of probability measures with density e^{-U}, U strictly convex, on a convex set in $\mathbb{R}^N$. The general theory of the Prékopa–Leindler and transportation cost inequalities as presented in [Le1] then shows that Q satisfies a Gaussian like concentration inequality for Lipschitz functions. The next statement describes the result.

Proposition 3.6. *Let Q be defined by (3.10) with $d\rho = \mathrm{e}^{-V}dx$ where V is symmetric and such that $V(x) - c\,(|x|^2/2)$ is convex for some $c > 0$. Then, for any 1-Lipschitz function $F : \mathbb{R}^N \to \mathbb{R}$ and any $r \geq 0$,*

$$Q\Big(\big\{|F - \textstyle\int F dQ| \geq r\big\}\Big) \leq 2\,\mathrm{e}^{-r^2/2c}.$$

Applied to the particular Lipschitz function given by $\max_{1 \leq i \leq N} x_i$, we recover Proposition 3.1 for the GUE, which thus applies to more general orthogonal of unitary ensembles with a strictly convex potential. Proposition 3.6 may also be used to cover the concentration inequalities for Lipschitz functions of the spectral measure of the preceding section. However, again, the Tracy–Widom rate does not seem to follow from this description. It actually appears that in distributions $dQ(x) = \frac{1}{Z}\,\big|\Delta_N(x)\big|^\beta d\rho(x)$, the important factor is the Vandermonde determinant Δ_N and not the underlying probability measure ρ, while in the concentration approach, we rather focus on ρ.

4 Hypercontractive Methods

We presented in Part 2 the known asymptotic exponential deviation inequalities on largest eigenvalues and last-passage times. As described there, these actually follow from quite refined methods and results. The aim of this part and the next one is to suggest some more accessible tools to reach some of these bounds (or parts of them) at the correct small deviation order. The tools developed here are of functional analytic flavour, with a particular emphasis on hypercontractive methods. They however still rely on the orthogonal polynomial representation.

In the first part, we present an elementary approach, relying on the hypercontractivity property of the Hermite semigroup, to the small deviation inequality of Proposition 2.1 for the largest eigenvalue of the GUE. We then investigate, following the recent contribution [B-K-S] by I. Benjamini, G. Kalai and O. Schramm, variance bounds for directed last-passage percolation with the same tool of hypercontractivity.

4.1 Upper Tails on the Right of the Mean

We first briefly describe the semigroup tools we will be using. Consider the Hermite of Ornstein–Uhlenbeck operator

$$\mathcal{L}f = \Delta f - x \cdot \nabla f$$

acting on smooth functions $f : \mathbb{R}^n \to \mathbb{R}$. It satisfies the integration by parts formula

$$\int f(-\mathcal{L}g)d\mu = \int \nabla f \cdot \nabla g d\mu \tag{4.1}$$

for smooth functions f, g on $\mathbb{R}^n$ with respect to the standard Gaussian measure μ on $\mathbb{R}^n$. The associated semigroup $\mathcal{P}_t = \mathrm{e}^{t\mathcal{L}}$, $t \geq 0$, solution of the heat equation $\frac{\partial}{\partial t} = \mathcal{L}$, is, in this case, explicitly described by the integral representation

$$\mathcal{P}_t f(x) = \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t}x + (1-\mathrm{e}^{-2t})^{1/2}y\big)d\mu(y), \quad t \geq 0,\ x \in \mathbb{R}^n. \tag{4.2}$$

Note that $\mathcal{P}_0 f = f$ and $\mathcal{P}_t f \to \int f d\mu$ (for suitable f's).

To illustrate $\mathcal{L}$ and $\mathcal{P}_t$ in the one-dimensional case, recall the generating function of the (normalized) Hermite polynomials P_ℓ, $\ell \in \mathbb{N}$, on the real line is given by

$$\mathrm{e}^{\lambda x - \lambda^2/2} = \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\sqrt{\ell!}} P_\ell(x), \quad \lambda, x \in \mathbb{R}.$$

Since for every $t \geq 0$ and $\lambda \in \mathbb{R}$,

$$\mathcal{P}_t(\mathrm{e}^{\lambda x - \lambda^2/2}) = \mathrm{e}^{(\lambda \mathrm{e}^{-t})x - (\lambda \mathrm{e}^{-t})^2/2},$$

it follows that $\mathcal{P}_t(P_\ell) = \mathrm{e}^{-\ell t}P_\ell$, $\ell \in \mathbb{N}$. Hence the Hermite polynomials are the eigenfunctions of $\mathcal{L}$, with eigenvalues $-\ell$, $\ell \in \mathbb{N}$ ($\mathcal{L}$ is sometimes called the number operator).

The central tool in this section is the celebrated hypercontractivity property of the Hermite semigroup first put forward by E. Nelson [Ne] in quantum field theory. It expresses that, for any function f (in L^p),

$$\|\mathcal{P}_t f\|_q \leq \|f\|_p \tag{4.3}$$

for every $1 < p < q < \infty$ and $t > 0$ such that $\mathrm{e}^{2t} \geq \frac{q-1}{p-1}$ (cf. [Ba]). L^p-norms are understood here with respect to the Gaussian measure μ.

For comparison, it might be worthwhile mentioning that hypercontractivity has been shown by L. Gross [Gr] to be equivalent to the logarithmic Sobolev inequality (3.7) (with $C = 1$) for the standard normal distribution μ, in actually the general setting of Markov operators (cf. [Bak]).

We now make use of hypercontractivity to reach small deviation inequalities for the largest eigenvalues of the GUE. We follow the note [Le2]. Recall thus X from the GUE with $\sigma^2 = \frac{1}{4N}$, with eigenvalues $\lambda_1^N, \ldots, \lambda_N^N$. The starting point is the representation (1.13) of the spectral measure μ^N in terms of the Hermite polynomials and the simple union bound

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq t\}\big) \leq N\mu^N\big([t,\infty)\big), \quad t \in \mathbb{R},\ N \geq 1. \tag{4.4}$$

Let $N \geq 1$. As a consequence, for every $\varepsilon > 0$ (recall $\sigma^2 = \frac{1}{4N}$),

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1 + \varepsilon\}\big) \leq \int_{2\sqrt{N}(1+\varepsilon)}^{\infty} \sum_{\ell=0}^{N-1} P_\ell^2 d\mu \tag{4.5}$$

(where μ is here the standard Gaussian measure on $\mathbb{R}$). Now, by Hölder's inequality, for every $r > 1$, and every $\ell = 0, \ldots, N-1$,

$$\begin{aligned}\int_{2\sqrt{N}(1+\varepsilon)}^{\infty} P_\ell^2 d\mu &\leq \mu\big([2\sqrt{N}(1+\varepsilon), \infty)\big)^{1-(1/r)} \, \|P_\ell\|_{2r}^2 \\ &\leq \mathrm{e}^{-2N(1+\varepsilon)^2(1-\frac{1}{r})} \, \|P_\ell\|_{2r}^2\end{aligned}$$

where we used the standard bound on the tail of the Gaussian measure μ. Since as we have seen $\mathcal{P}_t(P_\ell) = \mathrm{e}^{-\ell t} P_\ell$, it follows from the hypercontractivity property (4.3) that for every $r > 1$ and $\ell \geq 0$,

$$\|P_\ell\|_{2r} \leq (2r-1)^{\ell/2}.$$

Hence,

$$\begin{aligned}\int_{2\sqrt{N}(1+\varepsilon)}^{\infty} \sum_{\ell=0}^{N-1} P_\ell^2 d\mu &\leq \mathrm{e}^{-2N(1+\varepsilon)^2(1-\frac{1}{r})} \sum_{\ell=0}^{N-1} (2r-1)^\ell \\ &\leq \frac{1}{2(r-1)} \, \mathrm{e}^{-2N(1+\varepsilon)^2(1-\frac{1}{r}) + N\log(2r-1)}.\end{aligned}$$

Optimizing in $r \to 1$ then shows, after a Taylor expansion of $\log(2r-1)$ at the third order, that for some numerical constant $C > 0$ and all $0 < \varepsilon \leq 1$,

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1 + \varepsilon\}\big) \leq C\,\varepsilon^{-1/2}\,\mathrm{e}^{-N\varepsilon^{3/2}/C} \tag{4.6}$$

for $C > 0$ numerical. Up to some polynomial factor, we thus recover the content of Proposition 2.1. (The argument is easily extended to also include the large deviation behavior of the order of ε^2 (cf. (2.7) and Section 2.3).

The same strategy may be developed similarly for orthogonal polynomial ensembles which may be diagonalized by an hypercontractive operator [Le2]. This is the case for example of the Laguerre operator, so that this approach yields exponential deviation inequalities for the largest eigenvalue of Wishart matrices or last-passage times for exponential random variables. The class of interest seems however to be restricted to the classical examples of Hermite, Laguerre and Jacobi polynomials [Ma]. Even the application of the method to discrete orthogonal polynomial ensembles does not seem to be clear.

4.2 Variance Bounds

This section is devoted to the question of the variance growth of last-passage time functions for more general distributions than geometric and exponential. We follow here a recent contribution by I. Benjamini, G. Kalai and O.

Schramm [B-K-S] who proved sub-linear growth by means of hypercontractive tools. In connection with the growth models discussed in the preceding sections, we only investigate here the directed percolation model. Furthermore, for simplicity again, we restrict ourselves in the exposition of this result to a Gaussian setting. It actually holds for a variety of examples discussed at the end of the section.

Consider thus, as in Section 3.1, an array $(w_{ij})_{1\leq i\leq M, 1\leq j\leq N}$ of independent standard Gaussian random variables and let

$$W = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}$$

where the maximum runs over all up/right paths from $(1,1)$ to (M,N). Assume furthermore for simplicity that $M = N$. We saw from the general concentration bounds in Part 3 that $\mathrm{var}\,(W) \leq CN$ (while it is expected to be of the order of $N^{2/3}$ by Corollary 2.5). We provide here, following [B-K-S], a slight, but significant improvement.

For a suitably integrable function $f : \mathbb{R}^{N^2} \to \mathbb{R}$, denote by

$$\mathrm{var}_\mu(f) = \int f^2 d\mu - \left(\int f d\mu\right)^2$$

its variance with respect to the standard Gaussian measure μ on $\mathbb{R}^{N^2}$. When f is smooth enough, the heat equation for the Ornstein–Uhlenbeck semigroup $(\mathcal{P})_{t\geq 0}$ with generator $\mathcal{L}$ on $\mathbb{R}^{N^2}$ allows one to write

$$\begin{aligned}
\mathrm{var}_\mu(f) &= -\int_0^\infty dt\, \frac{d}{dt} \int (\mathcal{P}_t f)^2 d\mu \\
&= 2\int_0^\infty dt \int \mathcal{P}_t f(-\mathcal{L}\mathcal{P}_t f) d\mu \\
&= 2\sum_{i,j=1}^N \int_0^\infty dt \int (\partial_{ij}\mathcal{P}_t f)^2 d\mu.
\end{aligned}$$

From the integral representation (4.2) of the Ornstein–Uhlenbeck semigroup,

$$|\partial_{ij}\mathcal{P}_t f| \leq \mathrm{e}^{-t}\, \mathcal{P}_t\big(|\partial_{ij} f|\big), \quad i,j = 1,\ldots,N,\ t\geq 0. \tag{4.7}$$

Hence, together with hypercontractivity (4.3), for every $t\geq 0$ and every $i,j = 1,\ldots,N$,

$$\int (\partial_{ij}\mathcal{P}_t f)^2 d\mu \leq e^{-2t} \int \big[\mathcal{P}_t\big(|\partial_{ij} f|\big)\big]^2 d\mu \leq e^{-2t} \|\partial_{ij} f\|^2_{1+e^{-2t}}\,.$$

Setting $u = e^{-2t}$, and $v = u+1$,

$$\mathrm{var}_\mu(f) \leq \sum_{i,j=1}^N \int_0^1 \|\partial_{ij} f\|^2_{1+u} du = \sum_{i,j=1}^N \int_1^2 \|\partial_{ij} f\|^2_v\, dv. \tag{4.8}$$

By a simple upper-bound on the right-hand side, this inequality may also be written as

$$\mathrm{Var}_\mu(f) \le 4 \sum_{i,j=1}^{N} \frac{\|\partial_{ij} f\|_2^2}{1 + \log\left(\|\partial_{ij} f\|_2^2 / \|\partial_{ij} f\|_1^2\right)} \,. \tag{4.9}$$

Inequality (4.9) is actually due to M. Talagrand [Ta1] on the discrete cube, and was investigated in [B-H] in the Gaussian case as a dual version of the logarithmic Sobolev (3.7).

Recall now the (Lipschitz) function

$$F(x) = \max_{\pi} \sum_{(i,j)\in\pi} x_{ij}, \quad x = (x_{ij})_{0\le i,j\le N} \in \mathbb{R}^{N^2}.$$

Define, for each up/right path π from $(1,1)$ to (N,N),

$$A_\pi = \left\{ x : F(x) = \sum_{(i,j)\in\pi} x_{ij} \right\}.$$

The sets A_π are actually intersections of subsets of $\mathbb{R}^{N^2}$ delimitated by subspaces of lower dimension. In particular, since μ is absolutely continuous, $\mu(A_\pi \cap A_{\pi'}) = 0$ whenever $\pi \ne \pi'$. Hence, almost everywhere,

$$F = \sum_{\pi} \sum_{(i,j)\in\pi} x_{ij} \mathbf{1}_{A_\pi}.$$

On the interior of A_π,

$$\partial_{ij} F = \sum_{\pi \ni (i,j)} \mathbf{1}_{A_\pi}$$

for every $i,j = 1,\dots,N$. Now $\mu(A_\pi)$ is independent of π and thus equal to the inverse of the total number

$$\Pi = \binom{2N-2}{N-1}$$

of up/right paths from $(1,1)$ to (N,N). Furthermore, for every $p > 0$,

$$\int |\partial_{ij} F|^p d\mu = \sum_{\pi \ni (i,j)} \mu(A_\pi) = \frac{\Pi_{ij}}{\Pi}$$

where

$$\Pi_{ij} = \binom{i+j-2}{i-1} \binom{2N-i-j}{N-i}$$

is the number of up/right paths from $(1,1)$ to (N,N) going through (i,j). Therefore, by (4.8) applied to F,

$$\begin{aligned}\mathrm{var}_\mu(F) &\le \sum_{i,j=1}^{N} \int_1^2 \left(\frac{\Pi_{ij}}{\Pi}\right)^{2/v} dv \\ &\le 4 \sum_{i,j=1}^{N} \frac{\Pi_{ij}}{\Pi} \left(\log \frac{\Pi}{\Pi_{ij}}\right)^{-1} \\ &\le 4 \sum_{k=1}^{2N} \sum_{i=1}^{k-1} p_i^k (\log p_i^k)^{-1}\end{aligned}$$

where, for each $k = 1, \dots, 2N$,

$$p_i^k = \binom{2N-2}{N-1}^{-1} \binom{k-2}{i-1} \binom{2N-k}{N-i}, \quad i = 1, \dots, k-1,$$

are the probabilities of an hypergeometric distribution. Equivalently, one may use directly (4.9). Now, it is easily checked by Stirling's asymptotics that p_i^k is bounded above, for every $i = 1, \dots, k-1$, by a constant times

$$\left(\frac{N}{k(2N-k-2)}\right)^{1/2}.$$

We thus easily conclude to the main result of [B-K-S].

Proposition 4.1. *Let W be the directed last-passage time of an array of independent standard Gaussian random variables on the square from* $(1,1)$ *to* (N,N)*,* $N \ge 2$*. Then*

$$\mathrm{var}\,(W) \le \frac{CN}{\log N}$$

where $C > 0$ *is numerical.*

This result was actually established in [B-K-S] for general (not necessarily oriented) percolation of Bernoulli variables w_{ij}, the scheme of proof being similar with however a further twist at the level of the partition (A_π). The preceding approach applies more generally to examples where both the hypercontractive bound and the commutation property (4.7) may be applied. One instance would be the example of uniform random variables. A further example is the case of exponential variables, for which however the much stronger Corollary 2.5 is available.

5 Moment Methods

In this part, we take a somewhat different route from the one of Part 4 and concentrate on moments of the spectral distribution. Moment methods and combinatorial arguments are at the roots of the study of random matrix models, and for example are typically used in proofs of Wigner's theorem (cf.

Section 1.1). The combinatorial part has been significantly improved by A. Soshnikov in [So1] to reach fluctuation results. Here, we again make advantage of the orthogonal polynomial structure to derive recurrence equations for moments, which may be shown of interest in non-asymptotic deviation inequalities. The strategy is based on integration by parts for the underlying Markov operator of the orthogonal polynomial ensemble.

In the first paragraph, we derive in this way the moment equations of the GUE model using simple integration by parts arguments for the Hermite operator. We then emphasize their usefulness in non-asymptotic deviation inequalities on the largest eigenvalues. Below the mean, we follow an argument by H. Widom relying on a simple trace inequality.

5.1 Moment Recurrence Equations

Let μ denote again the standard Gaussian measure on $\mathbb{R}$. By integration by parts, for every smooth function f on $\mathbb{R}$,

$$\int xf d\mu = \int f' d\mu. \tag{5.1}$$

(This formula is actually a particular case of the integration by parts formula (4.1) applied to $g = P_1 = x$ the first eigenvector of the one-dimensional Ornstein–Uhlenbeck operator $\mathcal{L}f = f'' - xf'$ with eigenvalue 1.)

By (1.13), moments of the mean spectral measure (1.4) amounts to moments of orthogonal polynomial measures. In this direction, we examine first a reduced case. Let

$$a_p = a_p^N = \int x^{2p} P_N^2 d\mu, \quad p \in \mathbb{N},$$

where we recall that the Hermite polynomials P_ℓ, $\ell \in \mathbb{N}$, are normalized in $\mathrm{L}^2(\mu)$. (The odd moments are zero by symmetry.) By (5.1),

$$a_p = \int x\, x^{2p-1} P_N^2 d\mu = (2p-1)a_{p-1} + 2\int x^{2p-1} P_N P_N' d\mu. \tag{5.2}$$

Repeating the same step,

$$\begin{aligned} a_p - (4p-3)a_{p-1} &+ (2p-1)(2p-3)a_{p-2} \\ &= 2\int x^{2p-2} {P_N'}^2 d\mu + 2\int x^{2p-2} P_N P_N'' d\mu. \end{aligned} \tag{5.3}$$

Now, P_N is an eigenfunction of $-\mathcal{L}$ with eigenvalue N. Thus, by the integration by parts formula (4.1) for $\mathcal{L}$,

$$Na_p = \int x^{2p} P_N(-\mathcal{L}P_N) d\mu = 2p\int x^{2p-1} P_N P_N' d\mu + \int x^{2p} {P_N'}^2 d\mu.$$

Together with (5.2),

$$\int x^{2p} {P'_N}^2 d\mu = (N-p)a_p + p(2p-1)a_{p-1}. \tag{5.4}$$

In the same way, on the basis of (5.2),

$$\begin{aligned} N\big[a_p - (2p-1)a_{p-1}\big] &= 2\int x^{2p-1}(-\mathcal{L}P_N)P'_N d\mu \\ &= 2(2p-1)\int x^{2p-2}{P'_N}^2 d\mu + 2\int x^{2p-1}P'_N P''_N d\mu. \end{aligned}$$

Now, since $P'_N = \sqrt{N}\,P_{N-1}$ (which may be checked from the generating function of the Hermite polynomials) is eigenfunction of $-\mathcal{L}$ with eigenvalue $N-1$, we also have that

$$\begin{aligned} &(N-1)\big[a_p - (2p-1)a_{p-1}\big] \\ &= 2\int x^{2p-1}P_N(-\mathcal{L}P'_N)d\mu \\ &= 2(2p-1)\int x^{2p-2}P_N P''_N d\mu + 2\int x^{2p-1}P'_N P''_N d\mu. \end{aligned}$$

Substracting to the latter,

$$a_p - (2p-1)a_{p-1} = 2(2p-1)\int x^{2p-2}{P'_N}^2 d\mu - 2(2p-1)\int x^{2p-2}P_N P''_N d\mu,$$

so that, by (5.4),

$$\begin{aligned} 2(2p-1)\textstyle\int x^{2p-2}P_N P''_N d\mu = &-a_p + (2p-1)(2N-2p+1)a_{p-1} \\ &+(2p-1)(2p-2)(2p-3)a_{p-2}. \end{aligned} \tag{5.5}$$

Plugging (5.5) and (5.4) into (5.3) finally shows the recurrence equations

$$a_p = (4N+2)\,\frac{2p-1}{2p}\,a_{p-1} + \frac{(2p-1)(2p-2)(2p-3)}{2p}\,a_{p-2} \tag{5.6}$$

($a_0 = 1$, $a_1 = 2N+1$).

We now make use of the following elementary lemma that appears as a version in this context of the classical Christoffel–Darboux formula (1.15).

Lemma 5.1. *For every integer $k \geq 1$, and every $N \geq 1$,*

$$k\int x^{k-1}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu = \sqrt{N}\int x^k P_N P_{N-1} d\mu.$$

Proof. Let $\mathcal{A}$ be the first order operator $\mathcal{A}f = f' - xf$ acting on smooth functions f on the real line $\mathbb{R}$. The integration by parts formula for $\mathcal{A}$ (analogous to (4.1)) indicates that for smooth functions f and g,

$$\int g(-\mathcal{A}f)d\mu = \int g' f d\mu.$$

Since $-\mathcal{L}P_N = NP_N$ and $P'_N = \sqrt{N}P_{N-1}$ for every $N \geq 1$, the recurrence relation for the (normalized) Hermite polynomials P_N, takes the form

$$xP_N = \sqrt{N+1}\,P_{N+1} + \sqrt{N}\,P_{N-1}.$$

Hence,

$$\mathcal{A}(P_N^2) = P_N\big[2P'_N - xP_N\big] = \sqrt{N}\,P_N P_{N-1} - \sqrt{N+1}\,P_{N+1}P_N.$$

Therefore,

$$(-\mathcal{A})\bigg(\sum_{\ell=0}^{N-1} P_\ell^2\bigg) = \sqrt{N}\,P_N P_{N-1}$$

from which the conclusion follows from the integration by parts formula for $\mathcal{A}$. □

Recall now the GUE random matrix $X = X^N$ with $\sigma^2 = \frac{1}{4N}$ and $N \geq 1$ fixed. Set, for every integer p,

$$\begin{aligned} b_p = b_p^N &= \frac{1}{N}\,\mathbb{E}\big(\mathrm{Tr}\,(X^{2p})\big) = \mathbb{E}\bigg(\frac{1}{N}\sum_{i=1}^{N}\big(\lambda_i^N\big)^{2p}\bigg) \\ &= \int_{\mathbb{R}}\bigg(\frac{x}{2\sqrt{N}}\bigg)^{2p}\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2\,d\mu \end{aligned}$$

where we used (1.13). (The odd moments are zero by symmetry.) By Lemma 5.1,

$$(2p-1)\int x^{2p-2}\sum_{\ell=0}^{N-1} P_\ell^2\,d\mu = \int x^{2p-1}P_N P'_N d\mu$$

so that, by (5.2),

$$2^{2p-1}N^p(2p-1)b_{p-1} = a_p - (2p-1)a_{p-1}$$

for every $p \geq 1$. As a consequence of (5.6), we may then deduce the following recurrence equations on the moments of X.

Proposition 5.2. *For every integer $p \geq 2$,*

$$b_p = \frac{2p-1}{2p+2}\,b_{p-1} + \frac{2p-1}{2p+2}\cdot\frac{2p-3}{2p}\cdot\frac{p(p-1)}{4N^2}\,b_{p-2}$$

($b_0 = 1$, $b_1 = \frac{1}{4}$.)

This recurrence equation, reminiscent of the three-step recurrence equation for orthogonal polynomials, was first put forward in an algebraic context by J. Harer and D. Zagier [H-Z] (to determine the Euler characteristics of moduli spaces of curves). It is also discussed in the book by M. L. Mehta [Meh]. The proof above is essentially due to U. Haagerup and S. Thorbjørnsen [H-T]. Similar recurrence identities may be established, with the same strategy, for the Laguerre and Jacobi orthogonal polynomials, and thus the corresponding moments of Wishart and Beta matrices [H-T], [Le3].

It should be pointed out that the equation

$$\chi_p = \frac{2p-1}{2p+2}\,\chi_{p-1} = \frac{(2p)!}{2^{2p}p!(p+1)!} \tag{5.7}$$

is the recurrence relation of the (even) moments of the semicircle law (the so-called Catalan numbers, the number of non-crossing pair partitions of $\{1, 2, \dots, 2p\}$). In particular, Proposition 5.2 may then be used to produce a quick proof of the Wigner theorem, showing namely that $b_p^N \to \chi_p$ for every p. Moreover, for every fixed p and every $N \geq 1$,

$$\chi_p \leq b_p^N \leq \chi_p + \frac{C_p}{N^2}$$

where $C_p > 0$ only depends on p.

5.2 Upper Tails on the Right of the Mean

Next make use of the recurrence equations of Proposition 5.2 to recover the sharp exponential bounds on the probability that the largest eigenvalues of the GUE matrix exceeds its limiting value discussed by other means in the preceding sections.

We start again from (4.5). Together with Markov's inequality, for every $N \geq 1$, $\varepsilon > 0$ and $p \geq 0$,

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1+\varepsilon\}\big) \leq (1+\varepsilon)^{-2p} N b_p$$

where we recall that $b_p = b_p^N$ are the $2p$-moments of μ^N (or $X = X^N$). Now, by induction on the recurrence formula of Proposition 5.2 for b_p, it follows that, for every $p \geq 2$,

$$b_p \leq \left(1 + \frac{p(p-1)}{4N^2}\right)^p \chi_p. \tag{5.8}$$

By Stirling's formula,

$$\chi_p \leq \frac{C}{p^{3/2}}\,, \quad p \geq 1. \tag{5.9}$$

Hence, for $0 < \varepsilon \leq 1$ and some numerical constant $C > 0$ possibly changing from line to line below,

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1+\varepsilon\}\big) \leq CNp^{-3/2}\,\mathrm{e}^{-\varepsilon p + p^3/4N^2}.$$

Therefore, optimizing in $p \sim \sqrt{\varepsilon}\,N$, $0 < \varepsilon \leq 1$, we recover the sharp small deviation inequality

$$\mathbb{P}\big(\{\lambda_{\max}^N \geq 1+\varepsilon\}\big) \leq C\,\mathrm{e}^{-N\varepsilon^{3/2}/C},$$

$N \geq 1$, $0 < \varepsilon \leq 1$, $C > 0$ numerical, of Proposition 2.1. When $\varepsilon \geq 1$, the optimization is modified to recover the large deviation rate of the order of $N\varepsilon^2$. With respect to the hypercontractive approach of Part 4, no further polynomial factors have to be added.

As observed by S. Szarek in [Sza], the moment recurrence equation of Proposition 5.2 and the preceding argument may be used to reach the sharp upper bound (2.12) on $\mathbb{E}(\lambda_{\max}^N)$, and even

$$\mathbb{E}\Big(\max_{1\leq i\leq N}|\lambda_i^N|\Big) \leq 1 - \frac{1}{CN^{2/3}} \tag{5.10}$$

for some numerical $C > 0$ and all $N \geq 1$. Indeed, for every $p \geq 1$,

$$\mathbb{E}\Big(\max_{1\leq i\leq N}|\lambda_i^N|\Big) \leq \big(Nb_p\big)^{1/2p},$$

and thus, by (5.8),

$$\mathbb{E}\Big(\max_{1\leq i\leq N}|\lambda_i^N|\Big) \leq \big(N\chi_p\,\mathrm{e}^{p^3/4N^2}\big)^{1/2p}. \tag{5.11}$$

If $t > 0$ and $N \geq 1$ are such that $p = [tN^{2/3}] \geq 3$, then, together with (5.9),

$$\mathbb{E}\Big(\max_{1\leq i\leq N}|\lambda_i^N|\Big) \leq \bigg[\Big(\frac{C^{2/3}}{t}\Big)^{3/2t}\mathrm{e}^{t^2/4}\bigg]^{1/2N^{2/3}}.$$

The constant $C > 0$ in (5.9) may be taken to be $\pi^{-1/2}$ so that taking for example $t = C\sqrt{\mathrm{e}}$ shows that the bracket in the preceding inequality is strictly less than 1. Therefore (5.10) holds except for some few values of N which may be checked directly on the basis of (5.11).

As a consequence of (5.8), for every $t > 0$,

$$\sup_{N\geq 1} Nb^N_{[tN^{2/3}]} \leq Ct^{-3/2}\mathrm{e}^{Ct^3} \tag{5.12}$$

for the moments of the GUE. One important step in Soshnikov's extension [So1] of the Tracy–Widom theorem to more general (real or complex) Wigner matrices amounts to establish that $\limsup_{N\to\infty} Nb^N_{[N^{2/3}]} < \infty$, actually

$$\limsup_{N\to\infty} Nb^N_{[tN^{2/3}]} \leq Ct^{-3/2}\mathrm{e}^{Ct^3}$$

for every $t > 0$. This is accomplished through delicate combinatorial arguments on moments. It is however an open question so far whether its non-asymptotic version (5.12) also holds for these families of random matrices, which would then yield the expected deviation inequalities for every fixed $N \geq 1$. It would be of particular interest to study the case of Bernoulli entries.

Very recently, O. Khorunzhiy [Kh] developed Gaussian integration by parts methods at the level of traces, together with triangle recurrence schemes, to reach moment bounds on both the GUE and the GOE. The estimates provide exact expressions for the $1/N$-corrections of the moments, but do not allow yet for the sharp deviation inequalities on largest eigenvalues. This first step outside the orthogonal polynomial method might however be promising.

This strategy relying on integration by parts for Markov generators and moment equations may be used similarly for some other classical orthogonal polynomial ensembles of the continuous variables. For example, the Laguerre and Jacobi ensembles are studied along these lines in [Le3] to yield deviation inequalities at the Tracy–Widom rate of the largest eigenvalue of Wishart and Beta matrices. Discrete examples may also be considered, although not necessarily through recurrence equations, but rather the explicit expression for moments (this is actually also possible in the continuous variable). For example, integration by parts with respect to the negative binomial distribution (1.17) with parameters q and γ reads

$$\int x f d\mu = \frac{\gamma q}{1-q} \int f(x+1) d\mu$$

for any, say, polynomial function f on $\mathbb{N}$. It allows one to express the factorial moment (of order p) of the mean spectral measure of the Meixner Ensemble as

$$\int x(x-1)\cdots(x-p+1)\frac{1}{N}\sum_{\ell=0}^{N-1} P_\ell^2 d\mu = \Big(\frac{q}{1-q}\Big)^p \sum_{i=0}^{p} q^{-i} \binom{p}{i}^2 \frac{1}{N} \sum_{\ell=i}^{N-1} \frac{(\gamma+\ell)_{p-i}\, \ell\,!}{(\ell-i)!} \tag{5.13}$$

where P_ℓ are the associated normalized Meixner polynomials. Therefore, if Q is the Coulomb gas of the Meixner Ensemble, by the union bound inequality(4.4) and the representation (1.13) of the spectral measure, for every $t \geq 0$,

$$\begin{aligned} Q\Big(\big\{\max_{1\leq i\leq N} x_i \geq t\big\}\Big) &\leq \int_t^\infty \sum_{\ell=0}^{N-1} P_\ell^2 d\mu \\ &\leq \frac{(t-p)!}{t!} \Big(\frac{q}{1-q}\Big)^p \sum_{i=0}^{p} q^{-i} \binom{p}{i}^2 \sum_{\ell=i}^{N-1} \frac{(\gamma+\ell)_{p-i}\, \ell\,!}{(\ell-i)!}\,. \end{aligned}$$

Stirling's formula may then be used to control the right-hand side of the latter, and to derive exponential deviation inequalities on the rightmost charge. Together with Johansson's combinatorial formula (1.19), the conclusion of Proposition 2.2 on the random growth functions W may be recovered in this way, avoiding superadditivity and large deviation arguments. In the limit from the Meixner Ensemble to the length of the longest increasing subsequence L_n, it also covers the tail inequality (2.9) (cf. [Le4]). However, the key of the analysis still relies on the orthogonal polynomial representation.

5.3 Upper Tails on the Left of the Mean

We next turn, with the tool of moment identities, to the probability that the largest eigenvalue of the GUE is less than or equal to 1. As discussed in Part 2, bounds on this probability turn out to be much more delicate. We present here a simple inequality, in the context of the GUE, taken from the note [Wid2] by H. Widom.

We start from the determinantal description (1.11)

$$\mathbb{P}\big(\{\lambda_{\max}^N \le t\}\big) = \det\big(\mathrm{Id} - K\big)$$

where, for each $t \in \mathbb{R}$, $K = K_t$ is the symmetric $N \times N$ matrix

$$\big(\langle P_{\ell-1}, P_{k-1}\rangle_{\mathrm{L}^2((t/\sigma,\infty),d\mu)}\big)_{1\le k,\ell\le N}.$$

Since for any unit vector $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$,

$$0 \le \sum_{k,\ell=1}^N u_k u_\ell \langle P_{\ell-1}, P_{k-1}\rangle_{\mathrm{L}^2((t/\sigma,\infty),d\mu)} \le 1,$$

the eigenvalues $\rho_1, \ldots, \rho_N$ of K are all non-negative and less than or equal to 1. Hence,

$$\mathbb{P}\big(\{\lambda_{\max}^N \le t\}\big) = \prod_{i=1}^N (1-\rho_i) \le \mathrm{e}^{-\Sigma_{i=1}^N \rho_i}.$$

Now, by (1.13),

$$\sum_{i=1}^N \rho_i = \sum_{\ell=1}^N \langle P_{\ell-1}^2\rangle_{\mathrm{L}^2((t/\sigma,\infty),d\mu)} = N\mu^N\big((t,\infty)\big).$$

Therefore, for every $t \in \mathbb{R}$,

$$\mathbb{P}\big(\{\lambda_{\max}^N \le t\}\big) \le \exp\Big(-N\mu^N\big((t,\infty)\big)\Big). \tag{5.14}$$

Note that (5.14) would be the inequality that one would deduce if the λ_i^N's were independent. The latter are however strongly correlated so that (5.14) already misses a big deal of the interactions between the eigenvalues.

