Fast Simulation of Quasistatic Rod Deformations for VR Applications

J. Linn[1], T. Stephan[1], J. Carlsson[2], and R. Bohlin[2]

[1] Fraunhofer ITWM, Kaiserslautern
(linn@itwm.fraunhofer.de)
[2] Fraunhofer–Chalmers Research Center FCC, Göteborg
(www.fcc.chalmers.se)

Summary. We present a model of flexible rods – based on *Kirchhoff's* geometrically exact theory – which is suitable for the fast simulation of quasistatic deformations within VR or functional DMU applications. Unlike simple models of "*mass & spring*" type typically used in VR applications, our model provides a proper *coupling of bending and torsion.* The computational approach comprises a *variational formulation* combined with a *finite difference discretization* of the continuum model. Approximate solutions of the equilibrium equations for sequentially varying boundary conditions are obtained by means of *energy minimization* using a nonlinear CG method. The computational performance of our model proves to be sufficient for the interactive manipulation of flexible cables in assembly simulation.

1 Introduction

The handling of flexible objects in multibody simulation (MBS) models is both a long term research topic [1, 2] as well as an active area of current research within the MBS community [3–5]. A standard approach supported by most commercial software packages represents flexible bodies by means of vibrational modes (e.g. of *Craig–Bampton* type [6, 7]) computed by modal analysis within the framework of linear elasicity. The modal representation of a flexible structure usually yields a drastic reduction of the degrees of freedom and thereby provides a reduced model. However, such methods are suitable (as well as by definition restricted) to model forced oscillations effecting *small* deformations within a flexible structure.

If the flexible bodies of interest possess special geometrical properties characterising them as *slender* (or *thin*) structures (i.e. rods, plates or shells), their overall deformation in response to moderate external loads may become *large*, although locally the stresses and strains remain *small.* Therefore, models suitable to describe such *large deformations of slender structures* must be capable to account for *geometric nonlinearities.* Compared to object geometries that require fully three-dimensional volume modelling, the reduced dimensionality of rod or shell models is accompanied by a considerable reduction in the number of degrees of freedom, which makes the inclusion of appropriately discretised versions of the *full* models (in contrast to modally reduced ones) into a MBS framework [4] computationally feasible even for time critical simulation applications.

Modelling of Flexible Structures in VR Applications

The application aimed at within the framework of this article is the modelling of flexible cables or tubes (e.g. those externally attached to manufacturing robots) such that *quasistatic* deformations occurring during sufficiently slow motions of these cables can be simulated in *real time*. This capability is crucial for the seamless integration of a cable simulator module within VR (*virtual reality*) or FDMU (*functional digital mock up*) software packages used for *interactive simulation* (e.g. of assembly processes).

Although the dominant paradigm to assess the quality of an animation or simulation within these application areas – as well as related ones like computer games or movies – seems to be "*... It's good enough if it looks good ...*" [8], such that a mere "*fake*" [9] of structure deformation is considered to be acceptable (at least for those applications were "*... fooling of the eye ...*" [8] is the main issue), the need for "physics based" approaches increases constantly, and the usage of models that are more [12] or less [10,11,13] based on ideas borrowed from classical structural and rigid body mechanics is not uncommon, especially if the primary concern is not visual appearance but physical information (see e.g. [14]).

2 Cosserat and Kirchhoff Rod Models

In structural mechanics *slender objects* like cables, hoses, etc. are described by one-dimensional *beam* or *rod* models which utilise the fact that, due to the relative smallness of the linear dimension D of the cross-section compared to the length L of a rod, the local stresses and strains remain small and the cross sections are almost unwarped, even if the overall deformation of the rod relative to its undeformed state is large. This justifies kinematical assumptions that restrict the cross sections of the deformed rod to remain *plane and rigid*.

In the following we give brief introduction to rod models of *Cosserat* and *Kirchhoff* type, the latter being a special case of the former. We do not present the most general versions of these models, which are discussed at length in the standard references [16] and [18]. The approach we finally use as a basis for the derivation of a generalized "*mass & spring*" type model by finite difference discretisation is an extensible variant of Kirchhoff's original theory [15] as presented in [17] (see part II, 16–19) for a hyperelastic rod with symmetric cross-section subject to a constant gravitational body force.

