Relational Attribute Systems II: Reasoning with Relations in Information Structures

Ivo Düntsch[1,⋆], Günther Gediga[2], and Ewa Orłowska[3]

[1] Brock University, St. Catharines, Ontario, Canada
[2] University of Osnabrúck, Germany
[3] National Institute of Telecommunications, Warsaw, Poland

Abstract. We describe deduction mechanisms for various types of data bases with incomplete information, in particular, relational attribute systems, which we have introduced earlier in [8].

Keywords: Attribute, information relation, information system, relational attribute system, fuzzy information system, relational deduction, semantical framework.

1 Introduction

Rough sets were introduced by [29]; they served well as a vehicle for expressing dependencies in datatables, as well as for attribute reduction that depends on the equivalence classes induced by the attribute mappings. The information systems of the rough set model were single valued and could only express deterministic information. Already [15,16] had considered information systems where an object under an attribute function was allowed to take a set of values, which could also be empty; a similar road was taken by [26]. Common to both approaches is the replacement of an attribute function between objects and a single value of an attribute domain by an attribute relation where an object can be related to any set of attribute values. In [8] we have supplemented the notion of an indeterministic information system to a model of data called *relational attribute system* (RAS) in the spirit of non–invasive data analysis [7]. Its distinguishing feature is the provision of a semantical framework for the data table: Given an attribute a, an object x, and a set $a(x)$ of values which are associated with x, there are various ways in which $a(x)$ can be interpreted; for instance, as exemplified in [8],

1. $a(x)$ is interpreted conjunctively and exhaustively. If a is the attribute "speaking a language", then,

$$a(x) = \{\text{German, Polish, French}\}$$

   can be interpreted as

   x speaks German, Polish, and French and no other languages.

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2. $a(x)$ can also be interpreted conjunctively and non-exhaustively as in

   a speaks German, Polish, and French and possibly other languages.

3. $a(x)$ is interpreted disjunctively and exclusively. For example, a witness states that

   The car that went too fast was either a Mercedes or a Ford.

   Here, exactly one of the statements
   - The car that went too fast was a Mercedes.
   - The car that went too fast was a Ford.

   is true, but it is not known which one.

4. $a(x)$ is interpreted disjunctively and non-exclusively. If a is "cooperates with", then

   $$a(\text{Ivo}) = \{\text{Günther, Ewa}\}$$

   means that Ivo cooperates with Günther, or Ewa, or both.

The desired semantics can be given in the form of relational constraints, using machinery from the theory of relation algebras [33]. We have indicated the usefulness of the approach by two examples, one pertaining to interrater reliability, the other one to software usability. In a subsequent paper we will use the RAS model and the inference techniques described below to address in detail the practical aspects of our approach.

In this paper, our task is to develop a reasoning mechanism for the RAS model. We follow the general methodology for developing inference tools for information structures based on the object-property assignments, as surveyed in [5]. The specific feature of the methods presented in this paper is that, firstly, we define a class of algebras of relations suitable for the information structure under consideration, and, secondly, we develop deduction rules for this class of algebras. In applying this methodology we observe that in fact several other information structures besides our RAS model can be dealt with in a similar way. Thus, we present deduction systems for various information structures:

- *Information systems with incomplete information and no semantics*,
- *Relational attribute systems*,
- *Fuzzy information systems*,
- *Temporal information systems*.

Once an object–property assignment is given, with each object from the information structure under consideration there is associated a finite set which, in particular, may be empty or contain more than one value . Consequently, each information structure determines a family of sets specific for the structure, resulting from the assignment of the properties to the objects. The relationships among the objects can be articulated by comparing their sets of properties. The comparison is usually expressed in terms of binary set relations. This leads to the concept of information relations.

There are three fundamental ingredients of a definition of any information relation:

- A specification of a family of sets of properties of objects,
- A specification of set relations meaningful for this family,
- A specification of the information relation itself in terms of these set relations.

For the information structures listed above, we present deduction mechanisms for verification of constraints holding for those information relations. The deduction systems presented here belong to the family of *Rasiowa–Sikorski* (RS or dual tableau) style relational proof systems [31]; systems of such type were developed for a number of theories, for example, [11,17,9,25]. There are various implementations of RS systems: Relational attribute systems have been implemented in [3]; an implementation of more general relational proof systems can be found at [4]. The system presented there contains rules for deduction with binary relations as well as with typed relations, and thus, it is suitable for reasoning in relational databases [19]. The system is modular – a general feature of the RS style –, and the user can include specific deduction rules if needed. In particular, the deduction rules presented in the present paper can be incorporated. Some other implementations of relational deduction in nonclassical logics are presented in [10].

2 Deduction System for Standard Algebras of Binary Relations

In this Section we recall basic principles of relational proof systems and the deduction rules for standard algebras of binary relations [23]. The operations of Tarski's algebra of binary relations [33] are Boolean set operations of union ($\cup$), intersection ($\cap$) and complement ($-$), and relational operations of product (;), converse ($^{-1}$), and the constants $1'$ of the identity and 1 of universal relation. For binary relations R and S on a set U, $R\ ;\ S = \{(x,y) \in U \times U : \exists z \in U (x,z) \in R\ \wedge (z,y) \in S\}$[1] and $R^{-1} = \{(x,y) \in U \times U : (y,x) \in R\}$ A *relational term* is any expression built from relation variables and constants with these operations. If x,y are object variables and P is a relational term, then any expression of the form xPy is a relational formula.

The semantics of relational formulas is determined in terms of the notion of model and satisfiability of formulas. A *model* is a system $\mathscr{M} = (U,m)$, where U is a nonempty set (of objects) and m is a meaning function that provides an interpretation of relational terms, i.e. $m(P) \subseteq U \times U$ for any relation variable P, $m(1')$ is the identity relation on U, $m(1) = U \times U$, and m extends homomorphically to all terms. By a *valuation* in a model $\mathscr{M}$ we understand a function v that assigns objects from U to object variables, that is, $v(x) \in U$ for any object variable x. The *satisfiability relation* is defined by $\mathscr{M},v \models xPy$ iff $(v(x),v(y)) \in m(P)$. A formula xPy is *true* in a model $\mathscr{M}$ whenever $\mathscr{M},v \models xPy$ for every valuation v in $\mathscr{M}$, nd it is *valid* whenever is true in all models. Hence, validity of xPy amounts to saying that $P = 1$ holds in every algebra of binary relations. A finite sequence of relational formulas is said to be valid whenever universally quantifird disjunction of its members is valid in the classical first order logic.