Let us now apply (5.14) to $t = 1 - \varepsilon$, $0 < \varepsilon \leq 1$. By Wigner's theorem, $\mu^N((1-\varepsilon,\infty)) \to \nu((1-\varepsilon,1))$ where we recall that ν is the semicircle law (on $(-1,+1)$). It is easy to evaluate

$$\nu\big((1-\varepsilon,1)\big) \geq C^{-1}\varepsilon^{3/2}, \quad 0 < \varepsilon \leq 1,$$

where $C > 0$ is numerical. We expect that

$$\mu^N\big((1-\varepsilon,\infty)\big) \geq C^{-1}\varepsilon^{3/2}, \quad 0 < \varepsilon \leq 1, \tag{5.15}$$

at least for every $N \geq 1$ such that $CN^{-2/3} \leq \varepsilon \leq 1$. To this task, we could invoke a recent result of F. Götze and A. Tikhomirov [G-T] on the rate of convergence of the spectral measure of the GUE to the semicircle which implies that

$$\Big|\mu^N\big((1-\varepsilon,\infty)\big) - \nu\big((1-\varepsilon,1)\big)\Big| \leq \frac{C}{N}$$

for some $C > 0$ and all $N \geq 1$, and thus (5.15). While the proof of [G-T] requires quite a bit of analysis, we provide here an independent elementary argument to reach (5.15) using the moment equations.

Fix $N \geq 1$ and $0 < \varepsilon \leq 1$. For every $p \geq 1$,

$$b_{2p} = \int_{\mathbb{R}} x^{4p} d\mu^N(x) \leq (1-\varepsilon)^{2p}\, b_p + 2\int_{1-\varepsilon}^{\infty} x^{4p} d\mu^N(x).$$

From the recurrence equations put forward in Proposition 5.2, for every p,

$$b_{2p} \geq \chi_{2p}$$

while (cf. (5.8))

$$b_p \leq \left(1 + \frac{p(p-1)}{4N^2}\right)^p \chi_p \leq \mathrm{e}^{p^3/4N^2}\, \chi_p.$$

By the Cauchy–Schwarz inequality,

$$\int_{1-\varepsilon}^{\infty} x^{4p} d\mu^N(x) \leq \mu^N\big((1-\varepsilon,\infty)\big)^{1/2} b_{4p}^{1/2}.$$

Hence,

$$\mu^N\big((1-\varepsilon,\infty)\big) \geq 4^{-1}\, \mathrm{e}^{-16p^3/N^2}\, \chi_{4p}^{-1}\Big[\chi_{2p} - \mathrm{e}^{p^3/4N^2}\, \chi_p\Big]^2.$$

Choose then $p = [\varepsilon^{-1}]$ and assume that $N^{-2/3} \leq \varepsilon \leq 1$. Then

$$\mu^N\big((1-\varepsilon,\infty)\big) \geq \mathrm{e}^{-18}\, \chi_{4p}^{-1}\Big[\chi_{2p} - \mathrm{e}^{1/4}\, \chi_p\Big]^2.$$

Since by Stirling's formula $\chi_p \sim \pi^{-1/2}p^{-3/2}$ as $p \to \infty$, uniform bounds show that for some constant $C > 0$,

$$\mu^N\big((1-\varepsilon,\infty)\big) \geq C^{-1}\varepsilon^{3/2}.$$

Together with (5.14), we thus conclude that for some $C > 0$, every $N \geq 1$ and every ε such that $N^{-2/3} \leq \varepsilon \leq 1$,

$$\mathbb{P}\big(\{\lambda^N_{\max} \leq 1-\varepsilon\}\big) \leq C\,\mathrm{e}^{-N\varepsilon^{3/2}/C}. \tag{5.16}$$

Increasing if necessary C, the inequality easily extends to all $0 < \varepsilon \leq 1$. The deviation inequality (5.16) is weaker than the one of Proposition 2.4 and does not reflect the N^2 rate of the large deviation asymptotics. Its proof is however quite accessible, and gives a firm basis to $N^{-4/3}$ growth rate of Corollary 2.6.

The preceding argument may be extended to more general orthogonal polynomial ensembles provided the corresponding version of (5.15) can be established. In case for example of the Meixner Ensemble, the explicit expression (5.13) for the factorial moments of the mean spectral measure might be useful to this task.

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On the Euclidean Metric Entropy of Convex Bodies

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1 Introduction

In this note we study the Euclidean metric entropy of convex bodies and its relation to classical geometric parameters such as diameters of sections or mean widths. We provide an exact analysis of a classical Sudakov's inequality relating Euclidean covering numbers of a body to its mean width, and we obtain some new upper and lower bounds for these covering numbers.

We will explain the subject in a little more detail while briefly describing the organization of the paper. In order to be more precise, let B_2^n denote the unit Euclidean ball in $\mathbb{R}^n$. For a symmetric convex body $K \subset \mathbb{R}^n$ let $N(K, \varepsilon B_2^n)$ be the smallest number of Euclidean balls of radius ε needed to cover K, and finally let $M^*(K)$ be a half of the mean width of K (see (2.1) and (2.2) below).

Section 2 collects the notation and preliminary results used throughout the paper. Sudakov's inequality gives an upper bound for $N(K, tB_2^n)$ in terms of $M^*(K)$, and we show (in Section 3) that if this upper bound is essentially sharp, then diameters of all k-codimensional sections of K are large, for an appropriate choice of k. On the other hand, in Section 4 we discuss conditions that ensure that the covering can be significantly decreased by cutting the body K by a Euclidean ball of a certain radius, in which case "most" of the entropy of K lies outside of this Euclidean ball. In Section 5 this leads to further consequences of sharpness in Sudakov's inequality which turn out

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to be close to a well-known concept of M-position. Finally, in Section 6 we obtain lower estimates for covering numbers $N(K, B_2^n)$ in terms of diameters of sections of a body. It is worthwhile to point out here that the most satisfactory results involve a smaller body T intimately related to K, its skeleton. In Sections 5 and 6 we will use notions of random projections and sections, however we shall not try to specify any probability estimates, as they will be not needed.

2 Notation and Preliminaries

We denote by $|\cdot|$ the canonical Euclidean norm on $\mathbb{R}^n$, by $\langle \cdot, \cdot \rangle$ the canonical inner product and by B_2^n the Euclidean unit ball.

By a convex body we always mean a closed convex set with non-empty interior. By a symmetric convex body we mean centrally symmetric (with respect to the origin) convex body. Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior. The gauge of K is denoted by $\|\cdot\|_K$. The space $\mathbb{R}^n$ endowed with such a gauge is denoted by $(\mathbb{R}^n, \|\cdot\|_K)$ or just by $(\mathbb{R}^n, K)$. The radius of K is the smallest number R such that $K \subset RB_2^n$, and is denoted by $R(K)$. Note that if K is centrally symmetric then $2R(K)$ is the diameter of K.

Let K be a symmetric convex body in $\mathbb{R}^n$ and let $k \geq 1$. By $c_k(K)$ we denote the infimum of $R(K \cap E)$ taken over all $(k-1)$-codimensional subspaces $E \subset \mathbb{R}^n$. Clearly, $2c_1(K) = 2R(K)$ is the diameter of K and $2c_{k+1}(K)$ is the smallest possible diameter of k-codimensional section of K. Below we call $c_{k+1}(K)$ the k-diameter of K.

Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. We denote by $|K|$ the volume of K, and by K^0 the polar of K, i.e.

$$K^0 = \{x \mid \langle x, y \rangle \leq 1 \text{ for every } y \in K\}.$$

Given $\rho > 0$, we denote

$$K_\rho = K \cap \rho B_2^n \quad \text{and} \quad K_\rho^0 = (K_\rho)^0 .$$

Let X be a linear space and K, L be subsets of X. We recall that covering number $N(K, L)$ is defined as the minimal number N such that there exist vectors x_1, ..., x_N in X satisfying

$$K \subset \bigcup_{i=1}^{N} (x_i + L). \tag{2.1}$$

We also will use the notions of ε-net and ε-separated set. Let K, A be sets in $\mathbb{R}^n$ and $\varepsilon > 0$. The set A is called an ε-net for K if $K \subset A + \varepsilon B_2^n$; it is called an ε-separated set for K if $A \subset K$ and for any two different points x and y in A one has $|x - y| > \varepsilon$. It is well known (and easy to check) that any maximal (in sense of inclusion) ε-separated set is an ε-net and that any ε-net has cardinality not smaller than the cardinality of any (2ε)-separated set.

Following [MSz] we say that A is an ε-skeleton of K if $A \subset K$ and A is an ε-net for K. If, in addition, A is convex we say that A is a convex ε-skeleton.

We say that A is an ε-separated skeleton of K if A is a maximal ε-separated set for K. Note that every ε-separated skeleton of K is also an ε-skeleton of K. We say that A is a convex ε-separated skeleton of K if A is the convex hull of an ε-separated skeleton of K. We say that A is an absolute ε-separated skeleton of K if A is the absolute convex hull of an ε-separated skeleton of K.

Given a convex body $K \subset \mathbb{R}^n$ with the origin in its interior, we let

$$M_K = M(K) = \int_{S^{n-1}} \|x\|_K \, d\nu \quad \text{and} \quad \ell(K) = \mathbb{E}\Big\| \sum_{i=1}^n g_i e_i \Big\|_K ,$$

where $d\nu$ is the probability Lebesgue measure on S^{n-1}, and g_i's are $N(0,1)$ Gaussian random variables.

It is well known and easy to check that there exists a positive constant ω_n such that for every convex K one has $\ell(K) = \omega_n \sqrt{n}\, M(K)$. In fact

$$1 - \frac{1}{4n} < \omega_n = \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^n g_i^2 \right)^{1/2} < 1.$$

We also set

$$M_K^* = M^*(K) = M(K^0). \tag{2.2}$$

It is also well known that for any subspace E of $\mathbb{R}^n$ we have $(PK)^0 = K^0 \cap E$ where P is the orthogonal projection on E and the polar of PK is taken in E. In particular we have

$$\ell\big((PK)^0 \big) \le \ell\big(K^0\big)$$

hence,

$$M^*(PK) \le \frac{\omega_n}{\omega_m} \sqrt{\frac{n}{m}}\, M^*(K),$$

where $m = \dim E$.

Recall Urysohn's inequality (see e.g. [P])

$$\left(\frac{|K|}{|B_2^n|} \right)^{1/n} \le M^*(K). \tag{2.3}$$

An upper estimate for the k-diameters of K in terms of $M^*(K)$ originated in [M1]. Below we will use the result from [PT], with the best known constant proved in [Go2]. This estimate is also know as "lower M^*-estimate".

Theorem 2.1. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body and $\rho > 0$. Let $1 \le k \le n$ satisfy*

$$k > \left(\frac{\ell(K_\rho^0)}{\omega_k \, \rho} \right)^2 .$$

Then for a "random" k-codimensional subspace $E \subset \mathbb{R}^n$ one has

$$K \cap E \subset \rho B_2^n .$$

Let us also recall the following form of Dvoretzky's Theorem ([MSch], [P], [Go1]). The "moreover" part is one-sided estimate that follows from Milman's proof. The dependence on ε in both parts follows from Gordon's work.

Theorem 2.2. *Let $\varepsilon > 0$. Let $K \subset \mathbb{R}^n$ be a convex body with origin in its interior and let $R := R(K)$. Let $m \le \varepsilon^2 \left(M^*(K)/R\right)^2 n$. Then for "random" projection P of rank m one has*

$$\frac{1-\varepsilon}{1+\varepsilon} M^*(K) \, PB_2^n \subset PK \subset \frac{1+\varepsilon}{1-\varepsilon} M^*(K) \, PB_2^n .$$

Moreover, if $M^*(K) < A < R$ and $m \le \varepsilon^2 \left(A/R\right)^2 n$ then for "random" projection P of rank m one has

$$PK \subset \frac{1+\varepsilon}{1-\varepsilon} A \, PB_2^n .$$

We also will use Sudakov's inequality ([P], [Lif]).

Theorem 2.3. *Let $K \subset \mathbb{R}^n$ be a convex body with origin in its interior. Then for every $t > 0$ one has*

$$N(K, tB_2^n) \le \exp\left(\kappa \left(\ell(K^0)/t \right)^2 \right),$$

where $1 \le \kappa \le 4.8$ (in fact $\kappa \to 1$ very fast as $N(K, tB_2^n)$ grows).

Below we keep the notation κ for the constant from this theorem.

3 On the Sharpness of Sudakov's Inequality

Our starting point is a recent result from [LPT] valid for arbitrary symmetric bodies K and L.

Theorem 3.1. *Let $R > a > 0$ and $1 \le k \le n$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Let $K \subset RL$ and $K \cap E \subset aL$ for some k-codimensional subspace of $\mathbb{R}^n$. Then for every $r > a$ one has*

$$N(K, 2rL) \le 2^k \left(\frac{R+r}{r-a} \right)^k .$$

Remark. The factor 2^k above can be replaced by a better function of k (see [LPT]).

Theorem 3.1 was used in [LPT] as an upper bound for the covering numbers, here we would like to interpret it as a lower bound for k-diameters. In this form it will provide an additional insight into Sudakov's inequality.

Theorem 3.2. *Let $R > 1$, $\eta > 0$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$ such that $K \subset RL$. Assume that*

$$N(K, L) \geq \exp(\eta n).$$

Then for every k-codimensional subspace E of $\mathbb{R}^n$ with

$$k = \left[\frac{\eta\, n}{\ln(12R)}\right]$$

one has

$$K \cap E \not\subset \frac{1}{4} L.$$

Proof. Let $k = [(\eta\, n)/\ln(12R)]$. Denote by a the smallest real number $r \geq 0$ such that there exists a k-codimensional subspace E of $\mathbb{R}^n$ satisfying

$$K \cap E \subset r\, L.$$

Assume that $a < 1/2$ (otherwise the proof is finished). Then by Theorem 3.1 we have

$$\exp(\eta n) \leq N(K, L) \leq \left(\frac{2R+1}{1/2 - a}\right)^k.$$

Since $R > 1$ we obtain

$$a \geq \frac{1}{2} - \frac{2R+1}{\exp(\eta n/k)} > \frac{1}{2} - \frac{3R}{\exp(\eta n/k)} \geq \frac{1}{4}. \qquad \square$$

From now on, we restrict ourselves to the case $L = B_2^n$, which is our main interest in this paper. By Sudakov's inequality (Theorem 2.3) we then have $N(K, B_2^n) \leq \exp(\kappa (M_K^*)^2 n)$, and let us now assume that this inequality is almost sharp, i.e.,

$$N(K, B_2^n) \geq \exp\left(\varepsilon (M_K^*)^2 n\right), \tag{3.1}$$

for some $\varepsilon > 0$.

Applying Theorem 3.2 directly, for $k = k_0 := [(\varepsilon\,(M^*)^2 n)/\ln(12R(K))]$, we get that every k-codimensional section of K has diameter at least $1/4$.

This insight into geometry of K can be strengthen even further by considering truncations of the body K. Namely, assuming again that K satisfies (3.1), for some $\varepsilon > 0$, and letting $\beta > 1$, we have two distinct possibilities:

I. Either the covering number $N(K, B_2^n)$ can be significantly decreased by cutting K on the level β, i.e., $N(K \cap \beta B_2^n, B_2^n)$ is essentially smaller (which in turn means that "most" of the entropy of K comes from parts far from B_2^n);
II. Or every k'-codimensional section of K has large diameter, for an appropriate choice of $k' > k_0$ depending on β.

In the next section we study sufficient conditions for Case I to hold, while in Section 5 we return to consequences of essential sharpness of Sudakov's estimate.

4 Improving Sudakov's Inequality

In connection with Case I in Section 3, we discuss the behavior of the covering numbers of K_β $(= K \cap \beta B_2^n)$, when β varies. Sudakov's inequality relates these numbers to $M^*(K_\beta)$, while our point below is to replace $M^*(K_\beta)$ by smaller $M^*(K_\rho)$, for some parameter $\rho < \beta$.

To prepare the discussion we consider a general statement, which combines Theorems 3.1 and 2.1.

Theorem 4.1. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body. Let $\rho > 0$ and $\beta > 0$. Then for every $\gamma > \rho$ one has*

$$N(K_\beta, 2\gamma B_2^n) \le \left(2\,\frac{\beta + \gamma}{\gamma - \rho}\right)^{2(\ell(K_\rho^0)/\rho)^2} . \tag{4.1}$$

Remark. The proof below gives, actually, the exponent $k = \lceil \big(\ell(K_\rho^0)/\rho\big)^2 + 1/2 \rceil$ in (4.1).

Proof. Let $k = \lceil (\ell(K_\rho^0)/\rho)^2 + 1/2 \rceil$. Then, since $\omega_k^2 > 1 - 1/(2k)$, we have

$$\frac{\ell(K_\rho^0)}{\rho} \le \sqrt{k - \frac{1}{2}} < \omega_k \sqrt{k}.$$

By Theorem 2.1 there exists a k-codimensional subspace $E \subset \mathbb{R}^n$ such that

$$K \cap E \subset \rho B_2^n.$$

Applying Theorem 3.1 to the bodies K_β and B_2^n with $R = \beta$, $r = \gamma$, and $a = \rho$ we obtain the estimate announced in the Remark. Now if $\rho \ge R(K)$ then, clearly, $N(K_\beta, 2\gamma B_2^n) = 1$. If $\rho < R(K)$ then

$$\ell(K_\rho^0)/\rho = \ell\left(\left(B_2^n \cap \frac{1}{\rho} K\right)^0\right) \ge 1,$$

which means that $k \le 2(\ell(K_\rho^0)/\rho)^2$. It proves the theorem. □

The next corollary is a partial case of Theorem 4.1, where we fix some of the parameters, in order to compare the results with Sudakov's inequality.

Corollary 4.2. *Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Let $\rho > 0$ and $\beta \geq \rho/3$. Then*

$$N(K_\beta, 4\,\rho\,B_2^n) \leq \exp\left(2\left(\frac{\ell\big(K_\rho^0\big)}{\rho}\right)^2 \ln\frac{3\beta}{\rho}\right).$$

Proof. If $\beta \leq 4\rho$ then the estimate is trivial. Otherwise let $\gamma = 2\rho$. Then, since $\beta \geq 2\gamma$,

$$2\,\frac{\beta+\gamma}{\gamma-\rho} \leq \frac{3\beta}{\rho}.$$

Theorem 4.1 implies the desired result. □

First note that by Sudakov's inequality we have

$$N(K_\beta, 4\rho B_2^n) \leq \exp\left(\kappa\Big(\frac{\ell(K_\beta^0)}{4\rho}\Big)^2\right). \tag{4.2}$$

Therefore, if

$$4\sqrt{\frac{2}{\kappa}}\sqrt{\ln\frac{3\beta}{\rho}}\;\ell\left(K_\rho^0\right)\;\leq \ell\left(K_\beta^0\right),$$

then Corollary 4.2 improves Sudakov's inequality.

Now we shall consider coverings of the whole body, without additional truncations. Let K be a symmetric convex body. Given $\rho > 0$ define the function $F = F_K$ by

$$F(\rho) = \frac{\ell(K^0)}{\ell\left(K_\rho^0\right)}.$$

This function can be used to measure a possible gain in Sudakov's estimates. Rewriting Theorem 2.3 we get

$$N(K, 8\,\rho\,B_2^n) \leq \exp\left(\kappa\left(\frac{\ell(K_\rho^0)}{8\rho}\right)^2 F(\rho)^2\right),$$

which should be compared with the following:

Theorem 4.3. *Let K be a symmetric convex body and $\rho > 0$. Then*

$$N(K, 8\,\rho\,B_2^n) \leq \exp\left(2\left(\frac{\ell(K_\rho^0)}{\rho}\right)^2 \ln\big(6\,F(\rho)\big)\right).$$

Proof. It is known (and easy to check) that for every $t > 0$ one has $N(K, tB_2^n) = N(K, (2K) \cap tB_2^n)$. Therefore, for $\beta > 0$ and $\rho > 0$ we have

$$\begin{aligned} N(K, 8\,\rho\, B_2^n) &\le N\big(K, (2K) \cap 2\beta B_2^n\big)\, N\big((2K) \cap 2\beta B_2^n, 8\,\rho\, B_2^n\big) \\ &= N(K, 2\beta B_2^n)\, N(K_\beta, 4\,\rho\, B_2^n). \end{aligned}$$

Now we apply Sudakov's inequality to estimate the first factor and Corollary 4.2 to estimate the second one. We obtain

$$N(K, 8\,\rho\, B_2^n) \le \exp\left(\kappa \left(\frac{\ell(K^0)}{2\beta}\right)^2 + 2\left(\frac{\ell(K_\rho^0)}{\rho}\right)^2 \ln\frac{3\beta}{\rho}\right).$$

Notice that $F(\rho) \ge 1$ and choose

$$\beta = \frac{1}{2}\sqrt{\kappa}\, F(\rho)\rho \ge \frac{1}{2}\,\rho.$$

Then

$$N(K, 8\,\rho\, B_2^n) \le \exp\left(2\left(\frac{\ell\left(K_\rho^0\right)}{\rho}\right)^2 \ln\left(1.5\sqrt{e\,\kappa}\, F(\rho)\right)\right),$$

which proves the theorem, since $\kappa < 5$. □

5 Quasi M-Position

We are now prepared to obtain a further consequence of sharpness of Sudakov's inequality.

Theorem 5.1. *Let $\varepsilon \in (0,1)$. Let K be a symmetric convex body normalized in such a way that $M^*(K) = 1$. Assume that*

$$N(K, 8\, B_2^n) \ge \exp(\varepsilon n)\,.$$

Then there exist a constant $0 < c_\varepsilon < 1$ depending only on ε such that for a "random" projection P of rank $m = [c_\varepsilon^2 n]$ one has

$$c_\varepsilon P B_2^n \subset PK \quad \textit{and} \quad \left(\frac{|PK|}{|PB_2^n|}\right)^{1/m} \le \frac{1}{c_\varepsilon}. \tag{5.1}$$

We refer to property (5.1) by saying, informally, that PK has a finite volume ratio.

Proof. Let

$$\gamma := F(1) = \frac{M^*(K)}{M^*(K \cap B_2^n)} = \frac{\ell(K^0)}{\ell\left((K \cap B_2^n)^0\right)} = \frac{\omega_n \sqrt{n}}{\ell\left((K \cap B_2^n)^0\right)}.$$

Applying Theorem 4.3 with $\rho = 1$ and using the assumption of our Theorem we conclude that

$$2\,\ln(6\,\gamma) \ge \frac{\varepsilon\, n}{\left(\ell\left((K\cap B_2^n)^0\right)\right)^2} = \frac{\varepsilon\,\gamma^2}{\omega_n^2}.$$

Therefore there exists an absolute positive constant C such that

$$\gamma \le C'_\varepsilon := C\sqrt{\frac{1}{\varepsilon}\,\ln\left(\frac{2}{\varepsilon}\right)}.$$

Therefore $M^*(K\cap B_2^n) \ge 1/C'_\varepsilon$ and, applying Theorem 2.2 to the body $K\cap B_2^n$, we obtain that for "random" projection P of rank $m = [n/(2C'_\varepsilon)^2]$ one has

$$\frac{1}{3\,C'_\varepsilon} PB_2^n \subset PK \cap B_2^n \subset PK.$$

On the other hand, by Urysohn's inequality (2.3) we obtain

$$\left(\frac{|PK|}{|PB_2^n|}\right)^{1/m} \le M^*(PK) \le \frac{\omega_n\sqrt{n}}{\omega_m\sqrt{m}} M^*(K) \le 4C'_\varepsilon.$$

That completes the proof. □

Remark. The proof shows that c_ε can be taken as

$$c_\varepsilon = c_0\sqrt{\varepsilon\,/\ln\left(\frac{2}{\varepsilon}\right)},$$

where c_0 is an absolute positive constant.

The property (5.1) exhibited above appeared in the theory already long time ago, in the context of M-positions of convex bodies. The existence of M-position was first proved in [M2], and we refer the interested reader to [P] and references therein for the definition and properties of M-position. Here let us just recall that an arbitrary convex body K in M-position has this property (5.1), moreover, "random proportional projection" of K has "finite volume ratio" for any proportion $0 < \lambda < 1$ of the dimension n. This nowadays appears to be the main property of bodies in M-position used in applications. Theorem 5.1 shows that such a property for some proportion of n (with some dependence of parameters), is a consequence of some tightness of covering estimates. We feel that this property may be important for understanding the geometry of convex bodies, especially when we investigate covering numbers by ellipsoids. With this in mind we introduce a new (slightly informal) definition:

Definition. *Let K be a convex body in $\mathbb{R}^n$. We say that K is in a quasi M-position (for a proportion $0 < \lambda < 1$) if "random" proportional projection of K onto λn-dimensional subspace has finite volume ratio.*

The next corollary gives another example of bodies in quasi M-position. This is a variant of Theorem 5.1 in which the hypothesis about $M^*(K)$ is replaced by a weaker condition of an upper estimate for entropies.

Corollary 5.2. *Let $0 < \delta < \varepsilon < 1 < A$. Let K be a symmetric convex body. Assume that*

$$N(K, B_2^n) \geq \exp(\varepsilon n) \quad \text{and} \quad N(K, A\, B_2^n) \leq \exp(\delta n)\,.$$

Then there exist positive constants c, $\bar{c}$, C depending only on ε, δ, A such that for "random" projection P of rank $m = [cn]$ one has

$$\bar{c}\, PB_2^n \subset PK \quad \text{and} \quad \left(\frac{|PK|}{|PB_2^n|}\right)^{1/m} \leq C,$$

i.e. K is in a quasi M-position for a proportion c.

Proof. First note that the estimate for volumes follows immediately from covering estimates.

To show the existence of the desired projection note that we have

$$\begin{aligned} e^{\varepsilon\, n} &\leq N(K, B_2^n) \leq N\big(K, (2K) \cap A\, B_2^n\big)\, N\big((2K) \cap A\, B_2^n, B_2^n\big) \\ &\leq e^{\delta\, n}\, N\Big(K \cap \frac{A}{2}\, B_2^n, \frac{1}{2} B_2^n\Big). \end{aligned}$$

To estimate $(\varepsilon - \delta)n$ we can use either one of the two following ways:

[i] Sudakov's inequality implies

$$(\varepsilon - \delta) n \leq 4\kappa \left(\ell\left(\left(K \cap \frac{A}{2}\, B_2^n\right)^0\right)\right)^2.$$

[ii] Corollary 4.2 implies

$$(\varepsilon - \delta) n \leq 2\left(8\ell\left(\left(K \cap \frac{1}{8}\, B_2^n\right)^0\right)\right)^2 \ln(12A).$$

Now the result follows from Theorem 2.2 in the same way as in the proof of Theorem 5.1. □

Remark. The proof above shows that Corollary 5.2 holds with (at least) two choices of constants c, $\bar{c}$, C:

[i]

$$c = \frac{c_0\, (\varepsilon - \delta)}{A^2}, \quad \bar{c} = c_1 \sqrt{\varepsilon - \delta}, \quad C = A \exp\Big(2\frac{\delta}{c}\Big),$$

[ii]

$$c = \frac{c_0\, (\varepsilon - \delta)}{\ln(12A)}, \quad \bar{c} = \frac{c_1 \sqrt{\varepsilon - \delta}}{\sqrt{\ln(12A)}}, \quad C = A \exp\Big(2\frac{\delta}{c}\Big),$$

where c_0, c_1 are absolute positive constant.

6 Comparing k-Diameters and Covering Numbers

Here we discuss lower estimates for Euclidean covering numbers of a body in terms of k-diameters of its skeleton. More precisely, we get inequalities between k-diameters of a body (or its skeleton) and a covering number of K_β ($= K \cap \beta B_2^n$) for some β, by small balls. We have already seen (Theorem 3.1) that a small k-diameter of K implies an upper bound for covering of K_β. On the other hand we show here that if such a covering is small then, for some m, m-diameter of any absolute skeleton is small as well.

Theorem 6.1. *Let $1 \le k \le n$. Let K be a symmetric convex body in $\mathbb{R}^n$. Let $\beta > a := c_{k+1}(K)$. Let $\rho > 0$, $0 < \delta < 1$ be such that*

$$M^*(K_{\rho/2}) \le \frac{1}{8}\delta M^*(K_\beta). \tag{6.1}$$

Then

$$N\left(K_\beta, \rho B_2\right) \ge \Big(\frac{1}{\delta}\Big)^m,$$

where

$$m = \left[\frac{1}{9}\Big(\frac{M^*(K_\beta)}{\beta}\Big)^2 n\right] > \frac{1}{18}\Big(\frac{a}{\beta}\Big)^2 k - 1.$$

Remarks.

1. In fact we show that for every $0 < \varepsilon < 1$ and every $\rho > 0$, $0 < \delta < 1$ such that
$$2M^*(K_{\rho/2}) \le \delta\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^2 M^*(K_\beta), \tag{6.2}$$
one has the lower bound from the Theorem with
$$m = \left[\varepsilon^2\Big(\frac{M^*(K_\beta)}{\beta}\Big)^2 n\right] > \Big(\frac{\varepsilon\, a}{\beta}\Big)^2 \Big(k - \frac{1}{2}\Big) - 1.$$
2. Condition (6.1) can be viewed in two different ways. Firstly, if we fix ρ and let δ be the smallest satisfying (6.1), we obtain a lower bound for covering numbers in terms of the ratio of M^*'s. Secondly, if we fix δ and chose the best ρ satisfying (6.1), we get a lower estimate for the entropy number (see [P] for the precise definition).

Proof. We will show the estimate from Remark 1. The Theorem follows by taking $\varepsilon = 1/3$. Without loss of generality we assume that $\beta \le R(K)$.

Denote $N := N(K_\beta, \rho B_2^n)$. Then there are $x_i \in \mathbb{R}^n$, $i \le N$, such that

$$K_\beta \subset \bigcup_{i=1}^{N} x_i + \rho B_2^n. \tag{6.3}$$

Since we cover K_β by the Euclidean balls, without loss of generality we can assume $x_i \in K_\beta$. Therefore

$$K_\beta \subset \bigcup_{i=1}^{N} (x_i + \rho\, B_2^n) \cap K_\beta \subset \bigcup_{i=1}^{N} x_i + (\rho\, B_2^n) \cap (K_\beta - x_i). \tag{6.4}$$

Since K is centrally symmetric and, by (6.1), $\rho/2 < \beta$, we obtain

$$K_\beta \subset \bigcup_{i=1}^{N} x_i + (\rho\, B_2^n) \cap (2K_\beta) = \bigcup_{i=1}^{N} x_i + 2K_{\rho/2}.$$

Denote

$$m := \left[\varepsilon^2 \Big(\frac{M^*(K_\beta)}{\beta}\Big)^2 n\right] \leq \left[\varepsilon^2 \Big(\frac{M^*(K_{\rho/2})}{\rho/2}\Big)^2 n\right].$$

By Theorem 2.2 we obtain that for a random projection P of rank m one has

$$\frac{1-\varepsilon}{1+\varepsilon}\, M^*(K_\beta)\, PB_2^n \subset PK_\beta$$

and, by the "moreover" part of Theorem 2.2, for every $i \leq N$

$$P\left(2K_{\rho/2}\right) \subset 2\frac{1+\varepsilon}{1-\varepsilon}\, M^*\left(K_{\rho/2}\right)\, PB_2^n.$$

It implies that

$$\frac{1-\varepsilon}{1+\varepsilon}\, M^*(K_\beta)\, PB_2^n \subset \bigcup_{i=1}^{N} Px_i + 2\frac{1+\varepsilon}{1-\varepsilon}\, M^*\left(K_{\rho/2}\right)\, PB_2^n.$$

Thus, by comparison of volumes, for every ρ satisfying

$$2M^*(K_{\rho/2}) \leq \delta \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^2 M^*(K_\beta)$$

one has

$$N \geq \delta^{-m}.$$

Finally notice that, by Theorem 2.1,

$$a < \frac{M^*(K_\beta)\sqrt{n}}{\omega_k \sqrt{k}}, \tag{6.5}$$

which implies

$$\Big(\frac{\varepsilon\omega_k a}{\beta}\Big)^2 k < m+1.$$

Since $\omega_k^2 > 1 - 1/(2k)$, we obtain the desired result. □

The following theorem shows that the use of skeletons allows to avoid estimating the ratio of M^*'s (and the parameter ρ) in Theorem 6.1, as explained in Remark 2 after that theorem. Thus, it provides another lower estimate for covering numbers.

Theorem 6.2. *Let $\delta \in (0,1)$, and $\beta > 2a > 0$. There exists a constant α, depending only δ such that the following statement holds:*

Let K be a symmetric convex body such that $R(K) \geq \beta$. Let T be an absolute $(2\alpha a)$-separated skeleton of K_β and m be such that $c_{m+1}(T) \geq a$ (i.e. the m-diameter of T is not smaller than a). Then

$$\delta^{-m_0} \leq N\big(K_\beta, \alpha\, \beta\, B_2^n\big),$$

where

$$m_0 = \left[\frac{a^2\, m}{2\, \beta^2}\right].$$

Remarks.

1. In fact we will prove slightly stronger result, namely that for every $\varepsilon \in (0,1)$, $\delta \in (0,1)$, and $\beta > 2a > 0$ there exists a constant α, depending only on ε and δ such that the statement above holds with
$$m_0 = \left[\Big(\frac{\varepsilon\, a}{\beta}\Big)^2 \Big(m - \frac{1}{2}\Big)\right].$$
Moreover, our proof gives that α can be taken to be equal to $C_{\varepsilon\,\delta} \ln^{-3}(\frac{1}{C_{\varepsilon\,\delta}})$, where
$$C_{\varepsilon\,\delta} = \min\left\{c,\ \frac{\delta}{\sqrt{\ln(1/\delta)}}\ \frac{(1-\varepsilon)^2}{\varepsilon}\right\},$$
for some absolute constant $c \in (0, 1/4]$.
2. Taking $a = 1$, $\delta = 1/4$, $\beta > 2$, and ε close enough to 1 we obtain that if the m-diameter of T is not smaller than 1 then
$$2^{m/\beta^2} \leq N\big(K_\beta, c\, \beta\, B_2^n\big),$$
where c is an absolute positive constant.

In the proof of Theorem 6.2 we will use the following result by Milman–Szarek ([MSz]).

Theorem 6.3. *Let $m \geq 1$. Let $S \subset \mathbb{R}^n$ be a finite set of cardinality m and let T be the convex hull of S. Then for every $0 < r < R(T)$ one has*

$$M^*\big(T \cap (r\, B_2^n)\big) \leq Cr\Big(\ln \frac{2R(T)}{r}\Big)^3 \sqrt{\frac{\ln \max\{m, N\}}{n}},$$

where $N = N(T, (r\, B_2^n))$ and C is an absolute constant.

Proof of Theorem 6.2. Let α be of the form given in Remark 1 for some (small) positive constant c. Denote $\rho := \alpha\beta$ and $N = N\,(K_\beta, \rho\, B_2^n)$. We will argue by contradiction. Assume that $N < \delta^{-m_0}$.