2.1 Kinematics of Cosserat and Kirchhoff Rods

A *(special) Cosserat rod* [18] is a *framed curve* [21] formally defined as a mapping $s \mapsto (\boldsymbol{\varphi}(s), \hat{\mathbf{F}}(s))$ of the interval $\mathcal{I} = [0, L]$ into the configuration space $\mathbb{R}^3 \times \mathrm{SO}(3)$ of the rod, where L is the length of the undeformed rod. Its constituents are *(i)* a space curve $\boldsymbol{\varphi} : \mathcal{I} \to \mathbb{R}^3$ that coincides with the *line of centroids* piercing the cross sections along the deformed rod at their geometrical center, and *(ii)* an "curve of frames" $\hat{\mathbf{F}} : \mathcal{I} \to \mathrm{SO}(3)$ with the origin of each frame $\hat{\mathbf{F}}(s)$ attached to the point $\mathbf{x}_s = \boldsymbol{\varphi}(s)$. The matrix representation of the frame $\hat{\mathbf{F}}(s)$ w.r.t. a fixed global coordinate system $\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\}$ of $\mathbb{R}^3$ may be written as a triple of column vectors, i.e. $\hat{\mathbf{F}}(s) = \big(\mathbf{d}^{(1)}(s), \mathbf{d}^{(2)}(s), \mathbf{d}^{(3)}(s)\big)$, obtained as $\mathbf{d}^{(k)}(s) = \hat{\mathbf{F}}(s) \cdot \mathbf{e}^{(k)}$. By definition $\mathbf{d}^{(3)}$ coincides with the unit *cross section normal* vector located at $\boldsymbol{\varphi}(s)$.

For simplicity we assume the undeformed rod to be *straight and prismatic* such that its intial geometry relative to $\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\}$ is given by the direct product $\mathcal{A} \times \mathcal{I}$ with a constant cross section area $\mathcal{A}$ parallel to the plane spanned by $\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\}$. Introducing coordinates (ξ_1, ξ_2) in the plane of the cross section $\mathcal{A}$ relative to its geometrical center, we may parametrise the material points $\mathbf{X} \in \mathcal{A} \times \mathcal{I}$ of the undeformed rod geometry by $\mathbf{X}(\xi_1, \xi_2, s) = \sum_{k=1,2} \xi_k \mathbf{e}^{(k)} + s\, \mathbf{e}^{(3)}$, and the deformation mapping $\mathbf{X} \mapsto \mathbf{x} = \boldsymbol{\Phi}(\mathbf{X})$ is given by the formula $\mathbf{x}(\xi_1, \xi_2, s) = \boldsymbol{\varphi}(s) + \sum_{k=1,2} \xi_k \mathbf{d}^{(k)}(s)$. The kinematics of a framed curve as presented above determine the possible deformations of a *Cosserat* rod. These are *stretching* (in the direction of the curve tangent), *bending* (around an axis in the plane of the cross section), *twisting* (of the cross section around its normal) and *shearing* (i.e. tilting of the cross section normal w.r.t. the curve tangent).

Following Chouaieb and Maddocks [21] we denote a frame $\hat{\mathbf{F}}(s)$ as *adapted* to the curve $\boldsymbol{\varphi}(s)$ if $\mathbf{d}^{(3)}(s)$ coincides with the *unit tangent* vector $\mathbf{t}(s) = \partial_s \boldsymbol{\varphi}(s) / \|\partial_s \boldsymbol{\varphi}(s)\|$ along the curve. An adapted frame satisfies the *Euler–Bernoulli* hypothesis, which states that the cross sections remain always orthogonal to the centerline curve in a deformed state also. Curves with adapted frames describe the possible deformations of (extensible) *Kirchhoff* rods. Compared to Cosserat rods the kinematics of Kirchhoff rods are further restricted, as they do not allow for shear deformations. The *inextensibiliy* condition $\|\partial_s \boldsymbol{\varphi}\| = 1$ constitutes an additional kinematical restriction.

Measuring the slenderness of a rod of cross-section diameter D and length L in terms of the small parameter $\varepsilon = D/L$ and assuming (hyper)elastic material behaviour one may show that the potential energy terms corresponding to bending and torsion are of the order $\mathcal{O}(\varepsilon^4)$, while the energy terms corresponding to stretching and shearing scale are of the order $\mathcal{O}(\varepsilon^2)$. In this way the latter effectively act as *penalty terms* that enforce the kinematical restrictions $\|\partial_s \boldsymbol{\varphi}\| = 1$ and $\mathbf{d}^{(3)}(s) = \mathbf{t}(s)$. This explains how Kirchhoff rods appear as a natural limit case of Cosserat rods subject to moderate deformations provided ε is sufficiently small.

