Minimum Time for a Hybrid System with Thermostatic Switchings

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Abstract. In this paper we study a minimum time problem for a hybrid system subject to thermostatic switchings. We apply the Dynamic Programming method and the viscosity solution theory of Hamilton-Jacobi equations. We regard the problem as a suitable coupling of two minimum-time/exit-time problems. Under some controllability conditions, we prove that the minimum time function is the unique bounded below continuous function which solves a system of two Hamilton-Jacobi equations coupled via the boundary conditions.

1 Introduction

In this paper we study a minimum time problem for a hybrid system in $\mathbb{R}^n$ whose evolution y is subject to a switching parameter which may take the values, 1 and -1. In particular, the switching rule is subject to the evolution of an assigned component of the state y, and it is governed by a so-called thermostatic (or relay-type) hysteresis input-output relationship. That is, the switching between the two values occurs when such fixed component of y reaches (or better, gets over) some fixed thresholds, see Figure 1 for an example.

We are interested in applying the Dynamic Programming method to such kinds of problem, and then study the corresponding Hamilton-Jacobi-Bellman equation in the framework of the viscosity solutions theory. In this paper, under some controllability hypotheses, we prove the continuity of the minimum time function, we derive a Hamilton-Jacobi problem satisfied by the minimum time function in the viscosity sense, and prove that the latter is indeed the unique viscosity solutions. Such a Hamilton-Jacobi problem is given by a system of two Hamilton-Jacobi equations coupled by some part of the boundary conditions, which are also expressed in the viscosity sense. Our method consists in interpreting the problem as a coupling of two optimal control problems which present feature of minimum-time as well as exit-time problems.

Optimal control problems for systems with thermostatic behavior are of course important for applications. Many mechanical, physical, economical, biological systems have such a kind evolution (see for instance [12] for motivations from magnetism, [15] for a biological motivation, [14] from economics). Moreover, the thermostat is a simple example of hysteresis operator, which is also fundamental

A. Bemporad, A. Bicchi, and G. Buttazzo (Eds.): HSCC 2007, LNCS 4416, pp. 32–45, 2007.

to construct the so-called Preisach model of hysteresis, which is one of the most interesting and versatile analytical description of hysteresis phenomena (it is a superposition of a continuum of thermostats).

The dynamic programming method for the optimal control of systems with hysteresis feature is our main motivation. The present author has already studied, via dynamic programming method and viscosity solutions theory, some optimal control problems with hysteresis (see [2], [3], [4], [5]) (which lead to different Hamilton-Jacobi equations with respect to present one). Some possible directions of future investigations on the minimum time problem are: the case of a "discrete" Preisach operator, i.e. a finite sum of thermostats, see [3] for a one-dimensional infinite horizon problem in this framework; the case of the "true" Preisach operator, i.e. a superposition of a continuum of thermostats, see [4] for a one-dimensional finite horizon problem in this framework; the case of laking controllability conditions on the switching points, which leads to discontinuous solutions of Hamilton-Jacobi equations, see [5] for a multidimensional finite horizon problem in this framework. We also recall the works of Belbas and Mayergoyz on some applications of the Dynamic Programming method to optimal control problems with hysteresis (see for instance [8] and [9]).

Systems with thermostatic (or relay-type) switchings are also good examples of hybrid systems where the switchings are mandatory, when the state reaches some particular interdict zones (which is probably one of the most difficult behavior to treat in the framework of Dynamic Programming method and viscosity solutions). The application of the Dynamic Programming method to hybrid optimal control problems was first outlined by Branicky, Borkar, and Mitter in [11]. Bensoussan and Menaldi in [10] were the first to apply the viscosity solutions theory to a hybrid control problem, and they proved uniqueness of the value function as continuous viscosity solutions of the corresponding Hamilton-Jacobi problem. However, in the study in the Hamilton-Jacobi problem, they suppose that the system has no mandatory switchings. The present author, as already outlined, in [3] and [5], applied the viscosity solution theory to hybrid problems with thermostats, where the switchings are mandatory (although there are only mandatory switchings). In particular in [5] the case where the value function is discontinuous is addressed. Recently, other works on this subject have appeared. Dharmatti and Ramaswamy in [13] studied a problem with continuous value function, whereas Zhang and James in [18] studied a problem with discontinuous value function.

Finally we recall that the mathematical theory of hysteresis operators may be found in Visintin [17], and the theory of viscosity solutions for Hamilton-Jacobi-Bellman equations in Bardi-Capuzzo Dolcetta [7].

The present paper is organized as follows. In Section 2 we describe the delayed relay switching rule and introduce the minimum time problem; in Section 3 we prove the continuity of the minimum time function; in section 4 we prove that the minimum time function is the unique viscosity solution of the associated Hamilton-Jacobi problem; in the Appendix we give some results for an optimal control problem (without switchings) which is a combination of minimum time and exit-time features (which is, at least as formulation, rather new). In the Appendix. we also give the definitions of viscosity solutions and of boundary conditions in the viscosity sense.