The proof system consists of two groups of rules, namely, *decomposition rules* and *specific rules*. Decomposition rules enable us to decompose formulas into a finite sequence of (usually syntactically simpler) formulas, or a pair of finite sequences of formulas (and then we separate the sequences with — in a definition of the rule) while the specific rules enable us to modify a sequence to which they are applied; they have

[1] Using Tarski's existential quantifier $\mathbf{E}$ (*i.e.*, some) [34], the assertion $\exists z \in U (x,z) \in R$ can also be written $\mathop{\mathbf{E}}_{z}(z \in U (x,z) \in R)$.

Table 1. Decomposition rules

$(\cup)$	$\dfrac{K,x(P\cup Q)y,H}{K,xPy,xQy,H}$	$(-\cup)$	$\dfrac{K,x-(P\cup Q)y,H}{K,x(-Q)y,H \mid K,x(-Q)y,H}$
$(\cap)$	$\dfrac{K,x(P\cap Q),H}{K,xPy,H \mid K,xQy,H}$	$(-\cap)$	$\dfrac{K,x-(P\cap Q)y}{K,x(-P)y,x(-Q)y,H}$
$(^{-1})$	$\dfrac{K,xP^{-1}y,H}{K,yPx,H}$	$(-^{-1})$	$\dfrac{K,x(-P^{-1})y,H}{K,y(-P)x,H}$
$(--)$	$\dfrac{K,x(--P)y,H}{K,xPy,H}$		
$(\,;\,)$	$\dfrac{K,x(P;Q)y,H}{K,xPz,H,x(P;Q)y \mid K,zQy,H,x(P;Q)y}$	z is an object variable	
$(-\,;\,)$	$\dfrac{K,x-(P;Q)y,H}{K,x(-P)z,z(-Q)y,H}$	z is restricted	

Table 2. Specific rules

$1'_1$	$\dfrac{K,xPy,H}{K,x1'z,H,xPy \mid K,\ zPy,H,\ xPy}$	z is an object variable
$1'_2$	$\dfrac{K,xPy,H}{K,xPz,H,xPy \mid K,\ z1'y,\ H,xPy}$	
sym $1'$	$\dfrac{K,x1'y,H}{K,y1'x,H}$	

the status of structural rules. The role of axioms is played by what is called *axiomatic sequences*.

A proof system for Tarski's algebras of binary relations consists of the decomposition rules given in Table 1, where K and H denote finite, possibly empty, sequences of relational formulas; the specific rules are presented in Table 2. There, a variable is said to be *restricted in a rule* whenever it does not appear in any formula of the upper sequence in that rule. This system has been developed in [21].

The specific rules characterize the identity relation $1'$. Namely, $(1'_1)$ corresponds to the property that $1';R\subseteq R$ for any relation R. Similarly, $(1'_1)$ says that $R;1'\subseteq R$. Observe that the reverse inclusions also hold, since $1'$ is reflexive; thus, no more rules are needed for guaranteeing that $1'$ is a unit element of relational composition. (sym $1'$) expresses the symmetry of $1'$, and transitivity of $1'$ is an instance of $(1'_1)$.

A sequence of relational formulas is said to be *axiomatic* if it contains formulas of the following forms; here, P is a relational term, and x,y are object variables.

a1. $xPy,x(-P)y$.
a2. $x1y$.
a3. $x1'x$

(a2) reflects the fact that 1 is the universal relation, and (a3) says that $1'$ is reflexive. The rules listed in Table 1 and Table 2 are correct i.e., they preserve and reflect validity of the sequences of formulas: the upper sequence of the rule is valid if and only if all the lower sequences of this rule are valid. Axiomatic sequences are valid.

Although these rules and axiomatic sequences enable us to prove only that $1'$ is an equivalence relation, the given deduction system is complete with respect to the class of standard algebras of relations, where $1'$ is the identity. The proof uses the usual argument well known from first order logic. Namely, it can be shown that for every model of the relational language with $1'$ interpreted as an equivalence relation there is a model where $1'$ is an identity and both models verify the same formulas.

To check the validity of a relational formula, we successively apply decomposition and/or specific rules to it, thus obtaining a tree whose nodes consist of finite sequences of formulas. Such a tree is referred to as a decomposition tree. We stop applying the rules to the formulas of a node whenever the node contains an axiomatic sequence of formulas. A branch with such a node is declared closed. A decomposition tree is said to be closed whenever all of its branches are closed. The following soundness and completeness Theorem is well known (see e.g. [23,13]).

Theorem 1. *A relational formula is valid iff it possesses a closed decomposition tree.*

Hence, possession of a closed decomposition tree may be understood as provability. The proof of this theorem is based on the three lemmas. First of all, we assume that a decomposition tree of a formula is complete: if a rule is applicable to a node of the tree, then it has been applied. Then we prove a closed branch theorem which says that if a branch of the complete decomposition tree includes a node with the formula xRy and a node with the formula $x(-R)y$, where R is a relational term, and x,y are object variables, then this branch has also a node with and axiomatic sequence. This follows from the fact that the rules appropriately transfer the formulas from the upper sequence to the lower sequences. Next, for an open branch, say b, of a complete decomposition tree we construct what is called a branch model, M^b. It is constructed from the syntactic resources of the relational language. Its universe is the set of object variables. The meaning of a relation variable or a relation constant, say R, is a binary relation defined as $(x,y) \in m^b(R)$ iff formula xRy does not appear in any node of branch b. The second important lemma, referred to as a branch model theorem, says that a branch model constructed as above is a model of the relational language i.e., the relational constants admitted in the language are appropriately interpreted: $m^b(1)$ is the universal relation and $m^b(1')$ is an equivalence relation. The third lemma, referred to as a satisfaction in branch model theorem, says that if a formula is satisfied in a branch model M^b by an identity valuation v^b such that $v^b(x) = x$ for any object variable x, then it does not appear in any node of branch b. With these lemmas the completeness (validity implies provability) can be proved. The soundness (provability implies validity) follows from the correctness of the rules and from validity of axiomatic sequences.

If we extend the set of relational formulas to the first order language with binary predicates, then the appropriate deduction system can be obtained by adding the deduction rules of first order logic developed in [31].

