Positive Real Balancing for Nonlinear Systems

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Summary. We extend the positive real balancing procedure for passive linear systems to the nonlinear systems case. We show that, just like in the linear case, model reduction based on this technique preserves passivity.

Keywords—positive real, passive, energy functions, Hamilton-Jacobi equations, nonlinear balancing, truncation.

1 Introduction

Positive Real Balancing for linear systems is an attractive tool for passivity preserving model reduction [Ant05]. The method deals with the class of passive linear systems. It combines the useful properties of the balancing technique with the passivity theory. The latter provides a particular pair of energy functions to be balanced. The balanced form of the energy functions reveal the positive real singular values. They measure the energetic importance of the states. The less important states are omitted to obtain a reduced order system. If the full order system were passive then the reduced model would be passive too [Ant05].
The idea in this paper is to extend this method to the case of passive nonlinear systems. It is motivated by the wide range of applications such as power systems stability analysis and controller design, see e.g. [Giu05]. We use the nonlinear balancing method developed in [Sch93, Sch94] in combination with the passivity theory in [Wil72, vdS00]. In this case, the positive real singular values are nonlinear positive functions of the state, having the same significance as in the linear case, i.e. measure the energetic importance of the states.
In Section 2, a brief overview of the passivity and positive realness properties is given and the energy functions, the available storage and the required supply, will be defined. Section 3 shortly reviews the positive real balancing procedure for linear systems and the properties of the reduced model. Section 4 presents the energy functions as the solutions of a Hamilton Jacobi equation. Section 5 is an adaptation of the nonlinear balancing procedure to the positive real systems case. We define the positive real singular value functions. The outcome of it is used in Section 6, where the truncation itself is done and the reduced system will pe proved to be passive. Some conclusions and future work make up Section 7.
The nonlinear systems we treat are given in the state space representation as:

$$x = f(x) + g(x)u,\ y = h(x) + d(x)u, \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$, with $m = p$. x is called the state vector, u is the input and y is the output of the system. f, g, h are smooth nonlinear vectorfields depending on the state vector x. n is called the dimension of system (1). The input u will be considered to have finite energy, i.e. $u \in L_2(\mathbb{R}^p)$.

2 Passivity, Energy Functions and Positive Realness

In this section, we give a brief overview on the dissipativity theory as in [Wil72, Wil71, vdS00]. A function $w : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ will be called the supply rate. The dissipativity property is defined with respect to the supply rate w.

Definition 1 *[Wil72, vdS00] A system (1) is called dissipative with respect to the supply rate $w(u, y)$, if there exists a storage function $S : \mathbb{R}^n \to \mathbb{R}$, with the following properties:*

1. $S(x) \geq 0$
2. $S(x_0) + \int_{t_0}^{t_1} w(u, y)dt \geq S(x_1)$,

where $x_0 = x(t_0)$ and $x_1 = x(t_1)$. A particular case is when the supply rate represents the energy supplied at the terminals of the system, that is $w(u, y) = u^T y$. In this case the system is called passive. □

Remark 2 *If the inequality is strict, we will call the system strictly passive, that is the internal energy of the system is decreasing even when supplied at the terminals. In case of equality the system is called lossless. It means that the internal energy of the system is not changing. Property 2. can also be written in a differential form as:*

$$\frac{\partial S(x)}{\partial x}(f(x) + g(x)u) \leq u^T h(x) + u^T d(x)u \tag{2}$$

□

For our purpose, from the set of storage functions satisfying the definition or (2), two particular types of storage functions are of interest: the available storage and the required supply.