2 Models and Problems

2.1 The Delayed Relay Switching Rule

For more details on the argument of this subsection see Visintin [17]. Let us fix two thresholds $\rho_1, \rho_2 \in \mathbb{R}$ with $\rho_1 < \rho_2$, and write $\rho := (\rho_1, \rho_2)$. For every continuous input function $g : [0, +\infty[\to \mathbb{R}$, and for every initial state $w_0 \in \{1, -1\}$, we define the following discontinuous output function $z(\cdot) := h_\rho[g, w_0](\cdot) : [0, +\infty[\to \{1, -1\}$ by

$$z(0) := \begin{cases} 1 & \text{if } g(0) > \rho_2, \\ -1 & \text{if } g(0) < \rho_1 \\ w_0 & \text{if } \rho_1 \le g(0) \le \rho_2, \end{cases}$$

and, for $t > 0$, we define $X(t) := \left\{ \tau \in [0, t] \middle| g(\tau) < \rho_1 \text{ or } g(\tau) > \rho_2 \right\}$, and

$$\begin{cases} z(t) = w_0 & \text{if } X(t) = \emptyset, \\ z(t) = 1 & \text{if } X(t) \neq \emptyset, \text{ and } g(\sup X(t)) \ge \rho_2, \\ z(t) = -1 & \text{if } X(t) \neq \emptyset, \text{ and } g(\sup X(t)) \le \rho_1. \end{cases}$$

For instance, if $z(t) = 1$ (which of course implies that $g(t) \ge \rho_1$), then z will remain constantly equal to 1 until g will possibly get over (downward) the threshold ρ_1, and after that time, z will be switched on -1 until g will possibly get over (upward) the threshold ρ_2. For example, if $z(t) = -1$ and, in the time interval $[t, t']$, g strictly increases from a value $g_1 < \rho_2$ to a value $g_2 > \rho_2$, passing on the threshold ρ_2 at the time t'', then the output is $z \equiv -1$ in $[t, t'']$, $z \equiv 1$ in $]t'', t']$. If instead g, after reaching the threshold ρ_2 at the time t'', changes the monotonicity and stays below or equal to ρ_2, then $z \equiv -1$ in $[t, t']$.

2.2 The Delayed Controlled Dynamical System

Let us consider a set of constant controls $A \subset \mathbb{R}^m$, for some m, a function

$$f : \mathbb{R}^n \times \{-1, 1\} \times A \to \mathbb{R}^n,$$

a fixed unit vector $S \in \mathbb{R}^n$, and $\rho = (\rho_1, \rho_2)$ a couple of thresholds for a delayed relay h_ρ. Let us also define the set of measurable controls

$$\mathcal{A} := \left\{ \alpha : [0, +\infty[\to A \middle| \alpha \text{ is measurable} \right\}.$$

Then, we consider the following dynamical system (the dot "·" between vectors is the usual scalar product)

$$\begin{cases} y'(t) = f(y(t), z(t), \alpha(t)), \ t > 0, \\ z(t) = h_\rho[y(\cdot) \cdot S, w](t), \quad t \ge 0, \\ y(0) = x, \end{cases} \tag{1}$$

where $\alpha \in \mathcal{A}$, the initial state is the couple (x, w), and it is admissible if $w = 1$ and $x \cdot S \geq \rho_1$, or $w = -1$ and $x \cdot S \leq \rho_2$. Hence the solution $(y(\cdot), z(\cdot))$ of (1) (if it exists) can be view as a trajectory starting from (x, w) and evolving in the subset of $\mathbb{R}^n \times \mathbb{R}$

$$\overline{\mathcal{H}} := \overline{\mathcal{H}}_1 \cup \overline{\mathcal{H}}_{-1},$$

where,

$$\begin{aligned} \overline{\mathcal{H}}_1 &:= \left\{(x, 1) \in \mathbb{R}^n \times \{1\} \middle| x \cdot S \geq \rho_1\right\}, \\ \overline{\mathcal{H}}_{-1} &:= \left\{(x, -1) \in \mathbb{R}^n \times \{-1\} \middle| x \cdot S \leq \rho_2\right\}, \end{aligned} \tag{2}$$

with a rule for switching from one connected component $\overline{\mathcal{H}}_w$ to the other, given by the switching rule of the delayed relay. See Figure 1. In the sequel, for any $w \in \{-1, 1\}$, we will denote by $\mathcal{H}_w$ the set, defined as in (2), but with the strict inequality for $x \cdot S$; in some sense it is the "interior" of $\overline{\mathcal{H}}_w$. Moreover, with some abuse of notations, we will denote by $\partial\overline{\mathcal{H}}_w$ the set, defined as in (2), but with just the equality only for $x \cdot S$; in some sense it is the "boundary" of $\overline{\mathcal{H}}_w$ (it is one of the two "switching boundaries").

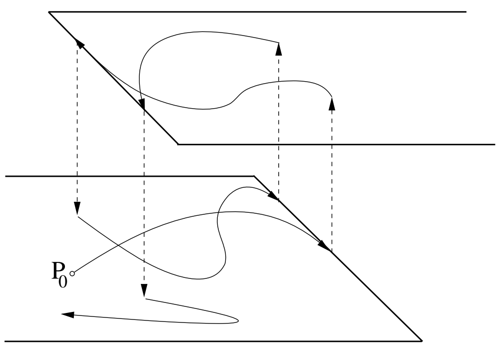

Fig. 1. Delayed switching evolution, starting from P_0

Some problems arise in defining a solution of (1): it even may not exist (see Alt [1] and also Bagagiolo [3] for a discussion on such a problem). We give the following definition of solution. For instance, let (x, w) be an initial state. We denote by $\tilde{y}(\cdot)$ the solution (if it exists) defined in $[0, +\infty[$ of the system

$$\begin{cases} \tilde{y}'(t) = f(\tilde{y}(t), w, \alpha(t)) \; t > 0, \\ \tilde{y}(0) = x. \end{cases}$$