Let S be a (2ρ)-separated set for K_β. Then, as is discussed in the first section, the cardinality of S does not exceed N and S is a (2ρ)-net for K. Let

T be the absolute convex hull of S. Denote $b = \beta - 2\rho > a$. Since $K \not\subset \beta B_2^n$ and S is a (2ρ)-net for K we obtain $T \not\subset b B_2^n$. Apply Theorem 6.1 to the body T_b with parameter b. We have that whenever ρ satisfies

$$2M^*(T_{\rho/2}) \leq \delta C_\varepsilon M^*(T_b), \tag{6.6}$$

where $C_\varepsilon = ((1-\varepsilon)/(1+\varepsilon))^2$, one has

$$N(T_b, \rho B_2) \geq \left(\frac{1}{\delta}\right)^{m_1},$$

where

$$m_1 = \left[\varepsilon^2 \left(\frac{M^*(T_b)}{b}\right)^2 n\right] \geq \left[\left(\frac{\varepsilon \omega_m a}{b}\right)^2 m\right] \geq m_0.$$

This would give a contradiction and thus prove the theorem. Therefore to complete the proof it is enough to verify (6.6) for our choice of ρ. First note that by (6.5) we have

$$M^*(T_b) \geq \omega_m a \sqrt{\frac{m}{n}}.$$

On the other hand, since $T = \text{conv}\,\{S, -S\} \subset \beta B_2^n$ and $N(T, 2\rho B_2^n) \leq N$, by Theorem 6.3,

$$2M^*(T_{\rho/2}) \leq 2\, M^*(T_{2\rho}) \leq 4\, C\, \rho \left(\ln \frac{\beta}{\rho}\right)^3 \sqrt{\frac{\ln(2N)}{n}},$$

where C is an absolute positive constant. Therefore to satisfy (6.6) it is enough to have for some absolute positive constant C_1

$$C_1\, \rho \left(\ln \frac{\beta}{\rho}\right)^3 \sqrt{\ln(2N)} \leq \delta\, C_\varepsilon \sqrt{m}\; a,$$

or, using the assumption $N < (1/\delta)^{m_0}$,

$$C_2\, \rho \left(\ln \frac{\beta}{\rho}\right)^3 \varepsilon \sqrt{\ln\left(\frac{1}{\delta}\right)} \leq \delta\, C_\varepsilon\, \beta$$

for some absolute positive constant C_2. Clearly there exists a choice of absolute constant c such that our ρ satisfies the last inequality. It proves the result. □

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Some Remarks on Transportation Cost and Related Inequalities

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Summary. We discuss transportation cost inequalities for uniform measures on convex bodies, and connections with other geometric and functional inequalities. In particular, we show how transportation inequalities can be applied to the slicing problem, and prove a new log-Sobolev-type inequality for bounded domains in $\mathbb{R}^n$.

1 Introduction

We work in $\mathbb{R}^n$ equipped with its standard inner product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$. $|A|$ also denotes the volume (Lebesgue measure) of a measurable set A. D_n is the Euclidean ball of volume one. For a measurable set A with $0 < |A| < \infty$, m_A denotes the uniform probability measure on A, that is, $m_A(B) = \frac{|A \cap B|}{|A|}$. The symbols μ and ν will always stand for Borel probability measures on $\mathbb{R}^n$.

We first introduce two different ways to quantify the difference between two probability measures. First, for $p \geq 1$, the (L_p) *Wasserstein distance* between μ and ν is

$$W_p(\mu, \nu) = \inf_{\pi} \left(\int |x - y|^p \, d\pi(x, y) \right)^{1/p},$$

where π runs over probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ and ν. We will be interested mainly in the special cases $p = 1, 2$. Second, if $\nu \ll \mu$, the *relative entropy* of ν with respect to μ is

$$H(\nu|\mu) = \int \log \left(\frac{d\nu}{d\mu} \right) d\nu.$$

The (L_p) *transportation cost constant* $\tau_p(\mu)$ is the largest constant τ such that

$$W_p(\mu, \nu) \leq \sqrt{\frac{2}{\tau} H(\nu|\mu)} \tag{1}$$

for every $\nu \ll \mu$. An inequality of the form of (1) is referred to as a *transportation cost inequality* for μ. Note that if $p \leq q$, then $W_p \leq W_q$ by Hölder's inequality, and hence $\tau_p(\mu) \geq \tau_q(\mu)$.

Transportation cost inequalities are by now well known as a method to derive measure concentration (cf. [L, Chapter 6]). In fact, as follows from Bobkov and Götze's dual characterization of the L_1 transportation cost inequality [BG], $\tau_1(\mu)$ is equivalent to the best constant α in the normal concentration inequality:

$$\mu\big(\{x \in \mathbb{R}^n : |F(x)| \geq t\}\big) \leq 2e^{-\alpha t^2} \text{ for } t > 0 \tag{2}$$

for all 1-Lipschitz functions F with $\int F d\mu = 0$.

In this paper we consider transportation cost inequalities for uniform measures on convex bodies. In the next section we show that such inequalities can be applied to the slicing problem. In the last section we discuss their relationship with Sobolev-type functional inequalities, and present a logarithmic Sobolev inequality with trace for bounded domains in $\mathbb{R}^n$.

2 Relation to the Slicing Problem

We recall the following definitions and facts about isotropic convex bodies (see [MiP]). A convex body K is called *isotropic* if

1. its centroid is 0,
2. $|K| = 1$, and
3. there is a constant $L_K > 0$ such that
$$\int_K \langle x, y\rangle^2 \, dx = L_K^2 |y|^2$$
for all $y \in \mathbb{R}^n$.

Every convex body K has an affine image $T(K)$ (unique up to orthogonal transformations) which is isotropic; the *isotropic constant* of K is defined as $L_K = L_{T(K)}$. The isotropic constant also has the extremal characterization

$$L_K = \min_T \left(\frac{1}{n|K|^{1+2/n}} \int_{T(K)} |x|^2 \, dx \right)^{1/2}, \tag{3}$$

where T runs over volume-preserving affine transformations of $\mathbb{R}^n$, with equality iff K is isotropic. The slicing problem for convex bodies asks whether there is a universal constant c such that $L_K \leq c$ for all convex bodies K; see [MiP] for extensive discussion and alternate formulations.

If $K, B \subset \mathbb{R}^n$ are convex bodies, the *volume ratio* of K in B is

$$\mathrm{vr}(B, K) = \min_T \left(\frac{|B|}{|T(K)|} \right)^{1/n},$$

where T runs over affine transformations of $\mathbb{R}^n$ such that $T(K) \subset B$. The following lemma indicates the relevance of transportation cost inequalities to the slicing problem.

Lemma 1. *Let $K, B \subset \mathbb{R}^n$ be convex bodies, with B isotropic. Then*

$$L_K \leq c\big(1 + \sqrt{\log v}\big)v\ \tau^{-1/2},$$

where $\tau = \tau_1(B)$, $v = \mathrm{vr}(B, K)$, and c is an absolute constant.

Proof. We may assume that $K \subset B$ and $|K| = v^{-n}$. If δ_0 denotes the point mass at $0 \in \mathbb{R}^n$, then by the triangle inequality for W_1,

$$\begin{aligned} \frac{1}{|K|}\int_K |x|\ dx = W_1(m_K, \delta_0) &\leq W_1(m_K, m_B) + W_1(m_B, \delta_0) \\ &\leq \sqrt{\frac{2}{\tau} H(m_K|m_B)} + \int_B |x|\ dx \\ &\leq \sqrt{\frac{2}{\tau}\log\frac{1}{|K|}} + \left(\int_B |x|^2\ dx\right)^{1/2} \\ &= \sqrt{\frac{2n}{\tau}\log v} + \sqrt{n}L_B. \end{aligned}$$

Now by applying (2) to a linear functional, $L_B \leq c\,\tau^{-1/2}$. On the other hand, by Borell's lemma (see e.g. [L, Section 2.2]), there is an absolute constant c such that

$$\left(\frac{1}{|K|}\int_K |x|^2\ dx\right)^{1/2} \leq c\frac{1}{|K|}\int_K |x|\ dx.$$

The claim now follows from the extremal characterization of L_K (3). □

An analogous estimate with $\tau = \tau_2(B)$ can be proved more directly, without Borell's lemma.

In light of the equivalence of L_1 transportation cost inequalities and normal concentration, Lemma 1 can also be thought of as an application of measure concentration to the slicing problem. Since the Euclidean ball is well known to have normal concentration, as an immediate corollary we obtain the following known fact.

Corollary 2. *If $\mathrm{vr}(D_n, K) \leq c$, then $L_K \leq c'$, where c' depends only on c.*

Recently, Klartag [Kl] introduced the following isomorphic version of the slicing problem: given a convex body K, is there a convex body B such that $L_B \leq c_1$ and $d(B, K) \leq c_2$, where d is Banach–Mazur distance? In the case that K and B are centrally symmetric, Klartag solved this problem in the affirmative, up to a logarithmic (in n) factor in c_2. Lemma 1 suggests approaching the slicing problem via a modified version of the isomorphic problem: given a convex body K, can one find a "similar" body B such that $\tau_1(B)$

is large when B is in isotropic position? Notice that while Klartag's result uses Banach–Mazur distance to quantify "similarity" of bodies, in Lemma 1 it is the weaker measure of volume ratio which is relevant. It also seems that this approach via transportation cost is less sensitive to central symmetry than more traditional methods of asymptotic convexity.

Finally, we remark that the real point of the proof of Lemma 1 is that moments of the Euclidean norm on convex bodies, thought of as functionals of the bodies, are Lipschitz with respect to Wasserstein distances on uniform measures. This suggests an alternative approach to the slicing problem, related to the one discussed above, of directly studying optimal (or near-optimal) probability measures π in the definition of $W_p(m_K, m_B)$ for $p = 1, 2$. Particularly in the case $p = 2$ a great deal is known about the optimal π; see [V] for an excellent survey.

Note. Since this paper was written, Klartag [B. Klartag, "On convex perturbations with a bounded isotropic constant", to appear in Geom. and Funct. Anal.] has solved the isomorphic slicing problem, without the logarithmic factor, for arbitrary convex bodies.

3 Functional Inequalities

The *entropy* of $f : \mathbb{R}^n \to \mathbb{R}_+$ with respect to μ is

$$\mathrm{Ent}_\mu(f) = \int f \log \left(\frac{f}{\int f d\mu} \right) d\mu,$$

and the *variance* of $f : \mathbb{R}^n \to \mathbb{R}$ with respect to μ is

$$\mathrm{Var}_\mu(f) = \int f^2 d\mu - \left(\int f d\mu \right)^2 .$$

The *logarithmic Sobolev constant* $\rho(\mu)$ is the largest constant ρ such that

$$\mathrm{Ent}_\mu(f^2) \le \frac{2}{\rho} \int |\nabla f|^2 \, d\mu \tag{4}$$

for all smooth $f \in L_2(\mu)$. The *spectral gap* $\lambda(\mu)$ is the largest constant λ such that

$$\mathrm{Var}_\mu(f) \le \frac{1}{\lambda} \int |\nabla f|^2 \, d\mu \tag{5}$$

for all smooth $f \in L_2(\mu)$. It is well known (cf. [L]) that a logarithmic Sobolev inequality for μ implies normal concentration (and hence an L_1 transportation cost inequality, by Bobkov and Götze's result [BG]) and a spectral gap inequality implies exponential concentration. A result of Otto and Villani [OV] shows further that

$$\rho(\mu) \leq \tau_2(\mu) \leq \lambda(\mu)$$

for any absolutely continuous μ. Thus transportation cost inequalities are somehow intermediate between these Sobolev-type functional inequalities, and it is of interest here to consider what is known about $\rho(K)$ and $\lambda(K)$ for a convex body K. We briefly review known results.

Kannan, Lovász, and Simonovits [KLS] showed that

$$\lambda(K) \geq c\left(\frac{1}{|K|}\int_K |x-z|^2 \, dx\right)^{-1},$$

where z is the centroid of K. It is easy to see that this is an optimal estimate in general. By testing (5) on linear functionals, one can see that $\lambda(\mu) \leq \alpha_1^{-1}$ for any μ, where α_1 is the largest eigenvalue of the covariance matrix of μ. If α_1 is much larger than the remaining eigenvalues (i.e., μ is close to being one-dimensional), then $\int |x|^2 \, d\mu(x) \approx \alpha_1$. However, this situation is far from isotropicity (in which all the eigenvalues are equal), and the authors of [KLS] conjecture that when K is isotropic,

$$\lambda(K) \geq cn\left(\int_K |x|^2 \, dx\right)^{-1} = \frac{c}{L_K^2}. \tag{6}$$

Bobkov [B] estimated $\rho(K)$ in terms of the $L_{\psi_2}(m_K)$ norm of $|\cdot|$; in the case that K is isotropic, this can be combined with a result of Alesker [A] to yield

$$\rho(K) \geq \frac{c}{nL_K^2}. \tag{7}$$

The estimate for $\tau_1(K)$ which follows from (7) also follows by combining Alesker's result with an L_1 transportation cost inequality proved recently by Bolley and Villani [BoV] in an extremely general setting. The estimate (7) misses the level of (6) by a factor of n, but in this case the estimate cannot be sharpened even when K is isotropic: if K is taken to be the ℓ_1^n unit ball, renormalized to have volume one, then exponential concentration correctly describes the behavior of a linear functional in a coordinate direction; it can in fact be shown that $\tau_1(K) \approx \frac{1}{n}$ in this case. However, in two concrete cases we have best possible estimates:

$$\rho(Q_n) \geq c \quad \text{and} \quad \rho(D_n) \geq c,$$

where Q_n is a cube of volume 1. The estimate for Q_n is probably folklore; the estimate for D_n is due to Bobkov and Ledoux [BL].

Finally, we present the following "doubly homogeneous L_p trace logarithmic Sobolev inequality" for uniform measures on bounded domains, inspired both by the search for good estimates on $\rho(K)$ for isotropic K, and by the recent work [MV] by Maggi and Villani on trace Sobolev inequalities. This seems not to be directly comparable to the classical logarithmic Sobolev inequality (4), but interestingly is completely insensitive to isotropicity or even convexity of the domain. For $p > 1$ we denote by q the conjugate exponent $q = \frac{p}{p-1}$, and $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the Euclidean unit ball.

Proposition 3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with locally Lipschitz boundary and let $p \geq 1$. Then*

$$\mathrm{Ent}_\Omega(|f|^p) \leq \left(\frac{p-1}{n+q}\right)^{p-1} \frac{1}{\omega_n^{p/n}|\Omega|^{1-p/n}} \int_\Omega |\nabla f|^p + \frac{1}{\omega_n^{1/n}|\Omega|^{1-1/n}} \int_{\partial\Omega} |f|^p$$

for every locally Lipschitz $f : \overline{\Omega} \to \mathbb{R}$, where $(\frac{p-1}{n+q})^{p-1}$ is interpreted as 1 if $p = 1$, and the integral over $\partial\Omega$ is with respect to $(n-1)$-dimensional Hausdorff measure.

In the case $p = 2$ and $f|_{\partial\Omega} = 0$, we obtain

$$\mathrm{Ent}_\Omega(f^2) \leq \frac{1}{(n+2)\omega_n^{2/n}|\Omega|^{1-2/n}} \int_\Omega |\nabla f|^2 \leq \frac{c}{|\Omega|^{1-2/n}} \int_\Omega |\nabla f|^2.$$

Notice that

$$\frac{|\Omega|^{2/n}}{(n+2)\omega_n^{2/n}} = \frac{|\Omega|^{2/n}}{n\omega_n^{1+2/n}} \int_{D_n} |x|^2 \; dx \leq \frac{1}{n|\Omega|} \int_\Omega |x|^2 \; dx$$

with equality only if Ω is a Euclidean ball. Therefore if one restricts the logarithmic Sobolev inequality (4) for $\mu = m_\Omega$ to functions which vanish on the boundary of Ω, one can improve the constant ρ to $2(n+2)\omega_n^{2/n}|\Omega|^{-2/n}$, which is always stronger by at least a factor of 2 than the best possible result for general f, and much stronger still in many cases.

Proof of Proposition 3. The proof is based on the results of Brenier and McCann on mass transportation via a convex gradient; we refer to [V] for details and references. To begin, we assume that $p > 1$; the case $p = 1$ follows the same lines and is slightly simpler. We also assume that f is smooth and nonnegative,

$$\frac{1}{|\Omega|} \int_\Omega f^p = 1,$$

and

$$|\Omega| = \left(\frac{n+q}{p-1}\right)^{n/q} \omega_n.$$

We will use the fact that there is a convex function φ such that $\nabla\varphi$ (the *Brenier map*) transports the probability measure $f^p dm_\Omega$ to the probability measure m_{B_R}, where $B_R = |\Omega|^{1/n} D_n$ is the Euclidean ball normalized so that $|B_R| = |\Omega|$.

By the results of McCann, the Monge–Ampère equation

$$f^p(x) = \det H_A\varphi(x)$$

is satisfied $f^p dm_\Omega$-a.e., where $H_A\varphi$ is the Aleksandrov Hessian of φ (i.e., the absolutely continuous part of the distributional Hessian $H\varphi$). Using the fact that φ is convex and $\log t \le t-1$ for $t>0$,

$$\log f^p(x) = \log\det H_A\varphi(x) \le \Delta_A\varphi(x) - n,$$

where $\Delta_A\varphi$ is the Aleksandrov Laplacian of φ (i.e., the trace of $H_A\varphi$). Integrating with respect to $f^p dm_\Omega$ yields

$$\frac{1}{|\Omega|}\int_\Omega f^p \log f^p \le \frac{1}{|\Omega|}\int_\Omega f^p \Delta_A\varphi - n \le \frac{1}{|\Omega|}\int_\Omega f^p \Delta\varphi - n, \tag{8}$$

since $\Delta_A\varphi \le \Delta\varphi$ as distributions, where $\Delta\varphi$ is the distributional Hessian of φ. Integrating by parts (cf. [MV] for a detailed justification),

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega f^p\Delta\varphi &= -\frac{1}{|\Omega|}\int_\Omega \langle\nabla\varphi, \nabla(f^p)\rangle + \frac{1}{|\Omega|}\int_{\partial\Omega}\langle\nabla\varphi,\sigma\rangle f^p \\ &= -\frac{p}{|\Omega|}\int_\Omega f^{p-1}\langle\nabla\varphi,\nabla f\rangle + \frac{1}{|\Omega|}\int_{\partial\Omega}\langle\nabla\varphi,\sigma\rangle f^p, \end{aligned}\tag{9}$$

where σ is the outer unit normal vector to $\partial\Omega$.

Now

$$\frac{1}{|\Omega|}\int_{\partial\Omega}\langle\nabla\varphi,\sigma\rangle f^p \le \frac{R}{|\Omega|}\int_{\partial\Omega} f^p = \frac{1}{\omega_n^{1/n}|\Omega|^{1-1/n}}\int_{\partial\Omega} f^p. \tag{10}$$

On the other hand, by Hölder's inequality, the definition of mass transport, and the arithmetic-geometric means inequality,

$$\begin{aligned} -\frac{p}{|\Omega|}\int_\Omega f^{p-1}\langle\nabla\varphi,\nabla f\rangle &\le p\left(\frac{1}{|\Omega|}\int_\Omega f^p|\nabla\varphi|^q\right)^{1/q}\left(\frac{1}{|\Omega|}\int_\Omega|\nabla f|^p\right)^{1/p} \\ &\le \frac{p-1}{|B_R|}\int_{B_R}|x|^q dx + \frac{1}{|\Omega|}\int_\Omega|\nabla f|^p \\ &= \frac{(p-1)n}{n+q}R^q + \frac{1}{|\Omega|}\int_\Omega|\nabla f|^p \\ &= n + \left(\frac{p-1}{n+q}\right)^{p-1}\frac{1}{\omega_n^{p/n}|\Omega|^{1-p/n}}\int_\Omega|\nabla f|^p. \end{aligned}\tag{11}$$

Combining (8), (9), (10), and (11) yields

$$\mathrm{Ent}_\Omega(f^p) \le \left(\frac{p-1}{n+q}\right)^{p-1}\frac{1}{\omega_n^{p/n}|\Omega|^{1-p/n}}\int_\Omega|\nabla f|^p + \frac{1}{\omega_n^{1/n}|\Omega|^{1-1/n}}\int_{\partial\Omega} f^p.$$

Both sides of this inequality have the same homogeneity with respect to both f and Ω, so the claim follows for general f and Ω by rescaling, approximation, and the fact that $|\nabla|f|| = |\nabla f|$ a.e. □

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A Comment on the Low-Dimensional Busemann–Petty Problem

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Summary. The generalized Busemann–Petty problem asks whether centrally-symmetric convex bodies having larger volume of all m-dimensional sections necessarily have larger volume. When $m > 3$ this is known to be false, but the cases $m = 2, 3$ are still open. In those cases, it is shown that when the smaller body's radial function is a $n-m$-th root of the radial function of a convex body, the answer to the generalized Busemann–Petty problem is positive (for any larger star-body). Several immediate corollaries of this observation are also discussed.

1 Introduction

Let $\mathrm{Vol}(L)$ denote the Lebesgue measure of a set $L \subset \mathbb{R}^n$ in its affine hull, and let $G(n,k)$ denote the Grassmann manifold of k dimensional subspaces of $\mathbb{R}^n$. Let D_n denote the Euclidean unit ball, and S^{n-1} the Euclidean sphere. All of the bodies considered in this note will be assumed to be centrally symmetric star-bodies, defined by a continuous radial function $\rho_K(\theta) = \max\{r \geq 0 \mid r\theta \in K\}$ for $\theta \in S^{n-1}$ and a star-body K.

The Busemann–Petty problem, first posed in [BP56], asks whether two centrally-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying:

$$\mathrm{Vol}(K \cap H) \leq \mathrm{Vol}(L \cap H) \;\; \forall H \in G(n, n-1) \tag{1.1}$$

necessarily satisfy $\mathrm{Vol}(K) \leq \mathrm{Vol}(L)$. For a long time this was believed to be true (this is certainly true for $n = 2$), until a first counterexample was given in [LR75] for a large value of n. In the same year, the notion of an *intersection-body* was first introduced by Lutwak in [Lut75] (see also [Lut88] and Section 2 for definitions) in connection to the Busemann–Petty problem. It was shown in [Lut88] (and refined in [Gar94a]) that the answer to the Busemann–Petty problem is equivalent to whether all convex bodies in $\mathbb{R}^n$ are intersection bodies. Subsequently, it was shown in a series of results ([LR75], [Bal88],

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[Bou91], [Gia90], [Pap92], [Gar94a], [Gar94b], [Kol98b], [Zha99], [GKS99]), that this is true for $n \leq 4$, but false for $n \geq 5$.

In [Zha96], Zhang considered a natural generalization of the Busemann–Petty problem, which asks whether two centrally-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying:

$$\mathrm{Vol}(K \cap H) \leq \mathrm{Vol}(L \cap H) \;\; \forall H \in G(n, n-k) \tag{1.2}$$

necessarily satisfy $\mathrm{Vol}(K) \leq \mathrm{Vol}(L)$, where k is some integer between 1 and $n-1$. Zhang showed that the *generalized k-codimensional Busemann–Petty problem* is also naturally associated to another class of bodies, which will be referred to as *k-Busemann-Petty bodies* (note that these bodies are referred to as *$n-k$-intersection bodies* in [Zha96] and *generalized k-intersection bodies* in [Kol00]), and that the generalized k-codimensional problem is equivalent to whether all convex bodies in R^n are k-Busemann–Petty bodies. Analogously to the original problem, it was shown in [Zha96] that if K and L are two centrally-symmetric star-bodies (not necessarily convex) satisfying (1.2), and if K is a k-Busemann–Petty body, then $\mathrm{Vol}(K) \leq \mathrm{Vol}(L)$.

It was shown in [BZ98] (see also a correction in [RZ04]), and later in [Kol00], that the answer to the generalized k-codimensional problem is negative for $k < n-3$, but the cases $k = n-3$ and $k = n-2$ still remain open (the case $k = n-1$ is obviously true). A partial answer to the case $k = n-2$ was given in [BZ98], where it was shown that when L is a Euclidean ball and K is convex and sufficiently close to L, the answer is positive. Several other generalizations of the Busemann–Petty problem were treated in [RZ04]. Our main observation in this note concerns the cases $k = n-2, n-3$ and reads as follows:

Theorem 1.1. *Let K denote a centrally-symmetric convex body in $\mathbb{R}^n$. For $a = 2, 3$, let K_a be the star-body defined by $\rho_{K_a} = \rho_K^{1/(n-a)}$. Then K_a is a $(n-a)$-Busemann–Petty body, implying a positive answer to the $(n-a)$-codimensional Busemann–Petty problem (1.2) for the pair K_a, L for any star-body L.*

The case $a = 1$ is also true, but follows trivially since it is easy to see (e.g. [Mil05]) that any star-body is an $n-1$-Busemann–Petty body. The case $a = 2$ follows from $a = 3$ by a general result from [Mil05], stating that if K is a k-Busemann–Petty body and L is given by $\rho_L = \rho_K^{k/l}$ for $1 \leq k < l \leq n-1$, then L is a l-Busemann–Petty body.

Theorem 1.1 has several interesting consequences. The first one is the following complementary result to the one aforementioned from [BZ98]. Roughly speaking, it states that any small enough perturbation K of the Euclidean ball, for which we have control over the second derivatives of ρ_K, satisfies the low-dimensional generalized Busemann–Petty problem (1.2) with *any* star-body L.

Corollary 1.2. *For any n, there exists a function $\gamma : [0,\infty) \to (0,1)$, such that the following holds: let φ denote a twice continuously differentiable function on S^{n-1} such that:*

$$\max_{\theta \in S^{n-1}} |\varphi(\theta)| \leq 1, \max_{\theta \in S^{n-1}} |\varphi_i(\theta)| \leq M, \max_{\theta \in S^{n-1}} |\varphi_{i,j}(\theta)| \leq M,$$

for every $i, j = 1, \ldots, n-1$, where φ_i and $\varphi_{i,j}$ denote the first and second partial derivatives of φ (w.r.t. any local coordinate system of S^{n-1}), respectively. Then the star-body K^ε defined by $\rho_{K^\varepsilon} = 1 + \varepsilon\varphi$ for any $|\varepsilon| < \gamma(M)$ is a $(n-a)$-Busemann–Petty body for $a = 2, 3$, implying a positive answer to the $(n-a)$-codimensional Busemann–Petty problem (1.2) for K^ε and any star-body L.

Note that the definition of K_a in Theorem 1.1 is highly non-linear with respect to K. Since the class of k-Busemann–Petty bodies is closed under certain natural operations (see [Mil05] for the latest known results), we can take advantage of this fact to strengthen the result of Theorem 1.1. For instance, it is well known (e.g. [GZ99], [Mil05]) that the class of k-Busemann–Petty bodies is closed under taking *k-radial sums*. The k-radial sum of two star-bodies L_1, L_2 is defined as the star-body L satisfying $\rho_L^k = \rho_{L_1}^k + \rho_{L_2}^k$. When $k = 1$ this operation will simply be referred to as radial sum. The space of star-bodies in $\mathbb{R}^n$ is endowed with the natural radial metric d_r, defined as $d_r(L_1, L_2) = \max_{\theta \in S^{n-1}} |\rho_{L_1}(\theta) - \rho_{L_2}(\theta)|$. We will denote by $\mathcal{RC}^n$ the closure in the radial metric of the class of all star-bodies in $\mathbb{R}^n$ which are finite radial sums of centrally-symmetric convex bodies. It should then be clear that:

Corollary 1.3. *Theorem 1.1 holds for any $K \in \mathcal{RC}^n$.*

Our last remark in this note is again an immediate consequence of Theorem 1.1 and the following characterization of k-Busemann–Petty bodies due to Grinberg and Zhang ([GZ99]), which generalizes the characterization of intersection-bodies (the case $k = 1$) given by Goodey and Weil ([GW95]):

Theorem (Grinberg and Zhang). *A star-body K is a k-Busemann–Petty body iff it is the limit of $\{K_i\}$ in the radial metric d_r, where each K_i is a finite k-radial sums of ellipsoids $\{\mathcal{E}_j^i\}$:*

$$\rho_{K_i}^k = \rho_{\mathcal{E}_1^i}^k + \ldots + \rho_{\mathcal{E}_{m_i}^i}^k .$$

Applying Grinberg and Zhang's Theorem to the bodies K_a from Theorem 1.1, we immediately have:

Corollary 1.4. *Let K denote a centrally-symmetric convex body in $\mathbb{R}^n$. Then for $a = 2, 3$, K is the limit in the radial metric d_r of star-bodies K_i having the form:*

$$\rho_{K_i} = \rho_{\mathcal{E}_1^i}^{n-a} + \ldots + \rho_{\mathcal{E}_{m_i}^i}^{n-a},$$

where $\{\mathcal{E}_j^i\}$ are ellipsoids.

2 Definitions and Notations

A star body K is said to be an intersection body of a star body L, if $\rho_K(\theta) = \mathrm{Vol}(L \cap \theta^\perp)$ for every $\theta \in S^{n-1}$. K is said to be an intersection body, if it is the limit in the radial metric d_r of intersection bodies $\{K_i\}$ of star bodies $\{L_i\}$, where $d_r(K_1, K_2) = \sup_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|$. This is equivalent (e.g. [Lut88], [Gar94a]) to $\rho_K = R^*(d\mu)$, where μ is a non-negative Borel measure on S^{n-1}, R^* is the dual transform (as in (2.1)) to the Spherical Radon Transform $R : C(S^{n-1}) \to C(S^{n-1})$, which is defined for $f \in C(S^{n-1})$ as:

$$R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi) d\sigma_{n-1}(\xi),$$

where σ_{n-1} the Haar probability measure on S^{n-2} (and we have identified S^{n-2} with $S^{n-1} \cap \theta^\perp$).

Before defining the class of k-Busemann–Petty bodies we shall need to introduce the m-dimensional Spherical Radon Transform, acting on spaces of continuous functions as follows:

$$\begin{aligned} R_m : C(S^{n-1}) &\longrightarrow C\big(G(n,m)\big) \\ R_m(f)(E) &= \int_{S^{n-1} \cap E} f(\theta) d\sigma_m(\theta), \end{aligned}$$

where σ_m is the Haar probability measure on S^{m-1} (and we have identified S^{m-1} with $S^{n-1} \cap E$). The dual transform is defined on spaces of *signed* Borel measures $\mathcal{M}$ by:

$$\begin{aligned} R_m^* : \mathcal{M}(G(n,m)) &\longrightarrow \mathcal{M}(S^{n-1}) \\ \int_{S^{n-1}} f R_m^*(d\mu) &= \int_{G(n,m)} R_m(f) d\mu \quad \forall f \in C(S^{n-1}), \end{aligned} \tag{2.1}$$

and for a measure μ with continuous density g, the transform may be explicitly written in terms of g (see [Zha96]):

$$R_m^* g(\theta) = \int_{\theta \in E \in G(n,m)} g(E) d\nu_m(E), \tag{2.2}$$

where ν_m is the Haar probability measure on $G(n-1, m-1)$.

We shall say that a body K is a k-Busemann–Petty body if $\rho_K^k = R_{n-k}^*(d\mu)$ as measures in $\mathcal{M}(S^{n-1})$, where μ is a non-negative Borel measure on $G(n, n-k)$. We shall denote the class of such bodies by $\mathcal{BP}_k^n$. Choosing $k = 1$, for which $G(n, n-1)$ is isometric to S^{n-1}/Z_2 by mapping H to $S^{n-1} \cap H^\perp$, and noticing that R is equivalent to R_{n-1} under this map, we see that $\mathcal{BP}_1^n$ is exactly the class of intersection bodies.

We will also require, although indirectly, several notions regarding Fourier transforms of homogeneous distributions. We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable test functions in $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier Transform $\hat{f}$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ, where $\hat{\phi}(y) = \int \phi(x) \exp(-i\langle x, y \rangle)dx$. A distribution f is called homogeneous of degree $p \in \mathbb{R}$ if $\langle f, \phi(\cdot/t) \rangle = |t|^{n+p} \langle f, \phi \rangle$ for every $t > 0$, and it is called even if the same is true for $t = -1$. An even distribution f always satisfies $(\hat{f})^\wedge = (2\pi)^n f$. The Fourier Transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n - p$.

We will denote the space of continuous functions on the sphere by $C(S^{n-1})$. The spaces of even continuous and infinitely smooth functions will be denoted $C_e(S^{n-1})$ and $C^\infty(S^{n-1})$, respectively.

For a star-body K (not necessarily convex), we define its Minkowski functional as $\|x\|_K = \min \{t \geq 0 \mid x/t \in K\}$. When K is a centrally-symmetric convex body, this of course coincides with the natural norm associated with it. Obviously $\rho_K(\theta) = \|\theta\|_K^{-1}$ for $\theta \in S^{n-1}$.