Definition 3 *[Wil72, vdS00] The available storage function of a system (1) is the energy function:*

$$S_a(x_0) = -\min_u \int_0^\infty u^T y \, dt, \; x(0) = x_0, \; x(\infty) = 0 \tag{3}$$

The required supply function of system (1) is the energy function:

$$S_r(x_0) = \min_u \int_{-\infty}^0 u^T y \, dt, \; x(0) = x_0, \; x(-\infty) = 0 \tag{4}$$

□

$S_a(x)$ represents the maximal amount of energy that can be extracted from the terminals of the system when starting at the initial state x_0. $S_r(x)$ represents the minimal amount of energy required to be supplied to the system in order to reach x_0 from the equilibrium.
The property of the system being reachable from x_0 is a condition for the existence and non-negativity of the energy functions defined above.

Lemma 4 *[Wil71] Let system (1) be passive as in Definition 1 and reachable from the state x_0. Then, the energy functions S_a and S_r as in Definition 3 exist and are nonnegative. Moreover, $S_a \leq S_r$.* □

Definition 5 *[BIW91] A system (1) is called positive real if, for all $u \in L_2(R^p)$,*

$$\int_0^t u(\tau)^T y(\tau) d\tau \geq 0. \tag{5}$$

□

Combined with Lemma 4, we obtain:

Proposition 6 *[BIW91] A passive system (1) is positive real. Conversely, a positive real system (1), that is reachable from the state x_0, is passive.* □

Remark 7 *If the inequality is strict, the system is strictly positive real.* □

3 Linear Systems Case

A linear system is a particular case of system (1), given as: $\dot{x} = Ax + Bu, \; y = Cx + Du$, where A, B, C, D are constant matrices of appropriate dimensions. The system is assumed to be reachable and observable (minimal) and $R = D + D^T > 0$. Then, strict positive realness, can be studied with the Kalman-Yakubovitch-Popov lemma, see e.g. [Ant05]. The energy functions are quadratic and related to a pair of matrices called the positive real Gramians of the system.

Theorem 8 *[Wil72] Assume that the linear system is strictly passive. Then $S_a(x) = \frac{1}{2}x^T K_{min} x$ and $S_r(x) = \frac{1}{2}x^T K_{max} x$, where K_{min} and K_{max} are the minimal, respectively maximal solution of the Positive Real Algebraic Riccati equation:*

$$KA + A^T K + (KB - C^T)R^{-1}(B^T K - C) = 0 \tag{6}$$

□

Definition 9 *[Ant05] A positive real linear system is called positive real balanced if $K_{min} = (K_{max})^{-1} = \mathrm{diag}(\pi_1 I_{s_1}, \pi_2 I_{s_2}, ..., \pi_q I_{s_q})$, where $1 \geq \pi_1 > \pi_2 > ... > \pi_q > 0$, $s_1 + s_2 + ...s_q = n$.* □

The positive real singular value π_k, $k = 1, ..., q$ represents the energetic measure of the state components $x_{s_1+...+s_{k-1}+1}, ..., x_{s_1+...s_k}$. If π_l is much larger than π_{l+1}, then the state vector can be truncated from $w = s_1 + ... + s_l + 1$ to n, i.e. $x_{s_1+...+s_l+1} = 0, ..., x_n = 0$. A reduced model of dimension $\hat{n} = s_1 + ... + s_l < n$ is obtained. Then:

Theorem 10 *Let the passive linear system be brought into the positive real balanced form (A_b, B_b, C_b, D_b). The reduced system obtained after truncation with dimension l, i.e. dim $\hat{x} = \hat{n}$, is minimal and passive.* □

4 Nonlinear Systems Case

In this section we consider a system (1), under the following assumptions:

1. 0 is an equilibrium point of the system and $h(0) = 0$;
2. the system is strictly positive real, i.e. $r(x) = d(x) + d^T(x) > 0$, and reachable from x_0;
3. $x \in Y$, where Y is a neghbourhood of 0.

Assumption 1 is made for the sake of simplicity, but generality is not lost. Assumption 2 is in accordance with the nonlinear version of the Kalman-Yakubovitch-Popov lemma which characterizes the property of (strict) positive realness [Moy74, HilMoy76]. We mention that the smoothness assumed in the definition of system (1) guarantees the existence of solutions to be introduced. This condition could be relaxed, but it is kept for convenience.

