# Integrable Nonlinear Wave Equations and Possible Connections to Tsunami Dynamics

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**Summary.** In this article we present a brief overview of the nature of localized solitary wave structures/solutions underlying integrable nonlinear dispersive wave equations with specific reference to shallow water wave propagation and explore their possible connections to tsunami waves. In particular, we will discuss the derivation of Korteweg-de Vries family of soliton equations in unidirectional wave propagation in shallow waters and their integrability properties and the nature of soliton collisions.

## 1 Introduction

The term 'tsunami' which was perhaps an unknown word even for scientists in countries such as India, Srilanka, Thailand, etc. till recently has become a house-hold word since that fateful morning of December 26, 2004. When a powerful earthquake of magnitude 9.1-9.3 on the Richter scale, epicentered off the coast of Sumatra, Indonesia, struck at 07:58:53 local time described as the 2004 Indian Ocean earthquake or Sumatra-Andaman earthquake it triggered a series of devastating tsunamis as high as 30 meteres that spread throughout the Indian Ocean killing about 2,75,000 people and inundating coastal communities across South and Southeat Asia, including parts of Indonesia, Srilanka, India and Thailand and even reaching as far as the east coast of Africa. The catastrophe is considered to be one of the deadliest diasters in modern history (see Figs. 1 and 2 for some details [1]) (1; 2).

Since this earthquake and consequent tsunamis, several other earthquakes of smaller and larger magnitudes keep occurring off the coast of Indonesia. Even as late as July 17, 2006 an earthquake of magnitude 7.7 on the Richter scale struck off the town of Pangadaran at 15.19 local time and set off a tsunami of 2m high which had killed more than 300 people.

These tsunamis, which can become monstrous tidal waves when they approach coastline, are essentially triggered due to the sudden vertical rise of the seabed by several meters (when earthquake occurs) which displaces massive volume of water.

[1] Fig. 1 http:// www.blogaid.org.uk
Fig. 2a) http:// www.hinduonnet.com/gallery/0071/007108.htm
Fig. 2b) http:// www.bhoomikaindia.org

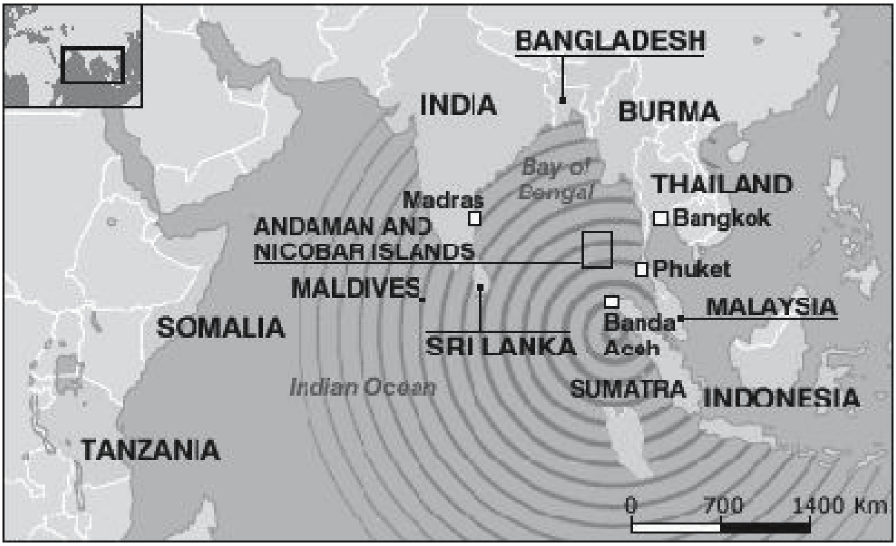


**Fig. 1.** December 2004 Tsunami in Indian Ocean

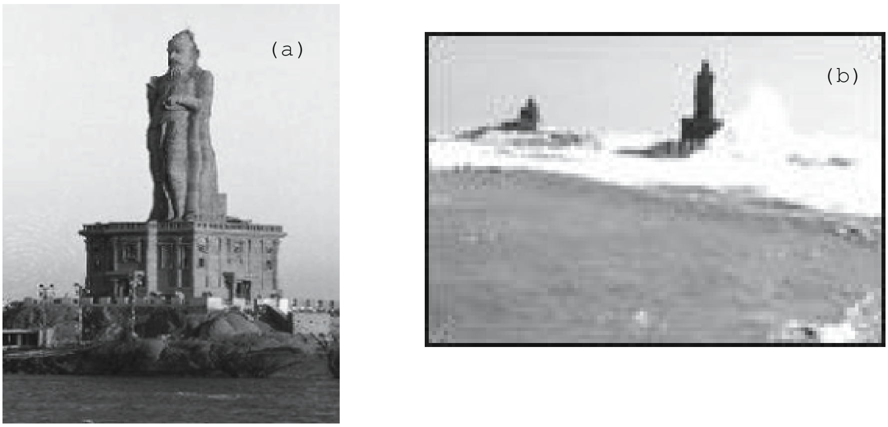


**Fig. 2.** (a) Thiruvalluvar statue in Kanyakumari (in the southern tip of peninsular India), height 133ft (b) Tsunami waves rising near the statue on December 26, 2004 (to almost its top)