3 Proofs of the Statements

Before we begin, we shall need to recall several known facts about the Spherical Radon Transform R, and its connection to the Fourier transform of homogeneous distributions. It is well known (e.g. [Gro96, Chapter 3]) that $R : C_e(S^{n-1}) \to C_e(S^{n-1})$ is an injective operator, and that it is onto a dense set in $C_e(S^{n-1})$ which contains $C_e^\infty(S^{n-1})$. The connection with Fourier transforms of homogeneous distributions was demonstrated by Koldobsky, who showed (e.g. [Kol98a]) the following:

Lemma 3.1. *Let L denote a star-body in $\mathbb{R}^n$. Then for all $\theta \in S^{n-1}$:*

$$(\|\cdot\|_L^{-n+1})^\wedge(\theta) = \pi(n-1)\mathrm{Vol}\, D_{n-1} R(\|\cdot\|_L^{-n+1})(\theta).$$

In particular $(\|\cdot\|_L^{-n+1})^\wedge$ is continuous, and of course homogeneous of degree -1. Hence, if we denote $\rho_K(\theta) = \|\theta\|_K^{-1} = (\|\cdot\|_L^{-n+1})^\wedge(\theta)$ for $\theta \in S^{n-1}$ and use $(\|\cdot\|_K^{-1})^\wedge(\theta) = (2\pi)^n \|\theta\|_L^{-n+1}$, we immediately get the following inversion formula for the Spherical Radon transform:

Lemma 3.2. *Let K denote a star-body in $\mathbb{R}^n$ such that ρ_K is in the range of the Spherical Radon Transform. Then for all $\theta \in S^{n-1}$:*

$$R^{-1}(\rho_K)(\theta) = \frac{\pi(n-1)\mathrm{Vol}\, D_{n-1}}{(2\pi)^n} (\|\cdot\|_K^{-1})^\wedge(\theta).$$

Koldobsky also discovered the following property of the Fourier transform of a norm of a *convex* body ([Kol00, Corollary 2]):

Lemma 3.3. *Let K be an infinitely smooth centrally-symmetric convex body in $\mathbb{R}^n$. Then for every $E \in G(n,k)$:*

$$\int_{S^{n-1}\cap E} (\|\cdot\|_K^{-n+k+2})^\wedge(\theta)d\theta \geq 0.$$

Since $C_e^\infty(S^{n-1})$ is in the range of the Spherical Radon Transform, applying Lemma 3.3 with $k = n-3$ and using Lemma 3.2, we have:

Proposition 3.4. *Let K be an infinitely smooth centrally-symmetric convex body in $\mathbb{R}^n$. Then for every $E \in G(n,n-3)$:*

$$\int_{S^{n-1}\cap E} R^{-1}(\rho_K)(\theta)d\theta \geq 0.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, assume that K is infinitely smooth and fix $\theta \in S^{n-1}$. Denote by $H_\theta \in G(n,n-1)$ the hyperplane $\theta^\perp$, and let σ_{H_θ} denote the Haar probability measure on $S^{n-1}\cap H_\theta$. Let η_{H_θ} denote the Haar probability measure on the homogeneous space $G^{H_\theta}(n,n-3) := \{E \in G(n,n-3) | E \subset H_\theta\}$, and let σ_E denote the Haar probability measure on $S^{n-1}\cap E$ for $E \in G(n,n-3)$. Then:

$$\begin{aligned}\rho_K(\theta) = R(R^{-1}(\rho_K))(\theta) &= \int_{S^{n-1}\cap H_\theta} R^{-1}(\rho_K)(\xi)d\sigma_{H_\theta}(\xi) \\ &= \int_{E\in G^{H_\theta}(n,n-3)} \int_{S^{n-1}\cap E} R^{-1}(\rho_K)(\xi)d\sigma_E(\xi)d\eta_{H_\theta}(E). \qquad (3.1)\end{aligned}$$

The last transition is explained by the fact that the measure $d\sigma_E(\xi)d\eta_{H_\theta}(E)$ is invariant under orthogonal transformations preserving H_θ, so by the uniqueness of the Haar probability measure, it must coincide with $d\sigma_{H_\theta}(\xi)$. Denoting:

$$g(F) = \int_{S^{n-1}\cap F^\perp} R^{-1}(\rho_K)(\xi)d\sigma_E(\xi)$$

for $F \in G(n,3)$, we see by Proposition 3.4 that $g \geq 0$. Plugging the definition of g in (3.1), we have:

$$\rho_K(\theta) = \int_{E\in G^{H_\theta}(n,n-3)} g(E^\perp)d\eta_{H_\theta}(E) = \int_{F\in G_\theta(n,3)} g(F)d\nu_\theta(F),$$

where ν_θ is the Haar probability measure on the homogeneous space $G_\theta(n,3) := \{F \in G(n,3) | \theta \in F\}$ and the transition is justified as above. By (2.2), we conclude that $\rho_K = R_3^*(g)$ with $g \geq 0$, implying that the body K_3 satisfying $\rho_{K_3}^{n-3} = \rho_K$ is in $\mathcal{BP}_{n-3}^n$.

As mentioned in the Introduction, the case $a = 2$ follows from $a = 3$ by a general result from [Mil05], but for completeness we reproduce the easy argument. Using double-integration as before:

$$\rho_K(\theta) = \int_{F\in G_\theta(n,3)} g(F)d\nu_\theta(F) = \int_{J\in G_\theta(n,2)} \int_{F\in G_J(n,3)} g(F)d\nu_J(F)d\mu_\theta(J),$$

where μ_θ and ν_J are the Haar probability measures on the homogeneous spaces $G_\theta(n,2) := \{J \in G(n,2) | \theta \in J\}$ and $G_J(n,3) := \{F \in G(n,3) | J \subset F\}$, respectively. Denoting:

$$h(J) = \int_{F\in G_J(n,3)} g(F)d\nu_J(F),$$

we see that $h \geq 0$ and $\rho_K = R_2^*(h)$, implying that the body K_2 satisfying $\rho_{K_2}^{n-2} = \rho_K$ is in $\mathcal{BP}_{n-2}^n$.

When K is a general convex body, the result follows by approximation. It is well known (e.g. [Sch93, Theorem 3.3.1]) that any centrally-symmetric convex body K may be approximated (for instance in the radial metric) by a series of infinitely smooth centrally-symmetric convex bodies $\{K^i\}$. Denoting by K_a^i the star-bodies satisfying $\rho_{K_a^i} = \rho_{K_i}^{1/(n-a)}$ for $a = 2,3$, we have seen that $K_a^i \in \mathcal{BP}_{n-a}^n$. Obviously the series $\{K_a^i\}$ tends to K_a in the radial metric, and since $\mathcal{BP}_{n-a}^n$ is closed under taking radial limit (see [Mil05]), the result follows. □

Remark 3.5 (Added in Proofs). After reading a version of this note posted on the arXiv, it was communicated to us by Profs. Boris Rubin and Gaoyong Zhang that Theorem 1.1 also follows from Theorems 4.3, 4.4 and 5.1 from [RZ04]. Instead of using Koldobsky's Lemma 3.3 which is formulated in the language of Fourier-transforms, these authors use the language of analytic families of operators to prove similar results to those of Koldobsky, which enable them to answer certain generalizations of the generalized Busemann-Petty problem.

We now turn to close a few loose ends in the proof of Corollary 1.3. Since $\mathcal{BP}_k^n$ is closed under k-radial sums, it is immediate that if K^1 and K^2 are two convex bodies, L is their radial sum, and $\rho_{T_a} = \rho_T^{1/(n-a)}$ for $T = K_1, K_2, L$, then:

$$\rho_{L_a}^{n-a} = \rho_L = \rho_{K_1} + \rho_{K_2} = \rho_{K_a^1}^{n-a} + \rho_{K_a^2}^{n-a},$$

and therefore $L_a \in \mathcal{BP}_{n-a}^n$. This argument of course extends to any finite radial sum of convex bodies, and since $\mathcal{BP}_k^n$ is closed under taking limit in the radial metric, the argument extends to the entire class $\mathcal{RC}^n$ defined in the Introduction.

It remains to prove Corollary 1.2.

Proof of Corollary 1.2. By Theorem 1.1, it is enough to show that for a small enough $|\varepsilon|$ (which depends on n and M), the star-bodies L_a^ε defined by $\rho_{L_a^\varepsilon} = \rho_{K^\varepsilon}^{n-a}$ are in fact convex. Since $\rho_{L_a^\varepsilon} = (1+\varepsilon\varphi)^{n-a}$, it is clear that for every $\theta \in S^{n-1}$:

$$\left|\rho_{L_a^\varepsilon}(\theta)\right| \leq f_0(\varepsilon, n), \left|(\rho_{L_a^\varepsilon})_i(\theta)\right| \leq f_1(\varepsilon, n, M), \left|(\rho_{L_a^\varepsilon})_{i,j}(\theta)\right| \leq f_2(\varepsilon, n, M),$$

for every $i, j = 1, \ldots, n-1$, where f_0 tends to 1 and f_1, f_2 tend to 0, as $\varepsilon \to 0$. It should be intuitively clear that the convexity of L_a^ε depends only on the behaviour of the derivatives of order 0,1 and 2 of $\rho_{L_a^\varepsilon}$, and since we have uniform convergence of these derivatives to those of the Euclidean ball as ε tends to 0, L_a^ε is convex for small enough ε. To make this argument formal, we follow [Gar94a], and use a formula for the Gaussian curvature of a star-body L whole radial function ρ_L is twice continuously differentiable, which was explicitly calculated in [Oli84, 2.5]. In particular, it follows that $M_L(\theta)$, the Gaussian curvature of ∂L (the hypersurface given by the boundary of L) at $\rho_L(\theta)\theta$, is a continuous function of the derivatives of order 0,1 and 2 of ρ_L at the point θ. Since the Gaussian curvature of the boundary of the Euclidean ball is a constant 1, it follows that for small enough ε, the boundary of L_a^ε has everywhere positive Gaussian curvature. By a standard result in differential geometry (e.g. [KN69, p. 41]), this implies that L_a^ε is convex. This concludes the proof. □

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Random Convex Bodies Lacking Symmetric Projections, Revisited Through Decoupling

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Summary. In 2001, E.D. Gluskin, A.E. Litvak and N. Tomczak-Jaegermann, using probabilistic methods inspired by some earlier work of Gluskin's, provided an example of a convex body $K \subset \mathbb{R}^n$ such that no suitably large rank projection of K is symmetric. We provide an alternate proof of the existence of such a body, the key ingredient of which is a decoupling result due to S.J. Szarek and Tomczak-Jaegermann.

1 Introduction

In problems that seem susceptible to probabilistic methods, independence is often desirable but not necessarily present within the given constraints. A recent decoupling result due to S.J. Szarek and N. Tomczak-Jaegermann allows one to overcome the obstacle of dependency, given that certain conditions are present. Originally applied to some problems in the asymptotic theory of normed spaces [ST2], the general statement in an arbitrary probabilistic setting appears in [ST1]. This note presents a natural application of said decoupling result in the theory of non-symmetric convex bodies. Namely, we show how it can be used to provide a new proof of a theorem due to E.D. Gluskin, A.E. Litvak and Tomczak-Jaegermann [GLT]. The theorem asserts there is a convex body $K \subset \mathbb{R}^n$ such that for any projection P, the Minkowski measure of symmetry of PK is at least $(\operatorname{rank} P)/c\sqrt{n \ln n}$, where c is a positive absolute constant. Besides using Gaussian random vectors instead of uniformly distributed random vectors on the sphere, we follow the same approach as the original proof. The constraints that arise in this approach are particularly well-suited to an application of the decoupling result, after which the argument can proceed using only the most elementary properties of Gaussian random vectors.

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While the theorem under consideration is a relatively recent result and deals with non-symmetric convex bodies, it is a natural descendent of some important work in the 1980s dealing with symmetric convex bodies. Namely, both proofs use non-symmetric analogues of the random convex bodies introduced by Gluskin. Initially used to give the asymptotic order of the Banach-Mazur compactum [G1], random bodies have played an important role in the development of the theory of symmetric convex bodies (see, e.g., [MT]) and, more recently, non-symmetric convex bodies (see, e.g., [GLT] and the references cited therein).

2 Preliminaries

We will adopt the standard terminology and notation from the asymptotic theory of normed spaces (see, e.g., [MT]). For $p = 1, 2$ the closed unit ball in ℓ_p^n will be denoted by B_p^n. The canonical Euclidean norm on $\mathbb{R}^n$ will be denoted by $\|\cdot\|_2$. The standard unit vector basis for $\mathbb{R}^n$ will be denoted by $(e_i)_{i=1}^n$. For $B \subset \mathbb{R}^n$, the convex hull of B shall be denoted by conv$B$; the absolute convex hull of B is the set absconv$B := \text{conv}(B \cup (-B))$.

Let K and L be convex bodies in $\mathbb{R}^n$, i.e., compact, convex sets with non-empty interior. We say that K is centrally symmetric if $K = -K$ and we denote the set of all centrally symmetric convex bodies in $\mathbb{R}^n$ by $\mathcal{C}^n$. If K and L belong to $\mathcal{C}^n$ then their Banach-Mazur distance is defined by

$$d(K, L) := \inf\{A \geq 1 : L \subset TK \subset AL\},$$

where the infimum is taken over all invertible linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. The Banach-Mazur distance between arbitrary convex bodies K and L is defined by

$$d(K, L) := \inf\big\{A \geq 1 : L - x \subset TK \subset A(L - x)\big\} \tag{1}$$

where the infimum is taken over all $x \in \mathbb{R}^n$ and all invertible affine transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. It is easy to verify that these notions coincide when K and L are centrally symmetric.

We shall be concerned with a measure of symmetry for convex bodies that aren't centrally symmetric. One such measure is the quantity $\delta(K) := \inf\{d(K, L) : L \in \mathcal{C}^n\}$. In fact, this is simply the Minkowski measure of symmetry of K, i.e.,

$$\delta(K) = \inf\big\{A \geq 1 : -(K - x) \subset A(K - x) \text{ for some } x \in K\big\}.$$

For other measures of symmetry we refer the reader to [Gr].

As with many proofs involving random convex bodies, we will use an approximation argument by means of an ε-net. Fix a normed space $X = (\mathbb{R}^n, \|\cdot\|)$ with unit ball B_X. Let $Y \subset X$ and let $\varepsilon > 0$. A subset $\mathcal{N}$ of Y is an ε-net for

Y if for any $y \in Y$ there exists $x \in \mathcal{N}$ such that $\|x - y\| \le \varepsilon$. Let $|\cdot|$ denote the cardinality of a finite set. The following fact is well-known.

Fact 2.1. If U is any subset of B_X then there exists an ε-net $\mathcal{N}$ for U such that $|\mathcal{N}| \le (1 + 2/\varepsilon)^n$.

Let us now recall some basic probabilistic tools. Let $(\Omega, \mathbb{P})$ be a probability space. If h is a random variable defined on Ω and B is a subset of its range, we will use the notation $\mathbb{P}(\{\omega \in \Omega : h(\omega) \in B\})$ and $\mathbb{P}(\{h \in B\})$ interchangeably. Let γ_i $(i = 1, \dots, n)$ be independent Gaussian random variables with $N(0, 1)$ distribution. Then the random variable $h : \Omega \to \mathbb{R}^n$ defined by $h = (1/\sqrt{n}) \sum_{i=1}^n \gamma_i e_i$ satisfies

$$\mathbb{P}\{h \in B\} = \left(\frac{n}{2\pi}\right)^{\frac{n}{2}} \int_B e^{-\frac{n\|x\|_2^2}{2}} dx_1 \dots dx_n$$

for any Borel set $B \subset \mathbb{R}^n$. In this case, we say that h is a normalized Gaussian vector. Our choice of normalization yields that the expected value of $\|h\|_2^2$ is equal to 1. The following elementary facts can be found, e.g., in [MT].

Fact 2.2. Let $g : \Omega \to \mathbb{R}^n$ be a normalized Gaussian vector. Then g exhibits the following properties:

(i) For every r-dimensional subspace $E \subset \mathbb{R}^n$, $\sqrt{\frac{n}{r}} P_E g$ is a normalized Gaussian vector in E, where P_E denotes the orthogonal projection onto E.
(ii) $\mathbb{P}\{\|g\|_2 \le 2\} \ge 1 - e^{-(1/2)n}$.
(iii) For every Borel set $B \subset \mathbb{R}^n$ we have

$$\mathbb{P}\{g \in B\} \le e^{n/2} \frac{\mathrm{vol}\,(B)}{\mathrm{vol}\,(B_2^n)}.$$

In light of the last fact, it will be convenient to have some volume formulas at hand. In particular, we will need $\mathrm{vol}\,(B_1^n) = 2^n/n!$ and $\mathrm{vol}\,(B_2^n) = \pi^{n/2}/\Gamma(n/2 + 1)$, where Γ denotes the Gamma function. We shall also make use of the following well-known result from [CP] and [G2] (see also [BP] for a simpler proof and [BaF] for a related result).

Theorem 2.3. *Let $1 \le n \le m$ and let $x_1, \dots, x_m \in \mathbb{R}^n$. Then*

$$\mathrm{vol}\left(\mathrm{absconv}\,\{(x_i)_{i=1}^m\}\right) \le \mathrm{vol}\left(3\alpha\sqrt{1 + \ln(m/n)} B_1^n\right),$$

where $\alpha := \max_{i \le m} \|x_i\|_2$.

3 Convex Bodies Lacking Symmetric Projections

The following theorem is due to Gluskin, Litvak and Tomczak-Jaegermann [GLT].

Theorem 3.1. *For any positive integer n, there exists a convex body $K \subset \mathbb{R}^n$ such that for any projection P, the lower bound*

$$\delta(PK) \geq \frac{\operatorname{rank} P}{c\sqrt{n \ln n}}$$

is satisfied, where $c > 0$ is an absolute constant.

Remark 3.2. For projections of proportional rank, the estimate of Theorem 3.1 is optimal, save a logarithmic factor. Indeed, for every convex body K in $\mathbb{R}^n$ there exists a projection P of rank $r = n/2$, say, such that $\delta(PK) \leq C\sqrt{n}$ where $C > 0$ is an absolute constant (see [LT]).

We will repeat several arguments from the original proof.

Proof. Since $\delta(\cdot) \geq 1$, we need only prove the theorem for those projections P with $\operatorname{rank} P > c\sqrt{n \ln n}$. Moreover, it suffices to prove the theorem for orthogonal projections. Indeed, let P be any projection and let Q be the orthogonal projection with the same kernel as P. Then Q and P have the same rank and $Q = QP$. Thus $\delta(PK) \geq \delta(Q[PK]) = \delta(QK)$.

Let $m = m(n)$ be an integer such that $c_0 n^2 \ln n \leq m \leq e^{(1/4)n}$, where $c_0 > 1$ is an absolute constant to be specified later and the upper bound on m need only hold for all $n \geq n_0$ for some fixed integer n_0. Let $g_1, \ldots, g_m$ be independent normalized Gaussian vectors defined on a probability space $(\Omega, \mathbb{P})$. Set

$$K(\omega) := \operatorname{conv}\{g_1(\omega), \ldots, g_m(\omega)\}.$$

For $r \leq n$, set $A_r = r/(c_1\sqrt{n \ln m})$ with $c_1 > 1$ an absolute constant to be specified later. It will be shown that for each $n \geq n_0$, the measure of the set of $\omega \in \Omega$ for which $K(\omega)$ satisfies $\delta(PK(\omega)) \geq \frac{1}{2}A_r$ for all projections P of rank $r > 2c_1\sqrt{n \ln m}$ is larger than $1 - e^{-(1/4)n} - e^{-(1/12)m}$. Hence for $m = c_0 n^3$, say, we will obtain an absolute constant $c = c(c_0, c_1)$ so that the theorem is true for all $n \geq n_0$; by then adjusting c we can of course claim the result for all n. Thus let us assume that $n \geq n_0$. It is sufficient to prove the following proposition.

Proposition 3.3. *The probability that there exists an orthogonal projection P of rank $r > c_1\sqrt{n \ln m}$ such that*

$$-Pg_i \in A_r \operatorname{conv}\left\{(Pg_j)_{j \neq i}, 0\right\} \tag{2}$$

for each $i \leq m$ is less than $e^{-(1/4)n} + e^{-(1/12)m}$.

Suppose for the moment that Proposition 3.3 is true. Take $\omega \in \Omega$ such that for any orthogonal projection P with $\operatorname{rank} P > c_1\sqrt{n \ln m}$ there is at least one $i \leq m$ such that (2) fails. Write $g_j = g_j(\omega)$ ($j = 1, \ldots, m$) and $K = K(\omega)$. Let P be an orthogonal projection of rank $r > 2c_1\sqrt{n \ln m}$ and set $A = \frac{1}{2}A_r$. If $\delta(PK) \leq A$ then there exists $a \in PK$ such that

$$-(Pg_i - a) \in A \operatorname{conv} \{(Pg_j - a)_{j \le m}\}$$

for all $i \le m$, which implies that

$$-(Pg_i - a) \in A \operatorname{conv} \{(Pg_j - a)_{j \ne i}, 0\}$$

for all $i \le m$. Let R be the orthogonal projection whose kernel is equal to $\operatorname{span}\{a\}$. Then for each $i \le m$ we have

$$-RPg_i \in A \operatorname{conv} \{(RPg_j)_{j \ne i}, 0\} \subset A_{r-1} \operatorname{conv} \{(RPg_j)_{j \ne i}, 0\}.$$

Observe now that RP is itself an orthogonal projection (we stress that $a \in PK$) and $\operatorname{rank}(RP) \ge r - 1 > c_1\sqrt{n \ln m}$, contradicting our choice of ω.

The annihilating projection argument used in the preceding proof is well-known to specialists.

Proof of Proposition 3.3. We will use a standard approximation technique. Let $\varepsilon = 1/(2\sqrt{n})$ and let $r_0 := c_1\sqrt{n \ln m}$. For $r > r_0$, let $\mathcal{N}_r$ be an ε-net of rank r orthogonal projections (in the operator norm) with minimal cardinality. It can be shown that $|\mathcal{N}_r| \le (C/\varepsilon)^{r(n-r)}$, where $C > 0$ is an absolute constant (see [S]). For our purpose, however, the weaker estimate $|\mathcal{N}_r| \le (3/\varepsilon)^{n^2}$, obtained from Fact 2.1, will be sufficient.

We will first concern ourselves with finding an upper bound for the probability that there exists $Q \in \mathcal{N}_r$ with $r > r_0$ such that

$$-Qg_i \in A_r \operatorname{conv} \{(Qg_j)_{j \ne i}, 0\} + 4A_r\varepsilon QB_2^n \tag{3}$$

for each $i \le m$.

Let us now fix $r > r_0$ and $Q \in \mathcal{N}_r$. For convenience of notation let $[m]$ denote the set $\{1, \dots, m\}$. For $I \subset [m]$ and $\omega \in \Omega$, let

$$\mathbf{K}_I(\omega) := A_r \operatorname{conv} \{\big(Qg_j(\omega)\big)_{j \in I}, 0\} + 4A_r\varepsilon QB_2^n.$$

Since our main estimate will involve a comparison of volumes, it will be useful to consider subsets of Ω on which the g_j's are uniformly bounded. For $I \subset [m]$ let

$$\Omega_I^0 := \bigcap_{j \in I} \{\omega \in \Omega : \|g_j(\omega)\|_2 \le 2\}$$

and set $\Omega^0 := \Omega_{[m]}^0$. Let Σ_I denote the σ-algebra generated by $\{g_j : j \in I\}$. Then, in particular, Ω_I^0 is Σ_I-measurable.

Let us also note that by Fact 2.2(ii), for each $j \in [m]$ the set $\Omega_{\{j\}}^0$ satisfies $\mathbb{P}(\Omega_{\{j\}}^0) \ge 1 - e^{-(1/2)n}$ and hence

$$\mathbb{P}(\Omega^0) \ge 1 - me^{-(1/2)n} \ge 1 - e^{-(1/4)n} \tag{4}$$

by our choice of m.

For $i \in [m]$ and $I \subset [m]$ let

$$\Omega_{i,I} := \big\{\omega \in \Omega : -Qg_i(\omega) \in \mathbf{K}_I(\omega)\big\}$$

and let

$$\widetilde{\Omega}_{i,I} := \Omega_{i,I} \cap \Omega_I^0.$$

We will first show that

$$\mathbb{P}\bigg(\bigcap_{i=1}^{m} \widetilde{\Omega}_{i,\{i\}^c}\bigg) \leq e^{-(1/6)m}. \tag{5}$$

The notation $\cdot^c$ denotes the complement of a set with respect to $[m]$.

Consider the family of events $\{\widetilde{\Omega}_{i,I} : i \in [m], I \subset [m]\}$ in the context of Theorem 2 from [ST1]. By Caratheodory's theorem, for each fixed ω, membership in the set $\mathbf{K}_I(\omega)$ depends on at most $r+1$ elements of the set $\{g_j(\omega) : j \in I\}$. Thus for any $i \in [m]$ and any $I \subset [m]$ we have the inclusion

$$\widetilde{\Omega}_{i,I} \subset \bigcup_{\substack{I' \subset I \\ |I'| \leq r+1}} \widetilde{\Omega}_{i,I'}.$$

Next, if I_1 and I_2 are disjoint subsets of $[m]$, the family $\{\widetilde{\Omega}_{i,I_1} : i \in I_2\}$ is Σ_{I_1}-conditionally independent by the independence of the collections $(g_i)_{i\in I_1}$ and $(g_i)_{i\in I_2}$.

Once we find constants p_i such that $\mathbb{P}(\widetilde{\Omega}_{i,\{i\}^c}|\Sigma_{\{i\}^c}) \leq p_i$ $(i \in [m])$, Theorem 2 from [ST1] will allow us to conclude that

$$\mathbb{P}\bigg(\bigcap_{i=1}^{m} \widetilde{\Omega}_{i,\{i\}^c}\bigg) \leq \sum_{J\in\mathcal{J}_\ell} \prod_{i\in J} p_i, \tag{6}$$

where ℓ is the smallest integer larger than $m/(3r)$ and $\mathcal{J}_\ell := \{J \subset [m] : |J| = \ell\}$. To obtain suitable p_i's, we will appeal to Fact 2.2(iii) and therefore need a volume estimate.

Lemma 3.4. *For any $i \in [m]$ and for any $\omega \in \Omega^0_{\{i\}^c}$, we have*

$$\operatorname{vol}\big(\mathbf{K}_{\{i\}^c}(\omega)\big) \leq \Big(24A_r\sqrt{\ln m}\Big)^r \operatorname{vol}(B_1^r).$$

Proof. Let $i \in [m]$ and let $\omega \in \Omega^0_{\{i\}^c}$. Note that $(1/\sqrt{n})B_2^n \subset B_1^n = \operatorname{absconv}\big\{(e_j)_{j=1}^n\big\}$ which implies

$$4A_r\varepsilon QB_2^n \subset A_r \operatorname{absconv}\big\{(2Qe_j)_{j=1}^n\big\}.$$

Thus if we set

$$B := A_r \operatorname{absconv}\Big\{\big(Qg_j(\omega)\big)_{j\neq i}, 0, (2Qe_j)_{j=1}^n\Big\},$$

we obtain $\mathbf{K}_{\{i\}^c}(\omega) \subset 2B$. Theorem 2.3 yields the desired result. □

Lemma 3.5. *The estimate*

$$\mathbb{P}\left(\widetilde{\Omega}_{i,\{i\}^c}|\Sigma_{\{i\}^c}\right) \le e^{-r}$$

is satisfied for any $i \in [m]$.

Proof. Let $i \in [m]$. Since we are concerned with the conditional probability given $\Sigma_{\{i\}^c}$, we may consider the probability space $\Omega_i \times \Omega_{\{i\}^c}$ equipped with σ-algebras $\Sigma_{\{i\}}$ and $\Sigma_{\{i\}^c}$, respectively, depending on the corresponding coordinates. If $\widetilde{\omega} \in \Omega^0_{\{i\}^c}$ is fixed, then by Fact 2.2(iii) and Lemma 3.4 we have

$$\begin{aligned}
&\mathbb{P}\Big(\big\{\omega_i \in \Omega_i : -Qg_i(\omega_i) \in \mathbf{K}_{\{i\}^c}(\widetilde{\omega})\big\}\Big) \\
&= \mathbb{P}\left(\big\{\omega_i \in \Omega_i : -\sqrt{n/r}Qg_i(\omega_i) \in \sqrt{n/r}\mathbf{K}_{\{i\}^c}(\widetilde{\omega})\big\}\right) \\
&\le e^{\frac{r}{2}} \frac{\mathrm{vol}\left(\sqrt{n/r}\mathbf{K}_{\{i\}^c}(\widetilde{\omega})\right)}{\mathrm{vol}\,(B_2^r)} \\
&\le \left(\sqrt{\frac{ne}{r}}\right)^r \left(24A_r\sqrt{\ln m}\right)^r \left(\sqrt{\frac{2e}{\pi r}}\right)^r \\
&\le (60/c_1)^r.
\end{aligned}$$

If $\widetilde{\omega} \notin \Omega^0_{\{i\}^c}$ then $\mathbb{P}(\widetilde{\Omega}_{i,\{i\}^c}|\Sigma_{\{i\}^c})(\widetilde{\omega}) = 0$. Thus for a suitable choice of c_1 (e.g. $c_1 = 60e$), we obtain the pointwise estimate $\mathbb{P}(\widetilde{\Omega}_{i,\{i\}^c}|\Sigma_{\{i\}^c}) \le e^{-r}$. □

We are now in a position to apply (6). First note that Stirling's formula implies the estimate $|\mathcal{J}_\ell| = \binom{m}{\ell} \le (em/\ell)^\ell \le \exp(\ell\ln(3er))$. Consequently, for sufficiently large r we obtain

$$\mathbb{P}\Big(\bigcap_{i=1}^m \widetilde{\Omega}_{i,\{i\}^c}\Big) \le \exp\big(\ell\ln(3er)\big)\exp(-r\ell) \le e^{-(1/6)m}.$$

Thus we have proven estimate (5) for a fixed $r > r_0$ and a fixed $Q \in \mathcal{N}_r$. Since the calculations so far do not depend on these fixed values, estimate (5) in fact holds for any $r > r_0$ and any $Q \in \mathcal{N}_r$.

Suppose now that there is an orthogonal projection P with $\mathrm{rank}\,P = r > r_0$ and $\omega \in \Omega$ such that (2) holds for each $i \le m$. Choose $Q \in \mathcal{N}_r$ such that $\|Q - P\| \le \varepsilon$. If $\omega \in \Omega^0$, note that $\max_i\|(Q-P)g_i(\omega)\|_2 \le 2\varepsilon$ and for each $i \le m$ we have

$$\mathrm{dist}\left(-Qg_i(\omega), A_r\,\mathrm{conv}\left\{\big(Qg_j(\omega)\big)_{j\neq i}, 0\right\}\right) \le 4A_r\varepsilon.$$

Observe now that

$$\bigcap_{i=1}^{m} \Big\{ -Pg_i \in A_r \operatorname{conv}\big\{(Pg_j)_{j\neq i}, 0\big\}\Big\} \subset \Omega\backslash\Omega^0 \cup \Big(\bigcap_{i=1}^{m} \Omega_{i,\{i\}^c} \cap \Omega^0 \Big)$$
$$\subset \Omega\backslash\Omega^0 \cup \bigcap_{i=1}^{m} \widetilde{\Omega}_{i,\{i\}^c}.$$

The latter inclusion, together with (4) and (5), implies that the probability there exists an orthogonal projection P of rank $r > r_0$ such that (2) holds for every $i \leq m$ is less than or equal to

$$e^{-(1/4)n} + n(3/\varepsilon)^{n^2} e^{-(1/6)m} \leq e^{-(1/4)n} + \exp\big(\ln n + n^2 \ln(6\sqrt{n}) - (1/6)m\big).$$

For a suitable choice of c_0 (e.g. $c_0 = 48$) the latter expression can be made less than $e^{-(1/4)n} + e^{-(1/12)m}$ for all $m \geq c_0 n^2 \ln n$. This proves the proposition. □

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The Random Version of Dvoretzky's Theorem in ℓ_∞^n

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Summary. We show that with "high probability" a section of the ℓ_∞^n ball of dimension $k \le c\varepsilon \log n$ ($c > 0$ a universal constant) is ε close to a multiple of the Euclidean ball in this section. We also show that, up to an absolute constant the estimate on k cannot be improved.

1 Introduction

Milman's version of Dvoretzky's theorem states that:
There is a function $c(\varepsilon) > 0$ such that for all $k \le c(\varepsilon) \log n$, ℓ_2^k $(1+\varepsilon)$-embeds into any normed space of dimension n.

See [Dv] for the original theorem of Dvoretzky (in which the dependence of k on n is weaker), [Mi] for Milman's original work, and [MS] and [Pi] for expository outlets of the subject (there are many others). It would be important for us to notice that the proof(s) of the theorem above actually give more: The vast majority of subspaces of the stated dimension are $(1+\varepsilon)$-isomorphic to ℓ_2^k.

The dependence of k on n in the theorem above is known to be best possible (for ℓ_∞^n) but the dependence on ε is far from being understood. The best known estimate is $c(\varepsilon) \ge c\varepsilon/(\log \frac{1}{\varepsilon})^2$ given in [Sc] (here and elsewhere in this paper c and C denote positive universal constants). However, the proof in [Sc] does not give the additional information that *most* subspaces are $(1+\varepsilon)$-isomorphic to ℓ_2^k. If one also want this requirement then the best estimate for $c(\varepsilon)$ that was known was $c(\varepsilon) \ge c\varepsilon^2$ ([Go]).

As an upper bound for $c(\varepsilon)$ one gets $C/\log \frac{1}{\varepsilon}$ for some universal C. Indeed, if ℓ_2^k $(1+\varepsilon)$ embed into ℓ_∞^n then $k \le C \log n / \log \frac{1}{\varepsilon}$. This is also the right order of k in the ℓ_∞ case: If $k \le c \log n/\log \frac{1}{\varepsilon}$ then ℓ_2^k $(1+\varepsilon)$ embed into ℓ_∞^n.

We show here that, in the ℓ_∞ case, if one is interested in the probabilistic statement of Dvoretzky theorem (i.e, that the vast majority of subspaces of

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ℓ_∞^n of a certain dimension are $(1+\varepsilon)$-isomorphic to Euclidean spaces) then the right estimate for $c(\varepsilon)$ is $c\varepsilon$.

Theorem 1. *For $k < c\varepsilon \log n$, with probability $> 1 - e^{-ck}$, the ℓ_∞^n norm and a multiple of the ℓ_2^n norm are $1+\varepsilon$ equivalent on a k dimensional subspace. Moreover, this doesn't hold anymore for k of higher order. i.e., For every a there is an A such that if, with probability larger than $1-e^{-ak}$, a k dimensional subspace satisfies that the ratio between the ℓ_∞^n norm and a multiple of the ℓ_2^n norm are $1+\varepsilon$ equivalent for all vectors in the subspace, then $k \le A\varepsilon \log n$.*

2 Computation of the Concentration of the Max Norm

Let $g_1, g_2, \dots$ be a sequence of standard independent Gaussian variables. fix n and let M be the median of $\|(g_1, g_2, \dots, g_n)\|_\infty$. In this section we compute some fine estimates on the probability of deviation of $\|(g_1, g_2, \dots, g_n)\|_\infty$ from M.

Claim 1.

$$\big(1 - 2^{-1/n}\big)\frac{\sqrt{\pi}M}{\sqrt{2}} \le e^{-M^2/2} \le (1 - 2^{-1/n})\frac{\sqrt{\pi}(M+1)}{\sqrt{2}(1 - e^{-\frac{1}{2}}e^{-M})}. \tag{1}$$

Proof.

$$\frac{1}{2} = P\Big(\max_{1\le i\le n} |g_i| < M\Big) = \left(1 - \sqrt{\frac{2}{\pi}}\int_M^\infty e^{-s^2/2} ds\right)^n.$$

Consequently,

$$\begin{aligned} 1 - 2^{-1/n} &= \sqrt{\frac{2}{\pi}}\int_M^\infty e^{-s^2/2} ds \ge \sqrt{\frac{2}{\pi}}\frac{1}{M+1}\int_M^{M+1} s e^{-s^2/2} ds \\ &\ge \sqrt{\frac{2}{\pi}}\frac{1}{M+1} e^{-M^2/2}(1 - e^{-\frac{1}{2}}e^{-M}), \end{aligned} \tag{2}$$

or

$$e^{-M^2/2} \le (1 - 2^{-1/n})\frac{\sqrt{\pi}(M+1)}{\sqrt{2}(1 - e^{-\frac{1}{2}}e^{-M})}. \tag{3}$$

Similarly,

$$1 - 2^{-1/n} = \sqrt{\frac{2}{\pi}}\int_M^\infty e^{-s^2/2} ds \le \sqrt{\frac{2}{\pi}}\frac{1}{M}\int_M^\infty s e^{-s^2/2} ds \le \sqrt{\frac{2}{\pi}}\frac{e^{-M^2/2}}{M},$$

or

$$e^{-M^2/2} \ge (1 - 2^{-1/n})\frac{\sqrt{\pi}M}{\sqrt{2}}. \tag{4}$$

□

Claim 2.

$$\frac{\log 2}{4+\log 2} e^{-3\varepsilon M^2/2} \le P\Big(\max_{1\le i\le n} |g_i| > (1+\varepsilon)M\big) \le \log 2\big(1+o(1)\Big) e^{-\varepsilon M^2} \quad (5)$$

where $o(1)$ means $a(n)$ with $a(n) \to 0$ as $n \to \infty$ independently of ε.