Hence, we let the relay switch exactly when $h_\rho[\tilde{y}(\cdot), w]$ should switch by its switching rule. If $\tilde{t}$ is the switching time, we define, as solution of (1) in $[0, \tilde{t}]$, $(y(\cdot), z(\cdot)) = (\tilde{y}(\cdot), w)$. For $t > \tilde{t}$, we consider the solution in $\mathbb{R}^n$ $\tilde{y}_1$ starting from $\tilde{y}(\tilde{t})$ with dynamics $f(\tilde{y}_1, -w, \alpha)$. Let t_1 be the (possible) switching time for $h_\rho[y_1(\cdot - \tilde{t}), -1]$. Then we define, as solution of (1) in $]\tilde{t}, \tilde{t}+t_1]$, $(y(\cdot), z(\cdot)) = (y_1(\cdot), -1)$. We go on in this way. Since $\rho_1 < \rho_2$, then, any possible switching time is larger than the previous one plus an independent quantity $\delta > 0$ (recall that f is bounded and hence the velocity of $y \cdot S$ is bounded). Hence, we guess that such a construction of solution is possible for all the time, and moreover it is a good definition. This is Proposition 1, whose proof is now easy. We suppose that

$$\begin{cases} A \text{ is compact}, \quad f \text{is continuous and bounded} \\ \exists L > 0 \text{ s.t. } |f(x_1, w, a) - f(x_2, w, a) \le L|x_1 - x_2| \\ \qquad\qquad \forall x_1, x_2 \in \mathbb{R}^n,\ w \in \{-1, 1\},\ a \in A. \end{cases} \tag{3}$$

Proposition 1. *Under the hypothesis (3), for every initial state* $(x, w) \in \overline{\mathcal{H}}$*, and for every measurable control* $\alpha \in \mathcal{A}$*, there exists a unique solution (in the sense given above)* $(y(\cdot), z(\cdot)) \in C^0([0, +\infty[; \mathbb{R}^n) \times L^\infty(0, +\infty; \mathbb{R})$ *of the system (1). We will denote such a solution by* $\big(y_{(x,w)}(\cdot; \alpha), z_{(x,w)}(\cdot; \alpha)\big)$.

Remark 1. Note that, in general, for the solution as above, it is not true that $z(t) = h_\rho[y(\cdot) \cdot S](t)$ for all $t \ge 0$. Indeed, a trajectory switches when, if it does not switch, $y \cdot S$ is going to cross the threshold. But it may happens that, as the trajectory switches, the new dynamics $f(\cdot, -w, \cdot)$ is such that $y \cdot S$ does not cross the threshold. Hence, the glued trajectory has never crossed the threshold, and so, for the true switching delayed relay rule, there should not be any switching. A discussion more detailed on such definition of solution, is reported in Bagagiolo [3]. Here, we only say that, if a "true solution" of (1) exists, then it must coincide we the one above. Moreover, as will be explained in the sequel, such a definition seems useful for transforming an hybrid optimal control problem with such a kind of switching, in an exit-time problem from a closed set, which is, in some sense, more stable than other exit-time problems.

2.3 The Minimum Time Problem

Let $\mathcal{T} \subset \overline{\mathcal{H}}$ be fixed. The minimum time problem is to reach the target $\mathcal{T}$ as quickly as possible. Hence, for every initial state $(x, w) \in \overline{\mathcal{H}}$, and for every control $\alpha \in \mathcal{A}$, we define the reaching time for the corresponding trajectory of (1)

$$t_{(x,w)}(\alpha) = \inf \left\{ t \ge 0 \middle| \big(y_{(x,w)}(t; \alpha), z_{(x,w)}(t; \alpha)\big) \in \mathcal{T} \right\},$$

with the convention $\inf \emptyset = +\infty$. The minimum time function is then defined in $\overline{\mathcal{H}}$ as

$$T(x, w) = \inf_{\alpha \in \mathcal{A}} t_{(x,w)}(\alpha).$$

We consider the following hypotheses

$$\begin{gathered} \mathcal{T} \text{ is the closure of its interior, has compact boundary } \partial\mathcal{T} \\ \text{which is also a } C^2 \text{ manifold,} \\ \forall (x, w) \in \partial\mathcal{T},\ \text{denoting by } n(x, w) \text{ the outer normal to } \partial\mathcal{T}, \\ \inf_{a \in A} f(x, w, a) \cdot n(x, w) < 0. \end{gathered} \tag{4}$$

Except for the compactness of $\partial\mathcal{T}$, the other regularity hypotheses on $\mathcal{T}$ may be changed, for instance in order to take account of a single point. Roughly speaking we need of the usual hypotheses of controllability on $\partial\mathcal{T}$ in order to have continuity of the minimum time function.

We also suppose the following controllability properties hold on the switching boundaries: for every $(x,w) \in \partial\overline{\mathcal{H}}_w$ there exist $a_1, a_2 \in A$ such that

$$f(x,w,a_1) \cdot S < -c < 0 < c < f(x,w,a_2) \cdot S, \tag{5}$$

with $c > 0$ independent from (x,w).

Let us define the controllable set

$$\mathcal{R} = \Big\{ (x,w) \Big| \exists \alpha \in \mathcal{A}, t_{(x,w)}(\alpha) < +\infty \Big\} \tag{6}$$

Finally note that, both $\mathcal{T}$ and $\mathcal{R}$, can be split in to the disjoint union of two sets $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_{-1}$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_{-1}$, where, for $w \in \{1,-1\}$

$$\mathcal{T}_w := \mathcal{T} \cap \overline{\mathcal{H}}_w, \quad \mathcal{R}_w = \mathcal{R} \cap \overline{\mathcal{H}}_w.$$

3 Continuity

Proposition 2. *Let (3), (4), (5) hold. Then T is continuous in $\mathcal{R}$.*

We first recall the following lemma. For the proof, which is quite standard, see for instance Soner [16] (see also Bagagiolo-Bardi [6])