Denote by $||v||_M^2 = v^T M v$, $(\forall) v \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}$.

The energy functions are computed as the stabilizing and antistabilizing solution, respectively, of a Hamilton-Jacobi equation, which is the nonlinear generalization of the Positive Real Algebraic Riccati equation, (6) from the previous section.

Theorem 11 *Let system (1) be, satisfying Assumptions 1-3. Then the Hamilton-Jacobi equation:*

$$\frac{\partial S(x)}{\partial x} f(x) + \frac{1}{2}\left(\frac{\partial S(x)}{\partial x} g(x) - h^T(x)\right) r^{-1}(x) \left(g^T(x)\frac{\partial S^T(x)}{\partial x} - h(x)\right) = 0 \tag{7}$$

has the smooth solution $S_a(x), S_a(0) = 0$, such that

$$f(x) + g(x) r^{-1}(x)\left(g^T(x)\frac{\partial S_a{}^T}{\partial x} - h(x)\right) \tag{8}$$

is asymptotically stable and the smooth solution $S_r(x), S_r(0) = 0$, such that

$$-\left(f(x) + g(x) r^{-1}(x)\left(g^T(x)\frac{\partial S_r{}^T}{\partial x} - h(x)\right)\right) \tag{9}$$

is asymptotically stable. □

Proof: Because system (1) is passive and reachable, according to Lemma 4, $S_a(x(t))$ and $S_r(x(t))$ exist and are nonnegative. We develop the proof for $S_r(x)$. The sequel follows the idea in Scherpen [Sch94], Section 3, Theorem 3.1.3. By definition, $S_r(x) = \min_{u, x(-\infty)=0}$

$\int_{-\infty}^{t} u^T(s)y(s)ds$. Because $S_r(x)$ exists, there exists an optimal input u^*, i.e. $S_r(x(t)) = \int_{-\infty}^{t} u^{*T}(s)y^*(s)ds$, where $y^*(s)$ is the output of the system with the input u^*. Differentiating $S_r(x(t))$ with respect to time we get:

$$\dot{S}_r(x(t)) = {u^*}^T y^* \Rightarrow \frac{\partial S(x)}{\partial x}(f(x) + g(x)u^*) - u^{*T}y^* = 0. \tag{10}$$

On the other hand, using completion of squares and (7), we have that

$$u^T y - \frac{1}{2}\left|\left|u - r^{-1}\left(g^T\frac{\partial S^T}{\partial x} - h\right)\right|\right|_r^2 = \frac{\partial S_r}{\partial x}gu - \frac{1}{2}\left(\frac{\partial S_r}{\partial x}g - h^T\right)r^{-1}\left(g^T\frac{\partial S_r^T}{\partial x} - h\right)$$
$$= \frac{\partial S_r}{\partial x}(f + gu) = \dot{S}_r. \tag{11}$$

Relation (10), can be written as $\frac{\partial S_r}{\partial x}f + \frac{\partial S_r}{\partial x}gu^* - {u^*}^T y^* = 0$. Relations (11), (7) give: ${u^*}^T y^* - \frac{\partial S_r^T}{\partial x}gu^* = -\frac{1}{2}\left(\frac{\partial S_r}{\partial x}g - h^T\right)r^{-1}\left(g^T\frac{\partial S_r^T}{\partial x} - h\right) + \frac{1}{2}\left|\left|u - r^{-1}\left(g^T\frac{\partial S^T}{\partial x} - h\right)\right|\right|_r^2$

$$\Rightarrow \frac{\partial S_r}{\partial x}f + \frac{1}{2}\left(\frac{\partial S_r}{\partial x}g - h^T\right)r^{-1}\left(g^T\frac{\partial S_r^T}{\partial x} - h\right) - \frac{1}{2}\left|\left|u^* - r^{-1}\left(g^T\frac{\partial S_r^T}{\partial x} - h\right)\right|\right|_r^2 = 0 \tag{12}$$