The tsunamis behave very differently in deep water than in shallow water as pointed out below. By no means the tsunami of 2004 and later ones are exceptional; More than two hundred tsunamis have been recorded in scientific literature since ancient times. The most notable earlier one is the tsunami triggered by the powerful earthquake ( 9.3 magnitude) off southern Chile on May 22, 1960 (3); Fifteen hours after the devastating earthquake, the tsunami hit Hawai (namely 10,000 kms away from the epicenter of earthquake), killing 61 people. Seven hours later, the Japanese islands of Honshu and Hokkaido were struck by a wall of water 21-feet high and 197 people drowned.

It is clear from the above events that the tsunami waves are fairly permanent and powerful waves, having the capacity to travel extraordinary distances without practically diminishing in size or speed. In this sense they seem to have considerable resemblance to shallow water nonlinear dispersive waves, particularly solitary waves and solitons. In particular, the Kortweg-de Vries family of nonlinear dispersive wave equations admit such solitary waves and solitons and describe unidirectional wave propagation in shallow waters, and it is appropriate to critically review the derivation of KdV and related equations. We will also briefly mention the nature of the solitary wave and soliton solutions and other integrability properties associated with KdV family of equations, including variable KdV and recently derived Camassa-Holm equation. Possible two dimensional generalizations will be also briefly touched.

The plan of the article is as follows. In sec.2, we will summarize the characteristic properties of tsunami waves. In sec.3, we will critically analyse how the Korteweg-de Vries equation was originally derived to describe the Scott-Russel phenomenon to describe shallow water wave propagation. In sec.4, we will discuss the properties of solitary waves and soliton solutions of the KdV equation. We will also briefly touch upon the complete integrability properties of the KdV equation. In sec.5, other interesting KdV type dispersive wave equations will be discussed. Finally, we summarize the discussion in sec.6.

## 2 Basics of Tsunami Waves

As noted above tsunami (tsu: harbour, nami: wave) waves of the type described earlier are essentially triggered by massive earthquakes which lead to vertical displacement of a large volume of water. Other possible reasons also exist for the formation and propagation of tsunami waves: underwater nuclear explosion, larger meteorites falling into the sea, volcano explosions, rockslides, etc. But the most predominanat cause of tsunamis appear to be large earthquakes as in the case of the Sumatra-Andaman earthquake of 2004. Then there are three major aspects associated with the tsunami dynamics:

1. Generation of tsunamis
2. Propagation of tsunamis
3. Tsunami run up and inundation

There exist rather successful models to approach the generation aspects of tsunamis when they occur due to the earthquakes (4). Using the available seismic data it is possible to reconstruct the permanent deformation of the sea bottom due to earthquakes and simple models have been developed (see for example, the article of F. Dias in this volume). Similarly the tsunami run up and unundation problems are extremely complex and they require detailed critical study from a practical point of view in order to save structures and lives when a tsunami strikes.

However, in this article we will be more concerned with the propagation of tsunami waves and their possible relation to wave propagation associated with nonlinear dispersive waves in shallow waters. In order to appreciate such a possible connection, we first look at the typical characteristic properties of tsunami waves as in the case of 2004 Indian Ocean tsunami waves or 1960 Chilean tsunamis.

Considering the Indian Ocean 2004 tsunami, satellite observations after a couple of hours after the earthquake establish an amplitude of approximately 60 cms in the open ocean for the waves. The estimated typical wavelength is about 200 kms (5). The maximum water depth $h$ is between 1 and 4 kms.

Consequently, one can identify the following small parameters ($\epsilon$ and $\delta^2$) of roughly equal magnitude:

$$\epsilon = \frac{a}{h} \approx 10^{-4} << 1, \tag{1}$$

$$\delta = \frac{h}{l} \approx 10^{-2} << 1 \tag{2}$$

As a consequence, it is possible that a nonlinear shallow water wave theory where dispersion also plays an important role has some relevance. However, we also wish to point out here that there are other points of view: Constantin and Johnson (6) estimate $\epsilon \approx 0.002$ and $\delta \approx 0.04$ and conclude that for both nonlinearity and dispersion to become significant the quantity $\delta\epsilon^{-\frac{3}{2}}\times$wavelength estimated as 90,000 kms is too large and shallow water equations with variable depth (without dispersion) should be used. However, it appears that these estimates can vary over a rather wide range and with suitable estimates it is possible that the range of 10,000 − 20,000 kms could be also possible ranges and hence taking into account the fact that both the Indian Ocean 2004 and Chilean 1960 tsunamis have travelled over 10 hours or more before encountering land mass appears to allow for the possibility of nonlinear dispersive waves as relevant features for the phenomena.