The above relational logic with its system of rules is complete both for the class RRA of representable relation algebras and the class RA of relation algebras. The system can also be applied to solve the three major logical tasks for a number of logics and classes of algebras, namely checking *validity*, *entailment*, *satisfiability*, and *truth in a model* (often referred to as *model checking*). The details can be found in [13]. Once a representation of formulas of a logic or the terms over a class of algebras is provided in the form of relational terms over a class, say C, of appropriate algebras of relations [22,24], the relational representation of these logical tasks is as follows: Checking validity amounts to verifying whether $R = 1$ holds in every algebra of relations from C, for some relation term R. Entailment is the problem of checking whether from a finite number of identities of the form $R_1 = 1, \ldots, R_n = 1$ we can infer that $R = 1$. According to the Tarski rule, this problem can be reduced to checking the identity $1 \,;\, -(R_1 \cap \cdots \cap R_n) \,;\, 1 \cup R) = 1$. The satisfaction problem of checking whether $\langle a, b \rangle \in R$ for some relation R and some objects a, b amounts to verifying whether $A \,;\, B^{-1} \subseteq R$, where A and B are the point relations representing the objects a and b, respectively, and they satisfy the usual point axioms $P \,;\, 1 = 1$, $P \,;\, P^{-1} \subseteq 1'$, and $P \neq \emptyset$ [32].

3 Relations Derived from Information Systems

In this Section we recall the notion of an information system [16,28], and relations derived from such a system; an exhaustive list of those relations can be found in [5].

By an *information system* we understand a structure $S = (\mathrm{OB}, \Omega, \{V_a : a \in \Omega\})$ such that U is a nonempty set of objects, Ω is a finite nonempty set of attributes, each V_a is a nonempty set of values of attribute a. An attribute is a function $a : U \mapsto \mathscr{P}(V_a)$ that assigns subsets of values of attributes to the objects. If for every $a \in \Omega$, $a(x)$ is a singleton set, then system S is said to be *deterministic*, otherwise S is *nondeterministic*.

Any set $a(x)$ can be viewed as a set of properties of an object x determined by attribute a. For example, if attribute a is 'color' and $a(x) = \{\text{green}\}$, then x possesses property of 'being green'. If a is 'age' and x is 25 years old, then $a(x) = \{25\}$ and this means that x possesses property of 'being 25 years old'. If a is 'languages spoken' and if a person x speaks, say, Polish (Pl), German (D), and French (F), then $a(x) = \{\text{Pl, D, F}\}$, and x possesses properties of 'speaking Polish', 'speaking German', and 'speaking French'. In this setting any set $a(x)$ is referred to as the set of a-properties of object x and its complement $V_a - a(x)$ is said to be the set of negative a-properties of x.

Let $S = (U, \Omega, \{V_a : a \in \Omega\})$ and $A \subseteq \Omega$. The following families of set information relations on set U are the subject of investigation in a number of papers:

Strong (weak) indiscernibility $(x,y) \in \mathrm{ind}_A$ iff $a(x) = a(y)$ for all (some) $a \in A$,
Strong (weak) similarity $(x,y) \in \mathrm{sim}_A$ iff $a(x) \cap a(y) \neq \emptyset$ for all (some) $a \in A$,
Strong (weak) forward inclusion $(x,y) \in \mathrm{fin}_A$ iff $a(x) \subseteq a(y)$ for all (some) $a \in A$,
Strong (weak) backward inclusion bin_A iff $a(y) \subseteq a(x)$ for all (some) $a \in A$,
Strong (weak) negative similarity $(x,y) \in \mathrm{nim}_A$ iff $-a(x) \cap -a(y) \neq \emptyset$ for all (some) $a \in A$,
Strong (weak) incomplementarity $(x,y) \in \mathrm{icom}_A$ iff $a(x) \neq -a(y)$ for all (some) $a \in A$,
Strong (weak) diversity $(x,y) \in \mathrm{div}_A$ iff $a(x) \neq a(y)$ for all (some) $a \in A$,
Strong (weak) disjointness $(x,y) \in \mathrm{dis}_A$ iff $a(x) \subseteq -a(y)$ for all (some) $a \in A$,
Strong (weak) exhaustiveness $(x,y) \in \mathrm{exh}_A$ iff $-a(x) \subseteq a(y)$ for all (some) $a \in A$,

Strong (weak) right negative similarity $(x,y) \in \mathrm{rnim}_A$ iff $a(x) \cap -a(y) \neq \emptyset$ for all (some) $a \in A$,

Strong (weak) left negative similarity $(x,y) \in \mathrm{lnim}_A$ iff $-a(x) \cap a(y) \neq \emptyset$ for all (some) $a \in A$,

Strong (weak) complementarity $(x,y) \in \mathrm{com}_A$ iff $a(x) = -a(y)$ for all (some) $a \in A$.

In all the above definitions, complement is taken with respect to set VAL_a. If $A = \{a\}$ is a singleton set, then we write R_a instead of $R_{\{a\}}$ for any information relation R. Observe that if $(x,y) \in \mathrm{dis}_a$, then $a(x) \cap a(y) = \emptyset$, and if $(x,y) \in \mathrm{exh}_a$, then $a(x) \cup a(y) = V_a$ which explains the names of the relations. In the earlier literature (e.g., [5]) the relations were referred to as right (resp. left) orthogonality.

The strong relations satisfy the following conditions for all $P, Q \subseteq \Omega$:

S1. $R_{P \cup Q} = R_P \cap R_Q$,
S2. $R_\emptyset = U \times U$.

The weak relations satisfy:

W1. $R_{P \cup Q} = R_P \cup R_Q$,
W2. $R_\emptyset = \emptyset$.

A specific family of sets associated to an information system $S = (U, \Omega, \{V_a : a \in \Omega\})$ is the family $\{a(x) : a \in \Omega, x \in U.\}$. It is easy to see that all the information relations defined above can be specified in terms of three families of set relations, namely, $\subseteq_a$, Σ_a, N_a, where $a \in \Omega$ and

$$\begin{aligned} x \subseteq_a y &\Longleftrightarrow a(x) \subseteq a(y), \\ x \Sigma_a y &\Longleftrightarrow a(x) \cap a(y) \neq \emptyset, \\ x N_a y &\Longleftrightarrow a(x) \cup a(y) \neq V_a. \end{aligned}$$

These relations can be extended in the usual way to the relations indexed with subsets of set Ω. Now, an information relation derived from an information system is any relation generated from $\subseteq_a, \Sigma_a, N_a$, for $a \in \Omega$, with the standard relational operations.

For example, the information relations relations determined by an attribute a are defined as follows:

$\mathrm{ind}_a = \subseteq_a \cap \subseteq_a^{-1}, \mathrm{sim}_a = \Sigma_a, \mathrm{fin}_a = \subseteq_a, \mathrm{bin}_a = \subseteq_a^{-1}, \mathrm{nim}_a = N_a, \mathrm{icom}_a = \Sigma_a \cup N_a$, $\mathrm{div}_a = - \subseteq_a \cup - \subseteq_a^{-1}, \mathrm{dis}_a = -\Sigma_a, \mathrm{exh}_a = -N_a, \mathrm{rnim}_a = - \subseteq, \mathrm{lnim}_a = - \subseteq_a^{-1}, \mathrm{com}_a = -\Sigma_a \cup -N_a$.