Now we show that $u^* = r^{-1}\left(g^T\frac{\partial S^T}{\partial x} - h\right)$. Let u be any continuous admissible control that steers the state from $x(t)$ to $x(-\infty) = 0$ (as the system is considered reachable). Let

$$\hat{u}(\bar{t}) = \begin{cases} u(\bar{t}), & t - \delta \le \bar{t} \le t \\ u^*(\bar{t}), & -\infty \le \bar{t} \le t - \delta \end{cases}$$

Denoting by $\hat{y}(s)$ the output of system (1) with input $\hat{u}(s)$ and by $J(\hat{u}) = \int_{-\infty}^{t}\hat{u}^T(s)\hat{y}(s)ds$, we have: $J(\hat{u}) = \int_{-\infty}^{t-\delta} u^{*T}(\bar{t})y^*(\bar{t})d\bar{t} + \int_{t-\delta}^{t}\hat{u}^T(\bar{t})\hat{y}(\bar{t})d\bar{t} = S_r(x(t-\delta)) + \int_{t-\delta}^{t}\hat{u}^T(\bar{t})\hat{y}(\bar{t})d\bar{t}$. The integral can be approximated as follows: $\int_{t-\delta}^{t}\hat{u}^T(\bar{t})\hat{y}(\bar{t})d\bar{t} = \delta\hat{u}^T(t)\hat{y}(t) + o(\delta)$, where $o(\delta)/\delta \to 0$, as $\delta \to 0$. By the smoothness of $S_r(x)$ we have that: $S_r(x(t)) = S_r(x(t-\delta)) + \delta\frac{dS_r(x(t))}{dt} + o(\delta) = S_r(x(t-\delta)) + \delta\frac{\partial S_r(x)}{\partial x}(f(x) + g(x)u) + o(\delta)$.
At the same time we know that $S_r(x) \le J(\hat{u})$ which leads to: $\frac{\partial S_r(x)}{\partial x}(f(x)+g(x)u) - u^T y \le 0$. Using relation (12) we conclude that $\frac{\partial S_r}{\partial x}f + \frac{1}{2}\left(\frac{\partial S_r}{\partial x}g - h^T\right)r^{-1}\left(g^T\frac{\partial S_r^T}{\partial x} - h\right)$ $-\frac{1}{2}\left|\left|u - r^{-1}\left(g^T\frac{\partial S_r^T}{\partial x} - h\right)\right|\right|_r^2 \le 0$ Taking into account equation (12) the equality holds for $u = u^* = r^{-1}\left(g^T\frac{\partial S_r^T}{\partial x} - h\right)$. Hence, because $S_r(0) = 0$ and u^* steers the state from t to $-\infty$ in 0, we conclude that $S_r(x)$ satisfies (7) such that (9) is asymptotically stable. The proof for $S_a(x)$ follows the exact same line. □
S_a and S_r are the minimal, respectively maximal solution of (7).

Remark 12 *If S_a and S_r were quadratic as in the linear systems case, everything would boil down to the Positive Real Algebraic Riccati equation and the positive real Gramians from Theorem 8, in Section 3.* □

Proposition 13 *If $S(x) \ge 0$ is a solution of (7), then $0 \le S_a \le S \le S_r$.* □
Proof: Follows the ideas of [Moy74]. □

Remark 14 *This result is in accordance with the ideas in [Wil72, vdS00].* □
The energy functions can be used according to [Wil72] as Lyapunov functions for system (1).

