# A Hierarchical Preconditioner within Edge Based BE-FE Coupling in Electromagnetism

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In this paper, a numerical algorithm solving large sparse linear systems that arise in electromagnetic field computation will be presented. It is based on hierarchical partitioning of the matrix and uses block-wise low-rank approximation in combination with element dropping in order to construct a preconditioner for iterative solution. Within the BE-FE coupling, this approximate factorisation is applied as preconditioner for the FE system. The treatment of multiply connected domains will also be described. The efficiency of the presented solver will be shown by means of an electromagnetic valve.

## 1 Introduction

In the design of electromagnetic components, numerical field computation of three-dimensional problems plays an important role. Efficient solver concepts are necessary to retrieve information about the components behaviour already at an early stage of development. The spatial discretisation is done by coupling of the boundary element method (BEM) and the finite element method (FEM) both based on edge elements. Fine discretisation of complex problems leads to large systems of equations. The BEM part is solved with asymptotically optimal complexity by using block-wise adaptive cross approximation (ACA) [2]. In larger problems, the main cost is then caused by the FEM part. In this paper, an efficient preconditioner for the large sparse FE matrix will be investigated.
The use of BE-FE coupling for complicated geometry can lead to multiply connected subdomains. Therefore, the discrete space approximating the boundary data needs to be extended in order to consider those degrees of freedom corresponding to the holes. This was already described in [10] for the Galerkin BEM. In this work, these degrees of freedom will be considered for the edge collocation method as described in [9, 11] (cf. Section 2). The discretisation then yields a regular non-symmetric system of equations consisting of sparse FE matrices and dense BE matrices, which is solved iteratively.
Due to the ill conditioning of the FE matrix, a preconditioner needs to be constructed. In [6], different strategies of hierarchical concepts solving the sparse FE system were presented. Especially, a non-recursive algorithm computing a preconditioner was developed that combines a block Cholesky decomposition with low-rank approximation and element dropping (Section 3). In this work, the efficiency of this preconditioner will be shown within the solution of the coupled BE-FE problem.
For this, a component of the fuel injection system is simulated using BE-FE coupling and considering multiply connected domains (Section 4). The FE stiffness matrix will be precon-

ditioned by the method described in Section 3 and two other preconditioning concepts. With this, an evaluation of the preconditioners will be carried out.

## 2 Discretisation and solver

Magnetostatic field problems in $\mathbb{R}^3$ can be described in terms of the magnetic vector potential $\mathbf{A}$ by

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} = \boldsymbol{\jmath}, \tag{1}$$

where $\boldsymbol{\jmath}$ is the electric current density. The material parameter $\mu$ describes the magnetic permeability and may depend on the magnetic field. It is assumed that $\operatorname{div} \boldsymbol{\jmath} = 0$. In order to use BE-FE coupling, the domain is decomposed into an inner domain $\Omega^-$ containing all conducting and magnetic materials of the component and an exterior infinite domain $\Omega^+ = \mathbb{R}^3 \backslash \Omega^-$ (cf. Figure 1). The FEM will be applied in $\Omega^-$ while the BEM is used in $\Omega^+$. At the coupling

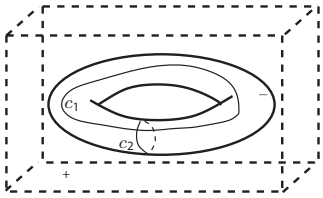


Fig. 1: Decomposition into $\Omega^+$ and $\Omega^-$ and two homology cycles $c_1$ and $c_2$ of the boundary

boundary $\Gamma$ of these two domains, the boundary data $\mathbf{A} \times \mathbf{n}$ and $\operatorname{curl} \mathbf{A} \times \mathbf{n}$ needs to be continuous. Here, $\mathbf{n}$ is the outer normal field on $\Gamma$.
The variational formulation of (1) is constructed in the Hilbert space $\mathbb{H}(\operatorname{curl}, \Omega^-)$ containing all square integrable functions with an existing $\operatorname{curl}$ in the weak sense. With $\mathbf{w} \in \mathbb{H}(\operatorname{curl}, \Omega^-)$ it reads

$$\int_{\Omega^-} \frac{1}{\mu} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \mathbf{w} \mathrm{d}x - \int_{\Gamma} \gamma_N \mathbf{A} \cdot \gamma_D \mathbf{w} \mathrm{d}S_x = \int_{\Omega^-} \boldsymbol{\jmath} \cdot \mathbf{w} \mathrm{d}x, \tag{2}$$