In an abstract setting, by an IS-frame (information system frame) we mean a system $(U, \{\leq_P : P \subseteq A\}, \{\sigma_P : P \subseteq A\}, \{\nu_P : P \subseteq A\})$, where U and A are nonempty sets, A is finite, and the following conditions are satisfied for all $x, y, z \in U$ and for every $p \in A$. For the sake of simplicity we write $\leq, \sigma, \nu$ instead of $\leq_p, \sigma_p, \nu_p$:

IS1. $\leq$ is reflexive, transitive , and antisymmetric.
IS2. σ is symmetric, and $1' \cap (\sigma \,;\, 1) \subseteq \sigma$ (weakly reflexive, i.e., $x\sigma y$ implies $x\sigma x$).
IS3. $\sigma \,;\, \leq \subseteq \sigma$, i.e., $x\sigma y$ and $y \leq z$ imply $x\sigma z$.
IS4. $x\sigma x$ or $x \leq y$.

IS5. ν is symmetric and weakly reflexive.
IS6. $\leq^{-1}\ ;\ \nu \subseteq \nu$, i.e., $x \leq^{-1} y$ and $y\nu z$ imply $x\nu z$.
IS7. $x\nu x$ or $x \leq^{-1} y$.
IS8. $-\sigma\ ;\ -\nu \subseteq \leq$ i.e., $x \leq y$ or $x\sigma z$ or $y\nu z$.
IS9. $x\sigma x$ or $x\nu x$.
IS10. $-\nu\ ;\ \leq \subseteq -\nu$

The above list of axioms is based on the axioms presented in [36]. Furthermore, we have to declare whether the relations are strong or weak by postulating the axioms (S1) and (S2) or (W1) and (W2).

By an IS-relation algebra we understand an algebra of relations generated by $\{\leq_p : p \in A\} \cup \{\sigma_p : p \in A\} \cup \{\nu_p : p \in A\}$ for some IS-frame $(U, \{\leq_P : P \subseteq A\}, \{\sigma_P : P \subseteq A\}, \{\nu_P : P \subseteq A\})$. In the following Section we present a deduction system for reasoning about properties of relations in IS-relation algebras.

4 Deduction in IS-Relation Algebras

The majority of deductive systems for reasoning about information relations derived from an information system are the appropriate systems of modal logics (for a survey see [5]). Modal approach enables us to study information operators, e.g., approximation operators or knowledge operators determined by information relations (see e.g. [35], [36], [37]). Here our aim is to develop a reasoning mechanism for verification of properties of plain information relations. The strategy is to design deduction rules for IS-frames and to adjoin them to the system of rules for the standard algebras of binary relations presented in Section 2, thus obtaining a deduction system for IS-relation algebras.

The formulas processed by the deduction system for IS-relation algebras are of the form xRy, where x, y are object variables and R is a term of an IS-relation algebra. For each $p \in A$ and for every $\leq_p, \sigma_p, \nu_p$ we assume the following rules. As usual we omit the index p in the names of the relations.

(ref $\leq$) $$\frac{K, x \leq y, H}{K, x1'y, H, x \leq y}$$

(tran $\leq$) $$\frac{K, x \leq y, H}{K, x \leq z, H, x \leq y \mid K, z \leq y, H, x \leq y}$$
z is any object variable

(antisym $\leq$) $$\frac{K, x(-\leq)y, y(-\leq x), H}{K, x(-1')y, H}$$

(sym σ) $$\frac{K, x\sigma y, H}{K, y\sigma x, H}$$

(wref σ) $$\frac{K, x\sigma y, H}{K, x1'y, H, x\sigma y \mid K, x\sigma z, H, x\sigma y}$$
z is any object variable

(rIS3) $$\frac{K, x\sigma y, H}{K, x\sigma z, H, x\sigma y \mid K, z \leq y, H, x\sigma y}$$
z is any object variable

(rIS4) $$\frac{K, x\sigma y, H}{K, x1'y, H, x\sigma y \mid K, x(-\leq)z, H, x\sigma y}$$
z is any object variable

(sym ν) and (wref ν) are analogous to (sym σ) and (wref σ), respectively.

(rIS6) $$\frac{K, x\nu y, H}{K, z \leq x, H, x\nu y \mid K, z\nu y, H, x\nu y}$$
z is any object variable

(rIS7) $$\frac{K, x\nu y, H}{K, x1'y, H, x\nu y \mid K, x(-\leq)z, H, x\nu y}$$
z is any object variable

(rIS8) $$\frac{K, x \leq y, H}{K, x(-\sigma)z, H, x \leq y \mid K, z(-\nu)y, H, x \leq y}$$
z is any object variable

(rIS9) $$\frac{K, x\sigma y, x\nu y, H}{K, x1'y, H, x\sigma y, x\nu y}$$

(rIS10) $$\frac{K, x(-\nu)y, H}{K, x(-\nu)z, H, x(-\nu)y \mid K, z \leq y, H, x(-\nu)y}$$
z is any object variable

For $R \in \{\leq_P : P \subseteq A\} \cup \{\sigma_P : P \subseteq A\} \cup \{\nu_P : P \subseteq A\}$, the characterization of strong relations is provided by the rules (rS1), (r-S1), and the axiomatic sequence (aS2):

(rS1) $$\frac{K, xR_{P\cup Q}y, H}{K, xR_P y, H \mid K, xR_Q y, H}$$

(r-S1) $$\frac{K, x(-R_{P\cup Q})y, H}{K, x(-R_P)y, x(-R_Q)y, H}$$

(aS2) $xR_\emptyset y$

The characterization of weak relations is given by the rules (rW1), (r-W1), and the axiomatic sequence (aW2):

(rW1) $$\frac{K, xR_{P\cup Q}y, H}{K, xR_P y, xR_Q y, H}$$

(r-W1) $$\frac{K, x(-R_{P\cup Q})y, H}{K, x(-R_P)y, H \mid K, x(-R_Q)y, H}$$

(aW2) $x(-R_\emptyset)y$

It is easy to verify that the rules presented above are correct in view of the properties of relational constants assumed in the models. The definition of a branch model is the same as described in Section 2. A completeness theorem analogous to Theorem 1 can be proved following the principles presented in Section 2, see also the general method described in [18].