Lemma 1. *With the same hypotheses as in Proposition 2, let $w \in \{-1,1\}$ and $K \subseteq \overline{\mathcal{H}}_w$ compact be fixed. Let us also fix $t \geq 0$. Then there exists a radius $r > 0$, and a constant $C > 0$ (both depending on t and K) such that, whenever for some $(x,w) \in K$ and for some $\alpha \in \mathcal{A}$ we have (here $B((x,w),r)$ is the ball of radius r around (x,w)),*

$$\big(y_{(x,w)}(\tau;\alpha), z_{(x,w)}(\tau;\alpha)\big) \in \overline{\mathcal{H}}_w \;\forall \tau \in [0,t],$$

then, for every $(\xi,w) \in B\left((x,w),r\right) \cap \overline{\mathcal{H}}_w$, there exists a control $\overline{\alpha}$ such that

$$\big(y_{(\xi,w)}(\tau;\overline{\alpha}), z_{(\xi,w)}(\tau;\overline{\alpha})\big) \in \overline{\mathcal{H}}_w, \quad \forall \tau \in [0,t],$$

and

$$|y_{(x,w)}(t;\alpha) - y_{(\xi,w)}(t;\overline{\alpha})| \leq C|x-\xi|. \tag{7}$$

Proof. (Of Proposition 2.) Let us sketch the proof that T is continuous in $\mathcal{R}_1$ (the other case is similarly treated). First of all note that $\mathcal{R}_1$ may be empty, and, in such a case there is nothing to prove. Hence, let us suppose that it is not empty (note that, by the hypotheses made, at least one of $\mathcal{R}_1$ and $\mathcal{R}_{-1}$ is not empty). Let $(x,1),(\xi,1)$ be two points of $\mathcal{R}_1$, and, for every $\varepsilon > 0$, let us take a control $\alpha_\varepsilon \in \mathcal{A}$ such that

$$T(x,1) \geq t_{(x,1)}(\alpha_\varepsilon) - \varepsilon.$$

Let r_ε be the number of switchings of the trajectory $\big(y_{(x,1)}(\cdot;\alpha_\varepsilon), z_{(x,1)}(\cdot;\alpha_\varepsilon)\big)$, in the time interval $[0, t_{(x,1)}(\alpha_\varepsilon)]$, and note that, by the fact that the switchings are delayed and that f is bounded, there exists $N > 0$ such that

$$r_\varepsilon \leq N, \ \forall \varepsilon > 0. \tag{8}$$

Let U be a bounded open neighborhood of $(x, 1)$ in $\overline{\mathcal{H}}_1$.

Let us first suppose that $r_\varepsilon = 0$. Then the trajectory starting from $(x, 1)$ with control α_ε does not switch up to the time $t_{(x,1)}(\alpha_\varepsilon)$. Hence, for a suitable ball $B \subseteq U$ around $(x, 1)$, and a suitable constant $C > 0$, applying Lemma 1, for every point $(\xi, 1) \in B \cap \overline{\mathcal{H}}_1$, we obtain a control $\overline{\alpha}_\varepsilon$ such that the corresponding trajectory starting from $(\xi, 1)$ does not switch, and (7) holds. Hence, since

$$\big(y_{(x,1)}(t_{(x,1)}(\alpha_\varepsilon); \alpha_\varepsilon), z_{(x,1)}(t_{(x,1)}(\alpha_\varepsilon); \alpha_\varepsilon)\big) \in \partial\mathcal{T},$$

by the controllability hypothesis on the boundary of the target, if $|x - \xi|$ is small enough, we obtain that

$$t_{(\xi,1)}(\overline{\alpha}_\varepsilon) = t_{(x,1)}(\alpha_\varepsilon) + O(|x - \xi|). \tag{9}$$

Now, let us suppose that $r_\varepsilon = 1$, and let t_1 be the switching instant. Again, for a suitable ball $B_1 \subseteq U$ around $(x, 1)$, and for a suitable constant $C_1 > 0$, for every point $(\xi, 1) \in B_1 \cap \overline{\mathcal{H}}_1$, we obtain a control $\overline{\alpha}^1_\varepsilon$ such that

$$\big(y_{(\xi,1)}(\tau; \overline{\alpha}^1_\varepsilon), z_{(\xi,1)}(\tau; \overline{\alpha}^1_\varepsilon)\big) \in \overline{\mathcal{H}}_1 \ \forall \tau \in [0, t_1],$$

and (7) holds with C_1 as constant. Since $\big(y_{(x,1)}(t_1; \alpha_\varepsilon), z_{(x,1)}(t_1; \alpha_\varepsilon)\big) \in \partial\mathcal{H}_w$, by the controllability hypotheses (refeq:switchingcondition), if $|x - \xi|$ is small enough, we can use a suitable control in order to make the trajectory starting from $(\xi, 1)$ switch in a lap of time of order $O(|x - \xi|)$. Then we have two new starting points on $\overline{\mathcal{H}}_{-1}$ which are $\big(y_{(x,1)}(t_1; \alpha_\varepsilon), -1\big)$ and, say, $(\xi_1, -1)$ with $|y_{(x,1)}(t_1; \alpha_\varepsilon) - \xi_1| = O(|x - \xi|)$. Since in the remaining time interval $[t_1, t_{(x,1)}(\alpha_\varepsilon)]$, the trajectory starting from $(x, 1)$ does not switch anymore, we eventually construct a control $\overline{\alpha}_\varepsilon$ for which (9) still holds, for $|x - \xi|$ sufficiently small.

Finally, repeating the previous steps r_ε times, for $r_\varepsilon > 1$, we obtain, for $|x-\xi|$ sufficiently small, let say less than $\mu > 0$, the same relation as in (9). In particular μ depends only on $\overline{U}$ and on N.