Lemma 15 *If the system (1) is passive and zero-state observable then any solution $S(x)$ of (7) is positive definite $(\forall)\ x \neq 0$.* □
Proof: This follows the line in [Moy74, Lemma 2]. □

Corollary 16 *A system (1) that is passive, with $r(x) > 0$ and zero state observable is asymptotically stable.* □

5 Nonlinear Balancing

In this section a system (1) is considered, under assumptions 1, 2, 3, and:
4. it is zero-state observable on Y;
5. $S_a(x)$ and $S_r(x)$ exist and are smooth on Y;
According to the previous section, the Assumptions 1-5 insure that S_a and S_r are the minimal, respectively the maximal positive definite solutions of the equation (7), for all $x \in Y$. The sequel follows the procedure in Scherpen [Sch93]. The goal is to find the coordinate transformation $\overline{z} = \xi(x)$ which brings the system into the positive real balanced form.

Theorem 17 *[Sch93] There exists a coordinate transformation $x = \phi(\overline{x}), \phi(0) = 0$ s.t., in the new coordinates $S_a(\phi(\overline{x})) = \frac{1}{2}\overline{x}^T\overline{x}$ and $S_r(\phi(\overline{x})) = \frac{1}{2}\overline{x}^T M(\overline{x})\overline{x}$, where $M(\overline{x})$ is an $n \times n$ symmetric matrix whose entries are smooth functions of $\overline{x}$.* □
Proof: See Lemma 3.2.2 in [Sch93], Chapter 3. □
For the sequel an extra assumption is needed
6. on Y, $M(\overline{x})$ has a constant number of distinct eigenvalues ([Sch94], Lemma 3.2.3)
According to Kato's result, [Ka82, Theorem 5.13a], Assumption 6 insures that $M(\overline{x})$ can be brought into a diagonal form, while leaving S_a in the same form.

Theorem 18 *[Sch93] Under assumptions 1-6, there exists a coordinate transformation $\overline{x} = \psi(z)$, s.t.*

$$S_a(z) = S_a(\psi(z)) = \frac{1}{2}z^T z, \tag{13}$$

and

$$S_r(z) = S_r(\psi(z)) = \frac{1}{2}z^T \text{diag}(v_1(z), ..., v_n(z))z \tag{14}$$

□

The nonlinear system (1) is brought in positive real balanced form using the following coordinate transformation $\overline{z} = \eta(z) = [\eta_1(z_1) \dots \eta_n(z_n)]^T$, where $\eta_i(z_i) = v_i(0, ..., z_i, ..., 0)^{\frac{1}{4}} z_i > 0$. Applying the transformation we get:

$$S_a(\overline{z}) = \frac{1}{2}\overline{z}^T \text{diag}(\pi_1(\overline{z}_1)^{-1}, ..., \pi_n(\overline{z}_n)^{-1})\overline{z} \tag{15}$$

$$S_r = \frac{1}{2}\overline{z}^T \text{diag}(\pi_1(\overline{z}_1)^{-1} v_1(\eta^{-1}(\overline{z})), ..., \pi_n(\overline{z}_n)^{-1} v_n(\eta^{-1}(\overline{z})))\overline{z} \tag{16}$$

$v_k(\eta^{-1}(\overline{z})) > 0$ for all k, can be called the positive real singular value functions of (1) and $\pi_k(\overline{z}_k) = \sqrt{v_k(0, ..., \eta_k^{-1}(\overline{z}_k), ..., 0)}$. Applying this coordinate transformation to (1), it becomes:

$$\dot{\overline{z}} = \overline{f}(\overline{z}) + \overline{g}(\overline{z})u, \quad \overline{y} = \overline{h}(\overline{z}) + \overline{d}(\overline{z}).$$