Example 1
We show that $-\sigma\,;\,\nu \subseteq \leq$. Since for any binary relations R,S we have $R \subseteq S$ iff $-R \cup S = 1$, we need to prove the formula (1) below:
(1) $x(-(-\sigma\,;\,-\nu) \cup \leq)y$.
Applying rule $(\cup)$ to (1) we get:
(2) $x(-(-\sigma\,;\,-\nu))y,\ x \leq y$.
Applying rule $(-\,;\,)$ with a restricted variable z, and rule (–) to (2) we obtain:
(3) $x\sigma z,\ z\nu y,\ x \leq y$.
Now we apply rule (rIS8) to $x \leq y$ choosing z as the new variable and we obtain two sequences (3.1) and (3.2):
(3.1) $x(-\sigma)z,\ x\sigma z,\ z\nu y,\ x \leq y$,
(3.2) $z(-\nu)y,\ x\sigma z,\ z\nu y,\ x \leq y$.
Both of them are axiomatic of the form (a1).

Example 2
We show that $-\nu\,;\,-\sigma\,;\,-\nu \subseteq -\nu$.
(1) $x(-(-\nu\,;\,-\sigma\,;\,-\nu) \cup -\nu)y$.
We apply rule $(\cup)$ and we get:
(2) $x(-(-\nu\,;\,-\sigma\,;\,-\nu))y,\ x(-\nu)y$.
Now we apply twice the rule $(-\,;\,)$ with restricted variables z and t, and then rule (–):
(3) $x\nu z,\ z\sigma t,\ t\nu y,\ x(-\nu)y$.
Rule (rIS10) applied to $x-\nu y$ with a new variable z yields two sequences (3.1) and (3.2):
(3.1) $x(-\nu)z,\ x\nu z,\ z\sigma t,\ t\nu y,\ x(-\nu)y$,
(3.2) $z \leq y,\ x\nu z,\ z\sigma t,\ t\nu y,\ x(-\nu)y$.
Sequence (3.1) is axiomatic of the type (a1). To the sequence (3.2) we apply rule (rIS8) with a new variable t and we get the following two sequences:
(3.2.1) $z(-\sigma)t,\ z \leq y,\ x\nu z,\ z\sigma t,\ t\nu y,\ x(-\nu)y$,
(3.2.2) $t(-\nu)y,\ z \leq y,\ x\nu z,\ z\sigma t,\ t\nu y,\ x(-\nu)y$.
Both of these sequences are axiomatic.

Example 3
Let $\{R_P\}_{P\subseteq A}$ be a family of strong relations and let R_p and R_q be transitive, that is the rules (tran R_p) and (tran R_q) analogous to the rule (tran $\leq$) presented above are admitted in a proof system. For the sake of simplicity we write R_p and R_q instead of $R_{\{p\}}$ and $R_{\{q\}}$, respectively. We show that $R_{\{p,q\}}$ is also transitive, i.e., $R_{\{p,q\}}\,;\,R_{\{p,q\}} \subseteq R_{\{p,q\}}$. Hence, we have to prove the formula:
(1) $x(-(R_{\{p,q\}}\,;\,R_{\{p,q\}}) \cup R_{\{p,q\}})y$
Applying rule $(\cup)$ we have:
(2) $x(-(R_{\{p,q\}}\,;\,R_{\{p,q\}}))y,\ xR_{\{p,q\}})y$

Now we apply rule $(-\,;\,)$ with a restricted variable z:
(3)$x(-(R_{\{p,q\}})z$, $z(-(R_{\{p,q\}})y$, $xR_{\{p,q\}}y$.
Applying rule (r-S1) twice we obtain:
(4) $x(-R_p)z$, $x(-R_q)z$, $z(-R_p)y$, $z(-R_q)y$, $xR_{\{p,q\}}y$.
Now we apply rule (rS1) and we get two sequences:
(4.1) xR_py, $x(-R_p)z$, $x(-R_q)z$, $z(-R_p)y$, $z(-R_q)y$, $xR_{\{p,q\}}y$,
(4.2) xR_qy, $x(-R_p)z$, $x(-R_q)z$, $z(-R_p)y$, $z(-R_q)y$, $xR_{\{p,q\}}y$.
We apply the rule (tran R_p) with a new variable z to the formula xR_py of (4.1) and the rule (tran R_q) also with variable z to the formula xR_qy of (4.2) which yield four sequences each of which is axiomatic of the form (a1).

5 Relations Derived from Temporal Information Systems

A temporal information system is an information system (U, $\{\text{time}\}$, VAL_{time}) whose set of attributes consists of a single attribute 'time', the set of values of this attribute is a set with a strict dense linear ordering $<$ without endpoints on it, and to every object x there is associated a closed time interval $\text{time}(x) = [t,t']$, where $t < t'$. Temporal information systems are useful, for example, in temporal scenario specification of multimedia objects, where the execution of a multimedia object is usually considered to be a temporal interval.

The family of sets specific for the temporal information systems is the underlying family of time intervals:

$$\{[t,t'] : t,t' \in V_{\text{time}}, \text{time}(x) = [t,t'] \text{ for some } x \in U \text{ and } t < t'\}.$$

The typical relations defined on this family of sets are the following [1]:

$1'$ (equals): $[t,t']\ 1'\ [u,u']$ iff $t = t'$ and $u = u'$,
P (precedes): $[t,t']\ P\ [u,u']$ iff $t' < u$,
D (during): $[t,t']\ D\ [u,u']$ iff $u < t$ and $t' < u'$,
O (overlaps): $[t,t']\ O\ [u,u']$ iff $t < u$ and $u < t'$,
M (meets): $[t,t']\ M\ [u,u']$ iff $t' = u$,
S (starts): $[t,t']\ S\ [u,u']$ iff $t = u$ and $t' < u'$,
F (finishes): $[t,t']\ F\ [u,u']$ iff $t' = u'$ and $u < t$.

By a TIS-frame (temporal information system frame) we mean a system $(U,<,1',P,D,O,M,S,F)$, where $<$ is a strict dense linear ordering on U without endpoints, and $1',P,D,O,M,S,F$ are the binary relations on the set $\{[t,t'] : t,t' \in U, t < t'\}$ as defined above.

By a *TIS-relation algebra* we understand a relation algebra generated by $1',P,D,O,M,S$, and F for some TIS-frame $(U,<,1',P,D,O,M,S,F)$. A TIS relation algebra has 13 atoms, namely the relations from the corresponding TIS-frame and their converses. Observe that $1'^{-1} = 1'$. Any TIS-relation algebra is isomorphic to the TIS-relation algebra whose universe is the set of real numbers. Detailed discussions of relation algebras for reasoning about time (and space) can be found in [20] and [6].