Recalling (8), we can say that, for every r_ε, we can use the same infinitesimal error-function O in (9).

Hence, if $(x, 1), (\xi, 1) \in U$ and $|x - \xi| \leq \mu$, then, supposing $T(x, 1) \leq T(\xi; 1)$ we get, for the arbitrariness of $\varepsilon > 0$,

$$0 \leq T(\xi, 1) - T(x, 1) \leq O(|x - \xi|).$$

Otherwise, if $T(x, 1) > T(\xi, 1)$, we exchange the role of x and ξ and note that all the previous estimates remain unchanged, with the same constant and error-functions O. In particular, the number of switchings r_ε cannot increase, since every switching requires a lap of time $\delta > 0$, and hence from (9), we would get an absurd. Hence we obtain

$$0 \leq T(x,1) - T(\xi,1) \leq O(|x-\xi|),$$

and we conclude. □

Remark 2. By the proof of Proposition 2 we also get the fact that $\mathcal{R}$ is open in $\overline{\mathcal{H}}$ (for the induced topology). Indeed, if $(x,w) \in \mathcal{R}_w$, then we have shown the existence of a ball B around it such that $B \cap \overline{\mathcal{H}}_w \subset \mathcal{R}_w$.

4 DPP and HJB

In this section we want to study a Hamilton-Jacobi-Bellman (HJB) problem for the minimum time function, which arises by applying the Dynamic Programming Principle (DPP). Let $(x,w) \in \mathcal{R} \setminus \mathcal{T}$, and a control $\alpha \in \mathcal{A}$ be fixed. Considering the corresponding trajectory, we define the *first exit time* from $\overline{\mathcal{H}}_w$ as

$$\tau_x^w(\alpha) := \inf \left\{ t \geq 0 \middle| y_{(x,w)}(t;\alpha) \notin \overline{\mathcal{H}}_w \right\},$$

which is nothing but the *first switching time* (for the trajectory). If $a, b \in [0,+\infty]$, then we are going to use the following convention: $\chi_{\{a \leq b\}} = 1$ if $a \leq b$, $\chi_{\{a \leq b\}} = 0$ otherwise (similarly for $\chi_{\{a<b\}}$).

Proposition 3. *For every* $(x,w) \in \overline{\mathcal{H}}$*, we have*

$$\begin{aligned} T(x,w) = \inf_{\alpha \in \mathcal{A}} \Big(&\min(t_{(x,w)}(\alpha), \tau_x^w(\alpha)) \\ &+ \chi_{\{t_{(x,w)}(\alpha) > \tau_x^w(\alpha)\}} T(y_{(x,w)}(\tau_x^w(\alpha);\alpha), -w)\Big). \end{aligned} \tag{10}$$

Proof. It can be easily proved by using the following lemma and dynamic programming techniques.. □

Lemma 2. *For every* $(x,w) \in \partial\overline{\mathcal{H}}_w$*, and for every* $\alpha \in \mathcal{A}$ *such that* $\tau_x^w(\alpha) = 0$*, we have*

$$T(x,w) \leq T(x,-w), \quad t_{(x,w)}(\alpha) \geq T(x,-w). \tag{11}$$

Proof. Let us prove the first inequality in (11). By the controllability hypothesis (5), there exists a control $\overline{\alpha} \in \mathcal{A}$ such that $\tau_x^w(\overline{\alpha}) = 0$ (i.e. there is an immediate switching). For every $\varepsilon > 0$, we have

$$T(x,w) \leq \varepsilon + T\left(y_{(x,w)}(\varepsilon;\overline{\alpha}), -w\right),$$

and, letting $\varepsilon \to 0^+$, we conclude by the continuity of T.

To prove the second inequality, for every $\varepsilon > 0$, we have

$$t_{(x,w)}(\alpha) = \varepsilon + t_{(y_{(x,w)}(\varepsilon;\alpha),-w)}(\alpha(\cdot+\varepsilon)) \geq \varepsilon + T\left(y_{(x,w)}(\varepsilon;\alpha), -w\right),$$

and, again, we conclude by the continuity of T. □

Remark 3. Note that, in (10), we can replace $t_{(x,w)}(\alpha)$ with the following instant

$$t_x^w(\alpha) := \inf\left\{t \geq 0 \middle| \left(y_{(x,w)}(t;\alpha), w\right) \in \mathcal{T}_w\right\}.$$

Hence, Proposition 3 says that, for every $w \in \{1, -1\}$ fixed, we can regard our problem in $\overline{\mathcal{H}}_w$ as the problem of minimizing the reaching time of $\mathcal{T}_w$, subject to stopping the process and paying the time elapsed plus an exit cost if we exit from $\overline{\mathcal{H}}_w$ before reaching $\mathcal{T}_w$. In particular, the exit cost is given by $T(\cdot, -w)$, i.e. our minimum time function evaluated on the point where we "switch down" after exit from $\overline{\mathcal{H}}_w$.

In the sequel, by $\partial_{\overline{\mathcal{H}}_w}\mathcal{R}$ we will denote the boundary of $\mathcal{R}$ with respect to the induced topology in $\overline{\mathcal{H}}_w$, and by $int_{\overline{\mathcal{H}}_w}\mathcal{R}$ the interior of $\mathcal{R}$, with respect to the same topology. In particular note that such interior may intersect the boundary $\partial\overline{\mathcal{H}}_w$ of $\overline{\mathcal{H}}_w$.