A system having the available storage and required supply of the form (15) and (16) is in the positive real balanced form.
So, given a system (1), by directly applying the coordinate change $\overline{z} = \xi(x) = (\eta \circ \psi^{-1} \circ \phi^{-1})(x)$, it is brought into positive real balanced form.
The available energy extracted at component $\overline{z}_k$ is given by the quantity $S_a(0, ..., \overline{z}_k, ...0) = \frac{1}{2}\overline{z}_k^2 \pi_k^{-1}(\overline{z}_k)$ and the energy supply required to reach component $\overline{z}_k$ is measured as $S_r(0, ..., \overline{z}_k, ..., 0) = \frac{1}{2}\overline{z}_k^2 \pi_k(\overline{z}_k)$. So, if $v_k(\overline{z}) \gg v_{k+1}(\overline{z})$, then $\pi_k^{-1}(\overline{z})v_k(\overline{z}) \gg \pi_{k+1}^{-1}(\overline{z})v_{k+1}(\overline{z})$. This means that to reach state component $\overline{z}_k$ less supply of energy is required than for the component $\overline{z}_{k+1}$ and at state component $\overline{z}_k$ is stored more energy available than at state component $\overline{z}_{k+1}$. This makes components $\overline{z}_1, ..., \overline{z}_k$ more important from energetic point of view than state components $\overline{z}_{k+1}, ..., \overline{z}_n$. Thus, model truncation can be applied, meaning that the $\overline{z}_{k+1}, ..., \overline{z}_n$ components can be made 0.

6 Model Reduction - Truncation

Partition the state vector $\overline{z}$ into $[\overline{z}^{1^T}, \overline{z}^{2^T}]^T$, where $\overline{z}^1 = [\overline{z}_1 \dots, \overline{z}_k]^T$ and $\overline{z}^2 = [\overline{z}_{k+1} \dots \overline{z}_n]^T$. Accordingly, the system can be partitioned into:

$$\overline{f}(\overline{z}) = \begin{bmatrix} \overline{f}_1(\overline{z}^1, \overline{z}^2) \\ \overline{f}_2(\overline{z}^1, \overline{z}^2) \end{bmatrix}, \overline{g}(\overline{z}) = \begin{bmatrix} \overline{g}_1(\overline{z}^1, \overline{z}^2) \\ \overline{g}_2(\overline{z}^1, \overline{z}^2) \end{bmatrix}, \overline{h}(\overline{z}) = \overline{h}(\overline{z}^1, \overline{z}^2), \overline{d}(\overline{z}) = \overline{d}(\overline{z}^1, \overline{z}^2).$$

According to the previous section, the energetic analysis of the state components tells that $\overline{z}^2$ is less important than $\overline{z}^1$. Hence, to reduce the system, we truncate i.e. we set $\overline{z}^2 = 0$. The reduced system is described by:

$$\dot{\overline{z}}^1 = \overline{f}_1(\overline{z}^1, 0) + \overline{g}_1(\overline{z}^1, 0)u, \quad \overline{y} = \overline{h}(\overline{z}^1, 0) + \overline{d}(\overline{z}^1, 0)u \tag{17}$$

The available storage of the reduced system is: $S_a(\overline{z}^1, 0)$. Because of the form in (15) we have that $\frac{\partial S_a}{\partial \overline{z}^2}(\overline{z}^1, 0) = 0$.
The Hamilton-Jacobi equation (7) is satisfied as follows:

$$\frac{\partial S_a}{\partial \overline{z}^1}(\overline{z}^1, 0)\overline{f}_1(\overline{z}^1, 0) + \frac{1}{2}\left(\frac{\partial S_a}{\partial \overline{z}^1}(\overline{z}^1, 0)\overline{g}_1(\overline{z}^1, 0) - \overline{h}^T(\overline{z}^1, 0)\right)\overline{r}^{-1}(\overline{z}^1, 0)\cdot \left(\overline{g}_1^T(\overline{z}^1, 0)\frac{\partial^T S_a}{\partial \overline{z}^1}(\overline{z}^1, 0) - \overline{h}(\overline{z}^1, 0)\right) = 0$$