2.2 Hyperelastic Kirchhoff Rods with Circular Cross-Section

As we are interested in a rod model that is suitable for the simulation of moderate cable deformations, both the Cosserat as well as the Kirchhoff approach would fit for our purpose. A characteristic feature of the Cosserat model consists in the description of the bending and torsion of the rod in terms of the frame variables, while the bending of the centerline curve $\boldsymbol{\varphi}(s)$ is produced only indirectly via shearing forces that try to align the curve tangent to the cross section normal. In contrast to that, Kirchhoff's model [20] encodes bending strain directly by the curvature of $\boldsymbol{\varphi}(s)$ and therefore provides a direct pathway to mass & spring type models formulated in terms of (discrete) dof of the centerline.

Averaging the normal Piola–Kirchhoff tractions and corresponding torques over the cross-section surface of the deformed rod located at $\boldsymbol{\varphi}(s)$ yields *stress resultants* $\mathbf{f}(s)$ and *stress couples* $\mathbf{m}(s)$, i.e. resultant force and moment vectors per unit reference length [19]. If the rod is in a *static equilibrium* state, these vectors satisfy the differential balance equations of forces and moments

$$\partial_s \mathbf{f} + \mathbf{G} = \mathbf{0}\,, \quad \partial_s \mathbf{m} + \partial_s \boldsymbol{\varphi} \times \mathbf{f} = \mathbf{0}\,, \tag{1}$$

where $\mathbf{G}$ represents a (not necessarily constant) body force acting along the rod, and we assumed that no external moment is applied in between the rod boundaries.

In the case of an extensible Kirchhoff rod which in its undeformed state has the form of a straight cylinder with circular cross section, the assumption of a hyperelastic material behaviour yields the expression [17]

$$\mathbf{m}(s) = EI\,\mathbf{t}(s) \times \partial_s \mathbf{t}(s) + GJ\,\Omega_t\,\mathbf{t}(s) \tag{2}$$

for the stress couple, where E is Young's modulus, G is the shear modulus, I measures the geometrical moment of inertia of the cross section ($I = \frac{\pi}{4}R^4$ for a circular cross section of radius R) and $J = 2I$. The quantities EI and GJ determine the stiffness of the rod w.r.t. bending and torsion. The strain measure related to the bending moment is given by the vector

$$\mathbf{t} \times \partial_s \mathbf{t} = \frac{\partial_s \boldsymbol{\varphi} \times \partial_s^2 \boldsymbol{\varphi}}{\|\partial_s \boldsymbol{\varphi}\|^2} = \|\partial_s \boldsymbol{\varphi}\|\,\kappa\,\mathbf{b} \tag{3}$$

which is proportional to the *Frenet curvature* $\kappa(s)$ of the centerline and (if $\kappa > 0$) points in the direction of the binormal vector $\mathbf{b}(s)$. The strain measure related to the torsional moment is determined by the *twist*

$$\Omega_t(s) = \mathbf{t}(s) \cdot [\mathbf{d}(s) \times \partial_s \mathbf{d}(s)]\,, \tag{4}$$

where $\mathbf{d}(s)$ is any unit normal vector field to the centerline given as a fixed linear combination $\mathbf{d} = \cos(\alpha_0)\,\mathbf{d}^{(1)} + \sin(\alpha_0)\,\mathbf{d}^{(2)}$ of the frame vectors $\mathbf{d}^{(1)}(s)$ and $\mathbf{d}^{(2)}(s)$ for some constant angle α_0. Note that the special constitutive relation (2) implies that in equilibrium the twist Ω_t is constant.

As a Kirchhoff rod is (by definition) unshearable, only the *tangential* component of the stress resultant $\mathbf{f}(s)$ is constitutively determined by the *tension*

$$\mathbf{t}(s) \cdot \mathbf{f}(s) \;=:\; T(s) \;=\; EA\,(\|\partial_s \boldsymbol{\varphi}\| - 1) \tag{5}$$

related to the elongational strain $(\|\partial_s \boldsymbol{\varphi}\| - 1)$. The resistance of the rod w.r.t. stretching is determined by EA where $A = |\mathcal{A}|$ is the size of the cross-section area (in our case $A = \pi R^2$). The *shearing force* acting parallel to the cross section is given by $\mathbf{f}_{sh}(s) = \mathbf{f}(s) - T(s)\,\mathbf{t}(s)$. It is not related to any strain measure but has to be determined from the equilibrium equations a Lagrange parameter corresponding to the internal constraint $\mathbf{d}(s) \cdot \mathbf{t}(s) = 0$.