with the Dirichlet and Neumann trace operators $\gamma_D$ and $\gamma_N$. The respective trace spaces are $\mathbb{H}_{\perp}^{-1/2}(\operatorname{curl}_{\Gamma}, \Gamma)$ and $\mathbb{H}_{\parallel}^{-1/2}(\operatorname{div}_{\Gamma} 0, \Gamma)$. The trace operators as well as the surface curl and the surface divergence operators are defined in [4]. The boundary integral in (2) prepares the coupling to $\Omega^+$. With the help of the fundamental solution of the Laplace operator $\mathbf{A}^*(\mathbf{x}, \mathbf{y}) = \mathbf{I}/4\pi|\mathbf{x} - \mathbf{y}|$, one can derive a representation formula [9]. In case of $\boldsymbol{\jmath} = 0$ in $\Omega^+$, it only contains boundary integrals:

$$\begin{aligned} \mathbf{A}(\mathbf{y}) = &\int_{\Gamma} (\gamma_N \mathbf{A}^*(\mathbf{x}, \mathbf{y}))^T \gamma_D \mathbf{A}(\mathbf{x}) \mathrm{d}S_x - \int_{\Gamma} \gamma_D \mathbf{A}^*(\mathbf{x}, \mathbf{y}) \gamma_N \mathbf{A}(\mathbf{x}) \mathrm{d}S_x \\ &+ \int_{\Gamma} \gamma (\operatorname{div} \mathbf{A}^*(\mathbf{x}, \mathbf{y}))^T \gamma_n \mathbf{A}(\mathbf{x}) \mathrm{d}S_x. \end{aligned} \tag{3}$$

The last integral, where $\gamma$ is a standard and $\gamma_n$ a normal trace operator, corresponds to a gauge potential which will be eliminated by the discretisation with collocation over cycles [9].

Let $\Omega_h^-$ be a discretisation of $\Omega^-$ having $k_E$ edges and $k_N$ nodes. The respective boundary mesh $\Gamma_h$ has $k_E^\Gamma$ edges and $k_N^\Gamma$ nodes. Equation (2) and (3) will be discretised by the use of Whitney $p$-forms $\mathcal{W}^p$, $p = 0, 1$. For $p = 0$ they are formed by the continuous Lagrangian nodal elements, and for $p = 1$ they are tangentially continuous vector fields defined along edges [3]. With the Whitney 1-forms $\boldsymbol{\omega}_i \in \mathcal{W}^1(\Omega_h^-)$, the approximation of $\mathbf{A}$ reads $\mathbf{A}_h = \sum_{i=1}^{k_E} \beta_i \boldsymbol{\omega}_i$. The approximation of the Dirichlet data contains the degrees of freedom $\beta_i^\Gamma$ corresponding to the boundary, so that $(\gamma_D \mathbf{A})_h = \sum_{i=1}^{k_E^\Gamma} \beta_i^\Gamma \boldsymbol{\omega}_i^\Gamma$ with $\boldsymbol{\omega}_i^\Gamma \in \mathcal{W}^1(\Gamma_h)$. As explained in [9], the Neumann data has a zero surface divergence $\mathrm{div}_\Gamma \gamma_N \mathbf{A} = 0$. In order to exploit this, the discrete kernel space

$$\ker(\mathrm{div}_\Gamma) = \mathbf{curl}_\Gamma \mathcal{W}^0(\Gamma_h) \cup \mathcal{K}^1(\Gamma_h) \subset \mathcal{W}^1(\Gamma_h)$$

is used to discretise $\gamma_N \mathbf{A}$. The operator $\mathbf{curl}_\Gamma$ acts on scalar functions and is the adjoint operator of $\mathrm{curl}_\Gamma$ [4]. Here, $\mathcal{K}^1(\Gamma_h)$ is a finite-dimensional space discretising cohomology fields due to the holes in the domain. The Betti number $b = \dim(\mathcal{K}^1(\Gamma_h))$ denotes its dimension, which is given by twice the number of holes of $\Omega_h^-$. The discrete representation of the cohomology group was described in [10]. Therefore, $b$ representative cycles on $\Gamma_h$ surrounding the holes need to be constructed (cf. $c_1$ and $c_2$ in Figure 1). The discrete space $\mathcal{K}^1(\Gamma_h)$ is spanned with the help of scalar functions $\psi_k$ being piecewise linear and continuous on $\Gamma_h$ except for a jump $[\psi_k]_{c_k} = 1$ across the corresponding homology cycle $c_k$. With $\widetilde{\mathbf{curl}}_\Gamma$ being the surface curl on $\Gamma \backslash c_k$, $\eta_k = \widetilde{\mathbf{curl}}_\Gamma \psi_k$, $k = 1, \ldots, b$ yields a basis of $\mathcal{K}^1(\Gamma_h)$. The discrete Neumann data reads