Substituting the required supply $S_r(\overline{z}^1, 0)$ from relation (16) it is obtained that:

$$\frac{\partial S_r}{\partial \overline{z}^1}(\overline{z}^1, 0)\overline{f}_1(\overline{z}^1, 0) + \frac{1}{2}\left(\frac{\partial S_r}{\partial \overline{z}^1}(\overline{z}^1, 0)\overline{g}_1(\overline{z}^1, 0) - \overline{h}^T(\overline{z}^1, 0)\right)\overline{r}^{-1}(\overline{z}^1, 0)\cdot \left(\overline{g}_1^T(\overline{z}^1, 0)\frac{\partial^T S_r}{\partial \overline{z}^1}(\overline{z}^1, 0) - \overline{h}(\overline{z}^1, 0)\right) + F\left(\frac{\partial S_r}{\partial \overline{z}^2}(\overline{z}^1, 0), \overline{g}_2(\overline{z}^1, 0), \overline{h}(\overline{z}^1, 0)\right) = 0$$

where

$$F = \frac{\partial S_r}{\partial \overline{z}^1}\overline{f}_2 + \left(\frac{\partial S_r}{\partial \overline{z}^1}\overline{g}_1 - \overline{h}^T\right)\overline{r}^{-1}\overline{g}_2^T\frac{\partial^T S_r}{\partial z^2} + \frac{\partial S_r}{\partial \overline{z}^2}\overline{g}_2\overline{r}^{-1}\overline{g}_2^T\frac{\partial^T S_r}{\partial \overline{z}^2}$$

The required supply of the reduced system does not equal the reduced required supply, unless an extra condition is fulfilled, i.e. $F = 0$.

Remark 19 *Being an input-output property, (strict) passivity is not affected by the coordinate transformation which brings the original system into (strictly) positive real balanced form. It means that the (strictly) positive real balanced system is again (strictly) passive.* □

Theorem 20 *The reduced order system is strictly passive.* □
Proof: We check if the strict passivity property in (2) is satisfied by the reduced system. We can write (2) for the full order strictly positive real balanced system:

$$\begin{bmatrix} \frac{\partial S_a}{\partial \overline{z}^1} & \frac{\partial S_a}{\partial \overline{z}^2} \end{bmatrix} \left(\begin{bmatrix} \overline{f}_1(\overline{z}^1, \overline{z}^2) \\ \overline{f}_2(\overline{z}^1, \overline{z}^2) \end{bmatrix} + \begin{bmatrix} \overline{g}_1(\overline{z}^1, \overline{z}^2) \\ \overline{g}_2(\overline{z}^1, \overline{z}^2) \end{bmatrix} u \right) < u^T\overline{h}(\overline{z}^1, \overline{z}^2) + u^T\overline{d}(\overline{z}^1, \overline{z}^2)u.$$

Setting $\overline{z}^2 = 0$ we have that $\frac{\partial S_a}{\partial \overline{z}^2}(\overline{z}^1, 0) = 0$, $S_a(\overline{z}^1, 0) > 0$. Substituting in the above inequality we get:

$$\frac{\partial S_a}{\partial \overline{z}^1}(\overline{z}^1, 0)(\overline{f}_1(\overline{z}^1, 0) + \overline{g}_1(\overline{z}^1, 0)u) < u^T\overline{h}(\overline{z}^1, 0) + u^T\overline{d}(\overline{z}^1, 0)u.$$

It means that the reduced order system satisfies inequality (2), hence the reduced system is strictly passive. □