To determine the deformation of the rod in static (or likewise quasistatic) equilibrium one has to solve the combined system of the equations (1)–(5) for a suitable set of *boundary conditions*, e.g. like those discussed in [20]. (This issue will not be discussed here.) Equivalently, one may obtain the centerline $\boldsymbol{\varphi}(s)$ and the unit normal vector field $\mathbf{d}(s)$ that represents the adapted frame of the rod by *minimization of the potential energy*

$$W_{\mathrm{pot}}[\boldsymbol{\varphi}, \mathbf{d}] \;=\; \int_0^L w_{\mathrm{el}}(s)\,\mathrm{d}s \;-\; \int_0^L \mathbf{G}(s) \cdot \boldsymbol{\varphi}(s)\,\mathrm{d}s. \tag{6}$$

According to (2)–(5) the *elastic energy density* is a quadratic form in the various strain measures given by

$$w_{\mathrm{el}}(s) \;=\; \frac{EI}{2}\,(\mathbf{t} \times \partial_s \mathbf{t})^2 \;+\; \frac{GJ}{2}\,\Omega_t^2(s) \;+\; \frac{EA}{2}\,(\|\partial_s \boldsymbol{\varphi}\| - 1)^2 \tag{7}$$

and determines the *stored energy function* $W_{\mathrm{el}}[\boldsymbol{\varphi}, \mathbf{d}] = \int_0^L w_{\mathrm{el}}(s)\,\mathrm{d}s$ containing the internal part of W_{pot}. A specific choice of boundary conditions may be accounted for by modified expressions for $w_{\mathrm{el}}(0)$ and $w_{\mathrm{el}}(L)$, which are obtained from (7) by fixing combinations of the kinematical variables $\boldsymbol{\varphi}(s)$ and $\mathbf{d}(s)$ and their derivatives at prescribed values (as required by the b.c.) and substituting these into $w_{\mathrm{el}}(s)$.

3 Discrete Rod Models of "Mass & Spring" Type

The final step of our approach towards a model of flexible rods suitable for the fast computation of quasistatic rod deformations is the discretisation of the potential energy by applying standard (e.g. central) finite difference stencils to the elastic energy density (7) and corresponding quadrature rules (e.g. trapezoidal) to the energy integrals (6). (Boundary conditions are treated in the way described at the end of the previous section.)

This procedure results in a discrete model of an extensible Kirchhoff rod that has a similar structure like the simple "mass & spring" type models presented in [11] and [13]. However, as a benefit of the systematic derivation procedure on the basis of a proper continuum model, our discrete rod model is able to capture the rather subtle coupling of bending and torsion deformation.

We compute approximate solutions of the equilibrium equations for sequentially varying boundary conditions by a minimization of the discrete potential energy using a nonlinear CG method [22]. The computational efficiency of our approach is illustrated by the typical results shown in Fig. 1 above. As the calculation times are comparable to those mentioned in Gregoire and Schomer [13], we estimate that our model is suitable for the interactive manipulation of flexible cables in assembly simulation (as indicated by preliminary tests with a software package developped at FCC.)