Using Remark 3 and Proposition 5 in the Appendix, we can say that the minimum function $T : \mathcal{R} \to [0, +\infty[$ solves the following problem in the unknown u:

$$\begin{cases} \text{for every } w \in \{-1, 1\},\ u \text{ is a viscosity solution of} \\ \begin{cases} \sup\limits_{a \in A} \{-\nabla u(x,w) \cdot f(x,w,a)\} = 1 \text{ in } (\mathcal{R}_w \cap \mathcal{H}_w) \setminus \mathcal{T}_w, \\ u = 0 & \text{on } \partial\mathcal{T}_w, \\ u(x,w) \to +\infty & \text{as } (x,w) \to \partial_{\overline{\mathcal{H}}_w}\mathcal{R}_w, \\ u(\cdot, w) = u(\cdot, -w) & \text{on } \left(int_{\overline{\mathcal{H}}_w}\mathcal{R}_w \cap \partial\overline{\mathcal{H}}_w\right) \setminus \mathcal{T}_w. \end{cases} \end{cases} \tag{12}$$

In particular, in (12), the last boundary condition has to be understood in the *viscosity sense*. Now, we prove that T is indeed the unique solution of (12).

Theorem 1. *The minimum time function is the unique bounded below continuous function from $\mathcal{R}$ to $\mathbb{R}$ which solves the problem (12).*

Proof. First of all, for every $w \in \{1, -1\}$, let us denote by $(12)_w$ the Hamilton-Jacobi boundary problem in $\overline{\mathcal{H}}_w$ which appear in (12). Note that, even if we have a uniqueness result for each single problem $(12)_w$, we cannot immediate conclude that we have uniqueness for the problem (12), since the boundary conditions are intrinsic to the problem: they are part of the solutions.

For every $w \in \{1, -1\}$ we define the set

$$S^w := \left\{(x,w) \in \overline{\mathcal{H}}_w \middle| (x,-w) \in \partial\overline{\mathcal{H}}_{-w}\right\},$$

and note that it is exactly the set where we arrive when we exit from $\overline{\mathcal{H}}_{-w}$.

Let $u : \mathcal{R} \to \mathbb{R}$ be a bounded below continuous solution of (12). For every $w \in \{1, -1\}$ we extend ("by continuity") u from $(S^w \cap \mathcal{R}) \setminus \mathcal{T}$ to $S^w \setminus \mathcal{T}$ by setting $u = +\infty$. From the uniqueness result Theorem 2 of the Appendix, we know that $u(\cdot, w)$ is the value function of the minimum/exit time problem in $\overline{\mathcal{H}}_w$, given by reaching the target $\mathcal{T}_w$ or exit from $\overline{\mathcal{H}}_w$ paying the spent time plus the cost $u(\cdot, -w)$. So, it possibly differs from the problem solved by the minimum time T on $\overline{\mathcal{H}}_w$, only for the exit cost. We are going to prove that, for every $w \in \{1,-1\}$ the two problems $(12)_w$ solved respectively by T and u have the same exit cost, and hence the thesis will be proved.

For every $\delta > 0$ let us define the set

$$\mathcal{R}(\delta) = \left\{(x,w) \in \overline{\mathcal{H}} \middle| T(x,w) \leq \delta\right\}.$$

Note that, since the dynamics f is bounded, and since the switchings are delayed, there exists $\delta > 0$ such that for every starting point $(x,-w) \in S^{-w}$ we need a time strictly larger than δ in order to exit from $\overline{\mathcal{H}}_{-w}$, whichever is the control we are using.

We first prove that $u \geq 0$. To this end we prove that $u \geq 0$ on $S^w \cup S^{-w}$, from which the claim follows by the interpretation of u as value function. Let us suppose that there exists $(x,w) \in S^w$ such that $u(x,w) < 0$. This means that there are trajectories starting from (x,w) which reach points of the boundary $(x',w) \in \partial\overline{\mathcal{H}}_w$ where the cost is $u(x',-w) < -\delta$. But, the same argumentation shows that there should exist trajectories starting from $(x',-w)$ which reach points of the boundary $(x'',-w) \in \partial\overline{\mathcal{H}}_{-w}$ where the cost is $u(x'',w) < -2\delta$. Iterating such procedure, we obtain a contradiction to the fact that u is bounded below.

For the point of the (possibly empty) set $S^{-w} \cap \mathcal{R}(\delta)$ the value of u does not depend on the exit cost, since, from those points, it is not convenient to reach the boundary and pay the exit cost because it needs a time larger than δ and the exit cost is nonnegative. Hence we have $u = T$ on $S^w \cap \mathcal{R}(\delta)$, Now, let us consider the point of the (possibly empty) set $S^w \cap \mathcal{R}(2\delta)$. For every such a point, the value of u is equal to T if $T(x) \leq \delta$ (since we do not switch), otherwise it may be conditioned by the value of the exit cost $u(\cdot,-w)$ on the points of $\partial\overline{\mathcal{H}}_w$. But for reaching such boundary points (x,w) from $S^w \cap \mathcal{R}(2\delta)$ we spent a time larger than δ and hence we certainly have $u(x,-w) = T(x,-w)$ since $(x,-w) \in S^{-w} \cap \mathcal{R}(\delta)$. Again, iterating such a process, we obtain that $u = T$ on $S^w \cup S^{-w}$, and we conclude. □

Remark 4. Arguing as in the Remark 5, we can say that the couple $(\mathcal{R}, T)$ is the unique couple given by an open set $\mathcal{O}$ in $\overline{\mathcal{H}}$ (for the induced topology) which contains $\mathcal{T}$, and by a bounded below continuous function $u : \mathcal{O} \to \mathbb{R}$ which solves the corresponding problem (12), where the set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_{-1}$ is replaced by $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_{-1}$.

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Appendix: On a Minimum/Exit Time Problem

For many results concerning the theory of viscosity solutions, adopted in this section, we refer the reader to the book Bardi-Capuzzo Dolcetta [7].