$$(\gamma_N \mathbf{A})_h = \sum_{i=1}^{k_N^\Gamma} \varphi_i \mathbf{curl}_\Gamma \lambda_i + \sum_{k=1}^{b} \varphi_{c_k} \sum_{(j,m) \in S_k} \mathbf{curl}_\Gamma \lambda_{j,m},$$

with $\lambda_i \in \mathcal{W}^0(\Gamma_h)$ and $\lambda_{j,m}$ being a restriction of $\lambda_j$ on the $m$-th element. The pair $(j, m)$ belongs to the index set $S_k$ if the $m$-th element lies on one side of the oriented cycle $c_k$ and the $j$-th node is contained in $c_k$.

The DeRham collocation method for boundary integral equations was considered in [9] for trivial domains. The evaluation of the discretised representation formula along $k_N^\Gamma$ closed cycles yields a fully populated unsymmetric system of equations. For multiply connected domains, $b$ additional cycles given by the homology paths of $\Gamma_h$ are used for the collocation. With this, $k_N^\Gamma + b$ linearly independent equations form the BE system. This discretisation of (2) and (3) yields the system

$$\begin{pmatrix} Q_{\Omega\Omega} & Q_{\Omega\Gamma} & 0 \\ Q_{\Gamma\Omega} & Q_{\Gamma\Gamma} & T \\ 0 & H & G \end{pmatrix} \begin{pmatrix} \beta^\Omega \\ \beta^\Gamma \\ \varphi \end{pmatrix} = \begin{pmatrix} g_\Omega \\ g_\Gamma \\ 0 \end{pmatrix} \tag{4}$$

with $k_E + k_N^\Gamma + b$ unknowns and equations.

The vector potential ansatz is unique up to gradient and cohomology fields. Because of this ambiguity, $Q$ is singular having a large kernel. The regularisation of the system by constructing a matrix of representative vectors spanning the kernel due to gradient fields is applied, so that $K = Q + UU^T$, $U \in \mathbb{R}^{k_E \times k_N}$. The kernel matrix contains discrete gradient fields given by the incidence matrix between edges and nodes of the mesh [3]. The kernel properties of the block matrices stated in [11] allow the regularisation of the system (4). The kernel due to the cohomology fields is eliminated by consideration of the cohomology in the BEM and therefore, the BE-FE system matrix is regular.

This regularised unsymmetric system is solved iteratively by the GMRES method. The BE matrices $H$ and $G$ are well conditioned, so that no preconditioning is required. In Section 3, the construction of a preconditioner for the FE stiffness matrix $K \in \mathbb{R}^{k_E \times k_E}$ will be described.

## 3 The hierarchical preconditioner

A recent development of numerical linear algebra is the application of hierarchical matrices ($\mathcal{H}$-matrices) to dense matrices arising from integral equations. $\mathcal{H}$-matrices are based on a geometrical clustering of the degrees of freedom so that the matrix can be partitioned into smaller blocks $A \in \mathbb{R}^{n\times m}$ where low-rank approximation $A = UV^T$ with $U \in \mathbb{R}^{n\times r}$ and $V \in \mathbb{R}^{m\times r}$, $r \ll n$ can be applied. In the context of sparse matrices, the idea of hierarchical approximation can be reused in order to approximate the much more populated matrix of the Cholesky decomposition. This was already done for general elliptic differential equations [1]. In [6], we investigated the $\mathcal{H}$-matrix based Cholesky decomposition under consideration of memory reducing clustering. The more promising method was given by a non-recursive way of an approximate decomposition, called HSILLT. It is based on low-rank approximation in combination with element dropping . For this algorithm, we will show the performance in Section 4 so that it is briefly explained.
By a hierarchical interface clustering of the degrees of freedom, a permuation is computed which reorders the system matrix $K \in \mathbb{R}^{k_E\times k_E}$ so that a block structure as shown in Figure 2 arises. The idea stems from a reordering strategy reducing the memory requirement of the Cholesky decomposition called nested-dissection [5]. This permutation is computed with the help of the geometry information corresponding to the degrees of freedom. The clustering algorithm consists of recursive repeats of the two steps: 1. Geometrical bisection, 2. Construction of the interface cluster.