Proof. (3) implies

$$\begin{aligned} &P\Big(\max_{1\le i\le n} |g_i| > (1+\varepsilon)M\Big) \\ &\le \sqrt{\frac{2}{\pi}} \frac{n}{(1+\varepsilon)M} e^{-(1+\varepsilon)^2 M^2/2} \\ &\le \frac{n}{(1+\varepsilon)M} (1-2^{-1/n}) \frac{M+1}{1-e^{-\frac{1}{2}} e^{-M}} e^{-\varepsilon M^2} e^{-\varepsilon^2 M^2/2} \end{aligned} \quad (6)$$

and, since M is of order $\sqrt{\log n}$, we get from this that

$$P\Big(\max_{1\le i\le n} |g_i| > (1+\varepsilon)M\Big) \le \log 2\big(1+o(1)\big) e^{-\varepsilon M^2}. \quad (7)$$

(For a fixed ε one can replace $\log 2(1+o(1))$ with a quantity tending to 0 with n.)

We now look for a lower bound on $P(\max_{1\le i\le n} |g_i| > (1+\varepsilon)M)$. Since for iid X_i-s,

$$\begin{aligned} P\Big(\max_{1\le i\le n} (X_i > t\Big) &= 1 - \big(1 - P(X_1 > t)\big)^n \ge 1 - e^{-nP(X_1>t)} \\ &\ge 1 - \frac{1}{1+nP(X_1>t)} = \frac{nP(X_1>t)}{1+nP(X_1>t)}, \end{aligned} \quad (8)$$

$$P\Big(\max_{1\le i\le n} |g_i| > (1+\varepsilon)M\Big) \ge \frac{nP(|g_1| > (1+\varepsilon)M)}{1+nP(|g_1| > (1+\varepsilon)M)}. \quad (9)$$

The right hand side is an increasing function of $P(|g_1| > (1+\varepsilon)M)$ and, by (4),

$$\begin{aligned} &P(|g_1| > (1+\varepsilon)M) \\ &\ge \sqrt{\frac{2}{\pi}} \frac{1}{(1+\varepsilon)M+1} e^{-(1+\varepsilon)^2 M^2/2} \big(1 - e^{-\frac{1}{2}} e^{-(1+\varepsilon)M}\big) \\ &= \sqrt{\frac{2}{\pi}} \frac{1}{(1+\varepsilon)M+1} e^{-M^2/2} e^{-\varepsilon M^2 - \varepsilon^2 M^2/2} \big(1 - e^{-\frac{1}{2}} e^{-(1+\varepsilon)M}\big) \\ &\ge \frac{M(1-2^{-1/n})}{(1+\varepsilon)M+1} e^{-\varepsilon M^2 - \varepsilon^2 M^2/2} \big(1 - e^{-\frac{1}{2}} e^{-(1+\varepsilon)M}\big) \\ &\ge \frac{\log 2}{4n} e^{-\varepsilon M^2 - \varepsilon^2 M^2/2} \ge \frac{\log 2}{4n} e^{-3\varepsilon M^2/2}, \end{aligned} \quad (10)$$

for $\varepsilon \le 1$ and n large enough (independently of ε). Using (9), we get

$$P\Big(\max_{1\le i\le n}|g_i|>(1+\varepsilon)M\Big)\ge\frac{\frac{\log 2}{4}e^{-3\varepsilon M^2/2}}{1+\frac{\log 2}{4}e^{-3\varepsilon M^2/2}}\ge\frac{\log 2}{4+\log 2}e^{-3\varepsilon M^2/2}. \quad (11)$$

□

Claim 3. For some absolute positive constants c, C and for all $0<\varepsilon<1/2$,

$$\exp\big(-Ce^{\varepsilon M^2}\big)\le P\Big(\max_{1\le i\le n}|g_i|<(1-\varepsilon)M\Big)\le C\exp\big(-ce^{3\varepsilon M^2/4}\big). \quad (12)$$

Proof.

$$\begin{aligned}
&P\Big(\max_{1\le i\le n}|g_i|<(1-\varepsilon)M\Big)\\
&=\Big(1-\sqrt{\frac{2}{\pi}}\int_{(1-\varepsilon)M}^{\infty}e^{-s^2/2}\Big)^n\le\Big(1-\sqrt{\frac{2}{\pi}}\int_{(1-\varepsilon)M}^{M}e^{-s^2/2}\Big)^n\\
&\le\Big(1-\sqrt{\frac{2}{\pi}}\frac{1}{M}\int_{(1-\varepsilon)M}^{M}se^{-s^2/2}\Big)^n\\
&=\Big(1-\sqrt{\frac{2}{\pi}}\frac{1}{M}\big(e^{-(1-\varepsilon)^2M^2/2}-e^{-M^2/2}\big)\Big)^n\\
&=\Big(1-\sqrt{\frac{2}{\pi}}\frac{1}{M}e^{-M^2/2}\big(e^{\varepsilon M^2-\varepsilon^2M^2/2}-1\big)\Big)^n\\
&\le\Big(1-(1-2^{-1/n})\big(e^{\varepsilon M^2-\varepsilon^2M^2/2}-1\big)\Big)^n \quad \text{by (4)}\\
&\le\exp\big(-n(1-2^{-1/n})(e^{\varepsilon M^2-\varepsilon^2M^2/2}-1)\big)\\
&\le 2\big(1+o(1)\big)\exp\big(-\log 2\big(1+o(1)\big)e^{3\varepsilon M^2/4}\big)
\end{aligned}$$

which proves the right hand side inequality in (12). As for the left hand side,

$$\begin{aligned}
&P\Big(\max_{1\le i\le n}|g_i|<(1-\varepsilon)M\Big)\\
&=\Big(1-\sqrt{\frac{2}{\pi}}\int_{(1-\varepsilon)M}^{\infty}e^{-s^2/2}\Big)^n\ge\Big(1-\sqrt{\frac{2}{\pi}}\frac{1}{(1-\varepsilon)M}\int_{(1-\varepsilon)M}^{\infty}se^{-s^2/2}\Big)^n\\
&=\Big(1-\sqrt{\frac{2}{\pi}}\frac{1}{(1-\varepsilon)M}\big(e^{-(1-\varepsilon)^2M^2/2}\Big)^n=\Big(1-\sqrt{\frac{2}{\pi}}\frac{e^{-M^2/2}}{(1-\varepsilon)M}e^{\varepsilon M^2-\varepsilon^2M^2/2}\Big)^n\\
&\ge\exp\Big(-\frac{2n(1-2^{-1/n})}{1-e^{-\frac{1}{2}-M}}\frac{M+1}{M}e^{\varepsilon M^2-\varepsilon^2M^2/2}\Big) \quad \text{by (3)}\\
&\ge\exp\big(-2\log 2\big(1+o(1)\big)e^{\varepsilon M^2}\big).
\end{aligned}$$

□

We summarize Claims 2 and 3 in a form that will be useful for us later in the following Proposition.

Proposition 1. *For some positive absolute constants c, C and for all $0<\varepsilon<1$ and $n \in \mathbb{N}$, denoting $g = (g_1, g_2, \ldots, g_n)$,*

$$ce^{-C\varepsilon \log n} \le P\Big(\|g\|_\infty < \frac{(1-\varepsilon)M}{\sqrt{n}}\|g\|_2 \quad or \quad \|g\|_\infty > \frac{(1+\varepsilon)M}{\sqrt{n}}\|g\|_2\Big)$$
$$\le Ce^{-c\varepsilon \log n}.$$

Proof. This follows easily from Claims 2 and 3 and the facts that $e^x > x$ for all x, M is of order $\sqrt{\log n}$ and

$$P\Big(\|g\|_2 < (1-\varepsilon)\sqrt{n} \;\text{ or }\; \|g\|_2 > (1+\varepsilon)\sqrt{n}\Big) < Ce^{-\varepsilon^2 n}.$$

3 Proof of the Theorem

The first part of the Theorem follows easily from the, by now well exposed, proof of Milman's version of Dvoretzky's theorem (see e.g, [MS] or [Pi]) with the improved concentration estimate in (the right hand side of the inequality in) Proposition 1 replacing the classical estimates. For the proof of the second part we need:

Lemma 1. *Let $\mathcal{A}$ be a subset of $G_{n,k}$ of $\mu_{n,k}$ measure a. Put $U_\mathcal{A} = \bigcup_{E\in\mathcal{A}} E$, then*

$$P\big((g_1, g_2, \ldots, g_n) \in U_\mathcal{A}\big) \ge a^{1/k}.$$

Proof. Let $X_1, X_2, \ldots, X_k$ be k independent random vectors distributed according to P, the canonical Gaussian measure on $\mathbb{R}^n$. Note that, since $\mu_{n,k}$ is the unique rotational invariant probability measure on $G_{n,k}$, the distribution of $\mathrm{span}\{\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{k}\}$ is $\mu_{n,k}$. Accordingly,

$$\begin{aligned} P(U_\mathcal{A})^k &= P(X_1, X_2, \ldots, X_k \in U_\mathcal{A}) \\ &\ge P\big(\mathrm{span}\{\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_\mathrm{k}\} \in \mathcal{A}\big) \\ &= \mu_{n,k}(\mathcal{A}). \end{aligned}$$ □

Remark 1. As we'll see below we use only a weak form of Lemma 1. We actually believe there is a much stronger form of it.

Proof of the moreover part in Theorem 1. Let $\mathcal{A} \subset G_{n,k}$ be such that every $E \in \mathcal{A}$ there is an M_E such that

$$M_E\|x\|_2 \le \|x\|_\infty \le (1+\varepsilon)M_E\|x\|_2$$

for all $x \in E$. Let $\mathcal{B}$ be the subset of $\mathcal{A}$ of all E for which $\frac{(1-3\varepsilon)M}{\sqrt{n}} \le M_E \le \frac{(1+\varepsilon)M}{\sqrt{n}}$, and let $\mathcal{C} = \mathcal{A} \setminus \mathcal{B}$. By Lemma 1,

$$\mu_{n,k}(\mathcal{C})^{1/k}$$
$$\leq P\left(\left\{x;\ \|x\|_\infty < \frac{(1+\varepsilon)(1-3\varepsilon)M}{\sqrt{n}}\|x\|_2 \text{ or } \|x\|_\infty > \frac{(1+\varepsilon)M}{\sqrt{n}}\|x\|_2\right\}\right)$$

and, by Proposition 1, this last quantity is smaller than $Ce^{-c\varepsilon\log n}$. It follows that

$$\mu_{n,k}(\mathcal{B}) > 1 - e^{-ak} - Ce^{-c\varepsilon k\log n}.$$

We may assume that $\varepsilon\log n$ is much larger than a so that the last term above is dominated by e^{-ak}. Applying Lemma 1 once more we get

$$P\left(\left\{x;\ (\frac{(1-3\varepsilon)M}{\sqrt{n}}\|x\|_2 \leq \|x\|_\infty \leq \frac{(1+\varepsilon)^2M}{\sqrt{n}}\|x\|_2\right\}\right) \geq \mu_{n,k}(\mathcal{B}) > 1-2e^{-ak}.$$

Using now the other part of Proposition 1 we get that

$$C\varepsilon\log n > ak.$$

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Tail-Sensitive Gaussian Asymptotics for Marginals of Concentrated Measures in High Dimension

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Summary. If the Euclidean norm $|\cdot|$ is strongly concentrated with respect to a measure μ, the average distribution of an average marginal of μ has Gaussian asymptotics that captures tail behaviour. If the marginals of μ have exponential moments, Gaussian asymptotics for the distribution of the average marginal implies Gaussian asymptotics for the distribution of most individual marginals. We show applications to measures of geometric origin.

1 Introduction

Let μ be a probability measure on $\mathbb{R}^n$; let $X = X_\mu$ be a random vector distributed according to μ.

We study the marginals $X^\xi = X_\mu^\xi = \langle X_\mu, \xi \rangle$ of X_μ ($\xi \in S^{n-1}$); let

$$F^\xi(t) = F_\mu^\xi(t) = \mathbb{P}\{X_\mu^\xi < t\}$$

be the distribution functions of X_μ^ξ. Consider also the average marginal X_μ^{av} defined by its distribution function

$$F^{\mathrm{av}}(t) = F_\mu^{\mathrm{av}}(t) = \int_{S^{n-1}} F_\mu^\xi(t)\, d\sigma(\xi)\,,$$

where $\sigma = \sigma_{n-1}$ is the rotation-invariant probability measure on S^{n-1}. If μ has no atom at the origin, the function F_μ^{av} is continuously differentiable (cf. the Brehm–Voigt formulæ in Section 2); denote $f_\mu^{\mathrm{av}} = \left(F_\mu^{\mathrm{av}}\right)'$.

It appears that, for certain classes of measures μ on $\mathbb{R}^n$, the distributions of X_μ^ξ (for many $\xi \in S^{n-1}$) and X_μ^{av} are approximately Gaussian. If $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_n$ is a tensor product of measures μ_i on the real line $\mathbb{R}$, this is the subject of classical limit theorems in probability theory.

The motivation for our research comes from a different family of measures: the (normalised) restrictions of the Lebesgue measure to convex bodies

$K \subset \mathbb{R}^n$. The behaviour of the marginals of these measures was studied recently by Anttila, Ball and Perissinaki, Bobkov and Koldobsky, Brehm and Voigt and others [ABP, BV, BK].

Let us state the problem more formally; denote as usual

$$\Phi(t) = \int_{-\infty}^{t} \phi(s)\, ds\,, \quad \phi(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,.$$

We wish to find sufficient conditions for proximity of distribution functions

$$1 - F^{\mathrm{av}}(t) \approx 1 - \Phi(t)\,, \quad 1 - F^{\xi}(t) \approx 1 - \Phi(t)\,, \tag{1}$$

or density functions:

$$f^{\mathrm{av}}(t) \approx \phi(t)\,, \quad f^{\xi}(t) \approx \phi(t)\,; \tag{2}$$

we discuss the exact meaning of proximity "≈" in the sequel. We refer to (1) as the *integral* problem and to (2) as the *local* problem.

Anttila, Ball and Perissinaki ([ABP]), Brehm and Voigt ([BV]), Bobkov and Koldobsky ([BK]), Romik ([R]) and others proposed to study these problems under the assumption that the Euclidean norm $|\cdot|$ is concentrated with respect to the measure μ.

These works provide a series of results, establishing (1) or (2) under assumptions of this kind. The assumptions can be verified for the geometric measures described above (see Anttila, Ball and Perissinaki [ABP]) for some classes of bodies $K \subset \mathbb{R}^n$.

However, these authors interpret "≈" in (1) and (2) as proximity in L_1 or L_∞ metrics[1]. These metrics fail to capture the asymptotics of the tails of the distribution of X^{av} beyond $t = O(\sqrt{\log n})$. We work with a stronger notion of proximity:

$$g \approx h \quad \text{if} \quad \sup_{0 \le t \le T} \left| \frac{g(t)}{h(t)} - 1 \right| \quad \text{is small,}$$

where T may be as large as some power of n.

In the classical case $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ this corresponds to limit theorems with moderate deviations in the spirit of Cramér, Feller, Linnik et al. (see Ibragimov and Linnik [IL]).

To obtain (1) or (2), we also assume concentration of Euclidean norm with respect to μ, but in a stronger form. That is, we reach a stronger conclusion under stronger assumptions.

Let us explain the results in this note. First, approach the question for average marginals (the first part of (1), (2)). It appears more natural to consider "spherical approximation":

$$1 - F^{\mathrm{av}}(t) \approx 1 - \Psi_n(t)\,, \quad f^{\mathrm{av}}(t) \approx \psi_n(t)\,,$$

[1] Recently H. Vogt [V] has proved some results concerning convergence in the W_2^k Wasserstein metric.

where

$$\Psi_n(t) = \int_{-\infty}^{t} \psi_n(s)\, ds\,,$$

$$\psi_n(t) = \frac{1}{\sqrt{\pi n}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \left(1 - \frac{t^2}{n}\right)^{\frac{n-3}{2}} \mathbf{1}_{[-\sqrt{n},\, \sqrt{n}]}(t)\,.$$

The geometric meaning of the distribution defined by these formulæ that justifies its name is it being the one-dimensional marginal of the uniform probability measure on the sphere (we explain this in the proof of the Brehm–Voigt formulæ in Section 2).

The following lemma shows the connection between Gaussian and spherical approximation:

Lemma 1. *For some constants C, C_1, $C_2 > 0$ and some sequence $\epsilon_n \searrow 0$ the following inequalities hold*[2] *for $0 < t < C\sqrt{n}$:*

$$\begin{aligned}
(1-\epsilon_n)\,\phi(t)\,\exp(-t^4/4n) &\le \psi_n(t) \\
&\le (1+\epsilon_n)\,\phi(t)\,\exp(-t^4/C_1 n)\,, \\
(1-\epsilon_n)\,\big(1-\Phi(t)\big)\,\exp(-t^4/C_2 n) &\le 1-\Psi_n(t) \\
&\le (1+\epsilon_n)\,\big(1-\Phi(t)\big)\,\exp(-t^4/C_1 n)\,.
\end{aligned}$$

Informally speaking, the lemma states that Gaussian approximation for the distribution of X^{av} is equivalent to spherical approximation if (and only if) the variable t is small with respect to $n^{1/4}$. We prove the lemma, together with other properties of spherical distributions, in Appendix A.

Now we formulate the main result for average marginals:

Theorem 2. *Suppose for some constants α, β, A, $B > 0$ we have*

$$\mathbb{P}\left\{\left|\frac{|X_\mu|}{\sqrt{n}} - 1\right| \ge u\right\} \le A\,\exp\left(-B\,n^\alpha\,u^\beta\right) \tag{3}$$

for $0 \le u \le 1$. Then

$$\left|\left\{1 - F_\mu^{av}(t)\right\} / \left\{1 - \Psi_n(t)\right\} - 1\right| < Ct^{2\max(\beta,\,1)}\,n^{-\alpha}\,; \tag{4}$$

$$\left|f_\mu^{av}(t)\,/\psi_n(t) - 1\right| < Ct^{2\max(\beta,\,1)}\,n^{-\alpha} \tag{5}$$

for $t > 0$ s.t. $t^{2\max(\beta,\,1)}\,n^{-\alpha} < c$; the constants c, C depend only on A, B, α, β.

In other words, the distribution of X^{av} has spherical asymptotics for $t = o(n^\gamma)$, where $\gamma = \alpha/\left(2\max(\beta,1)\right)$, and hence also Gaussian asymptotics for $t = o(n^{\min(\gamma,\,1/4)})$.

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[2] The constant 4 in the first inequality is written explicitly since it is sharp.

We prove this theorem in Section 2.

Then we approach the individual marginals X^ξ. Suppose the measure μ satisfies a property resembling (4):

$$\begin{aligned}(1-\epsilon)\left(1-\Phi(t)\right) \leq \int_{S^{n-1}} &\left(1-F^\eta(t)\right) d\sigma(\eta) \\ &\leq (1+\epsilon)\left(1-\Phi(t)\right) \quad \text{for } 0 \leq t \leq T.\end{aligned} \tag{6}$$

Suppose also that the measure μ has ψ_1 marginals:

$$\mathbb{P}\left\{\langle X_\mu, \theta\rangle > s\right\} \leq C \exp(-cs), \quad s \in \mathbb{R}^+, \quad \theta \in S^{n-1}. \tag{7}$$

The following inequality due to Borell (see eg. Giannopoulos [G, Section 2.1] or Milman–Schechtman [MS]) shows that this property holds for an important class of measures.

Definition 1. *A measure μ on $\mathbb{R}^n$ is called isotropic if*

$$\mathrm{Var}\,\langle \mathrm{X}_\mu, \xi\rangle = 1 \quad \textit{for} \quad \xi \in S^{n-1} \quad . \tag{8}$$

Definition 2. *A measure μ on $\mathbb{R}^n$ is called log-concave if*

$$\mu\left(\frac{A+B}{2}\right) \geq \sqrt{\mu(A)\,\mu(B)} \quad \textit{for} \quad A,\, B \subset \mathbb{R}^n. \tag{9}$$

Proposition (Borell). *Every isotropic, log-concave, even measure μ on $\mathbb{R}^n$ has ψ_1 marginals* (7).

Remark. Actually, the isotropicity condition is too rigid, and measures satisfying a weaker condition

$$\mathrm{Var}\,\langle \mathrm{X}_\mu, \xi\rangle \leq \mathrm{C}' \quad \text{for} \quad \xi \in S^{n-1} \tag{10}$$

also have ψ_1 marginals, with constants C and c in (7) depending on C'. Such measures are called *(C-)subisotropic*.

Our aim is to show that for most $\xi \in S^{n-1}$

$$(1-10\epsilon)\left(1-\Phi(t)\right) \leq 1-F^\xi(t) \leq (1+10\epsilon)\left(1-\Phi(t)\right) \quad \text{for} \quad 0 \leq t \leq T; \tag{11}$$

of course, the constant 10 has no special meaning (but influences the meaning of "most").

This should be compared with classical results on concentration of marginal distributions of isotropic measures.

To the extent of the author's knowledge, the earliest result of this kind is due to Sudakov ([Su], see also von Weizsäcker [W]). It states that if $n \geq n_0(\epsilon)$ and μ is a general isotropic measure on $\mathbb{R}^n$, then

$$\sigma\left\{\xi \in S^{n-1} \;\middle|\; \left\|F^\xi - F^{\mathrm{av}}\right\|_1 > \epsilon\right\} \leq \epsilon\,.$$

Anttila, Ball and Perissinaki have considered isotropic measures μ that are normalised restrictions of the Lebesgue measure to convex bodies $K \subset \mathbb{R}^n$; their work extends to general isotropic log-concave measures. The result in [ABP] states that in this case

$$\sigma\left\{\xi \in S^{n-1} \mid \left\|F^\xi - F^{\text{av}}\right\|_\infty > \delta\right\} \leq C\sqrt{n}\,\log n\,\exp\left(-cn\delta^2\right)\,.$$

Bobkov ([B]) improved both aforementioned results. In the log-concave case he proved that for some constant $b > 0$

$$\sigma\Big\{\xi \in S^{n-1} \mid \sup_{t\in\mathbb{R}} e^{bt}\left|F^\xi(t) - F^{\text{av}}(t)\right| > \delta\Big\} \leq C\sqrt{n}\,\log n\,\exp\left(-cn\delta^2\right)\,.$$

Note that the metric that appears in this inequality takes the tails of the distributions into account. Moreover, it seems reasonable that the term e^{bt} can not be replaced by $e^{bt^{1+\epsilon}}$ without additional assumptions.

On the other hand, the Gaussian case (6) is of special interest (see [ABP, B, R, W]). The cited results allow to deduce (11) from (6) only for $T = O(\log^{1/2} n)$.

Our results show that in fact (6) implies (11) for T as large as a certain power of n. Let us formulate the exact statements.

We consider even measures with ψ_1 marginals.

Theorem 3. *There exists $\epsilon_0 > 0$ such that if for some $\epsilon < \epsilon_0$*

$$\begin{aligned}(1-\epsilon)\left(1-\Phi(t)\right) \leq \int_{S^{n-1}} &(1 - F_\mu^\eta(t))\,d\sigma(\eta)\\ &\leq (1+\epsilon)\left(1-\Phi(t)\right)\,, \quad 0 \leq t \leq T\,,\end{aligned}$$

then

$$\begin{aligned}\sigma\Big\{\xi \in S^{n-1} \,\Big|\, \exists\, 0 \leq t \leq T,\ \Big|\frac{1-F_\mu^\xi(t)}{1-\Phi(t)} - 1\Big| > 10\epsilon\Big\}&\\ \leq \frac{CT^8}{n\epsilon^4}\,\exp\left(-c\,n\epsilon^2\,T^{-6}\right)\,.&\end{aligned} \tag{12}$$

The constants C, c, c_1, ϵ_0, ... in this theorem, as well as the constants in the following theorem and all other constants in this note, depend neither on μ nor on the dimension n.

Corollary 4. *If under assumptions of Theorem 3*

$$0 \leq T \leq \left\{\frac{c_1 n\epsilon^2}{\log n + \log\frac{1}{\epsilon} + \log\frac{1}{\zeta}}\right\}^{1/6}\,, \tag{13}$$

then

$$\sigma\Big\{\xi \in S^{n-1} \,\Big|\, \exists\, 0 \leq t \leq T,\ \Big|\frac{1-F_\mu^\xi(t)}{1-\Phi(t)} - 1\Big| > 10\epsilon\Big\} \leq \zeta\,.$$

Proof of Corollary. Substitute (13) into (12). We obtain:

$$\begin{aligned}\sigma\{\cdots\} &\le \frac{C}{n\epsilon^4}\left\{\frac{c_1 n\epsilon^2}{\log\frac{n}{\epsilon\zeta}}\right\}^{4/3}\exp\left\{-\frac{c}{c_1}\log\frac{n}{\epsilon\zeta}\right\}\\ &= C c_1^{4/3} n^{1/3-c/c_1}\epsilon^{-4/3+c/c_1}\zeta^{c/c_1}\log^{-4/3}\frac{n}{\epsilon\zeta}\,.\end{aligned}$$

If c_1 is small enough, this expression is less than ζ. □

We also prove a local version of the theorem. Suppose F_μ^η are concave on $\mathbb{R}_+$; then $f_\mu^\eta = \left(F_\mu^\eta\right)'$ are defined a.e. and

$$f_\mu^{\mathrm{av}}(t) = \int_{S^{n-1}} f_\mu^\eta(t)\,d\sigma(\eta)\,.$$

Theorem 5. *Suppose*

$$(1-\epsilon)\,\phi(t) \le \int_{S^{n-1}} f_\mu^\eta(t)\,d\sigma(\eta) \le (1+\epsilon)\,\phi(t)\,, \quad 0 \le t \le T\,.$$

Then

$$\sigma\left\{\xi\in S^{n-1}\,\middle|\,\exists\, 0\le t\le T,\ \left|\frac{f_\mu^\xi(t)}{\phi(t)}-1\right| > 10\epsilon\right\} \le \frac{CT^8}{n\epsilon^7}\exp\left(-c_1\,n\epsilon^4\,T^{-6}\right)\,.$$

Corollary 6. *If under assumptions of Theorem 5*

$$0 \le T \le \left\{\frac{c_1 n\epsilon^4}{\log n + \log\frac{1}{\epsilon} + \log\frac{1}{\zeta}}\right\}^{1/6}\,,$$

then

$$\sigma\left\{\xi\in S^{n-1}\,\middle|\,\exists\, 0\le t\le T,\ \left|\frac{f_\mu^\xi(t)}{\phi(t)}-1\right| > 10\epsilon\right\} \le \zeta\,.$$

The Corollary follows from Theorem 5 exactly as Corollary 4 follows from Theorem 3. Note that the only essential difference between the local and the integral versions is in the dependence on ϵ.

We prove the theorems in Section 3. Finally, in Section 4 we apply our results from Sections 2, 3 to measures associated with convex bodies $K \subset \mathbb{R}^n$; these examples are parallel to those by Anttila, Ball and Perissinaki [ABP].

We devote Appendix A to proofs of some properties of the spherical distribution that we use in Section 2.

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2 Average Marginals

We commence with explicit formulæ for $1 - F^{\mathrm{av}}(t)$, $f^{\mathrm{av}}(t)$, due to Brehm and Voigt ([BV], see also Bobkov and Koldobsky [BK]). Then we develop these formulæ to obtain the estimate in Proposition 8 (below). Finally, we bound the integrals that appear in the estimate to conclude the proof of Theorem 2.

Denote by μ^* the normalised radial projection

$$\mu^*(r) = \mathbb{P}\big\{|X_\mu| \leq \sqrt{n}r\big\} = \mu\big\{B(0;\sqrt{n}r)\big\}\,.$$

Proposition (Brehm–Voigt). *For any Borel probability measure μ on $\mathbb{R}^n$ with $\mu(\{0\}) = 0$, $1 - F^{av} \in C^1(\mathbb{R})$ and*

$$1 - F^{av}(t) = \int_0^\infty \left\{1 - \Psi_n\left(\frac{t}{r}\right)\right\} d\mu^*(r) \tag{14}$$

$$f^{av}(t) = \int_0^\infty \frac{1}{r}\,\psi_n\left(\frac{t}{r}\right) d\mu^*(r)\,. \tag{15}$$

For completeness, we prove this proposition.

Proof of Proposition. Proof of (14): First, let us verify the formula for $\mu = \sigma_{n-1}$. Let us project σ_{n-1} onto the x-axis; let $x_0 = \sin\theta_0$. Then

$$\mathbb{P}\left\{x < x_0\right\} = \frac{\int_{-\pi/2}^{\theta_0} \cos^{n-2}\theta d\theta}{\int_{-\pi/2}^{\pi/2} \cos^{n-2}\theta d\theta}\,.$$

Let $x = \sin\theta$, $dx = \cos\theta d\theta$; then the numerator equals

$$\int_{-\pi/2}^{\theta_0} \cos^{n-3}\theta\,\cos\theta\,d\theta = \int_{-1}^{x_0} (1 - x^2)^{(n-3)/2}\,dx\,.$$

The denominator is just a constant, and the correct one, since both Ψ_n and the marginal of σ_{n-1} are probability distributions. This proves the proposition for σ_{n-1}.

Next, let μ be a rotation-invariant measure. Then we can approximate μ by a convex combination of dilations of σ_{n-1}; these combinations satisfy (14). Now we can pass to the limit by the dominated convergence theorem.

Finally, both sides of (14) are equal for μ and its symmetrisation $\widetilde{\mu} = \int_{O(n)} T^*(\mu)\,d\sigma(T)$ (here σ is the translation-invariant measure on the orthogonal group $O(n)$), and hence the formula extends to arbitrary probability measures.

Proof of (15): Apply (14) to $\mu_B = \mu(B)^{-1}\mu|_B$ for Borel sets B; (15) follows by use Fubini's theorem. To see that f^{av} is continuous, it suffices to check that

$$\int_0^\infty \left|\frac{d}{dt}\psi_n(t/r)\right| dt < \infty\,.$$

This condition can be verified by straightforward computation (cf. second statement in Lemma 7 in the sequel). □

We develop the integral formula (14) needed for the proof of (4); note that without loss of generality μ has no atom at the origin. The computations for the local version (5) are rather similar; we prove all the needed technical lemmata in both versions. Anyway, at the end of the computations both questions reduce to asymptotics of the same integral (17).

First, split the domain of integration in (14) into 3 parts:

$$1 - F^{\mathrm{av}}(t) = \int_0^1 \left\{1 - \Psi_n\left(\frac{t}{r}\right)\right\} d\mu^*(r) \\ - \left[\int_1^2 + \int_2^\infty\right] \left\{1 - \Psi_n\left(\frac{t}{r}\right)\right\} d\big[1 - \mu^*(r)\big] .$$

Integrating by parts, we deduce:

$$1 - F^{\mathrm{av}}(t) = \left\{1 - \Psi_n\left(\frac{t}{r}\right)\right\} \mu^*(r)\Big|_0^1 - \left\{1 - \Psi_n\left(\frac{t}{r}\right)\right\} \big(1 - \mu^*(r)\big)\Big|_1^2 \\ - \int_0^1 \frac{t}{r^2}\, \psi_n\left(\frac{t}{r}\right) \mu^*(r)\, dr + \int_1^2 \frac{t}{r^2}\, \psi_n\left(\frac{t}{r}\right) \big(1 - \mu^*(r)\big)\, dr \\ + \int_2^\infty \left\{1 - \Psi_n\left(\frac{t}{r}\right)\right\} d\big[1 - \mu^*(r)\big]$$

and hence

$$\{1 - F^{\mathrm{av}}(t)\} - \{1 - \Psi_n(t)\} \\ = -\big\{1 - \Psi_n(t/2)\big\} \big(1 - \mu^*(2)\big) - \int_0^1 \frac{t}{r^2}\, \psi_n\left(\frac{t}{r}\right) \mu^*(r)\, dr \\ + \int_1^2 \frac{t}{r^2}\, \psi_n\left(\frac{t}{r}\right) \big(1 - \mu^*(r)\big)\, dr + \int_2^\infty \left\{1 - \Psi_n\left(\frac{t}{r}\right)\right\} d\big[1 - \mu^*(r)\big] .$$

Now we need to estimate $1 - \Psi_n(t)$. We formulate the needed property in a lemma that we prove in Appendix A.

Lemma 7.

$$0 < C^{-1} \le \tfrac{1-\Psi_n(t)}{t^{-1}\psi_n(t)} \le C \quad \textit{for} \quad 8t^2 < n,$$
$$\tfrac{\psi_n(t)}{t^{-1}\psi_n'(t)} = 1 - t^2/n\,,$$

where C is a universal constant.

This yields the following proposition:

Proposition 8. *The following inequality holds for any Borel probability measure μ on $\mathbb{R}^n$:*

$$\left|\frac{1-F^{av}(t)}{1-\Psi_n(t)}-1\right| \le \big(1-\mu^*(2)\big)\,\frac{Ct}{\psi_n(t)} + Ct^2\left\{\int_0^1 \frac{1}{r^2}\,\frac{\psi_n\left(\frac{t}{r}\right)}{\psi_n(t)}\,\mu^*(r)\,dr + \int_1^2 \frac{1}{r^2}\,\frac{\psi_n\left(\frac{t}{r}\right)}{\psi_n(t)}\,\big(1-\mu^*(r)\big)\,dr\right\}.$$

Now we can conclude the proof of Theorem 2.