6 Deduction in TIS-Relation Algebras

Deduction system for TIS-relation algebras processes formulas built either with relations $1', P, D, O, M, S, F$, acting on temporal intervals or with relation $<$ acting on time points. For the sake of uniformity, we can preprocess the interval formulas by replacing $1'$ (acting on temporal intervals), P, D, O, M, S, F by their definitions in terms of $1'$ (acting on time points) and $<$. A deduction system for TIS-relation algebras consists of the deduction rules and axiomatic sequences for point relations presented in Section 2, the rules of the same form for interval relations, and the following specific rules:

$$(\text{irref} <) \quad \frac{K, x(-1')y, H}{K, x < y, H, x(-1')y}$$

$$(\text{lin} <) \quad \frac{K}{K, x(-1')y \mid K, x(-<)y \mid K, y(-<)x}, \quad x, y \text{ are any variables}$$

$$(\text{tran} <) \quad \frac{K, x < y, H}{K, x < z, H, x < y \mid K, z < y, H, x < y}, \quad z \text{ is any variable}$$

The following rules reflect the property that $<$ does not have endpoints and is discrete:

$$(\text{nomin} <) \quad \frac{K}{K, x(-<)z}, \quad z \text{ is a restricted variable}$$

$$(\text{nomax} <) \quad \frac{K}{K, z(-<)x}, \quad z \text{ is a restricted variable}$$

$$(\text{dense}) \quad \frac{K}{K, x < y \mid K, x(-<)z, z(-<)y}, \quad z \text{ is a restricted variable}$$

The rules that provide definitions of the relations $1'$ acting on the intervals P, D, O, M, S, and F in terms of $<$ and $1'$ acting on time points are:

$$(1') \quad \frac{K, [t,t']\ 1'\ [u,u'], H}{K, t1'u, H \mid K, t'1'u', H} \qquad (-1') \quad \frac{K, [t,t']\ (-1')\ [u,u'], H}{K, t(-1')u, t'(-1')u', H}$$

$$(P) \quad \frac{K, [t,t']\ P\ [u,u'], H}{K, t' < u, H} \qquad (-P) \quad \frac{K, [t,t']\ (-P)\ [u,u'], H}{K, t'(-\ <)u, H}$$

$$(D) \quad \frac{K, [t,t']\ D\ [u,u'], H}{K, u < t, H \mid K, t' < u', H} \qquad (-D) \quad \frac{K, [t.t']\ (-D)\ [u,u'], H}{K, u(-<)t, t'(-<)u', H}.$$

$$(O) \quad \frac{K, [t,t']\ O\ [u,u'], H}{K, t < u, H \mid K, u < t', H} \qquad (-O) \quad \frac{K, [t,t']\ (-O)\ [u,u'], H}{K, t(-<)u, u(-<)t', H}$$

$$(M) \quad \frac{K, [t,t']\ M\ [u,u'], H}{K, t'1'u, H} \qquad (-M) \quad \frac{[t,t']\ (-M)\ [u,u']}{K, t'(-1')u, H}$$

$$(S)\ \frac{K,[t,t']\ S\ [u,u'],H}{K,t1'u,H \mid K,t'<u',H} \qquad (-S)\ \frac{K,[t,t']\ (-S)\ [u,u'],H}{K,t(-1')u,t'(-<)u',H}$$

$$(F)\ \frac{K,[t,t']\ F\ [u,u'],H}{K,t'1'u',H \mid K,u<t,H} \qquad (-F)\ \frac{K,[t,t']\ (-F)\ [u,u'],H}{K,t'(-1')u',u<t,H}$$

The decomposition rules for the compound interval relations generated by the relations listed in Section 4 with the standard relational operations and the specific rules for $1'$ on the set $\{[t,t'] : t,t' \in U, t<t'\}$ and for $1'$ on U are analogous to the corresponding rules in Section 2. Abusing the notation is harmless, because the arguments of the relation indicate on which set it is defined. A completeness Theorem analogous to Theorem 1 holds for the deduction system presented here. The details can be found in [2].

Example 4
We show that $F\ ;\ P \subseteq P$.
(1) $[t,t']\ (-(F\ ;\ P) \cup P)\ [u,u']$.
After application of rule $(\cup)$ we obtain:
(2) $[t,t']\ (-(F\ ;\ P))\ [u,u']$, [t,t'] P [u,u'].
Now we apply rule $(-\ ;\ )$ with a restricted interval $[z,z']$:
(3) $[t.t']\ (-F)\ [z,z']$, $[z,z']\ (-P)\ [u,u']$, $[t,t']\ P\ [u,u']$.
To (3) we apply rules (P) and (-P) which yield:
(4) $[t,t']\ (-F)\ [z,z']$, $z'(-<)u$, $t'<u$.
After application of rule (-F) we have:
(5) $t'(-1')z'$, $z<t$, $z'(-<)u$, $t'<u$.
Applying rule (tran $<$) with a new variable z' to $z<t$ we get two sequences:
(5.1) $t'1'z'$, $t'(-1')z'$, $z<t$, $z'(-<)u$, $t'<u$,
(5.2) $z'<u$, $t'(-1')z'$, $z<t$, $z'(-<)u$, $t'<u$.
Both of them are axiomatic of the form (a1).

7 Information Relations Derived from Relational Attribute Systems

Relational attribute systems [8] expand the notion of an information system in order to make explicit various conditions that are implicitly assumed in connection with information systems. By a relational attribute system (RAS) we mean a system $(U,\Omega,\{V_a : a\in\Omega\},\{\mathrm{Rel}_a : a\in\Omega\},\Delta)$, where

- U is a nonempty set of objects,
- Ω is a finite nonempty set of attributes, and, for each $a\in\Omega$, V_a is a nonempty set of values of attribute a,
- Each attribute is a function $a : U \mapsto \mathscr{P}(V_a)$.
- Each $R \in \mathrm{Rel}_a$ is a binary relation $R \subseteq U \times V_a$, and
- Δ is a set of constraints on relations from Rel_a's.

An appropriate choice of the families Rel_a of relations and constraints Δ enables us to explicitly specify various types of information structures.

For example, if an information system is assumed to be deterministic, then we postulate that for each attribute $a\in\Omega$ there is a relation $I_a \subseteq U \times V_a$ with the intuition that xI_av iff $v \in a(x)$ and whenever xI_av and xI_av' hold, then $v = v'$. If we additionally postulate that for every $x \in U$ there is exactly one $v \in V_a$ such that xI_av, then such an information system does not have missing values. To represent these constraints relationally, we additionally introduce relations $1'_a$ of identity on V_a and $1_a = U \times V_a$ for each $a \in \Omega$. Then, our constraints can be represented as the following two properties:

(Δ1) $I_a^{-1};I_a \subseteq 1'_a$, I_a is functional,
(Δ2) $I_a;1_a = 1_a$, I_a is total.

The family of sets specific to deterministic relational attribute systems is $\{I_a(x) : a \in \Omega, x \in U\}$.