Theorem 21 *If* $F\left(\frac{\partial S_r}{\partial \overline{z}^2}(\overline{z}^1, 0), \overline{g}_2(\overline{z}^1, 0), \overline{h}(\overline{z}^1, 0)\right) = 0$ *for all* $\overline{z}^1$ *around* 0*, then the reduced system is in strictly positive real balanced form having the singular value functions:* $v_1(z^1, 0) \geq ... \geq v_k(z^1, 0)$*, for* $z^1 = \eta^{-1}(\overline{z}^1, 0)$. □
Proof: If the condition on F is satisfied, then $S_r(\overline{z}^1, 0)$ as in (16) is the required supply of the reduced system. $S_a(\overline{z}^1, 0)$ as in (15) satisfies directly the Hamilton-Jacobi equation (7), so it is the available storage of the reduced system. Thus, the system is in positive real balanced form with the positive real singular value functions $v_1(z^1, 0) \geq \ldots \geq v_k(z^1, 0)$, where $z^1 = [\eta^{-1}(\overline{z}_1) \ldots \eta^{-1}(\overline{z}_k)]^T$. □

Remark 22 *If the singular value functions are independent of* $\overline{z}^2$*, then* $\frac{\partial S_r}{\partial z^2}(\overline{z}^1, 0) = 0$*. Then immediately* $F\left(\frac{\partial S_r}{\partial \overline{z}^2}(\overline{z}^1, 0), \overline{g}_2(\overline{z}^1, 0), \overline{h}(\overline{z}^1, 0)\right) = 0$ *follows.* □

7 Future Work

We present a passivity preserving model reduction technique, based on positive real balanced truncation. The results in Section 5 are coordinate dependent, leading to the fact that the balanced representation and the singular value functions are not unique, i.e. the choice of different sets of singular value functions gives different reduced systems. For future research, developments such as in [FujSch05], is to be taken into account for the nonlinear positive real balancing case.
If the system is not strictly positive real, but is positive real, there is no Hamilton-Jacobi equation to solved. However, if one can compute S_a and S_r in a different way, the balancing procedure and the results of this paper can still be applied. Additionally, for physical systems, such as port-Hamiltonian systems (see [vdS00]) it may be useful to preserve besides passivity, an additional energy/power-based structure in the model for control purposes. This is also a topic for future research.

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References

[Ant05] A. C. Antoulas: Approximation of Large-Scale Dynamical Systems. SIAM (2005).
[BIW91] C. I. Byrnes, A. Isidori, J. C. Willems: Passivity, feedbackequivalence, and the global stabilization of minimum phase nonlinear systems. IEEE Trans. Automat. Contr., **36**, 1228-1240 (1991).
[FujSch05] K. Fujimoto and J. M. A. Scherpen: Nonlinear input-normal realizations based on the differential eigenstructure of Hankel operators. IEEE Trans. on Aut. Contr., **50**, 2-18 (2005).
[Giu05] A. Giusto: On Transient Stabilization of Power Systems: A Power-Shaping Solution for Structure-Preserving Models , submitted to IEEE Conf on Decision and Control, Dec (2006).
[HilMoy76] D. Hill and P. J. Moylan: The stability of nonlinear dissipative systems, IEEE Trans. on Aut. Contr., **21**, 708-711 (1976).
[Ka82] T. Kato: A Short Introduction to Perturbation Theory for Linear Operators. Springer-Verlag (1982).
[Moy74] P. J. Moylan, Implications of passivity in a class of nonlinear systems, IEEE Trans. on Aut. Contr., **19**, 373-381 (1974).
[Sch93] J. M. A. Scherpen: Balancing for nonlinear systems. System & Control Letters, **21**, 143-153 (1993).
[Sch94] J. M. A. Scherpen: Balancing for nonlinear systems. PhD Thesis, Univ. of Twente (1994).
[vdS00] A. J. van der Schaft, L_2-gain and Passivity Techniques in Nonlinear Control. Springer-Verlag (2000).
[Wil71] J. C. Willems: The generation of Lyapunov functions for input-output stable systems. J. SIAM Contr., **9**, 105-134 (1971).
[Wil72] J. C. Willems: Dissipative Dynamical Systems - Part I: General theory, - Part II: Linear systems with quadratic supply rates. Archive for Rational Mechanics and Analysis, **45**, 321-393 (1972).