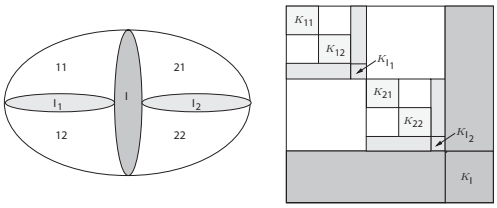


Fig. 2: Hierarchical interface clustering, geometry (left), matrix partitioning (right)

With the application of the clustering, the matrix can be organised in block rows as follows:

$$\left(\begin{array}{c|ccc} K_{11} & \multicolumn{3}{c}{K_{*1}^T} \\ \hline K_{*1} & \ddots & \multicolumn{2}{c}{\cdots} \\ \hline & \vdots & K_{(i-1)(i-1)} & K_{*(i-1)}^T \\ & & K_{*(i-1)} & \begin{array}{c|c} K_{ii} & K_{*i}^T \\ \hline K_{*i} & \ddots \end{array} \end{array}\right).$$

The Cholesky decomposition works block column wise with two accuracies $\varepsilon_{\mathrm{drop}}$ and $\varepsilon_{\mathrm{appr}}$. The first one controls the zero bound of small sub-diagonal block rows and the second one is the bound for low-rank approximation of the Schur complement.
The exact decomposition of the first block column would yield

$$\begin{pmatrix} K_{11} & K_{*1}^T \\ K_{*1} & K \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{*1} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & K - L_{*1}L_{*1}^T \end{pmatrix} \begin{pmatrix} L_{11}^T & L_{*1}^T \\ 0 & I \end{pmatrix},$$

where $K_{11} \in \mathbb{R}^{k_1\times k_1}$ and $K_{*1} \in \mathbb{R}^{(n-k_1)\times k_1}$. Here, $K_{11} = L_{11}L_{11}^T$ and $L_{*1}^T = L_{11}^{-1}K_{*1}^T$. The low-rank approximation of the matrix $L_{*1}^T$ is given by a reduced QR-decomposition with

accuracy $\varepsilon_{\mathrm{appr}}$ so that $L_{*1}^T \approx U_1 V_1$. Here, $U_1 \in \mathbb{R}^{k_1 \times r}$ contains $r$ orthonormal columns, i.e. $U_1^T U_1 = I \in \mathbb{R}^{r \times r}$ and $V_1 \in \mathbb{R}^{r \times (n-k_1)}$. This yields

$$\begin{pmatrix} K_{11} & K_{*1}^T \\ K_{*1} & K \end{pmatrix} \approx \begin{pmatrix} L_{11} & 0 \\ V_1^T U_1^T & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & K - V_1^T V_1 \end{pmatrix} \begin{pmatrix} L_{11}^T & U_1 V_1 \\ 0 & I \end{pmatrix}.$$

Since the block $K_{*1}^T$ is column sparse with only $k_2$ non-zero columns, a column sparse structure of $V_1$ results respectively. Thus, only the non-zero columns of this matrix will be stored as a fully populated matrix together with an additional information about the indices of the non-zero columns.
The computation of the Schur complement $K - V_1^T V_1$ will be postponed until the decomposition of the $i$th column is done. There the additional fill-in appears, and it is important to explain how we deal with it.
In the $i$th elimination step, there are $(i-1)$ matrices $V_1, \ldots, V_{i-1}$ which must be used to compute the Schur complement updates of the matrices $K_{ii}$ and $K_{*i}^T$. Thus, some additional non-zero columns will arise in $K_{*i}^T$. Here, the second accuracy $\varepsilon_{\mathrm{drop}}$ is used. It allows this fill-in only if the norm of the additional column is larger than the norm of the diagonal block times $\varepsilon_{\mathrm{drop}}$.

**Algorithm 1** HSILLT

1: **for all** block columns $i$ **do**
2: Compute the updates for the matrices $K_{ii}$ and $K_{*i}$ arising from the previous Schur complements,
3: Compute the Cholesky decomposition of the diagonal block $K_{ii} = L_{ii} L_{ii}^T$,
4: Compute the sub-diagonal block
$k_2 < k_1 : L_{*i} = K_{*i} L_{ii}^{-T}$,
$k_2 \geq k_1 : with\ K_{*i} \approx \widetilde{V}_i \widetilde{U}_i,\ \widetilde{V}_i^T \widetilde{V}_i = I,\ compute\ L_{*i} = \widetilde{V}_i (\widetilde{U}_i L_{ii}^{-T})$,
5: $k_2 < k_1$: Compute a low-rank approximation of $L_{*i} \approx V_i U_i,\ U_i^T U_i = I$,
$k_2 \geq k_1$: Do postcompression for $\widetilde{V}_i (\widetilde{U}_i L_{ii}^{-T}) \approx V_i U_i,\ U_i^T U_i = I$.
6: **end for**

This algorithm has a memory complexity of $\mathcal{O}(rn \log_2 n)$ and the number of operations is $\mathcal{O}(r^3 n \log_2^2 n)$ [6].