Let $\Omega \subset \mathbb{R}^n$ be an open set, $\mathcal{T} \subseteq \overline{\Omega}$ be a closed set satisfying (4), and $\psi : \partial\Omega \to [0, +\infty]$ be a continuous function (for the usual topology on $[0, +\infty]$). Using the same notations as before for controls, considering a bounded continuous function $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ satisfying the analogous of (3) (neglecting w), and the analogous on $\partial\Omega$ of (5) (again neglecting w), we consider the optimal control problem of minimizing the time spent for reaching $\mathcal{T}$ or the time spent plus a cost for exit from $\overline{\Omega}$, subject to the controlled (non switching) dynamical system

$$\begin{cases} y'(t) = f(y(t), \alpha(t)), \, t > 0, \\ y(0) = x, \end{cases} \tag{13}$$

We denote by $y_x(\cdot; \alpha)$ the solution of (13). We define the following instants

$$t_x(\alpha) := \inf \left\{ t \geq 0 \middle| y_x(t; \alpha) \in \mathcal{T} \right\}, \text{ reaching time},$$
$$\tau_x(\alpha) := \inf \left\{ t \geq 0 \middle| y_x(t; \alpha) \notin \overline{\Omega} \right\} \text{ exit time},$$

and define the value function

$$V(x) := \inf_{\alpha \in \mathcal{A}} \left(\min(t_x(\alpha), \tau_x(\alpha)) + \chi_{\{t_x(\alpha) > \tau_x(\alpha)\}} \psi(y_x(\tau_x(\alpha); \alpha)) \right)$$

Let us define the set

$$\mathcal{R} := \left\{ x \in \overline{\Omega} \middle| V(x) < +\infty \right\}.$$

Proposition 4. *With all the hypotheses made before, the set $\mathcal{R}$ is open in $\overline{\Omega}$ for the induced topology, and the value function V is continuous in $\mathcal{R}$.*

Proof. It is similar to that of Proposition 2. □

For every $x \in \mathcal{R}$, and for every $\alpha \in \mathcal{A}$ we define

$$\epsilon_x(\alpha) := \min(t_x(\alpha), \tau_x(\alpha)) = \inf \left\{ t \geq 0 \middle| y_x(t; \alpha) \notin \overline{\Omega} \setminus \mathcal{T} \right\}.$$

We have the following Dynamic Programming Principle: for every $x \in \mathcal{R}$ and for every $t \geq 0$

$$V(x) = \inf_{a \in A} \left\{ \min(t, \epsilon_x(\alpha)) + V(y_x(\min(t, \epsilon_x(\alpha)); \alpha)) \right\}. \tag{14}$$

Proposition 5. *Let all the hypotheses of Proposition 4). Then the value function V is a viscosity solution of the following problem in the unknown u:*

$$\begin{cases} \sup_{a \in A} \{-\nabla u(x) \cdot f(x,a)\} = 1, & \textit{in } (\mathcal{R} \cap \Omega) \setminus \mathcal{T}, \\ u = 0, & \textit{on } \partial \mathcal{T}, \\ u(x) \to +\infty, & \textit{as } x \to \partial_{\overline{\Omega}} \mathcal{R}, \\ u(x) = \psi(x), & \textit{on } (\textit{int}_{\overline{\Omega}} \mathcal{R} \cap \partial \Omega) \setminus \mathcal{T}. \end{cases} \tag{15}$$

In particular, the last boundary condition of (15), has to be understood in the viscosity sense.

A continuous function $u : \mathcal{R} \to \mathbb{R}$, satisfying the second boundary condition (the limit one) is a viscosity solution of (15) if it is a subsolution and a supersolution. Being a subsolution (respectively: a supersolution) means that $u \leq 0$ (respectively $u \geq 0$) on $\partial \mathcal{T}$, and moreover for every C^1 test function $\varphi : \overline{\Omega} \to \mathbb{R}$, and for every $x \in \mathcal{R} \setminus \mathcal{T}$ such that $u - \varphi$ has a local maximum in x (respectively: a local minimum) with respect to $\overline{\Omega}$, the following holds

$$\begin{cases} \sup_{a \in A} \{-\nabla \varphi(x) \cdot f(x,a)\} \leq 1, \\ \text{if either } x \in (\mathcal{R} \cap \Omega) \setminus \mathcal{T} \text{ or } x \in (\textit{int}_{\overline{\Omega}} \mathcal{R} \cap \partial \Omega) \setminus \mathcal{T} \text{ and } u(x) > \psi(x); \\ \Big[\text{respectively: } \sup_{a \in A} \{-\nabla \varphi(x) \cdot f(x,a)\} \geq 1, \\ \text{if either } x \in (\mathcal{R} \cap \Omega) \setminus \mathcal{T} \text{ or } x \in (\textit{int}_{\overline{\Omega}} \mathcal{R} \cap \partial \Omega) \setminus \mathcal{T} \text{ and } u(x) < \psi(x) \Big]. \end{cases}$$