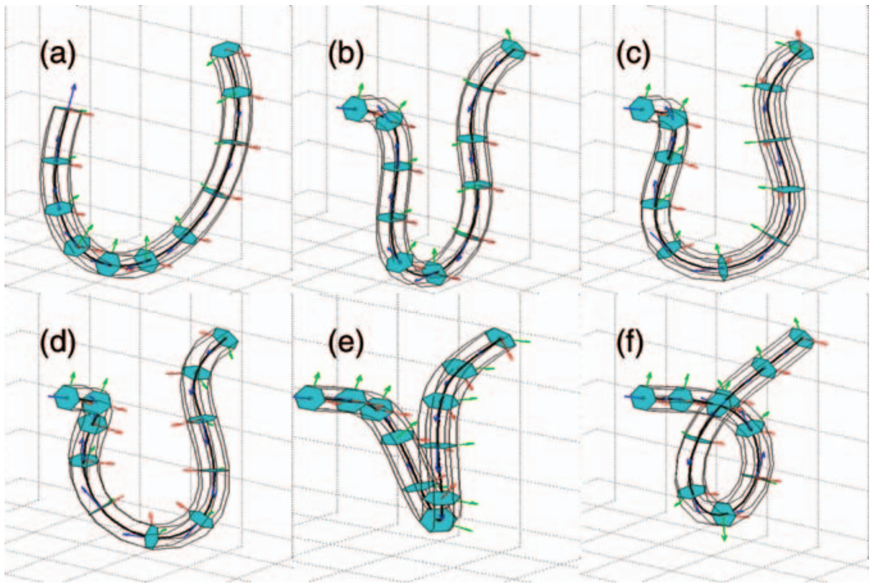

Fig. 1. Sequential deformation of a discrete, hyperelastic Kirchhoff rod of symmetric cross section: (**a**) Starting from a circle segment, the tangents of the boundary frames are bent inward to produce (**b**) an (upside down) Ω-shaped deformation of the rod at zero twist. To demonstrate the effect of mutual coupling of bending and torsion in the discrete model, the boundary frame at $s = L$ is twisted counterclockwise by an angle of 2π while the other boundary frame at $s = 0$ is held fixed. The pictures (**c**)–(**f**) show snapshots of the deformation state taken at multiples of $\pi/2$. The overall deformation from (**a**)–(**f**) was split up into a sequence of 25 consecutive changes of the boundary conditions defined by the terminal frames of the rod. For a discretization of the cable into 10 segments, the simulation took 150 ms on 1 CPU of an AMD 2.2 GHz double processor PC, which amounts to an average computation time of 6 ms per step

References

1. W.O. Schiehlen: Multibody system dynamics: Roots and perspectives, *Multibody System Dynamics* **1**, p. 149–188 (1997)
2. A.A. Shabana: Flexible multibody dynamics: Review of past and recent developments, *Multibody System Dynamics* **1**, p. 189–222 (1997)
3. B. Simeon: Numerical Analysis of Flexible Multibody Systems, *Multibody System Dynamics* **6**, p. 305–325 (2001)
4. P. Betsch: *Computational Methods for Flexible Multibody Dynamics*, Habilitationsschrift (2002)
5. A.A. Shabana: *Dynamics of Multibody Systems (Third edition)*, Cambridge University Press (2005)
6. R.R. Craig Jr. and M.C.C. Bampton: Coupling of Substructures for Dynamic Analysis, *AIAA Journal* Vol. **6**, No. 7, July 1968
7. R.R. Craig: *Structural Dynamics*, Wiley (1981)
8. D. Roble and T. Chan: Math in the Entertainment Industry, p. 971–990 in B. Engquist and W. Schmidt (ed.): *Mathematics Unlimited — 2001 and Beyond*, Springer (2001)
9. R. Barzel: Faking dynamics of ropes and springs, *IEEE Comput. Graph Appl.* **17**(3), p. 31–39 (1996)
10. D. Baraff and A. Witkin: Large steps in cloth simulation, p. 43–54 in *Proceedings of SIGGRAPH 98*, Computer Graphics Proceedings ed. by M. Cohen, Addison Wesley (1998)
11. A. Look and E. Schömer: A virtual environment for interactive assembly simulation: from rigid bodies to deformable cables, in *5th World Multiconference on Systemics, Cybernetics and Informatics (SCI'01)*, Vol. 3 (Virtual Engineering and Emergent Computing), p. 325–332 (2001)
12. D.K. Pai: Strands: Interactive simulation of thin solids using Cosserat models, *Comput. Graph. Forum* **21**(3), p. 347–352 (2002)
13. M. Gregoire and E. Schömer: Interactive simulation of one-dimensional felxible parts, *ACM Symposium on Solid and Physical Modeling (SPM'06)*, p. 95–103 (2006)
14. H. Baaser: Längenoptimierung von Bremsschläuchen und deren FE–Analyse auf Mikroebene, *VDI–Berichte* Nr. **1967**, p. 13–22 (2006)
15. G. Kirchhoff: Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes, *J. Reine Angew. Math. (Crelle)* **56**, p. 285–343 (1859)
16. A.E.H. Love: *A Treatise on the Mathematical Theory of Elasticity (4th edition)*, Cambridge University Press (1927), reprinted by Dover (1944)
17. L.D. Landau and E.M. Lifshitz: *Theory of Elasticity (Course of Theoretical Physics, Vol. 7, 3rd edition)*, Butterworth Heinemann (1986)
18. S. Antman: *Nonlinear Problems of Elasticity (2nd Edition)*, Springer (2005)
19. J.C. Simo: A finite strain beam formulation: the three dimensional dynamic problem – Part I, *Comp. Meth. Apl. Mech. Eng.* **49**, p. 55–70 (1985)
20. J.H. Maddocks: Stability of nonlinearly elastic rods, *Arch. Rational Mech. & Anal.* **85**, p. 311–354 (1984)
21. N. Chouaieb and J.H. Maddocks: Kirchhoff's problem of helical equilibria of uniform rods, *J. Elasticity* **77**, p. 221–247 (2004)
22. J.R. Shewchuck: An Introduction to the Conjugate Gradient Method Without the Agonizing Pain, `www.cs.berkeley.edu/jrs/` (1994)