If the system is nondeterministic, a set of attribute values assigned to an object may have several intuitive meanings, as mentioned in the Introduction. For example, to distinguish between disjunctive and conjunctive interpretation of nondeterministic information we consider relations $I_a, B_a \subseteq U \times V_a$ for each $a \in \Omega$ with the intuition that xI_av iff object x certainly possesses property v and xB_av iff object x possibly possesses property v (see also [8]). For every $a \in \Omega$, these relations are assumed to satisfy the constraint

(Δ3) $I_a \cap B_a = \emptyset$,

which says that I_a and B_a are incompatible. The family of sets specific to nondeterministic relational attribute systems is $\{I_a(x) : a \in \Omega, x \in U\} \cup \{B_a(x) : a \in \Omega, x \in U\}$.

Concerning information relations derived from the RAS's defined above we note that the building stones are the relations $\{1', \subsetneq, \supsetneq, P, D\}$, each of which is defined on 2^{VAL} for $\mathrm{VAL} = \bigcup\{V_a : a \in \Omega\}$. The relations P (*partial overlap*) and D (*disjointness*) are defined by

$$xPy \iff x \cap y \neq \emptyset,\ x \not\subseteq y,\ y \not\subseteq x, \text{ and } xDy \iff x \cap y = \emptyset.$$

An *information relation* derived from a RAS $= (U, \Omega, \{V_a : a \in \Omega\}, \{\mathrm{Rel}_a : a \in \Omega\}, \Delta)$ is a binary relation on U having the form

$$R';\rho;S'^{-1},$$

where $\rho \in \{1', \subsetneq, \supsetneq, P, D\}$, and R', S' are extensions of $R, S \in \mathrm{Rel}_a$, defined on $U \times 2^{\mathrm{VAL}_a}$ by

$$xR'_aA \iff R_a(x) = A.$$

Hence, $xR';\rho;S'^{-1}y$ iff $\langle R(x), S(y)\rangle \in \rho$.

By an (abstract) RAS-frame we understand a relational system

$$(U, V, \{\mathrm{Rel}_a : a \in A\}, 1', <, >, \pi, \delta),$$

where

- U, V and A are nonempty sets,
- $\mathrm{Rel}_a \subseteq 2^{U \times V}$,
- $<, >, \pi, \delta$ are binary relations on V,
- $1'$ is the identity on V,

and the following constraints are satisfied:

$(\Delta 4)$ $1' \cup < \cup > \cup \pi \cup \delta = 1$,
$(\Delta 5)$ Any two of $1', <, >, \pi, \delta$ are disjoint,
$(\Delta 6)$ δ is irreflexive and symmetric,
$(\Delta 7)$ $<$ and $>$ are irreflexive, transitive, and $> = <^{-1}$.

Clearly, the relations $1', \subset, \supset, P, D$ satisfy the constraints $(\Delta 4), \ldots, (\Delta 7)$.

By a RAS–relation algebra we understand an algebra of relations generated by the relations of the form

$$R; \rho; S^{-1},$$

where $R, S \in \mathrm{Rel}_a$, and $\rho \in \{1', <, >, \pi, \delta\}$. Any particular class of RAS–relation algebras is obtained by specifying axiomatically the family $\{Rel_a : a \in A\}$.

8 Deduction in RAS-Relation Algebras

The deduction system for RAS-relation algebras processes formulas built with relational terms involving both the relations from the families $\mathrm{Rel}_a, a \in A$, and relations $1', <, >, \pi, \delta$. Depending on the corresponding relations, the variables in the formulas may be either variables from U or variables from V.

The system consists of the rules for standard algebras of binary heterogenous relations and the rules reflecting the constraints from Δ. The rules for the operations of the algebras of heterogenous relations are analogous to the rules for the standard algebras of binary relations presented in Section 2 with an obvious restriction on domains of the left and right arguments of the relations. Below we present the exemplary rules reflecting the constraints discussed above. The rules for the constraints $(\Delta 1), (\Delta 2), (\Delta 3), (\Delta 4)$, and $(\Delta 5)$ are as follows. We assume that U-variables range over elements of set U, and V-variables over elements of V.

$(r\Delta 1)$ $$\frac{K, v1'v', H}{K, xI_a v, H, v1'v' \mid K, xI_a v', H, v1'v'}$$
x is any U-variable

$(r\Delta 2)$ $$\frac{K}{K, x - I_a v}$$
x is any U-variable, v is a new V-variable

$(r\Delta 3)$ $$\frac{K}{K, xB_a v, \mid K, xI_a v'}$$
x is any U-variable, v, v' are any V-variables

$(r\Delta 4)$ $$\frac{K}{K, x - 1'y \mid K, x(- \ <)y \mid K, x(- >)y \mid K, x - \pi y \mid K, x - \delta y}$$
where x, y are any V-variables

$(r\Delta 5)$ $$\frac{K}{K, xRy \mid K, xSy}$$
where $R, S \in \{1', <, >, \pi, \delta\}$, $R \neq S$, and x, y are any V-variables

The rules corresponding to the constraints $(\Delta 6)$ and $(\Delta 7)$ are analogous to the respective constraints for the relations of the preceding sections. Irreflexivity of δ, $<$, and $>$ is reflected by the rules obtained as the irreflexivity rule in Section 6. Symmetry of

δ is as in Section 4. The transitivity of $<$ and $>$ is as in Section 4. It is easy to extend the proof of completeness of the basic system described in Section 2 to the system expanded with the rules presented above. Namely, we check that the specific rules reflecting properties of the constants $\mathrm{Rel}_a, a \in A$, and $1', <, >, \pi, \delta$ are correct. Next, the definition of branch model is extended to include the meaning of these constants and the branch model theorem and the satisfaction in branch model theorem are proved in a standard way. These lemmas enable us to prove soundness and completeness of the RAS-theory whose family $\{R_a : a \in \Omega\}$ consists of relations I_a, B_a for $a \in \Omega$, satisfying the axioms $\Delta 1, \ldots, \Delta 7$. Any other RAS-theory can be obtained by choosing a family $\{R_a : a \in \Omega\}$ of relations and by postulating some axioms on the relations from that family.