## 4 Numerical example

In order to evaluate the presented method by an industrial application, a magnetic valve as an essential component of a fuel injection system is simulated by BE-FE coupling. It consists of a ringshaped coil to carry the exciting current, a core and a yoke as well as a moving armature. The discretisation of the three-dimensional domain is performed with the help of tetrahedral and prismal edge elements. Due to the toroidal geometry, its discretisation has one hole along the $z$-axis of the mesh (cf. Figure 3). Therefore, two additional degrees of freedom are added in order to construct the correct discretisation space for the Neumann data as explained in section 2. The domain has 4000 boundary elements and 31000 finite elements. The excitation is given by a current of $10A$ and the coil has 100 windings. We assume a non-linear material with an approximated magnetisation curve. Because of this, the Newton-Raphson method is applied. In every Newton step, the resulting linear system of equations is solved by the GMRES method.
For increasing problem size, the FE matrix $K$ is getting more ill-conditioned. Therefore, a preconditioner is constructed by HSILLT (cf. Section 3). For the comparison, two other solver

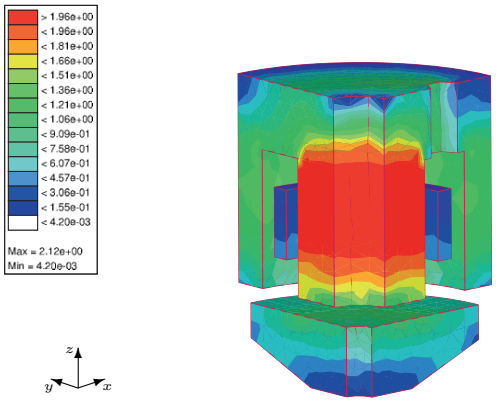


Fig. 3: A quarter of the valve geometry, where only material components and the coil are shown. The colour scale indicates the magnitude of the magnetic induction.

| | GMRES-Iter | $Mem_L$ [MB] | $t_{LL^T}$ [min] | $t_{\text{tot}}$ [min] |
|---|---|---|---|---|
| HSILLT | 1255 | 232 | 9 | 34 |
| Kaporin | 1305 | 227 | 130 | 171 |
| Taucs | 1287 | 258 | 36 | 64 |

Table 1: Comparison of different preconditioner concepts for the magnetic valve.

concepts are also applied: an incomplete factorisation method by Kaporin [7] and one given by the Taucs library [12]. The preconditioner HSILLT uses the interface clustering for reordering the matrix. In the other cases, we choose a reordering also based on a nested-dissection strategy given by the Metis library [8]. The accuracies of the preconditioners are chosen in order to get similar convergence of the iterative method.

With all three preconditioners, the solution as shown in Figure 3 was computed. The magnitude of the magnetic induction in the material domain is shown in a quarter of the valve. The maximum field can be found in the centre of the core and its value is 2.1 Tesla.

In Table 1, the performance of the preconditioners is compared. The number of Newton iteration steps is 10 for all three preconditioners in order to reach a Newton residual of $10^{-9}$. In every Newton step the GMRES method terminates with a residual of $10^{-12}$. The complete number of required GMRES steps is stated in the first column. The memory required for storing the incomplete Cholesky factor $L$ is given by $Mem_L$, and the required time for its computation is $t_{LL^T}$. Moreover, the total computation time $t_{\text{tot}}$ can be seen.

The factorisation time of HSILLT is four times faster than that of Taucs. Taucs computes a decomposition column-wise by dropping elements via an accuracy criterion, whereas HSILLT uses a low-rank approximation of sub-diagonal blocks. The block structure of those sub-diagonal entries allows us to use fast level-3-blas matrix operations (cf. Algorithm 1). The Kaporin method is said to yield high quality preconditioners. However, it doesn't use a block structure and requires a lot of index searches, so that it operates very slowly. A comparison for more complicated materials should be done in the future.

## 5 Conclusions

In this paper, we have presented a method to compute magnetic field problems on multiply connected domains with edge based BE-FE coupling and the time efficient preconditioner HSILLT. The efficiency of HSILLT was succesfully demonstrated by means of the numerical example of the fuel injection system.

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