Proof of Theorem 2. Apply Proposition 8 and denote

$$\mathrm{TERM}_1 = (1-\mu^*(2))\,\frac{Ct}{\psi_n(t)}\,,$$
$$\mathrm{TERM}_2 = \int_0^1 \frac{1}{r^2}\,\frac{\psi_n\left(\frac{t}{r}\right)}{\psi_n(t)}\,\mu^*(r)\,dr\,,$$
$$\mathrm{TERM}_3 = \int_1^2 \frac{1}{r^2}\,\frac{\psi_n\left(\frac{t}{r}\right)}{\psi_n(t)}\,\big(1-\mu^*(r)\big)\,dr\,.$$

By Lemma 1 and the concentration condition (3) (used with $u=1$),

$$\mathrm{TERM}_1 \le A'\,t\,\exp(B't^2 - B''n^\alpha) \tag{16}$$

with $A' = AC$, $B' = B/2$, $B'' = 2^\beta B$; this expression surely satisfies the bound (4).

Introduce a new variable $u = (r-1)$ in TERM_3 and use (3) once again. We obtain:

$$\mathrm{TERM}_3 \le \int_0^1 \frac{\psi_n\left(\frac{t}{1+u}\right)}{\psi_n(t)} \times A\exp(-Bn^\alpha u^\beta)\,du\,.$$

Now we use one more property of spherical distributions which we also prove in Appendix A.

Lemma 9. *There exist constants C_1 and C_2 such that for $2t^2 < n$*

$$\exp(C_1ut^2) \le \frac{\psi_n(t)}{\psi_n((1+u)t)}$$

and for $2(1+u)^2\,t^2 < n$

$$\frac{\psi_n(t)}{\psi_n((1+u)t)} \le \exp(C_2ut^2)\,.$$

By the lemma for $2t^2 < n$

$$\mathrm{TERM}_3 \le A\int_0^1 \exp\big(C_0t^2u - Bn^\alpha u^\beta\big)\,du\,. \tag{17}$$

The computations in their local version would lead us to the same integral.

Now we study the integral

$$I(K;\, L) = \int_0^1 \exp(Ku - Lu^\beta)\, du\,,$$

where K and L are large parameters, K much smaller than L.

The exponent $E(u) = Ku - Lu^\beta$ is concave for $\beta > 1$ and convex for $\beta \le 1$. Let us consider these cases separately.

Case 1. ($\beta > 1$) The maximum of the concave function $E(u) = Ku - Lu^\beta$ is achieved at the point $u_0 = (K/\beta L)^{1/(\beta-1)}$ inside the domain of integration; $E(u_0) = C_\beta (K^\beta/L)^{1/(\beta-1)}$, where $C_\beta = [\beta^{-1/(\beta-1)} - \beta^{-\beta/(\beta-1)}]$. Let $R > 1$ be fixed later (so that $Ru_0 \le 1$).

First, consider the integral from 0 to Ru_0.

$$\begin{aligned}
&\int_0^{Ru_0} \exp(Ku - Lu^\beta)\, du \\
&\le Ru_0 \exp E(u_0) = (K/\beta L)^{1/(\beta-1)}\, R\, \exp\left\{C_\beta (K^\beta/L)^{1/(\beta-1)}\right\} \qquad (18)\\
&= R\, \frac{(K^\beta/L)^{1/(\beta-1)}}{K}\, \frac{\exp\left\{C_\beta (K^\beta/L)^{1/(\beta-1)}\right\}}{\beta^{1/(\beta-1)}}\,.
\end{aligned}$$

Next, for $u \ge Ru_0$ we have:

$$\begin{aligned}
E(u) &\le E(u_0) \,+\, E'(Ru_0)\,(u - Ru_0)\\
&= C_\beta\, (K^\beta/L)^{1/(\beta-1)} \,-\, K\,(R^{\beta-1} - 1)\,(u - Ru_0) \qquad (19)
\end{aligned}$$

and hence

$$\int_{Ru_0}^1 \le \frac{\exp\left(C_\beta (K^\beta/L)^{1/(\beta-1)}\right)}{K(R^{\beta-1} - 1)}\,.$$

For $K^\beta/L < 1/2$ choose $R = (L/K^\beta)^{1/\beta(\beta-1)}$; then both (18) and (19) are bounded by a constant times

$$\frac{K^\beta/L}{K}\, \exp\Big(C_\beta (K^\beta/L)^{1/(\beta-1)}\Big) \,\le\, C'_\beta\, \frac{K^\beta/L}{K}\,.$$

Case 2. ($\beta \le 1$) For $K/L < 1/2$ the inequality $Ku \le Lu^\beta/2$ holds in the interval $[0,\, 1]$; hence

$$\begin{aligned}
I(K;\, L) &\le \int_0^1 \exp(-Lu^\beta/2)\, du \;=\; \beta^{-1}\, (2/L)^{1/\beta} \int_0^{L/2} \exp(-v)\, v^{1/\beta}\, dv\\
&\le\; \beta^{-1}\, (2/L)^{1/\beta}\, \Gamma(1/\beta) \;=\; 2^{1/\beta}\, \Gamma(\beta^{-1} + 1)\, \frac{K/L}{K}\,.
\end{aligned}$$

We have proved the following proposition:

Proposition 10. *If $K^{\max(\beta,\,1)}/L < 1/2$, then*

$$K \int_0^1 \exp(Ku - Lu^\beta)\, du \,\le\, C\, \frac{K^{\max(\beta,\,1)}}{L}\,,$$

where C depends only on β.

Taking $K = C_0 t^2$, $L = Bn^\alpha$ we arrive at the desired estimate for TERM_3. The integral TERM_2 is even smaller, since $\psi_n(t/r)/\psi_n(t) < 1$ for $r < 1$. □

3 Individual Marginals

Along the remainder of this note, we only deal with the upper bounds in Theorems 3 and 5. The same technique works also for lower bounds. Note that these bounds do not depend on each other: the left side inequality in (6) implies the left side inequality in (11), and similarly for the right side inequalities.

Also, all the measures μ in this section are assumed even with ψ_1 marginals; we reiterate that all the constants do not depend on μ nor on the dimension n.

Let us explain the idea of the proof (of the integral theorem). Let A be the set of directions $\eta \in S^{n-1}$ such that $1 - F^\eta(t-s)$ is not too large. Markov's inequality combined with the bound (6) for the average marginal shows that the measure of A is not too small.

Now use the triangle inequality in the following form:

$$\begin{aligned} 1 - F^\eta(t+s) - \mathbb{P}\left\{X^\xi - X^\eta > s\right\} &\le 1 - F^\xi(t) \\ &\le 1 - F^\eta(t-s) + \mathbb{P}\left\{X^\xi - X^\eta > s\right\}\,; \end{aligned} \tag{20}$$

we need the right side for the upper bounds.

Consider directions ξ in the δ-extension of A

$$\{A\}_\delta = \left\{\xi \in S^{n-1} \,\middle|\, \exists\, \eta \in A,\ |\xi - \eta| \le \delta\right\}.$$

For such ξ, the term $1 - F^\eta(t-s)$ is not too large; the term $\mathbb{P}\left\{X^\xi - X^\eta > s\right\}$ can be bounded in terms of δ using the ψ_1 condition (7).

Finally, we use the spherical isoperimetric inequality to show that $\{A\}_\delta$ covers most of the sphere.

Now we pass to rigorous exposition of the idea explained above. Define the set of "good directions"

$$A\,(t;\,\epsilon) = \left\{\eta \in S^{n-1} \;\mid\; \big(1 - F^\eta(t) \le (1 - \Phi(t))\big)\,(1+\epsilon)\right\}\,.$$

Our first aim is to prove the following proposition:

Proposition 11. *There exist $c, \epsilon_0 > 0$ (that depend neither on μ nor on n) such that for every $t > 1$ there exists $t' < t$ satisfying*

$$\left\{ A\,(t';\,\epsilon)\, \right\}_{c\epsilon t^{-3}} \subset A_{t;\,4\epsilon} \quad \textit{for} \quad 0 \le \epsilon \le \epsilon_0\,.$$

Remark. Note that all the results for $0 \le t \le 1$ follow from the known results (for example, [ABP]), and hence we may restrict ourselves to $1 \le t$ along all the proofs.

Proof of Proposition 11. Suppose $1 - F^{\eta}(t-s) \le (1 - \Phi(t-s))\,(1+\epsilon)$. Combining (20) with the ψ_1 condition (7), we deduce:

$$1 - F^{\xi}(t) \le \big(1 - \Phi(t-s)\big)\,(1+\epsilon) + C\,\exp(-cs\delta^{-1})\,, \tag{21}$$

where $\delta = |\xi - \eta|$.

Now we need to use properties of the Gaussian distribution that are summarized in the following elementary lemma (cf. Lemmata 7 and 9):

Lemma 12. *The following inequalities hold:*

$$1 - \Phi(t-s) \le (1 - \Phi(t))\,\exp(st)\,; \tag{22}$$

$$1 - \Phi(t) \le C\,t^{-1}\,\exp(-t^2/2)\,. \tag{23}$$

Substituting these inequalities into (21), we obtain:

$$\frac{1 - F^{\xi}(t)}{1 - \Phi(t)} \le (1+\epsilon)\,e^{st} + C_1 t\,\exp\left[t^2/2 - c\,s\,\delta^{-1}\right]\,. \tag{24}$$

This inequality holds for any $s > 0$, and s does not appear on its left side. To conclude the proof, we optimise over s in a rather standard way. Denote

$$a(s) = (1+\epsilon)\,e^{st} + C_1 t\exp\left[t^2/2 - c\,s\,\delta^{-1}\right]\,.$$

Then

$$a'(s) = t\,\left\{(1+\epsilon)\,e^{st} - C_1 c\,\delta^{-1}\,\exp\left[t^2/2 - c\,s\,\delta^{-1}\right]\right\}$$

and hence the minimum is obtained at s_0 such that

$$(1+\epsilon)\,e^{s_0 t} = C_1 c\,\delta^{-1}\,\exp\left[t^2/2 - c\,s_0\,\delta^{-1}\right]\,,$$

or:

$$e^{s_0 t} = \left(\frac{C_1 c\,e^{t^2/2}}{(1+\epsilon)\,\delta}\right)^{\left[1 + c\,\delta^{-1}\,t^{-1}\right]^{-1}}\,. \tag{25}$$

Hereby

$$a(s_0) = (1+\epsilon)\left(1 + \frac{t\delta}{c}\right)\left(\frac{C_1 c\,e^{t^2/2}}{(1+\epsilon)\,\delta}\right)^{\left[1 + c\,\delta^{-1}\,t^{-1}\right]^{-1}}\,.$$

Extracting logarithms, we see that

$$\log a(s_0) \le \epsilon + \frac{t\delta}{c} + \frac{C_2}{1 + c\delta^{-1}t^{-1}} + \frac{t^2/2}{1 + c\delta^{-1}t^{-1}} \,.$$

If $\delta < c_3 \epsilon t^{-3}$, the fourth term is bounded by ϵ. If $t \ge 1$, the preceding two terms are ignorable (and in particular their sum is bounded by $2\epsilon - 8\epsilon^2$). Finally, exploiting the inequality $\exp(u - u^2/2) \le 1 + u$ we deduce that $a(s_0) \le 4\epsilon$.

Hence $t' = t - s_0$ satisfies the requirements of the proposition. □

We also outline the proof of a local version of Proposition 11. Define

$$B\,(t;\,\epsilon) = \left\{ \eta \in S^{n-1} \;\mid\; f^\eta(t) \le \phi(t)\,(1+\epsilon) \right\} \,.$$

Proposition 13. *Suppose F^η_μ are concave on $\mathbb{R}_+$. Then there exist constants $c, \epsilon_0 > 0$ (that depend neither on μ nor on n) such that for every $t > 1$ there exists $t' < t$ satisfying*

$$\left\{ B\,(t';\,\epsilon)\, \right\}_{c\epsilon^2 t^{-3}} \subset B\,(t;\,4\epsilon) \quad \textit{for } 0 \le \epsilon \le \epsilon_0 \,.$$

Sketch of proof. Choose two small parameters, $1 \gg h \gg s > 0$. By the intermediate value theorem

$$\begin{aligned} h f^\xi(t) &\le F^\xi(t) - F^\xi(t-h) \\ &\le F^\eta(t+s) - F^\eta(t-h-s) + 2\mathbb{P}\big\{ \mathbb{P}\langle X,\, \xi - \eta\rangle > s \big\} \\ &\le [h + 2s]\, f^\eta(t-h-s) + 2C\, \exp\big(-c\,\delta^{-1} s\big)\,; \end{aligned}$$

therefore if $f^\eta(t-h-s) < (1+\epsilon)\,\phi(t-h-s)$,

$$\frac{f^\xi(t)}{\phi(t)} \le \left[1 + 2sh^{-1}\right]\, (1+\epsilon)\, \exp\big(t(h+s)\big) + 2C \exp\big(t^2/2 - c\delta^{-1} s\big) \,.$$

Take $s = C_1 \epsilon^2 t^{-1}$, $h = C_2 \epsilon t^{-1}$. For appropriate choice of the constants C_1, C_2 we deduce: $f^\xi(t)/\phi(t) \le 4\epsilon$. □

Now we are ready to prove Theorems 3 and 5. The proofs of these theorems are rather similar; let us prove for example the (upper bound in) Theorem 3.

Proof of Theorem 3. First, apply Markov's inequality to the right side of (6). We deduce:

$$\sigma\left\{ \eta \;\mid\; (1 - F^\eta(t) \le \big(1 - \Phi(t)\big)\,(1 + 2\epsilon) \right\} \ge \frac{\epsilon}{1 + 2\epsilon} \ge \epsilon/2 \,.$$

Surely, this inequality also holds with t' instead of t. Now we need to transform Proposition 11 into a lower bound on the measure of $A\,(t;\,8\epsilon)$.

Let $1 \le t_1 \le \cdots \le t_I = T$ be an increasing sequence of points such that

$$\sigma\left\{ \xi \;\mid\; 1 - F^\xi(t_i) \ge \big(1 - \Phi(t)\big)\,(1 + 8\epsilon) \right\} \le \zeta_i \,, \quad 1 \le i \le I \,.$$

Then

$$\sigma\left\{\xi \;\middle|\; \exists 1 \le i \le I,\ 1 - F^{\xi}(t_i) \ge \big(1 - \Phi(t)\big)\,(1 + 8\epsilon)\right\} \le \sum_{i=1}^{I} \zeta_i\,.$$

The function F^{ξ} is monotone for every ξ; hence for $t_i \le t \le t_{i+1}$ we have $1 - F^{\xi}(t) \le 1 - F^{\xi}(t_i)$. Applying Lemma 12, we conclude:

$$\sigma\big\{\xi \;\mid\; \exists 1 \le t_i \le t \le t_{i+1} \le t_I,\\ 1 - F^{\xi}(t) \ge \exp\big(t_{i+1}(t_{i+1} - t_i)\big)\big(1 - \Phi(t)\big)\,(1 + 8\epsilon)\big\} \le \sum_{i=1}^{I} \zeta_i\,.$$

Choose $t_i = \sqrt{C\epsilon i}$ with C such that $\exp\left((t_{i+1} - t_i)\, t_{i+1}\right) \le \epsilon$. Then

$$\sigma\big\{\xi \;\mid\; \exists 1 \le t \le T, 1 - F^{\xi}(t) \ge \exp\big(1 - \Phi(t)\big)\,(1 + 10\epsilon)\big\} \le \sum_{i=1}^{I} \zeta_i\,.$$

Now we use the concentration inequality on the sphere in the following form:

Proposition (Concentration on the sphere). *For $A \subset S^{n-1}$*

$$\sigma(A)\ \big[1 - \sigma\big(\{A\}_\gamma\big)\big] \le \exp\big(-(n-1)\,\gamma^2/4\big)\ . \tag{26}$$

This is a standard corollary of the isoperimetric inequality on the sphere due to P. Lévy that can be verified applying the concentration inequality as in Milman–Schechtman [MS] to the function $x \mapsto \inf_{y \in A} d(x,\, y)$.

Proposition 11 combined with the concentration inequality yields

$$\zeta_i = \frac{C}{\epsilon}\ \exp\left[-c_1 n \epsilon^{-1} i^{-3}\right]\ ;$$

hence

$$\sigma\left\{\xi \,\middle|\, \exists 1 \le t \le T,\ 1 - F^{\xi}(t) \ge \exp\big(t_{i+1}(t_{i+1} - t_i)\big)\big(1 - \Phi(t)\big)\,(1 + 10\epsilon)\right\}\\ \le \sum_{i=1}^{I} \frac{C}{\epsilon} \exp\left[-c_1 n \epsilon^{-1} i^{-3}\right] \le \frac{C}{\epsilon} \int_0^I \exp\left(-c_1 n \epsilon^{-1} x^{-3}\right)\, dx\,;$$

the second inequality is justified since the function $i \mapsto \exp\left(-c_1 n \epsilon^{-1} i^{-3}\right)$ is monotone decreasing.

Continuing the inequality and replacing $y^{-4/3}$ with its value at the left end of the integration domain, we obtain:

$$\cdots \le \frac{C_1}{\epsilon^{4/3}} \int_{\epsilon^2 T^{-6}}^{\infty} \exp(-c_1 n y)\, y^{-4/3}\, dy \le C_2 \epsilon^{-4} T^8 n^{-1}\, \exp\left(-c_1 n \epsilon^2 T^{-6}\right)\ .$$

□

We conclude with a remark.

Remark. One can generalise the conclusion of Theorems 3 and 5 to measures μ satisfying the ψ_α property

$$\mathbb{P}\big\{\langle X,\,\theta\rangle > s\big\} \le C\,\exp(-cs^\alpha)\,,\quad s\in\mathbb{R}^+ \tag{27}$$

for some $0<\alpha\le 2$. In this case we use (27) instead of (7) in the proofs of Propositions 11, 13. This yields $t^{-1-2/\alpha}$ instead of t^{-3} in these propositions, leading to exponent $2+4\alpha^{-1}$ instead of 6 in the theorems.

4 Examples

Let us show some examples where our results apply. Our examples have geometric motivation, hence we recall some geometric notions.

Let $K\subset\mathbb{R}^n$ be a symmetric convex body; denote its boundary by ∂K. Define three measures associated with K, called the *volume measure*, the *surface measure* and the *cone measure* and denoted by $\mathcal{V}_K$, $\mathcal{S}_K$ and $\mathcal{C}_K$ respectively:

$$\begin{aligned}\mathcal{V}_K(A) &= \frac{\mathrm{Vol\,A\cap K}}{\mathrm{Vol\,K}}\,;\\ \mathcal{S}_K(A) &= \lim_{\epsilon\to+0}\frac{\mathrm{Vol}\,\{\mathrm{A}\cap\partial\mathrm{K}\}_\epsilon}{\mathrm{Vol}\,\{\partial\mathrm{K}\}_\epsilon}\,;\\ \mathcal{C}_K(A) &= \frac{\mathrm{Vol}\,\big\{\mathrm{x}\in\mathrm{K}\,\big|\,\mathrm{x}/\|\mathrm{x}\|_\mathrm{K}\in\mathrm{A}\big\}}{\mathrm{Vol\,K}}\,.\end{aligned}$$

Here subscript denotes metric extension in $\mathbb{R}^n$:

$$\{A\}_\epsilon = \big\{a\in\mathbb{R}^n\,\big|\,\exists\,x\in A, |x-a|\le\epsilon\big\}\;.$$

Remark. The Brunn–Minkowski inequality (see [G, MS]) shows that the measure $\mathcal{V}_K$ is log-concave for any convex body K.

Definition 3. *The body K is called isotropic (subisotropic) if the measure $\mathcal{V}_K$ is isotropic (subisotropic).*

We are mainly interested in the volume measure $\mathcal{V}_K$; however, sometimes it is easier to verify the concentration condition (3) for $\mathcal{S}_K$ or $\mathcal{C}_K$. As well known, the difference is insignificant:

Proposition 14. *Suppose one of the following two inequalities holds:*

$$\mathcal{S}_K\left\{\big||X|-1\big|\ge u\right\}\le A\exp\left(-Bn^\alpha u^\beta\right), \qquad 0\le u\le 1 \tag{28}$$

$$\mathcal{C}_K\left\{\big||X|-1\big|\ge u\right\}\le A\exp\left(-Bn^\alpha u^\beta\right), \qquad 0\le u\le 1\,. \tag{29}$$

Then

$$\mathcal{V}_K\left\{\big||X|-1\big|\ge u\right\}\le A'\exp\big(-B'n^{\min(\alpha,\,1)}u^{\max(\beta,\,1)}\big)\,,\quad 0\le u\le 1\,, \tag{30}$$

where A', B' depend only on A, B, α, β.

Proof. Let X be distributed according to $\mathcal{V}_K$; then $X/\|X\|_K$ is distributed according to $\mathcal{C}_K$ and $\mathbb{P}\{\|X\|_K \le r\} = r^n$ for $0 \le r \le 1$.

Similarly, if Y is distributed according to $\mathcal{S}_K$ and R is a (scalar) random variable that does not depend on Y such that

$$\mathbb{P}\{R \le r\} = r^n \quad \text{for } 0 \le r \le 1,$$

then RY is distributed according to $\mathcal{V}_K$.

Therefore

$$\begin{aligned} \mathcal{V}_K\{|x| < 1-u\} &\le \mathcal{V}_K\big\{|x| < (1-u/2)^2\big\} \\ &\le \mathcal{S}_K\big\{|x| < (1-u/2)\big\} + (1-u/2)^n \\ &\le \mathcal{S}_K\big\{|x| < (1-u/2)\big\} + \exp(-nu/2) \end{aligned} \tag{31}$$

and also

$$\mathcal{V}_K\{|x| < 1-u\} \le \mathcal{C}_K\big\{|x| < (1-u/2)\big\} + \exp(-nu/2)\,. \tag{32}$$

On the other hand,

$$\mathcal{V}_K\big\{|x| > 1+u\big\} \le \mathcal{S}_K\big\{|x| > (1+u)\big\}\,, \tag{33}$$

$$\mathcal{V}_K\big\{|x| > 1+u\big\} \le \mathcal{C}_K\big\{|x| > (1+u)\big\}\,. \tag{34}$$

Combining (28) with (31) and (33) or (29) with (32) and (34), we arrive at (30). □

We also note that sometimes for K in natural normalisation we get

$$\mathcal{V}_K\left\{x \in \mathbb{R}^n \;\Big|\; \left|\frac{|x|}{C_K\sqrt{n}} - 1\right| \ge u\right\} \le A' \exp\left(-B' n^\alpha u^\beta\right) \tag{35}$$

for $0 \le u \le 1$, instead of (3). Then we obtain spherical asymptotics for the distribution of $X^{\mathrm{av}}_{\mathcal{V}_K}/C_K$ instead of $X^{\mathrm{av}}_{\mathcal{V}_K}$.

4.1 The l_p Unit Balls

The result of this subsection is

Corollary 15. *For $1 \le p \le \infty$ the average marginal of $\mathcal{V}_{B_p^n}$ has Gaussian asymptotics for $t = o\left(n^{1/4}\right)$. Almost all marginals of $\mathcal{V}_{B_p^n}$ have Gaussian asymptotics for*

$$t = o\left(\left[\frac{n}{\log n}\right]^{1/\{2+4/\min(p,\,2)\}}\right).$$

Remark (Rigorous meaning of Corollary 15).

1. Writing "Gaussian asymptotics of a random variable X", we really mean Gaussian asymptotics for X/C for some $C > 0$. The power $1/\{2 + 4/\min(p, 2)\}$ is between 1/6 and 1/4.

2. It seems natural to take $C = \sqrt{\operatorname{Var} X}$; however, strictly speaking, this can not be done under general assumptions (for a general body K). In the special case of B_p^n one can combine the inequality (42)(below) with an inequality for $u \geq 1$ and then use the methods described in Milman–Schechtman [MS, Appendix V] to show that one can take $C_p = \sqrt{\operatorname{Var} X^{\xi}_{\mathcal{V}_{B_p^n}}}$ in (42) without loss of generality. Here ξ is of no importance, since $\operatorname{Var} X^{\xi}_{\mathcal{V}_{B_p^n}}$ does not depend on $\xi \in S^{n-1}$. We pay no further attention to these issues.
3. The rigorous meaning of the expression "almost all marginals" is as in Theorems 3 and 5.

To prove the first part of this corollary, we verify (3) (or, rather, (35)) for $\mathcal{C}_K$, where $K = B_p^n$ is the l_p^n unit ball.

For $2 \leq p < \infty$, a reasonable estimate can be obtained using the representation of $\mathcal{C}_{B_p^n}$ found by Schechtman and Zinn and independently by Rachev and Rüschendorf ([SZ1, RR]; see Barthe, Guédon, Mendelson and Naor [BGMN] for an extension to $\mathcal{V}_{B_p^n}$).

Theorem (Schechtman–Zinn, Rachev–Rüschendorf). *Let $g_1, \ldots, g_n$ be independent identically distributed random variables with density*

$$\big(2\Gamma(1+p^{-1})\big)^{-1} \exp(-|t|^p)\,.$$

Denote $G = (g_1, \ldots, g_n)$ and consider the random vector $V = G/\|G\|_p$. Then V is distributed according to $\mathcal{C}_{B_p^n}$.

Corollary. *For $2 \leq p < \infty$ the inequality*

$$\mathcal{C}_{B_p^n}\left\{ \left| \|V\|_2 \Big/ \frac{(\mathbb{E}g^2)^{1/2}\, n^{1/2}}{(\mathbb{E}g^p)^{1/p}\, n^{1/p}} - 1 \right| > u \right\} \leq A \exp(-Bnu^2)\,. \tag{36}$$

holds for $0 \leq u \leq 1$.

Proof of Corollary. The inequality

$$(1 + u/4) \leq (1 - u/2)(1 + u) \quad 0 \leq u \leq 1$$

implies

$$\mathbb{P}\left\{ \|V\|_2 > \frac{(\mathbb{E}g^2)^{1/2}\, n^{1/2}}{(\mathbb{E}g^p)^{1/p}\, n^{1/p}}\,(1+u) \right\} \leq \mathbb{P}\left\{ \frac{\|G\|_2}{\|G\|_p} > \frac{(\mathbb{E}g^2)^{1/2}\, n^{1/2}\,(1+u/4)}{(\mathbb{E}g^p)^{1/p}\, n^{1/p}\,(1-u/2)} \right\}.$$

Then,

$$\mathbb{P}\left\{ \|G\|_2 > (\mathbb{E}g^2)^{1/2}\, n^{1/2}\,(1+u/4) \right\} \leq \mathbb{P}\left\{ \sum_i (g_i^2 - \mathbb{E}g_i^2) > \frac{\mathbb{E}g_i^2}{2}\, nu \right\},$$

$$\mathbb{P}\left\{ \|G\|_p < (\mathbb{E}g^p)^{1/p}\, n^{1/p}\,(1-u/2) \right\} \leq \mathbb{P}\left\{ \sum_i (g_i^p - \mathbb{E}g_i^p) < -\frac{\mathbb{E}g_i^p}{2}\, nu \right\}.$$

Now we need an inequality due to S. N. Bernstein ([Be]; see Bourgain, Lindenstrauss and Milman [BLM] for available reference).

Theorem (S. Bernstein). *Suppose $h_1, \ldots, h_n$ are independent random variables such that*

$$\mathbb{E}h_j = 0\,; \quad \mathbb{E}\exp\left(h_j/C\right) \le 2\,. \tag{37}$$

Then

$$\mathbb{P}\left\{h_1 + \cdots + h_n > \epsilon n\right\} \le \exp\left(-\frac{\epsilon^2 n}{16C^2}\right)\,, \quad 0 \le \epsilon \le c\sqrt{n}\,.$$

It is easy to verify that $g_i^2 - \mathbb{E}g_i^2$ and $-g_i^p + \mathbb{E}g_i^p$ satisfy (37) (with some constant C); this yields

$$\mathbb{P}\left\{\|V\|_2 > \frac{(\mathbb{E}g^2)^{1/2}\, n^{1/2}}{(\mathbb{E}g^p)^{1/p}\, n^{1/p}}\,(1+u)\right\} \le \frac{A}{2}\, \exp(-Bnu^2)$$

for some constants A and B. A bound for the probability of negative deviation can be obtained in a similar way.

The estimate (36) follows. □

For $1 \le p \le 2$, we use the following theorem due to Schechtman and Zinn ([SZ2]):

Theorem (Schechtman–Zinn). *There exist positive constants C, c such that if $1 \le p \le 2$ and $f : \partial B_p^n \to \mathbb{R}$ satisfies*

$$|f(x) - f(y)| \le |x - y| \quad \textit{for all } x, y \in B_p^n \tag{38}$$

then, for all $u > 0$,

$$\mathcal{C}_{B_p^n}\left\{x \,\Big|\, \left|f(x) - \int f\, d\mathcal{C}_{B_p^n}\right| > u\right\} \le C\exp\left(-cnu^p\right)\,. \tag{39}$$

The condition (38) surely holds for $f = |\cdot|$; hence $|\cdot|$ satisfies (39). For correct normalisation recall that

$$c_1 n^{1/2-1/p} \le \int |x|\, d\mathcal{C}_{B_p^n}(x) \le c_2 n^{1/2-1/p}\,; \tag{40}$$

hence for $1 \le p \le 2$

$$\mathcal{C}_{B_p^n}\left\{x \,\Big|\, \left||x| \Big/ \int |x|\, d\mathcal{C}_{B_p^n}(x) - 1\right| > u\right\} \le C\exp\left(-cn^{p/2}u^p\right)\,. \tag{41}$$

Corollary (Concentration of $|\cdot|$ with respect to $\mathcal{V}_{B_p^n}$). *For $1 \le p \le \infty$ there exist $A_p, B_p, C_p > 0$ such that the inequality*

$$\mathcal{V}_{B_p^n}\left\{x \,\middle|\, \left||x|/C_p - 1\right| > u\right\} \le A \exp\left(-Bn^{\min(p,\,2)/2}\, u^{\min(p,\,2)}\right) \tag{42}$$

holds for $0 \le u \le 1$.

Proof of Corollary. For $1 \le p \le 2$ combine (41) with Proposition 14. For $2 \le p < \infty$ combine (36) with Proposition 14.

For $p = \infty$ the coordinates of a random vector $X = (X_1, \cdots, X_n)$ distributed according to $\mathcal{V}_{B_p^n}$ are independent; hence Bernstein's inequality for $X_i^2 - \mathbb{E}X_i^2$ yields the result. □

Now we can prove Corollary 15:

Proof of Corollary 15. For the first statement, apply Theorem 2 using (42) for $\widetilde{\mathcal{V}}_{B_p^n}$,

$$\widetilde{\mathcal{V}}_{B_p^n}(A) = \mathcal{V}_{B_p^n}(C_p A)\,.$$

For the second statement, note that the measure $\widetilde{\mathcal{V}}_{B_p^n}$ satisfies the ψ_α condition (27) with $\alpha = \min(p, 2)$. Applying Theorems 3 and 5 (combined with the concluding remark in Section 3) we obtain the result. □

Remark. Note that Corollary 15 does not capture the change of asymptotic behaviour that probably occurs around $t = n^{1/4}$. This is because the bound (42) is not sharp.

To emphasise this point, let us consider the case $p = 2$. The surface measure $\mathcal{C}_{B_2^n}$ surely satisfies (28) with any $\alpha, \beta > 0$; hence $\mathcal{V}_{B_2^n}$ satisfies (30) with $\alpha = \beta = 1$ (as we could have also verified by direct computation). Applying Theorem 2, we obtain spherical asymptotics for $t = o\left(n^{1/2}\right)$; in particular, we capture the breakdown of Gaussian asymptotics around $t \approx n^{1/4}$ (recall Lemma 1).

Remark. In fact, the bound (39) for the concentration of Euclidean norm with respect to $\mathcal{C}_{B_p^n}$ is not sharp. Schechtman and Zinn proved a better bound for $p = 1$ (for $f = |\cdot|$) in the same paper [SZ2], and Naor ([N]) extended their results to all $1 \le p \le 2$.

Unfortunately, these bounds do not suffice to improve the result in Corollary 15. On the other hand, the bounds in [SZ2, N] were proved exact only on part of the range of u; this makes it tempting to conjecture spherical approximation for $t = o(n^{(p^2-p+2)/8})$, $1 \le p \le 2$.

This would be an improvement of Corollary 15 for all $1 < p < 2$; in particular, we would be able to capture the breakdown of Gaussian asymptotics around $t = n^{1/4}$ for all these p.

Now we compare these results to limit theorems with moderate deviations for independents random variables. This allows to analyse the sharpness of the result in Corollary 15 for the common case $p = \infty$.

The following more general statement follows from our results:

Theorem 16. *Let $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ be a tensor product of 1-dimensional even measures that satisfy*

$$\int_{-\infty}^{+\infty} \exp\left(x^2/C^2\right)\, d\mu_i(x) \leq 2\,. \tag{43}$$

Then the average marginal of μ has Gaussian asymptotics for $t = o\left(n^{1/4}\right)$. Almost all marginals of μ have Gaussian asymptotics for $t = o((n/\log n)^{1/4})$.

Remark. The remarks 1 and 3 after Corollary 15 are still valid. On the other hand, the variance of the approximating Gaussian variable is "correct" in this case.

The classical limit theorems with moderate deviations (see Feller [F, Chapter XV] or Ibragimov–Linnik [IL] for a more general treatment) assume a weaker assumption

$$\int_{-\infty}^{+\infty} \exp\left(x/C\right)\, d\mu_i(x) \leq 2 \tag{44}$$

and establish Gaussian asymptotics of $1 - F^\xi(t)$ and $f^\xi(t)$ for $t = o(\|\xi\|_\infty^{-1/2})$; these results are sharp. The l_∞ norm of a typical vector $\xi \in S^{n-1}$ is of order $\sqrt{\log n/n}$; hence the asymptotics for random marginals in Theorem 16 is valid for $t = o(\sqrt[4]{n/\log n})$ and our results are sharp. In particular, this is true for $p = \infty$ in Corollary 15.