Example 5
We show that in our exemplary RAS-theory if $I_a \ ; \ 1' \ ; \ I_a^{-1} = 1$, then $I_a \ ; \ -\delta \ ; \ I_a^{-1} = 1$. We use the principle of proving entailment explained in Section 2. According to that principle we have to prove the following formula:
(1) $x(1 \ ; \ (-(I_a \ ; \ 1' \ ; \ I_a^{-1})) \ ; \ 1 \cup I_a \ ; \ -\delta \ ; \ I_a^{-1})y$.
We apply the rule $(\cup)$, then twice the rule $(\ ; \ )$ with new variables x and y, and then twice the rule $(- \ ; \ )$ with restricted variables z and t, and we get:
(2) $x(-I_a)z$, $z(-1')t$, $t(-I_a)y$, $x(I_a \ ; \ -\delta \ ; \ I_a^{-1})y$ and two axiomatic sequences containing formulas $x1x$ and $y1y$.
We apply rule $(\ ; \ )$ with new variable z to the first composition and we obtain two sequences:
(2.1) containing xI_az and $x(-I_a)z$, so it is axiomatic, and
(2.2) containing $z(-\delta \ ; \ I_a^{-1})y$.
Decomposing the composition in (2.2) with rule $(\ ; \ )$ with new variable t we obtain two sequences:
(2.2.1) containing $z(-\delta)t$, $z(-1')t$,
(2.2.2) containing $tI_a^{-1}y$, yI_at, $y(-I_a)t$ which is axiomatic.
We apply rule (irref δ) to (2.2.1) obtaining a sequence with $z\delta t$ and $z(-\delta)t$ which is axiomatic.

9 Information Relations Derived from Fuzzy Information Systems

Fuzzy information systems differ from the ordinary information systems in that the sets of properties assigned to the objects are fuzzy sets. To define fuzzy sets we need, first of all, to establish a range of fuzziness, that is an algebra whose elements serve as degrees of membership of the elements to fuzzy sets. In this paper we assume that the range of fuzziness is modelled by a class of commutative doubly residuated lattices ([27], [5]).

A commutative *doubly residuated lattice* is a structure of the form

$$(W, \vee, \wedge, \otimes, \rightarrow, \oplus, \leftarrow, 1, 0),$$

where

- $(W, \vee, \wedge, 1, 0)$ is a lattice with the top element 1 and the bottom element 0;
- $(W, \otimes, 1)$ and $(W, \oplus, 0)$ are monoids;
- $\otimes$ and $\oplus$ are commutative;

- $\rightarrow$ is a *residuum* of $\otimes$, that is $z \leq x \rightarrow y$ iff $x \otimes z \leq y$ for all $x, y, z \in W$;
- $\leftarrow$ is a *dual residuum* of $\oplus$, that is $x \leftarrow y \leq z$ iff $y \leq x \oplus z$ for all $x, y, z \in W$.

The operations $\otimes$ and $\oplus$ are referred to as *product* and *sum*, respectively. They are intended to be abstract counterparts to t-norms and t-conorms, respectively. Condition (4) is referred to as a *residuation condition* and condition (5) is a *dual residuation condition*. Clearly, $\rightarrow$ and $\leftarrow$ are uniquely determined by the residuation condition and the dual residuation condition, respectively. Furthermore, it follows from these conditions that $x \rightarrow y$ is the greatest element in the set $\{z : x \otimes z \leq y\}$ and $x \leftarrow y$ is the least element in the set $\{z : y \leq x \oplus z\}$.

Given a doubly residuated lattice $L = (W, \vee, \wedge, \otimes, \rightarrow, \oplus, \leftarrow, 1, 0)$ and a universe U of objects, any mapping $X : U \mapsto L$ is an *L-fuzzy subset* of U. The family of L-fuzzy subsets of U is denoted by $F_L\mathscr{P}(U)$. By a *fuzzy set* we understand an L-fuzzy set for some doubly residuated lattice L. The operations on L-fuzzy sets are defined in the following way. Let $X, Y \in F_L\mathscr{P}(U)$, then:

$$(X \cup_L Y)(x) = X(x) \oplus Y(x),$$

$$(X \cap_L Y)(x) = X(x) \otimes Y(x).$$

The empty fuzzy set is defined as $\emptyset_L(x) = 0$, and the full set is $1_L(x) = 1$.

L-inclusion and *L-equality* of L-fuzzy sets are defined as follows:

$X \subseteq_L Y$ iff for every $x \in U, X(x) \leq Y(x)$, where $\leq$ is the natural ordering of lattice L;

$X =_L Y$ iff $X \subseteq_L Y$ and $Y \subseteq_L X$.

An *L-fuzzy binary relation* on U is a mapping $R : U \times U \mapsto L$. The family of all L-fuzzy binary relations on U is denoted by $F_L\mathrm{Rel}(U)$. By a *fuzzy relation* we understand an L-fuzzy relation for some doubly residuated lattice L. Clearly, every fuzzy relation on a set U is a fuzzy subset of $U \times U$, so the operations on fuzzy sets apply to fuzzy relations.

A *fuzzy information system* is a structure of the form

$$(U, L, \Omega, \{V_a : a \in \Omega\}),$$

where U is a non-empty set of objects, L is a commutative doubly residuated lattice, every attribute $a \in \Omega$ is a mapping $a : U \mapsto F_L\mathscr{P}(V_a)$ which assigns an L-fuzzy subset of V_a to an object. Intuitively, $a(x)(v)$ is a degree to which an object x assumes the value v of the attribute a.

We define several L-fuzzy binary relations on a family $F_L\mathscr{P}(U)$, for any set U. These relations provide patterns for information relations derived from a fuzzy information system. Let $X, Y \in F_L\mathscr{P}(U)$, then:

In_L (L-inclusion):	$In_L(X,Y) = \inf\{X(x) \rightarrow Y(x) : x \in U\}$,
Ni_L (L-non-inclusion):	$Ni_L(X,Y) = \sup\{Y(x) \leftarrow X(x) : x \in U\}$,
Sim_L (L-similarity):	$Sim_L(X,Y) = sup\{X(x) \otimes Y(x) : x \in U\}$,
Exh_L (L-exhaustiveness):	$Exh_L(X,Y) = \inf\{X(x) \oplus Y(x) : x \in U\}$,

For a discussion of fuzzy information relations see also [30].

A specific family of fuzzy sets associated to a fuzzy information system $S=(U,L,\Omega,\{V_a : a\in\Omega\})$ is the family:

$$\{a(x) : a\in\Omega \text{ and } x\in U\}.$$

A most basic algebra of fuzzy relations is just an algebra $(F_L\mathrm{Rel}(U),\cup_L,\cap_L,\emptyset_L,1_L)$. FIS-relation algebra (relation algebra of fuzzy information systems) is an algebra of fuzzy relations generated by the set relations In_L, Ni_L, Sim_L, and Exh_L defined above.

A construction of a Rasiowa–Sikorski style or a Gentzen style deduction system for fuzzy relations, and in particular for FIS-relation algebras is an open problem. So the only available means of reasoning about these relations is the equational reasoning within doubly residuated lattices. For the arithmetic of doubly residuated lattices see [27].

In an abstract setting, fuzzy algebras and fuzzy relation algebras are presented and investigated, among others, in [12,14,38,39].

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