Proof. The proof is almost standard, we only check the last boundary condition. Note that, by the controllability hypothesis on $\partial\Omega$, we may only have $V \leq \psi$ on $\partial_{\overline{\Omega}}\mathcal{R} \cap \partial\Omega$. Let us fix $x \in int_{\overline{\Omega}}\mathcal{R} \cap \partial\Omega$, we have only to analyze the case $V(x) < \psi(x)$. Since $x \in \mathcal{R} \setminus \mathcal{T}$, then there exists $\delta > 0$ such that, for every t small, there is a minimizing sequence of controls α_n for (14) with $\epsilon_x(\alpha_n) \geq \delta$. Hence, for every test function $\varphi \in C^1(\overline{\Omega})$ such that $V - \varphi$ has a local minimum in x with respect to $\overline{\Omega}$, by the usual technique we get

$$\sup_{a \in A} \{-\nabla\varphi(x) \cdot f(x,a)\} \geq 1. \qquad \square$$

We now suppose the following "internal cone condition" in $\overline{\Omega}$: there exists a constant $c > 0$ and a uniformly continuous function $\eta : \overline{\Omega} \to \mathbb{R}^n$ such that, for every $x \in \overline{\Omega}$ ($B(x + s\eta(x), cs)$ is the ball around $x + s\eta(x)$ with radius cs)

$$B(x + s\eta(x), cs) \subseteq \Omega, \ \forall 0 < s \leq c. \tag{16}$$

Theorem 2. *Let the hypotheses Proposition 4 and (16) hold. Then the value function is the unique continuous bounded below function $u : \mathcal{R} \to \mathbb{R}$ which is a viscosity solution of the problem (15).*

Proof. We sketch a standard technique. We first introduce the so-called Kruzkov transformation. Let $u : \mathcal{R} \to \mathbb{R}$ be a continuous bounded below function satisfying the first two boundary conditions in (15). Then, we define the bounded continuous function $\tilde{u} : \overline{\Omega} \to \mathbb{R}$ as follows

$$\tilde{u}(x) := \begin{cases} 1 - e^{-u(x)} & \text{if } x \in \mathcal{R}, \\ 1 & \text{if } x \in \overline{\Omega} \setminus \mathcal{R}. \end{cases}$$

We also denote by $\tilde{\psi} : \partial\Omega \to \mathbb{R}$ the Kruzkov transform of the boundary datum ψ

$$\tilde{\psi}(x) := \begin{cases} 1 - e^{-\psi(x)} & \text{if } \psi(x) \in \mathbb{R}, \\ 1 & \text{if } \psi(x) = +\infty. \end{cases}$$

If u is a continuous bounded below viscosity solutions of (15), then $\tilde{u} : \overline{\Omega} \to \mathbb{R}$ is a continuous bounded viscosity solution of the problem in the unknown v

$$\begin{cases} v + \sup_{a \in A} \{-\nabla v \cdot f(x,a)\} = 1 \text{ in } \Omega \setminus \mathcal{T}, \\ v = 0 \qquad \text{on } \partial\mathcal{T}, \\ v = \tilde{\psi} \qquad \text{on } \partial\Omega \setminus \mathcal{T}, \end{cases} \tag{17}$$

where the last boundary condition has to be understood in the viscosity sense.

Hence, our thesis will come from a uniqueness result for problem (17). Such a problem is a little bit different from the usual ones found in the literature, since it is a minimum/exit time problem. In particular, the target $\mathcal{T}$ and the exit boundary $\partial\Omega$ may intersect. Since, the open set on which the Hamilton-Jacobi equation must be verified is $\Omega \setminus \mathcal{T}$, it is not in general true that a internal cone condition, similar to (16) should hold for the closure of such a set. It depends on how $\mathcal{T}$ and $\partial\Omega$ intersect. And the internal cone condition is in general necessary for the uniqueness result where the boundary conditions are in the viscosity sense. However, in our case, the condition on $\partial\mathcal{T}$ is not in the viscosity sense, it is just a classical boundary condition. Moreover, since $\partial\mathcal{T}$ is compact by hypothesis, we can say that for every open (in $\overline{\Omega}$) neighborhood $\mathcal{U}$ of $\partial\mathcal{T}$, a sort of (uniformly) internal cone condition holds for the set $\overline{\Omega}\setminus\mathcal{U}$, in the following sense: there exists a constant $c' > 0$ (depending on $\mathcal{U}$), such that, for every $x \in \overline{\Omega}\setminus\mathcal{U}$, we have

$$B(x + s\eta(x), c's) \subseteq \Omega \setminus \mathcal{T} \quad \forall 0 < s \leq c'. \tag{18}$$

As usual, the uniqueness is proved by a comparison result between a continuous subsolution v_1 and a continuous supersolution v_2 (i.e. by proving that $v_1 \leq v_2$ in $\overline{\Omega}\setminus\mathcal{T}$). But, on $\partial\mathcal{T}$, they must satisfy $v_1 \leq 0 \leq v_2$, and then, for every $\delta > 0$ we found a neighborhood $\mathcal{U}_\delta$ of $\partial\mathcal{T}$ such that $v_1 - v_2 \leq \delta$ in $\mathcal{U}_\delta$. This permits to use the standard double variables/penalization technique. Indeed, one usually suppose by absurd that there exists $\tilde{x}$ such that $v_1(\tilde{x}) - v_2(\tilde{x}) = \delta > 0$. In our case, we certainly have $\tilde{x} \in \overline{\Omega}\setminus\mathcal{U}_\delta$. Hence, the usual machinery, and especially (18) may be used.

In the end, we get that there exists a unique bounded continuous viscosity solution of (17). □

Remark 5. Let us suppose that $\mathcal{R}' \subseteq \overline{\Omega}$ is another open set in $\overline{\Omega}$ (for the induced topology), which contains $\mathcal{T}$, and that $u' : \mathcal{R}' \to \mathbb{R}$ is a continuous bounded below viscosity solutions of the corresponding problem (15), where $\mathcal{R}$ is replaced by $\mathcal{R}'$. Then, the Kruzkov transformation applied to u' leads to the same problem (17). Hence, by uniqueness, we can say that the couple $(\mathcal{R}, T)$ is the unique couple given by an open set in $\overline{\Omega}$ containing $\mathcal{T}$, and by a bounded below continuous function on such an open set, which solves the corresponding problem (15).