4.2 Uniformly Convex Bodies Contained in Small Euclidean Balls

Let $K \subset \mathbb{R}^n$ be a convex body; define the modulus of convexity

$$\delta_K(\epsilon) = \min\left\{1 - \frac{\|x+y\|_K}{2} \;\Big|\; \|x\|_K = \|y\|_K = 1,\ \|x-y\|_K \geq \epsilon\right\}.$$

The following concentration property was proved by Gromov and Milman ([GM], see also Arias de Reyna, Ball and Villa [ABV]):

Theorem 17 (Gromov–Milman). *If $A \subset K$ has positive measure, and $d_K(x, A)$ is the distance from x to A (measured in the norm with unit ball K), then*

$$\mathcal{V}_K\left\{x \;\middle|\; d_K(x, A) > \epsilon\right\} < \frac{e^{-2n\delta_K(\epsilon)}}{\mathcal{V}_K(A)}\,. \tag{45}$$

Corollary 18. *Suppose an isotropic body K satisfies*

$$K \subset Cn^\nu B_2^n, \quad \delta_K(\epsilon) \geq c\epsilon^\mu \quad \text{and} \quad \mathcal{V}_K\left\{|x| < m\right\} > 1/2$$

for some constants C, c, m. Then

1. the average marginal of $\mathcal{V}_K$ *has spherical asymptotics for*

$$t = o\big(n^{(1/2+\mu^{-1}-\nu)/2}\big)\,;$$

2. almost all marginals of $\mathcal{V}_K$ *have Gaussian asymptotics for*

$$t = o\left(\left[\frac{n}{\log n}\right]^{\min\left(1/6,\,(1/2+\mu^{-1}-\nu)/2\right)}\right).$$

Proof of Corollary. Following [ABP] we show that (45) implies concentration of the Euclidean norm. Really, one can estimate the probability of deviation from the median $\mathcal{M}$:

$$\begin{aligned}\mathcal{V}_K\,\{|x| \le \mathcal{M}-\epsilon\} &\le \mathcal{V}_K\big\{d_2\,\big(x, \{y \le \mathcal{M}\}\big) > \epsilon\big\}\\ &\le \mathcal{V}_K\,\Big\{d_K\big(x, \{y \le \mathcal{M}\}\,\big) > \frac{\epsilon}{Cn^\nu}\Big\} \le 2\exp\Big(-\frac{2c}{C^\mu}n^{1-\mu\nu}\epsilon^\mu\Big)\,; \quad (46)\end{aligned}$$

$$\begin{aligned}\mathcal{V}_K\,\{|x| \ge \mathcal{M}+\epsilon\} &\le \mathcal{V}_K\,\big\{d_2\big(x, \{y \ge \mathcal{M}\}\,\big) > \epsilon\big\}\\ &\le \mathcal{V}_K\Big\{d_K\big(x, \{y \ge \mathcal{M}\}\,\big) > \frac{\epsilon}{Cn^\nu}\Big\} \le 2\exp\Big(-\frac{2c}{C^\mu}n^{1-\mu\nu}\epsilon^\mu\Big)\,; \quad (47)\end{aligned}$$

conclude with Theorems 2, 3 and 5 as in the proof of Corollary 15. □

A Proofs of Technical Lemmata

Here we prove Lemmata 1, 7 and 9; the proofs are also rather technical.

Proof of Lemma 1. First,

$$\begin{aligned}\lim_{n\to\infty} \frac{\Gamma(n/2)}{\sqrt{\pi n}\,\Gamma((n-1)/2)} &= \lim \frac{(n/2e)^{n/2}}{\sqrt{\pi n}\,((n-1)/2)^{(n-1)/2}}\\ &= \lim \left(\frac{n}{n-1}\right)^{(n-1)/2} \times \sqrt{\frac{n}{2e\pi n}} = (2\pi)^{-1}\end{aligned}$$

by Stirling's formula.

Now,

$$\exp(-\epsilon - \epsilon^2/2) \le 1-\epsilon \le \exp\big(-\epsilon - \epsilon^2/\big(2(1-\epsilon)^2\big)\big)\,;$$

hence

$$\begin{aligned}&(1-t^2/n)^{(n-3)/2}\,e^{t^2/2}\\ &= \big\{(1-t^2/n)\,e^{t^2/(n-3)}\big\}^{(n-3)/2}\\ &\le \big\{\exp\big(-t^2/n - t^4/(2n^2(1-t^2/n)^2)\big)\ \exp\big(t^2/(n-3)\big)\big\}^{(n-3)/2}\\ &\le \exp\left(\frac{3}{2n}\,t^2 - \frac{n-3}{16n^2}\,t^4\right) \le \exp\left(\frac{3}{2n}\,t^2 - \frac{1}{64n}\,t^4\right)\end{aligned}$$

for $n \geq 4$. For $t \leq 16$ and n large enough $3t^2/2n \leq (1+\epsilon)$; for $t \geq 16$ we have $3t^2/2n \leq 3t^4/256n$ and hence $3t^2/2n - t^4/64n < -t^4/256n$.

Similarly,

$$\begin{aligned}(1-t^2/n)^{(n-3)/2}\, e^{t^2/2} &= \left\{(1-t^2/n)\, e^{t^2/(n-3)}\right\}^{(n-3)/2} \\ &\geq \left\{\exp\left(-t^2/n - t^4/(2n^2)\right)\, \exp\left(t^2/(n-3)\right)\right\}^{(n-3)/2} \\ &\geq \exp(-t^4/4n)\,.\end{aligned}$$

This proves the first pair of inequalities; thereby

$$\begin{aligned}1-\Psi_n(t) \;=\; \int_t^\infty \psi_n(u)\,du \;&\leq\; (1+\epsilon_n)\int_t^\infty \phi(u)\,\exp(-u^4/256n)\,du \\ &\leq (1+\epsilon_n)\,\exp(-t^4/256n)\int_t^\infty \phi(u)\,du \\ &\leq\; (1+\epsilon_n)\,\big(1-\Phi(t)\big)\,\exp(-t^4/256n)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}&1-\Psi_n(t)-(1-\epsilon_n)\,(1-\Phi(t))\,\exp(-t^4/324n) \\ &\geq (1-\epsilon_n)\int_t^\infty \phi(u)\,\big[\exp(-u^4/4n)-\exp(-t^4/324n)\big]\,du \\ &= (1-\epsilon_n)\left[\int_t^{2t} + \int_{2t}^{3t} + \int_{3t}^\infty\right]\,.\end{aligned}$$

The second integral is positive; integrating by parts, we see that the third integral equals

$$\begin{aligned}-\int_{3t}^\infty \big(1-\Phi(u)\big)\,\exp(-u^4/4n)\,\frac{u^3}{n}\,du& \\ \geq -\exp(-t^4/324n)\int_{3t}^\infty \big(1-\Phi(u)\big)\,\frac{u^3}{n}\,du\,.&\end{aligned}$$

The first one is at least

$$\begin{aligned}&\big[\exp(-t^4/64n)-\exp(-t^4/324n)\big]\int_t^{2t}\phi(u)\,du \\ &\geq \big[\exp(-t^4/64n)-\exp(-t^4/324n)\big]\int_t^{2t}\big(1-\Phi(u)\big)\,u\,du \\ &\geq \big[\exp(-t^4/64n)-\exp(-t^4/324n)\big]\int_t^{2t}\big(1-\Phi(u)\big)\,\frac{u^3}{n}\,du\end{aligned}$$

for $t^2 < n/4$. If $t \geq t_0$ this proves the remaining inequality (t_0 does not depend on n). $1-\Psi_n(t) \rightrightarrows 1-\Phi(t)$ on $[0,\, t_0]$ and for these t one can ignore $\exp(-t^4/n)$ in all the expressions; hence the inequality also holds. □

Now we prove Lemma 7. The proof uses Lemma 9 that is proved further on (without using Lemma 7).

Proof of Lemma 7. By definition,

$$\frac{1-\Psi_n(t)}{\psi_n(t)} = \int_t^\infty \frac{\psi_n(s)}{\psi_n(t)}\,ds = t\int_0^\infty \frac{\psi_n((1+u)t)}{\psi_n(t)}\,du\,.$$

To obtain the upper bound, just note that if $2t^2 < n$, then by Lemma 9

$$\psi_n((1+u)t)/\psi_n(t) \le \exp(-C_1\,u\,t^2)$$

and hence the integral is bounded by $(C_1t^2)^{-1}$.

For the lower bound restrict the integral to $[0,\,1]$; if $8t^2 < n$, $2(1+u)^2t^2 < n$ and the subintegral expression is bounded from below by $\exp(-C_2ut^2)$ on this interval. Hence the integral is not less than $(1-\exp(-C_2t^2))/(C_2t^2)$. This concludes the proof for $t > t_0$ (for a constant t_0 independent of n); for $0 < t_0$ use Gaussian approximation for ψ_n and Ψ_n (Lemma 1) to verify the inequality.

The second statement can be verified by formal differentiation.

Proof of Lemma 9.

$$\begin{aligned}
\psi_n(t)\,/\,\psi_n\big((1+u)\,t\big) &= \left(\frac{1-t^2/n}{1-(1+u)^2t^2/n}\right)^{(n-3)/2}\\
&= \left(1+\frac{t^2\,(2u+u^2)}{n-(1+u)^2\,t^2}\right)^{(n-3)/2}\\
&\le \exp\left(\frac{(u+u^2/2)\,t^2}{1-(1+u)^2t^2/n}\right)\\
&\le \exp(3ut^2) \quad \text{for } \frac{(1+u)^2\,t^2}{n} < 1/2
\end{aligned}$$

and hence the second inequality holds. On the other hand,

$$\begin{aligned}
\psi_n\big((1+u)\,t\big)\,/\,\psi_n(t) &= \left(\frac{1-t^2\,\frac{(1+u)^2}{n}}{1-t^2/n}\right)^{(n-3)/2}\\
&= \left(1-\frac{t^2}{n-t^2}\,(2u+u^2)\right)^{(n-3)/2}\\
&\le \exp\left(-(u+u^2/2)\,t^2\,\frac{1-3/n}{1-t^2/n}\right)\\
&\le \exp(-6ut^2) \quad \text{for } \frac{t^2}{n} < 1/2 \text{ and } n > 6
\end{aligned}$$

and therefore the first inequality holds as well. □

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Decoupling Weakly Dependent Events

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In this note we discuss a probabilistic statement which conceptualizes arguments from recent papers [ST1, ST2]. Its framework appears to be sufficiently general to permit applications also beyond the original context.

The setting is as follows: we have a sequence of events the probability of each of which is rather small and we would like to deduce that the probability of their intersection is *very* small. This is of course straightforward if the events are independent; our approach allows to obtain comparable upper bounds on probabilities when the dependence is not "too strong." The precise formulation of the statement is somewhat technical, but its gist can be described as follows. Suppose that our probability space is a product space and that our events/sets are defined in terms of independent coordinates in a "local" way, i.e., while membership in each of the sets may depend on many or even all coordinates, it may be verified by checking a series of conditions each of which involves just a few coordinates (for example, to verify whether a sequence of vectors is orthogonal it is enough to look at just two elements of that sequence at a time). Then the probability of the intersection of these sets can "almost" be estimated as if they were independent. More precisely, the upper bound is not a product of their probabilities, but a homogeneous polynomial in the probabilities, the degree of which is high while the number of terms is controlled.

Problems similar in spirit if not in details were considered by many authors in probabilistic combinatorics and theoretical computer science. See, for example, [JR1, JR2] and their references; particularly [J] seems to exhibit many formal similarities to our setting. (We thank M. Krivelevich for helping us navigate the combinatorics literature.) Let us note, however, that while the results cited above have, as a rule, a "large deviation feel," applications of our scheme go in the direction of "small ball" estimates, and no "dictionary"

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relating the other results to ours is apparent. On the other hand, all these statements can be considered as counterparts to Local Lovász Lemma (see e.g., [AS], Chapter 5) which, again under assumptions similar in spirit but different in detail, gives *lower* bounds on probabilities of intersections.

For $s \in \mathbb{N}$, we use the notation $[s]$ for the set $\{1, \dots, s\}$. For a set J we denote by $|J|$ the cardinality of J.

We will present our result in two separate theorems. However, the first theorem is actually a special case of the second one and is stated here primarily for pedagogical reasons.

Theorem 1. *Let $d, N \in \mathbb{N}$. Consider a family of events $\{\Theta_{j,B} : j \in [N], B \subset [N]\}$ such that for any $j \in [N]$ and $B \subset [N]$ we have*

$$\Theta_{j,B} \subset \bigcup_{B' \subset B, |B'| \le d} \Theta_{j,B'}. \tag{1}$$

For $j \in [N]$ set $\Theta_j := \Theta_{j,\{j\}^c}$, and for $\ell \in [N]$ set $\mathcal{J}_\ell = \{J \subset [N] : |J| = \ell\}$. Then for any $\ell \le \lceil N/(2d+1) \rceil$, we have

$$\bigcap_{j=1}^{N} \Theta_j \subset \bigcup_{J \in \mathcal{J}_\ell} \bigcap_{j \in J} \Theta_{j,J^c}. \tag{2}$$

If additionally for any $I, J \subset [N]$ with $I \cap J = \emptyset$ the events $\{\Theta_{j,I} : j \in J\}$ are independent, then setting $p_j = \mathbb{P}(\Theta_j)$ for $j \in [N]$ we get

$$\mathbb{P}\Big(\bigcap_{j=1}^{N} \Theta_j\Big) \le \sum_{J \in \mathcal{J}_\ell} \prod_{j \in J} \mathbb{P}(\Theta_{j,J^c}) \le \sum_{J \in \mathcal{J}_\ell} \prod_{j \in J} p_j. \tag{3}$$

The following more general formulation – substituting conditional independence for independence in the hypothesis – appears to be more easily applicable to problems which come up naturally in convex geometry and combinatorics. To state it, we will use the following concept: a family $\{\Sigma_B : B \subset [N]\}$ of σ-algebras is nested if $B' \subset B$ implies $\Sigma_{B'} \subset \Sigma_B$.

Theorem 2. *In the notation of Theorem 1, assume that the family $\{\Theta_{j,B} : j \in [N], B \subset [N]\}$ satisfies condition* (1). *Let $\{\Sigma_B : B \subset [N]\}$ be a nested family of σ-algebras. Assume further that for any $I, J \subset [N]$ with $I \cap J = \emptyset$ the events $\{\Theta_{j,I} : j \in J\}$ are Σ_I-conditionally independent and that $\mathbb{P}(\Theta_j \mid \Sigma_{\{j\}^c}) \le p_j$ for $j \in [N]$. Then*

$$\mathbb{P}\Big(\bigcap_{j=1}^{N} \Theta_j\Big) \le \sum_{J \in \mathcal{J}_\ell} \prod_{j \in J} p_j. \tag{4}$$

Proofs of the Theorems are based on the following combinatorial lemma.

Lemma 3. *Assume that a sequence $B_1, B_2, \dots, B_N$ of subsets of $[N]$ satisfies $|B_j| \le d$ and $j \notin B_j$ for $j = 1, 2, \dots, N$. Then there exists $J \subset [N]$ such that $|J| \ge N/(2d+1)$ and*

$$J \cap \bigcup_{j \in J} B_j = \emptyset.$$

Consider the $N \times N$ $\{0,1\}$-matrix $\Lambda = (\lambda_{i,j})$ defined by $\lambda_{i,j} = 1$ if $i \in B_j$ and $\lambda_{i,j} = 0$ if $i \notin B_j$, for $j \in [N]$. Then the lemma follows immediately from a result on suppression of matrices due to K. Ball (cf., [BT], Theorem 3.1). We recall the statement of this result as we feel it might be applicable in variety of contexts; for example in [ST2] it was used to control probabilities via analytic considerations rather than combinatorial ones.

Proposition 4. *Let $A = (a_{i,j})$ be an $N \times N$ matrix such that $a_{i,j} \ge 0$ for all i, j, $\sum_{i=1}^N a_{i,j} \le 1$ and $a_{j,j} = 0$ for all j. Then for every integer $t \ge 1$ there is a partition $\{J_s\}_{s=1}^t$ of $[N]$ such that for $s = 1, \dots, t$,*

$$\sum_{i \in J_s} a_{i,j} \le 2/t \qquad \text{for } j \in J_s.$$

In the setting of Lemma 3 we apply Proposition 4 to the matrix $A = (1/d)\Lambda$ and $t = 2d+1$.

We shall also provide an elementary proof of (a variant of) Lemma 3 that gives the estimate $|J| \ge N/(6d+2)$, which is sufficient for most applications. Alternatively, various variants of the lemma may be also derived from various forms of Turán's theorem, cf. [AS], p. 81–82. (We note, however, that Turán's theorem concerns *undirected* graphs, which correspond to symmetric matrices in the language of Proposition 4, and so the derivation requires some additional – even if not difficult – steps similar to the first part of the argument presented below.)

Proof. Fix $a \ge 1$. Set $I := \{i : \sum_{j=1}^N \lambda_{i,j} \le a\, d\}$ and let $m := |I|$. Since the sum of all entries of Λ is $\le d\,N$, then $N - m = |I^c| \le N/a$, and so $m \ge (1 - 1/a)N$.

Let $J \subset I$ be a maximal subset of I such that the corresponding $|J| \times |J|$ submatrix consists only of 0's, and set $|J| =: k$. To facilitate visualizing the argument the reader may think of $J = [k]$. By maximality of J we have that for each $i \in I \setminus J$ there is $j \in J$ such that $\lambda_{i,j} + \lambda_{j,i} \ge 1$. Summing up over $i \in I \setminus J$ we get

$$S := \sum_{i \in I \setminus J} \sum_{j \in J} (\lambda_{i,j} + \lambda_{j,i}) \ge |I \setminus J| = m - k.$$

We will now get two lower estimates on k from considerations in two separate rectangles. Set

$$t := \frac{1}{S} \sum_{i \in I \setminus J} \sum_{j \in J} \lambda_{j,i},$$

so that the number of 1's in the "upper-right" rectangle $J \times (I \setminus J)$ is equal to tS. On the other hand, for each $j \in J$, the number of 1's in the j'th row is less than or equal to $a\,d$, therefore $t\,(m-k) \le tS \le k(a\,d)$. Similarly, considering the "lower-left" rectangle $(I \setminus J) \times J$, and calculating the number of 1's in two different ways we get $(1-t)S \le k\,d$, hence $(1-t)(m-k) \le (1-t)S \le k\,d$. Adding up the obtained inequalities we get $m - k \le k\,d(a+1)$, which yields

$$k \ge \frac{a-1}{a\,(ad+d+1)}\,N.$$

Setting, for example, $a = 2$, gives $k \ge N/(6d+2)$. □

We are now ready for the proofs of the Theorems.

Proof of Theorem 1. Observe that by (1), $B' \subset B \subset [N]$ implies $\Theta_{j,B'} \subset \Theta_{j,B}$ for any $j \in [N]$. Fix $\ell \le \lceil N/(2d+1) \rceil$. To show (2), let $\omega \in \bigcap_{j=1}^N \Theta_j$. Using (1) for each $j = 1, \dots, N$ again we get sets $B_j \not\ni j$ (which may depend on ω) with $|B_j| \le d$ such that $\omega \in \bigcap_{j=1}^N \Theta_{j,B_j}$. If $J \in \mathcal{J}_\ell$ is the set from Lemma 3 then, by the first observation above, we have $\omega \in \bigcap_{j=1}^N \Theta_{j,J^c}$. The set J may depend on ω as well, but since $J \in \mathcal{J}_\ell$, the inclusion (2) follows.

If the additional independence assumption is satisfied then the family $\{\Theta_{j,J^c} : j \in J\}$ is independent, hence $\mathbb{P}(\bigcap_{j \in J} \Theta_{j,J^c}) = \prod_{j \in J} \mathbb{P}(\Theta_{j,J^c})$. Since $\Theta_{j,J^c} \subset \Theta_j$, the last inequality follows as well. □

Proof of Theorem 2. By Theorem 1 the inclusion (2) holds.

Next, by the conditional independence assumption we have, for every $J \subset [N]$,

$$\mathbb{P}\Big(\bigcap_{j \in J} \Theta_{j,J^c} \,|\, \Sigma_{J^c}\Big) = \prod_{j \in J} \mathbb{P}\big(\Theta_{j,J^c} \,|\, \Sigma_{J^c}\big).$$

In turn, for $j \in J$,

$$\mathbb{P}\left(\Theta_{j,J^c} \,|\, \Sigma_{J^c}\right) \le \mathbb{P}\left(\Theta_j \,|\, \Sigma_{J^c}\right) = \mathbb{E}\big(\mathbb{P}\left(\Theta_j \,|\, \Sigma_{\{j\}^c}\right) \,\big|\, \Sigma_{J^c}\big) \le p_j,$$

with the last estimate following from the (pointwise) upper bound on the random variable $\mathbb{P}\left(\Theta_j \,|\, \Sigma_{\{j\}^c}\right)$. Accordingly,

$$\mathbb{P}\Big(\bigcap_{j \in J} \Theta_{j,J^c}\Big) = \mathbb{E}\,\left(\mathbb{P}\Big(\bigcap_{j \in J} \Theta_{j,J^c} \,|\, \Sigma_{J^c}\Big)\right) \le \prod_{j \in J} p_j.$$

Therefore, by (2), we conclude that for $\ell = \lceil N/(2d+1) \rceil$,

$$\mathbb{P}\Big(\bigcap_{j=1}^N \Theta_j\Big) \le \sum_{J \in \mathcal{J}_\ell} \mathbb{P}\Big(\bigcap_{j \in J} \Theta_{j,J^c}\Big) \le \sum_{J \in \mathcal{J}_\ell} \prod_{j \in J} p_j,$$

that is, (4) holds. □

Even though the hypotheses of the theorems seem quite abstract, there exists a variety of natural probabilistic settings in which they are satisfied. We will now describe some such settings. While our examples make the appearance of conditional independence quite clear, securing uniform estimates for conditional probabilities often requires some additional technicalities which we will ignore here as they are only marginally related to the decoupling procedure.

Let $D_1, \dots, D_N, L_1, \dots, L_N$ be random convex subsets of $\mathbb{R}^n$ such that the family of pairs $\{(D_j, L_j)\}_{j=1}^N$ is independent. Set $E_j = \operatorname{span} L_j \subset \mathbb{R}^n$ and let P_{E_j} denote the orthogonal projection on E_j, for $j = 1, \dots, N$. For $B \subset [N]$, we let Σ_B be the σ-algebra generated by $\{D_i, L_i\}_{i \in B}$.

In a typical setting the sets D_i, L_i will be symmetric and $\operatorname{conv}_{i\in[N]} D_i =: K$ will be a (random) symmetric convex body in $\mathbb{R}^n$. We will then define a normed space X as $\mathbb{R}^n$ endowed with the norm whose unit ball is K. We will be interested in particular in the character and complementability of subspaces of X, especially those determined by the E_j's.

For the first illustration of our scheme assume that there is $p \in (0,1)$ such that we have upper bounds for conditional probabilities

$$\mathbb{P}\Big(E_j \cap \operatorname*{conv}_{i\neq j} D_i \not\subset L_j \mid \Sigma_{\{j\}^c}\Big) \le p \qquad \text{for all } j \in [N]. \tag{5}$$

In the simplest case when $D_i = L_i$ for $i \in [N]$, the complement of the event appearing in (5) can be alternatively described by the equality $E_j \cap K = D_j$, that is, the unit ball in E_j considered as a subspace of X (with the induced norm) being exactly D_j.

For $j \in [N]$ and $B \subset [N]$, let $\Theta_{j,B} = \{E_j \cap \operatorname{conv}_{i\in B} D_i \not\subset L_j\}$, then the corresponding Θ_j's are exactly the sets appearing in (5). By Caratheodory's theorem condition (1) is satisfied with $d = n+1$. (In fact, we do have here, and in examples that follow, equality of the sets. We note, however, that in actual applications one often needs the weaker hypothesis involving inclusion.) Also, if $I \cap J = \emptyset$ then events $\{\Theta_{j,I} : j \in J\}$ are Σ_I-conditionally independent. Therefore by Theorem 2 we get, with $\ell = \lceil N/(2n+3) \rceil$,

$$\mathbb{P}\Big(\bigcap_{j=1}^N \Theta_j\Big) \le \binom{N}{\ell} p^\ell \le \big(ep(2n+3)\big)^\ell. \tag{6}$$

For another illustration we assume the following upper bound on the conditional probabilities

$$\mathbb{P}\Big(P_{E_j}\big(\operatorname*{conv}_{i\neq j} D_i\big) \not\subset L_j \mid \Sigma_{\{j\}^c}\Big) \le p \qquad \text{for all } j \in [N]. \tag{7}$$

Modulo some minor technicalities sets of this form were considered in [ST1]. Again, in the case when $D_i = L_i$ for $i = 1, \dots, N$, the complement of the event from (7) can be described as follows: the subspace E_j of X has the unit ball equal to D_j *and* is 1-complemented in X via the orthogonal projection.

Similarly as before, let $\Theta'_{j,B} = \{P_{E_j}(\operatorname{conv}_{i \in B} D_i) \not\subset L_j\}$, for $j \in [N]$ and $B \subset [N]$. Then the corresponding Θ'_j's are exactly the sets appearing in (7). Now, condition (1) is clearly satisfied with $d = 1$, and if $I \cap J = \emptyset$ then events $\{\Theta'_{j,I} : j \in J\}$ are Σ_I-conditionally independent. Using Theorem 2 with $\ell = \lceil N/3 \rceil$ we then obtain

$$\mathbb{P}\Big(\bigcap_{j=1}^{N} \Theta'_j\Big) \le \sum_{J \in \mathcal{J}_\ell} \mathbb{P}\Big(\bigcap_{j \in J} \Theta'_{j,J^c} \mid \Sigma_{J^c}\Big) \le \binom{N}{\ell} p^\ell \le (3ep)^\ell . \qquad (8)$$

For more elaborated geometric interpretations of our scheme let us assume that $D_i \subset L_i$ for $i \in [N]$ and consider a random symmetric convex body $L \subset \mathbb{R}^n$. By Y denote the space $\mathbb{R}^n$ with the norm for which L is the unit ball; consider the formal identity operator $id_{X,Y} : X \to Y$ and let $k := \max \operatorname{codim} E_i$.

If $L_j = E_j \cap L$ for $j \in [N]$, then the complement of the set $\bigcap_{j=1}^N \Theta_j$ (appearing in (6)) is connected with an upper bound for the Gelfand numbers of $id_{X,Y}$. More precisely, $\omega \notin \bigcap_{j=1}^N \Theta_j$ implies that $c_k(id_{X,Y}) \le 1$. Similarly, with L_j's of the same form, the complement of the set appearing in (8) relates to the approximation numbers of $id_{X,Y}$, namely $\omega \notin \bigcap_{j=1}^N \Theta'_j$ implies that $a_k(id_{X,Y}) \le 1$.

Finally, if $L_j = P_{E_j} L$ for $j \in [N]$, then $\omega \notin \bigcap_{j=1}^N \Theta'_j$ implies that the k's Kolmogorov number of $id_{X,Y}$ satisfies $d_k(id_{X,Y}) \le 1$.

Still another application of the present scheme can be found in [P] which provides a simpler and more structured proof of the result from [GLT] concerning highly asymmetric convex bodies. The sets corresponding to Θ_j's that appear in that paper are roughly of the form

$$\Big\{g_j \in t \operatorname{conv}\big(\{g_i\}_{i \ne j}, 0\big) \quad \text{for all} \quad j \in [N]\Big\},$$

where the g_j's are i.i.d. Gaussian vectors and $t > 0$ is a constant. For more details we refer the reader to [P].

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The Square Negative Correlation Property for Generalized Orlicz Balls

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Summary. Recently Anttila, Ball and Perissinaki proved that the squares of coordinate functions in l_p^n are negatively correlated. This paper extends their results to balls in generalized Orlicz norms on $\mathbb{R}^n$. From this, the concentration of the Euclidean norm and a form of the Central Limit Theorem for the generalized Orlicz balls is deduced. Also, a counterexample for the square negative correlation hypothesis for 1-symmetric bodies is given.

1 Introduction

Given a convex, central-symmetric body $K \subset \mathbb{R}^n$ of volume 1, consider the random variable $X = (X_1, X_2, \ldots, X_n)$, uniformly distributed on K. We are interested in determining whether the vector has the *square negative correlation*, i.e. if

$$\mathrm{cov}(X_i^2, X_j^2) := \mathbb{E}(X_i^2 X_j^2) - \mathbb{E}X_i^2 \mathbb{E}X_j^2 \leq 0.$$

We assume that K is in *isotropic position*, i.e. that

$$\mathbb{E}X_i = 0 \quad \text{and} \quad \mathbb{E}X_i \cdot X_j = L_K^2 \delta_{ij},$$

where δ_{ij} is the Kronecker delta and L_K is a positive constant. Since any convex body not supported on an affine subspace has an affine image which is in isotropic position, this is not a restrictive assumption.

The motivation in studying this problem comes from the so-called central limit problem for convex bodies, which is to show that most of the one-dimensional projections of the uniform measure on a convex body are approximately normal. It turns out that the bounds on the square correlation can be crucial to estimating the distance between the one-dimensional projections and the normal distribution (see for instance [ABP], [MM]). A related problem is to provide bounds for the quantity σ_K, defined by

$$\sigma_K^2 = \frac{\mathrm{Var}(|X|^2)}{nL_K^4} = \frac{n\mathrm{Var}(|X|^2)}{(\mathbb{E}|X|^2)^2},$$

where X is uniformly distributed on K. It is conjectured (see for instance [BK]) that σ_K is bounded by a universal constant for any convex symmetric isotropic body. Recently Anttila, Ball and Perissinaki (see [ABP]) observed that for $K = l_p^n$ the covariances of X_i^2 and X_j^2 are negative for $i \neq j$, and from this deduced a bound on σ_K in this class.

In this paper we shall study the covariances of X_i^2 and X_j^2 (or, more generally, of any functions depending on a single variable) on a convex, symmetric and isotropic body. We will show a general formula to calculate the covariance for given functions and K, and from this formula deduce the covariance of any increasing functions of different variables, in particular of the functions X_i^2 and X_j^2, has to be negative on generalized Orlicz balls. Then we follow [ABP] to arrive at a concentration property and [MM] to get a Central Limit Theorem variant for generalized Orlicz balls.

The layout of this paper is as follows. First we define notations which will be used throughout the paper. In Section 2 we transform the formula for the square correlation into a form which will be used further on. In Section 3 we use the formula and the Brunn–Minkowski inequality to arrive at the square negative correlation property for generalized Orlicz balls. In Section 4 we show the corollaries, in particular a central-limit theorem for generalized Orlicz balls. Section 5 contains another application of the formula from Section 2, a simple counterexample for the square negative correlation hypothesis for 1-symmetric bodies.

Notation

Throughout the paper $K \subset \mathbb{R}^n$ will be a convex central-symmetric body of volume 1 in isotropic position. Recall that by isotropic position we mean that for any vector $\theta \in S^{n-1}$ we have $\int_K \langle \theta, x \rangle^2 \, dx = L_K^2$ for some constant L_K. For $A \subset \mathbb{R}^n$ by $|A|$ we will denote the Lebesgue volume of A. For $x \in \mathbb{R}^n$, $|x|$ will mean the Euclidean norm of x. We assume that $\mathbb{R}^n$ is equipped with the standard Euclidean structure and with the canonic orthonormal base $(e_1, \ldots, e_n)$. For $x \in \mathbb{R}^n$ by x_i we shall denote the ith coordinate of x, i.e. $\langle e_i, x \rangle$. We will consider K as a probability space with the Lebesgue measure restricted to K as the probability measure. If there is any danger of confusion, then $\mathbb{P}_K$ will denote the probability with respect to this measure, $\mathbb{E}_K$ will denote the expected value with respect to $\mathbb{P}_K$, and so on. By X we will usually denote the n-dimensional random vector equidistributed on K, while X_i will denote its ith coordinate. By the covariance $\mathrm{cov}(Y, Z)$ for real random variables Y, Z we mean $\mathbb{E}(YZ) - \mathbb{E}Y\mathbb{E}Z$. By an 1-symmetric body K we mean one that is invariant under reflections in the coordinate hyperplanes, or equivalently, such a body that $(x_1, x_2, \ldots, x_n) \in X \Longleftrightarrow (\varepsilon_1 x_1, \varepsilon_2 x_2, \ldots, \varepsilon_n x_n \in X)$ for any choice of $\varepsilon_i \in \{-1, 1\}$. The parameter σ_K, as in [BK], will be defined by

$$\sigma_K^2 = \frac{\mathrm{Var}(|X|^2)}{nL_K^4} = \frac{n\mathrm{Var}(|X|^2)}{(\mathbb{E}|X|^2)^2}.$$

For any $n \geq 1$ and convex increasing functions $f_i : [0,\infty) \to [0,\infty)$, $i = 1,\ldots,n$ satisfying $f_i(0) = 0$ (called the Young functions) we define the generalized Orlicz ball $K \subset \mathbb{R}^n$ to be the set of points $x = (x_1,\ldots,x_n)$ satisfying

$$\sum_{i=1}^{n} f_i(|x_i|) \leq 1.$$

This is easily proven to be convex, symmetric and bounded, thus

$$\|x\| = \inf\{\lambda : x \in \lambda K\}$$

defines a norm on $\mathbb{R}^n$. In the case of equal functions f_i the norm is called an *Orlicz norm*, in the general case a *generalized Orlicz* norm. Examples of Orlicz norms include the l_p norms for any $p \geq 1$ with $f(t) = |t|^p$ being the Young functions. The generalized Orlicz spaces are also referred to as modular sequence spaces (I thank the referee for pointing this out to me).

2 The General Formula

We wish to calculate $\operatorname{cov}(f(X_i), g(X_j))$, where f and g are univariate functions, $i \neq j$ and X_i, X_j are the coordinates of the random vector X, equidistributed on a convex, symmetric and isotropic body K. For simplicity we will assume $i = 1$, $j = 2$ and denote X_1 by Y and X_2 by Z. For any $(y,z) \in \mathbb{R}^2$ let $m(y,z)$ be equal to the $n-2$-dimensional Lebesgue measure of the set $(\{(y,z)\} \times \mathbb{R}^{n-2}) \cap K$. We set out to prove:

Theorem 2.1. *For any symmetric, convex body K in isotropic position and any functions f, g we have*

$$\operatorname{cov}\big(f(Y), g(Z)\big) =$$
$$\int_{\mathbb{R}^4, |y|>|\bar{y}|, |z|>|\bar{z}|} \big(m(y,z)m(\bar{y},\bar{z}) - m(y,\bar{z})m(\bar{y},z)\big)\big(f(y) - f(\bar{y})\big)\big(g(z) - g(\bar{z})\big)\ .$$

Furthermore, for 1-symmetric bodies and symmetric functions we will have the following corollary:

Corollary 2.2. *For any symmetric, convex, unconditional body K in isotropic position and symmetric functions f, g we have*

$$\operatorname{cov}\big(f(Y), g(Z)\big) =$$
$$16 \int_{\mathbb{R}^4, y>\bar{y}>0, z>\bar{z}>0} \big(m(y,z)m(\bar{y},\bar{z}) - m(y,\bar{z})m(\bar{y},z)\big)\big(f(y) - f(\bar{y})\big)\big(g(z) - g(\bar{z})\big)\ .$$

The corollary is a simple consequence of the fact that for symmetric functions f and g and an 1-symmetric body K the integrand is invariant under the change of the sign of any of the variables, so we may assume all of them are positive.

As concerns the sign of $\mathrm{cov}(f, g)$, which is what we set out to determine, we have the following simple corollary:

Corollary 2.3. *For any central-symmetric, convex, 1-symmetric body K in isotropic position and symmetric functions f, g that are non-decreasing on $[0, \infty)$ if for all $y > \bar{y} > 0$, $z > \bar{z} > 0$ we have*

$$m(y, \bar{z})m(\bar{y}, z) \geq m(y, z)m(\bar{y}, \bar{z}), \tag{1}$$

then

$$\mathrm{cov}(f, g) \leq 0.$$

Similarly, if the opposite inequality is satisfied for all $y > \bar{y} > 0$ and $z > \bar{z} > 0$, then the covariance is non-negative.

Proof. The second and third bracket of the integrand in Corollary 2.2 is positive under the assumptions of Corollary 2.3. Thus if we assume the first bracket is negative, then the whole integrand is negative, which implies the integral is negative, and vice-versa. □

Proof of Theorem 2.1. We have

$$\mathrm{cov}\big(f(Y), g(Z)\big) = \mathbb{E}f(Y)g(Z) - \mathbb{E}f(Y)\mathbb{E}g(Z).$$

From the Fubini theorem we have

$$\mathbb{E}f(Y)g(Z) = \int_{R^2} m(y, z)f(y)g(z),$$

and similar equations for $\mathbb{E}f(Y)$ and $\mathbb{E}g(Z)$.

For any function h of two variables $a, b \in A$ we can write $\int_{A^2} h(a, b) = \int_{A^2} h(b, a) = \frac{1}{2}\int_{A^2} h(a, b) + h(b, a)$. We shall repeatedly use this trick to transform the formula for the covariance of f and g into the required form:

$$\begin{aligned}\mathbb{E}f(Y)\mathbb{E}g(Z) &= \int_{\mathbb{R}^2} m(y, z)f(y) \int_{\mathbb{R}^2} m(\bar{y}, \bar{z})g(\bar{z}) \\ &= \int_{\mathbb{R}^4} m(y, z)m(\bar{y}, \bar{z})f(y)g(\bar{z}) = \int_{\mathbb{R}^4} m(\bar{y}, \bar{z})m(y, z)f(\bar{y})g(z) \\ &= \frac{1}{2}\int_{\mathbb{R}^4} m(\bar{y}, \bar{z})m(y, z)\big(f(\bar{y})g(z) + f(y)g(\bar{z})\big).\end{aligned}$$

We repeat this trick, exchanging z and $\bar{z}$ (and leaving y and $\bar{y}$ unchanged):

$$\begin{aligned}\mathbb{E}f(Y)\mathbb{E}g(Z) = \frac{1}{4}\int_{\mathbb{R}^4} & m(\bar{y}, \bar{z})m(y, z)\big(f(y)g(\bar{z}) + f(\bar{y})g(z)\big) \\ & + m(\bar{y}, z)m(y, \bar{z})\big(f(y)g(z) + f(\bar{y})g(\bar{z})\big)\ .\end{aligned}$$

We perform the same operations on the second part of the covariance. To get a integral over $\mathbb{R}^4$ we multiply by an $\mathbb{E}1$ factor (this in effect will free us from the assumption that the body's volume is 1):

$$\begin{aligned}\mathbb{E}f(Y)g(Z)\mathbb{E}1 &= \int_{\mathbb{R}^4} m(y,z)m(\bar{y},\bar{z})f(y)g(z)\\ &= \frac{1}{4}\int_{\mathbb{R}^4} m(y,z)m(\bar{y},\bar{z})\big(f(y)g(z)\\ &\quad + f(\bar{y})g(\bar{z})\big) + m(y,\bar{z})m(\bar{y},z)\big(f(y)g(\bar{z}) + f(\bar{y})g(z)\big).\end{aligned}$$

Thus:

$$\begin{aligned}&\operatorname{cov}\big(f(Y),g(Z)\big) = \mathbb{E}(f(Y)g(Z))\mathbb{E}1 - \mathbb{E}f(Y)\mathbb{E}g(Z)\\ &= \frac{1}{4}\bigg(\int_{\mathbb{R}^4} m(y,z)m(\bar{y},\bar{z})\big(f(y)g(z) + f(\bar{y})g(\bar{z})\big)\\ &\quad + m(y,\bar{z})m(\bar{y},z)\big(f(y)g(\bar{z}) + f(\bar{y})g(z)\big)\\ &\quad - m(\bar{y},\bar{z})m(y,z)\big(f(y)g(\bar{z}) + f(\bar{y})g(z)\big)\\ &\quad - m(\bar{y},z)m(y,\bar{z})\big(f(y)g(z) + f(\bar{y})g(\bar{z})\big)\bigg)\\ &= \frac{1}{4}\int_{\mathbb{R}^4} \Big(\big(m(y,\bar{z})m(\bar{y},z) - m(y,z)m(\bar{y},\bar{z})\big)\big(f(y)g(\bar{z}) + f(\bar{y})g(z)\big)\\ &\quad + \big(m(y,z)m(\bar{y},\bar{z}) - m(\bar{y},z)m(y,\bar{z})\big)\big(f(y)g(z) + f(\bar{y})g(\bar{z})\big)\Big)\\ &= \frac{1}{4}\int_{\mathbb{R}^4} \big(m(y,\bar{z})m(\bar{y},z) - m(y,z)m(\bar{y},\bar{z})\big)\\ &\quad \cdot \big(f(y)g(\bar{z}) + f(\bar{y})g(z) - f(y)g(z) - f(\bar{y})g(\bar{z})\big)\\ &= \frac{1}{4}\int_{\mathbb{R}^4} \big(m(y,\bar{z})m(\bar{y},z) - m(y,z)m(\bar{y},\bar{z})\big)\big(f(y) - f(\bar{y})\big)\big(g(\bar{z}) - g(z)\big)\ .\end{aligned}$$

Finally, notice that if we exchange y and $\bar{y}$ in the above formula, then the formula's value will not change — the first and second bracket will change signs, and the third will remain unchanged. The same applies to exchanging z and $\bar{z}$. Thus

$$\operatorname{cov}(f,g) = \int_{\mathbb{R}^4,|y|>|\bar{y}|,|z|>|\bar{z}|} \big(m(y,z)m(\bar{y},\bar{z}) - m(y,\bar{z})m(\bar{y},z)\big)\big(f(y) - f(\bar{y})\big)\big(g(z) - g(\bar{z})\big)\ .$$

□

3 Generalized Orlicz Spaces

Now we will concentrate on the case of symmetric, non-decreasing functions on generalized Orlicz spaces. We will prove the inequality (1):

Theorem 3.1. *If K is a ball in an generalized Orlicz norm on $\mathbb{R}^n$, then for any $y > \bar{y} > 0$ and $z > \bar{z} > 0$ we have*

$$m(y, \bar{z})m(\bar{y}, z) \geq m(y, z)m(\bar{y}, \bar{z}). \tag{2}$$

From this Theorem and Corollary 2.3 we get

Corollary 3.2. *If K is a ball in an generalized Orlicz norm on $\mathbb{R}^n$ and f, g are symmetric functions that are non-decreasing on $[0, \infty)$, then $\mathrm{cov}_K(f, g) \leq 0$.*

It now remains to prove the inequality (2).

Proof of Theorem 3.1. Let f_i denote the Young functions of K. Let us consider the ball $K' \subset \mathbb{R}^{n-1}$, being an generalized Orlicz ball defined by the Young functions $\Phi_1, \Phi_2, \ldots, \Phi_{n-1}$, where $\Phi_i(t) = f_{i+1}(t)$ for $i > 1$ and $\Phi_1(t) = t$ — that is, we replace the first two Young functions of K by a single identity function.

For any $x \in \mathbb{R}$ let P_x be the set $(\{x\} \times \mathbb{R}^{n-2}) \cap K'$, and $|P_x|$ be its $n-2$-dimensional Lebesgue measure. K' is a convex set, thus, by the Brunn–Minkowski inequality (see for instance [G]) the function $x \mapsto |P_x|$ is a logarithmically concave function. This means that $x \mapsto \log |P_x|$ is a concave function, or equivalently that

$$|P_{tx+(1-t)y}| \geq |P_x|^t \cdot |P_y|^{1-t}.$$

In particular, for given real positive numbers a, b, c we have

$$|P_{a+c}| \geq |P_a|^{b/(b+c)}|P_{a+b+c}|^{c/(b+c)},$$
$$|P_{a+b}| \geq |P_a|^{c/(b+c)}|P_{a+b+c}|^{b/(b+c)},$$

and as a consequence when we multiply the two inequalities,

$$|P_{a+b}| \cdot |P_{a+c}| \geq |P_a| \cdot |P_{a+b+c}|. \tag{3}$$

Now let us consider the ball K. Let us take any $y > \bar{y} > 0$ and $z > \bar{z} > 0$. Let $a = f_1(\bar{y}) + f_2(\bar{z})$, $b = f_1(y) - f_1(\bar{y})$, and $c = f_2(z) - f_2(\bar{z})$. The numbers a, b and c are positive from the assumptions on y, z, $\bar{y}$ and $\bar{z}$ and because the Young functions are increasing. Then $m(\bar{y}, \bar{z})$ is equal to the measure of the set

$$\left\{x_3, x_4, \ldots, x_n : f_1(\bar{y}) + f_2(\bar{z}) + \sum_{i=3}^{n} f_i(x_i) \leq 1\right\}$$
$$= \left\{x_3, x_4, \ldots, x_n : a + \sum_{i=2}^{n} \Phi_i(x_i) \leq 1\right\} = P_a \ .$$

Similarly $m(y, \bar{z}) = |P_{a+b}|$, $m(\bar{y}, z) = |P_{a+c}|$ i $m(y, z) = |P_{a+b+c}|$.

Substituting those values into the inequality (3) we get the thesis:

$$m(y, \bar{z})m(\bar{y}, z) \geq m(y, z)m(\bar{y}, \bar{z}) \ . \qquad \square$$

4 The Consequences

For the consequences we will take $f(t) = g(t) = t^2$. The first simple consequence is the concentration property for generalized Orlicz balls. Here, we follow the argument of [ABP] for l_p balls.

Theorem 4.1. *For every generalized Orlicz ball $K \subset \mathbb{R}^n$ we have*

$$\sigma_K \leq \sqrt{5}.$$

Proof. From the Cauchy-Schwartz inequality we have

$$n^2 L_K^4 = \left(\sum_{i=1}^{n} \mathbb{E}_K X_i^2 \right)^2 = \left(\mathbb{E}_K |X|^2 \right)^2 \leq \mathbb{E}_K |X|^4.$$

On the other hand from Corollary 3.2 we have

$$\begin{aligned} \mathbb{E}_K |X|^4 = \mathbb{E}_K \left(\sum_{i=1}^{n} X_i^2 \right)^2 &= \sum_{i=1}^{n} \mathbb{E}_K X_i^4 + \sum_{i \neq j} \mathbb{E}_K X_i^2 X_j^2 \\ &\leq \sum_{i=1}^{n} \mathbb{E}_K X_i^4 + \sum_{i \neq j} \mathbb{E}_K X_i^2 \mathbb{E}_K X_j^2 \\ &= \sum_{i=1}^{n} \mathbb{E}_K X_i^4 + n(n-1) L_K^4. \end{aligned}$$

As for 1-symmetric bodies the density of X_i is symmetric and log-concave, we know (see e.g. [KLO], Section 2, Remark 5)

$$\mathbb{E}_K X_i^4 \leq 6 \left(\mathbb{E}_K X_i^2 \right)^2 = 6 L_K^4,$$

thence

$$n^2 L_K^4 \leq \mathbb{E}_K |X|^4 \leq (n^2 + 5n) L_K^4.$$

This gives us

$$\operatorname{Var}(|X|^2) = \mathbb{E}_K |X|^4 - n^2 L_K^4 \leq 5n L_K^4,$$

and thus

$$\sigma_K^2 = \frac{\operatorname{Var}|X|^2}{n L_K^4} \leq 5.$$

□

Corollary 4.2. *For every generalized Orlicz ball $K \subset \mathbb{R}^n$ and for every $t > 0$ we have*

$$\mathbb{P}_K \left(\left| \frac{|X|^2}{n} - L_K^2 \right| \geq t \right) \leq \frac{5 L_K^4}{n t^2}$$

and

$$\mathbb{P}_K \left(\left| \frac{|X|}{\sqrt{n}} - L_K \right| \geq t \right) \leq \frac{5 L_K^2}{n t^2}.$$

Proof. From the estimate on the variance of $|X|^2$ and Chebyshev's inequality we get

$$t^2\mathbb{P}_K\left(\left|\frac{|X|^2}{n}-L_K^2\right|\geq t\right)\leq \mathbb{E}_K\left(\frac{|X|^2}{n}-L_K^2\right)^2\leq \frac{1}{n^2}\mathrm{Var}(|X|^2)\leq \frac{5}{n}L_K^4.$$

For the second part let $t>0$. We have

$$\begin{aligned}\mathbb{P}_K\big(|X|-\sqrt{n}L_K|\geq t\sqrt{n}\big)&\leq \mathbb{P}_K\big(|X|^2-nL_K^2|\geq tnL_K\big)\\&\leq \frac{5L_K^4}{t^2nL_K^2}=\frac{5L_K^2}{t^2n}\ .\end{aligned}$$

□

This result confirms the so-called *concentration hypothesis* for generalized Orlicz balls. The hypothesis, see e.g. [BK], states that the Euclidean norm concentrates near the value $\sqrt{n}L_K$ as a function on K. More precisely, for a given $\varepsilon>0$ we say that K satisfies the *ε-concentration hypothesis* if

$$\mathbb{P}_K\left(\left|\frac{|X|}{\sqrt{n}}-L_K\right|\geq \varepsilon L_K\right)\leq\varepsilon.$$

From Corollary 4.2 we get that the class of generalized Orlicz balls satisfies the ε-concentration hypothesis with $\varepsilon=\sqrt{5}n^{-1/3}$.

A more complex consequence is the Central Limit Property for generalized Orlicz balls. For $\theta\in S^{n-1}$ let $g_\theta(t)$ be the density of the random variable $\langle X,\theta\rangle$. Let g be the density of $\mathcal{N}(0,L_K^2)$. Then for most θ the density g_θ is very close to g. More precisely, by part 2 of Corollary 4 in [MM] we get

Corollary 4.3. *There exists an absolute constant c such that*

$$\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t}\big(g_\theta(s)-g(s)\big)ds\right|\leq c\|\theta\|_3^{3/2}.$$

5 The Counterexample for 1-Symmetric Bodies

It is generally known that the negative square correlation hypothesis does not hold in general in the class of 1-symmetric bodies. However, the formula from section 2 allows us to give a counterexample without any tedious calculations. Let $K\subset\mathbb{R}^3$ be the ball of the norm defined by

$$\|(x,y,z)\|=|x|+\max\{|y|,|z|\}.$$

The quantity $m(y,z)$ considered in Corollary 2.3, defined as the volume of the cross-section $(\mathbb{R}\times\{y,z\})\cap K$ is equal to $2(1-\max\{|y|,|z|\})$ for $|y|,|z|\leq 1$ and 0 for greater $|y|$ or $|z|$. To check the inequality (1) for $y>\bar{y}>0$ and $z>\bar{z}>0$ we may assume without loss of generality that $y\geq z$ (as K is invariant under the exchange of y and z). We have

$$
\begin{aligned}
& m(y,\bar{z})m(\bar{y},z) - m(y,z)m(\bar{y},\bar{z}) \\
&= 4\big(1-\max\{y,\bar{z}\}\big)\big(1-\max\{\bar{y},z\}\big) - 4\big(1-\max\{y,z\}\big)\big(1-\max\{\bar{y},\bar{z}\}\big) \\
&= 4(1-y)\big(1-\max\{\bar{y},z\}\big) - 4(1-y)\big(1-\max\{\bar{y},\bar{z}\}\big) \\
&= 4(1-y)\big(\max\{\bar{y},\bar{z}\} - \max\{\bar{y},z\}\big)\ .
\end{aligned}
$$

As $y \leq 1$ all we have to consider is the sign of the third bracket. However, as $z > \bar{z}$, the third bracket is never positive, and is negative when $z > \bar{y}$. Thus from Corollary 2.3 the covariance $\mathrm{cov}(f,g)$ is positive for any increasing symmetric functions $f(Y)$ and $g(Z)$, in particular for $f(Y) = Y^2$ and $g(Z) = Z^2$.

References

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[MM] Meckes, E., Meckes, M.: The central limit problem for random vectors with symmetries. Preprint, available at http://arxiv.org/abs/math.PR/0505618

Israel GAFA Seminar (2003)

(The following was mistakenly omitted from the previous volume)

Friday, January 10, 2003

1. *K. Ball* (London): Entropy growth in presence of a spectral gap (joint work with S. Artstein, F. Barthe and A. Naor)
2. *V. Milman* (Tel Aviv): Some recent mathematical news (counter example to Knaster conjecture by Kashin and Szarek and some other news)
3. *Y. Shalom* (Tel Aviv): Isometric actions on Hilbert spaces and applications

Israel GAFA Seminar (2004-2006)

Friday, April 30, 2004

1. *B. Klartag* (Tel Aviv): Geometry of log-concave functions and measures (joint work with V. Milman)
2. *H. Furstenberg* (Jerusalem): Eigenmeasures, multiplicity of β expansions, and a problem in equidistribution

Friday, May 21, 2004

1. *A. Naor* (Microsoft): Stochastic metric partitions and the Lipschitz extension problem (joint work with J.R. Lee)
2. *M. Rudelson* (Columbia, Missouri): Euclidean embeddings in spaces of finite volume ratio via random matrices (joint work with A.E. Litvak, A. Pajor, N. Tomczak-Jaegermann and R.Vershynin)

Friday, October 29, 2004

1. *N. Alon* (Tel Aviv): Quadratic forms on graphs (joint work with K. Makarychev, Y. and A. Naor)
2. *M. Sodin* (Tel Aviv): Sign and area in nodal geometry of Laplace–Beltrami eigenfunctions (joint work with F. Nazarov and L. Polterovich)

Friday, December 24, 2004

1. *G.M. Zaslavsky* (Courant Institute, NYU): Chaotic and pseudochaotic field lines
2. *A. Elgart* (Stanford): Localization for random Schroedinger operator

Friday, December 31, 2004

1. *A. Sodin* (Tel Aviv): Central limit theorem for convex bodies and large deviations
2. *S. Artstein* (Princeton): Two geometric applications of Chernoff inequality: A zigzag approximation for balls and random matrices (joint work with O. Friedland and V. Milman)

Friday, January 7, 2005

1. *A. Shapira* (Tel Aviv): Recent applications of Szemeredi's regularity lemma
2. *B. Klartag* (I.A.S. Princeton): On Dvoretzky's theorem and small ball probabilities
3. *A. Samorodnitsky* (Jerusalem): Hypergraph linearity tests for Boolean functions
4. *G. Kalai* (Jerusalem): Discrete isoperimetric inequalities

Friday, April 1, 2005

1. *Y. Ostrover* (Tel Aviv): On the extremality of Hofer's metric on the group of Hamiltonian diffeomorphisms (joint work with R. Wagner)
2. *A. Naor* (Microsoft): How to prove non-embeddability results?

Friday, June 3, 2005

1. *E. Ournycheva* (Jerusalem): Composite cosine transforms on Stiefel manifolds (joint work with B. Rubin)
2. *S. Artstein* (Princeton): A few remarks concerning reduction of diameter and Dvoretzky's theorem for special classes of operators

Friday, December 16, 2005

1. *S. Artstein* (Princeton University and IAS): Logarithmic reduction of randomness in some probabilistic geometric constructions (joint work with V. Milman)
2. *S. Mendelson* (Technion and ANU, Canberra): Controlling weakly bounded empirical processes

Friday, December 23, 2005

1. *M. Sodin* (Tel Aviv): Transportation to random zeroes by the gradient flow (joint work with F. Nazarov and A. Volberg)
2. *O. Schramm* (Microsoft): The Gaussian free field and its level lines (based on joint work with Scott Sheffield)

Friday, January 13, 2006

1. *B. Sudakov* (Princeton University and IAS): Embedding nearly-spanning bounded degree trees (joint work with N. Alon and M. Krivelevich)
2. *S. Bobkov* (Minneapolis): Large deviations over convex measures with heavy tails

Friday, January 27, 2006

1. *M. Krivelevich* (Tel Aviv): Sphere packing in R^n through graph theory (joint work with S. Litsyn and A. Vardy)
2. *A. Giannopoulos* (Athens): Random 0-1 polytopes (joint work with D. Gatzouras and N. Markoulakis)

Midrasha Mathematicae: Connection Between Probability and Geometric Functional Analysis

(Jerusalem, June 14-19, 2005)

(Organizers: A. Szankowski and G. Schechtman)

This summer school was composed of the following eight short courses:

1. *N. Alon* (Tel Aviv): Semidefinite programming and Grothendieck type inequalities
2. *B. Bollobas* (Cambridge and Memphis): Discrete and continuous percolation
3. *N. Kalton* (Columbia, Missouri): R-boundedness: an introduction
4. *R. Latala* (Warsaw): Inequalities for Gaussian measures
5. *M. Ledoux* (Toulouse): Small deviation inequalities for largest eigenvalues
6. *K. Oleszkiewicz* (Warsaw): Comparison of moments for the sums of random vectors
7. *S.J. Szarek* (Paris and Case Western): Random normed spaces: from Gluskin spaces to the saturation phenomenon
8. *M. Talagrand* (Paris): The generic chaining

Contemporary Ramifications of Banach Space Theory In Honor of Joram Lindenstrauss and Lior Tzafriri

(Jerusalem, June 20–24, 2005)

(Organizers and program committee: W.J. Johnson, H. Koenig, V. Milman, G. Schechtman, A. Szankowski, M. Zippin)

Monday, June 20

1. *V. Milman* (Tel Aviv): Analytic form of some geometric inequalities
2. *M. Rudelson* (Columbia, Missouri): Geometric approach to error correcting codes
3. *R. Schneider* (Freiburg): Projective Finsler spaces and zonoid geometry
4. *A. Koldobsky* (Columbia, Missouri): The geometry of L_0
5. *N.J. Nielsen* (Odense): Rosental operator spaces
6. *A. Zvavitch* (Kent, Ohio): The Busemann–Petty problem for arbitrary measures
7. *A. Lima* (Kristiansand): A weak metric approximation property
8. *B. Klartag* (IAS): Geometric inequalities for logarithmically concave functions
9. *E. Oja* (Tartu): The approximation property and its weak bounded version
10. *D. Yost* (Ballarat): Quasilinear mappings and polyhedra
11. *V. Lima* (Kristiansand): Ideals of operators and the weak metric approximation property
12. *O. Maleva* (London): Bi-Lipschitz invariance of the class of cone null sets
13. *C. Read* (Leeds): Hypergraphs, probability and the non-amenability of $B(l^1)$

Tuesday, June 21

1. *W.J. Johnson* (College Station, Texas): A survey of non-linear Banach space theory
2. *D. Preiss* (London): Differentiability of Lipschitz maps between Banach spaces
3. *N. Kalton* (Columbia, Missouri): Extending Lipschitz and linear maps into $C(K)$-spaces

Wednesday, June 22

1. *M. Talagrand* (Paris): Functional Analysis problems related to the Parisi theory
2. *S. Szarek* (Paris): Geometric questions related to quantum computing and quantum information theory
3. *G. Schechtman* (Rehovot): L_p Spaces, Large and Small
4. *K. Oleszkiewicz* (Warsaw): Invariance principle and noise stability for functions with low influences
5. *A. Pajor* (Marne-la-Vallee): Diameter of random sections and reconstruction
6. *B. Cascales* (Murcia): The Bourgain property and Birkhoff integrability
7. *G. Godefroy* (Paris): Smoothness and weakly compact generation: Joram Lindenstrauss' question 30 years later
8. *M. Girardi* (South Carolina): Martingale transforms by operator-valued predictable sequences
9. *S. Artstein* (Princeton): Some results regarding sign matrices
10. *S. Argyros* (Athens): Indecomposable and sequentially unconditional Banach spaces
11. *A. Hinrichs* (Jena): Optimal Weyl inequalities in Banach spaces
12. *C. Zanco* (Milano): Around Corson's theorem
13. *A. Giannopoulos* (Athens): Lower bound for the maximal number of facets of a 0/1 polytope
14. *J. Orihuela* (Murcia): Renormings of $C(K)$ spaces

Thursday, June 23

1. *E. Lindenstrauss* (Princeton): Eigenfunctions of the Laplacian on finite volume arithmetic manifolds
2. *A. Naor* (Microsoft): The Johnson–Lindenstrauss extension paper: 23 years later
3. *N. Tomczak-Jaegermann* (Edmonton, Alberta) Saturating normed spaces
4. *A. Pelczynski* (Warsaw): Structure of complemented subspaces of special Banach spaces
5. *A. Lindenstrauss* (Bloomington Indiana): Goodwillie calculus in algebraic topology
6. *M. Larsen* (Bloomington Indiana): Spectra of field automorphisms acting on elliptic curves
7. *A. Defant* (Oldenburg): A logarithmical lower bound for multidimensional Bohr radii
8. *I. Doust* (Sidney): The spectral type of sums of operators on non-Hilbertian Banach lattices

Friday, June 24 (Dead Sea, joint with the next conference)

1. *H. Koenig* (Kiel): Spherical design techniques in Banach spaces
2. *A. Naor* (Microsoft): Metric cotype and some of its applications
3. *J. Bourgain* (Institute for Advanced Study): Localization for the Anderson Bernoulli model and unique continuation

Asymptotic Geometric Analysis In Honor of Nicole Tomczak-Jaegermann

(Dead Sea, June 24-27, 2005)

(Organizers and scientific committee: J. Bourgain, E. Gluskin, Y. Gordon, P. Mankiewicz, V. Milman, E. Odell, G. Pisier)

Friday, June 24
Morning (joint with previous conference)

1. *H. Koenig* (Kiel): Spherical design techniques in Banach spaces
2. *A. Naor* (Microsoft): Metric cotype and some of its applications
3. *J. Bourgain* (Institute for Advanced Study): Localization for the Anderson Bernoulli model and unique continuation

Friday, June 24
Afternoon

4. *N. Tomczak-Jaegermann* (Edmonton), Random subspaces and quotients of finite-dimensional spaces
5. *B. Klartag* (Institute for Advanced Study): On John type ellipsoids
6. *S. Mendelson* (Canberra): Random projections and empirical processes

Sunday, June 26

1. *S.J. Szarek* (Paris): Entropy duality over the years
2. *A. Pajor* (Marne-la-Vallee): Geometry of random (-1,+1)-polytopes
3. *N. Kalton* (Columbia, Missouri): The complemented subspace problem revisited
4. *P. Mankiewicz* (Warsaw): Low dimensional sections versus projections (joint work with N. Tomczak-Jaegermann)
5. *R. Latala* (Warsaw): Estimates of moments and tails of Gaussian chaoses
6. *S. Alesker* (Tel Aviv): Quaternionic pluripotential theory and its applications in convexity
7. *K. Oleszkiewicz* (Warsaw): Small ball probability estimates in terms of width – on two conjectures of R. Vershynin (joint work with R. Latala)

Monday, June 27

1. *A. Litvak* (Edmonton): Diameters of sections and coverings of convex bodies (joint work with N. Tomczak-Jaegermann and A. Pajor)
2. *T. Schlumprecht* (College Station, Texas): A separable reflexive Banach space which contains all separable uniformly convex spaces

Asymptotic Theory of the Geometry of Finite Dimensional Spaces

(Erwin Schrödinger Institute, Vienna, July 10 - August 5, 2005)

(Organizers: V. Milman, A. Pajor, C. Schütt)

Educational Talks:

Tuesday, July 12

1. *P.M. Gruber* (Vienna): Principles of classical discrete geometry
2. *S. Artstein* (Princeton): Metric entropy and coverings-duality

Wednesday, July 13

1. *I. Barany* (Budapest and London): On the power of linear dependencies
2. *C. Buchta* (Salzburg): What is the number of vertices of the convex hull of N randomly chosen points?
3. *K. Böröczky* (Budapest): Stability of affine invariant geometric inequalities

Thursday, July 14

1. *B. Klartag* (Clay Institute): Diameters of sections of convex bodies
2. *A. Koldobsky* (Columbia, Missouri):
3. *F. Barthe* (Toulouse): Entropy of spherical marginals

Friday, July 15

1. *W. Weil* (Karlsruhe): Boolean models and convexity
2. *R. Schneider* (Freiburg): Simplices I

Monday, July 18

1. *R. Schneider* (Freiburg): Simplices II
2. *N. Tomczak-Jaegermann* (Edmonton): Decoupling weakly dependent events

Tuesday, July 19

1. *A. Giannopoulos* (Athens): Random $0-1$-polytopes
2. *G. Kalai* (Jerusalem): Fourier analysis of Boolean functions

Thursday, July 28

1. *L. Pastur* (Kharkov): A simple approach to the global regime of random matrix theory
2. *H. König* (Kiel): Spherical functions and Grothendieck's inequality
3. *M. Shcherbina* (Kharkov): Universality of local eigenvalue statistics for matrix models

Friday, July 29

1. *D. Cordero-Erausquin* (Marne-la-Vallee): L^2-methods for Prekopa's theorem
2. *L. Pastur* (Kharkov): A simple approach to the global regime of random matrix theory

Monday, August 1

1. *R. Latala* (Warsaw): On majorizing measures
2. *K. Oleszkiewicz* (Warsaw): Kwapien's theorem

Thursday, August 4

1. *A. Stancu* (Montreal): Floating bodies

First Annual Conference of the EU Network "Phenomena in High Dimensions" Conference on Convex Geometry and High Dimensional Phenomena

(Vienna, July 20-27, 2005)

(Scientific Committee: P. Gruber, M. Ludwig, V. Milman, M. Reitzner, C. Schuett)

Wednesday, July 20

1. *S. Alesker* (Tel Aviv): Theory of valuations on manifolds
2. *B. Klartag* (Tel Aviv): From isomorphic to almost-isometric problems in asymptotic convex geometry
3. *Z. Füredi* (Illinois and Budapest): Sets of few distances in highdimensional normed spaces
4. *Y. Gordon* (Haifa): Probabilistic min-max theorems revisited and applications to geometry
5. *R. Latala* (Warsaw): Moments and tail estimates for Gaussian chaoses
6. *H. Vogt* (Dresden): Central limit theorems in the W_2^k-norm for one-dimensional marginal distributions
7. *O. Guédon* (Paris): L_p moments of random vectors via majorizing measure
8. *J. Bastero* (Zaragoza): Upper estimates for the volume and the diameter of sections of symmetric convex bodies
9. *J. Bernués* (Zaragoza): Averages of k-dimensional marginal densities

Thursday, July 21

1. *L. Pastur* (Kharkov): Limiting laws of fluctuations of linear eigenvalue statistics of matrix models
2. *S. Szarek* (Paris and Cleveland): Tensor products of convex sets
3. *R. Vershynin* (Davis, California): Signal processing: geometric and probabilistic perspectives
4. *M. Shcherbina* (Kharkov): Double scaling limit for matrix models with non analytic potential
5. *P. Salani* (Firenze): A Brunn–Minkowski inequality for the Monge-Ampere eigenvalue
6. *M. Meckes* (Stanford): The central limit problem for random vectors with symmetries

7. *E. Meckes* (Stanford): Normal approximation under continuous symmetries
8. *A. Hinrichs* (Jena): Optimal geometric design of high dimensional cubature formulas
9. *K. Marton* (Budapest): Logarithmic Sobolev inequality for weakly dependent spin systems
10. *I. Ryshkova* (Kharkov): Nonlinear oscillation of a plate in a potential gas flow in the presence of thermal effects
11. *A.S. Shcherbina* (Kharkov): Solutions of dissipative Zakharov system

Friday, July 22

1. *I. Barany* (Budapest and London): Recent results on random polytopes
2. *R. Schneider* (Freiburg): Limit shapes in random mosaics and isoperimetric inequalities
3. *A. Koldobsky* (Columbia, MO): On the road from intersection bodies to polar projection bodies
4. *A. Giannopoulos* (Athens): Asymptotic formulas for proportional sections of convex bodies
5. *K. Boroczky, Jr.* (Budapest): Approximation of smooth convex bodies by circumscribed polytopes with respect to the surface area
6. *C. Peri* (Milano): Discrete tomography: Point X-rays of convex lattice sets
7. *G. Bianchi* (Firenze): The covariogramm of 2-, 3- and 4-dimensional convex polytopes
8. *M.A. Hernandez Cifre* (Murcia): The Steiner polynomial and a problem by Hadwiger
9. *K. Bezdek* (Calgary and Budapest): On the illumination parameters of smooth convex bodies
10. *B.V. Dekster* (New Brunswick): The total angle around a point in Mink-owski plane
11. *J.M. Aldaz* (Rioja): Behavior of the maximal function in high dimensions

Monday, July 25

1. *K. Ball* (London and Redmond): Markov type and the non-linear Maurey Extension Theorem
2. *A. Colesanti* (Firenze): A functional inequality related to the Rogers-Shephard inequality
3. *M. Fradelizi* (Marne-la-Vallee): On some functional forms of Santalo inequality
4. *G. Paouris* (Paris): Concentration of mass on the Schatten classes
5. *N. Markoulakis* (Heraklion): -1/1 polytopes with many facets
6. *F. Schuster* (Vienna): Geometric inequalities for rotation equivariant additive mappings

7. *P. Pivovarov* (Edmonton): A convex body lacking symmetric projections
8. *V. Yaskin* (Columbia, MO): The Busemann–Petty problem in hyperbolic and spherical spaces

Tuesday, July 26

1. *F. Barthe* (Toulouse): Functional approach to isoperimetry and concentration
2. *G. Pisier* (Paris): Similarity problems and amenability for groups and operator algebras
3. *J. Matousek* (Prague): Challenges of combinatorial linear programming
4. *G. Aubrun* (Paris and Athens): Sampling convex bodies: a random matrix approach
5. *V. Vengerovskyi* (Tel Aviv): Eigenvalue distribution of some ensembles of sparse random matrices
6. *T. Muller* (Budapest): The chromatic number of random geometric graphs
7. *E. Milman* (Rehovot): Using dual mixed-volumes to bound the isotropic constant
8. *G. Averkov* (Chemnitz and Firenze): Convex bodies with critical cross-section measures
9. *M. Naszodi* (Calgary): Ball-polytopes in Euclidean spaces

Wednesday, July 27

1. *Y. Brenier* (Nice): Optimal transportation of currents
2. *L. Lovasz* (Budapest and Redmond): Graph limits, Szemeredi's Regularity Lemma, and some Banach spaces
3. *M. Krivelevich* (Tel Aviv): Smoothed analysis in graphs and boolean formulas
4. *G. Kalai* (Jerusalem): Noise sensitivity and noise stability, some recent results
5. *A. Pajor* (Marne-la-Vallee): Reconstruction and subgaussian operators

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