From this point of view in the next section we discuss the shallow water wave theory to deduce KdV equation.

## 3 Scott Russel Phenomenon and KdV Equation

It is a folklore that the first scientificaly recorded observation of a solitary wave was made by the Scottish naval engineer John Scott Russel in the year 1837 (7) when he identified a large solitary heap of water travelling with undiminished speed or shape over a distance in the Union Canal connecting the cities of Edinburg and Glasgow in Scotland. He went on to repeat the phenomenon at the laboratory in a rectangular channel of water by dropping wieghts at one end. By measuring the velocity and height of the wave he also established a phenomenological relation connecting these quantities which has stood the test of time.

In 1895, Kortweg and de Vries (8) considered the wave phenomenon underlying the observations of Scott Russel, from first principles of fluid dynamics. The basic features of their analysis can be summarized as follows (9; 10).

Consider the one-dimensional ($x$-direction) wave motion of an incompressible and inviscid fluid (water) in a shallow channel of height $h$, and of sufficient width with uniform cross-section leading to the formation of a solitary wave propagating under gravity. Let the length of the wave be $l$ and the maximum value of its amplitude, $\eta(x,t)$, above the horizontal surface be $a$ (see Fig.3). Then we can introduce two natural small parameters into the problem $\epsilon$ and $\delta$ as defined in Eqs. (1) and (2). Then we can proceed with the analysis as follows.

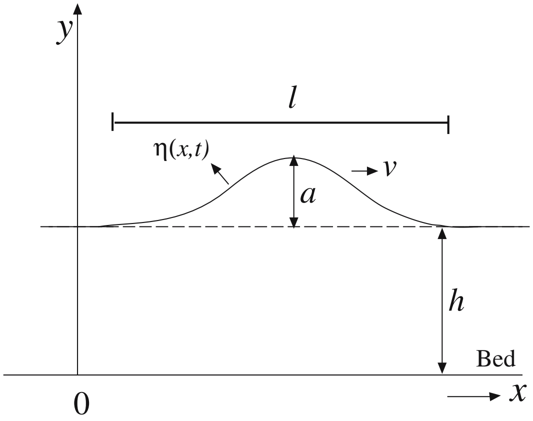


**Fig. 3.** One-dimensional wave motion in a shallow channel

## A. *Equation of Motion*

The fluid motion can be described by the velocity vector

$$\mathbf{V}(x, y, t) = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j} \,, \tag{3}$$

where **i** and **j** are the unit vectors along the horizontal and vertical directions, respectively. As the motion is irrotational, we have

$$\boldsymbol{\nabla} \times \mathbf{V} = 0 \,. \tag{4}$$

Consequently, we can introduce the velocity potential $\phi(x, y, t)$ by the relation

$$\mathbf{V} = \boldsymbol{\nabla}\phi \,. \tag{5}$$

### (i) *Conservation of Density*

The system obviously admits the following conservation law for the mass density $\rho(x,\, y,\, t)$ of the fluid,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \rho_t + \boldsymbol{\nabla} \cdot (\rho \mathbf{V}) = 0 \,, \tag{6}$$

where $\mathbf{V}(x, y, t)$ is the velocity vector of the fluid. As $\rho$ is a constant, from (6) we have

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0 \,. \tag{7}$$

Then using (5) in (7), we find that $\phi$ obeys the Laplace equation

$$\nabla^2 \phi(x, y, t) = 0 \,. \tag{8}$$

### (ii) *Euler's Equation*

As the density of the fluid $\rho = \rho_0$ =constant, using Newton's law for the rate of change of momentum, we can write

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\,\mathbf{V} = -\frac{1}{\rho_0}\boldsymbol{\nabla} p - g\mathbf{j}\,, \tag{9}$$

where $p = p(x, y, t)$ is the pressure at the point $(x, y)$ and $g$ is the acceleration due to gravity, which is acting vertically downwards (here $\mathbf{j}$ is the unit vector along the vertical direction). Making use of (5) in (9), we obtain (after one integration)

$$\phi_t + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{p}{\rho_0} + gy = 0\,. \tag{10}$$

### (iii) *Boundary Conditions*

The above two equations (8) and (9) or (10) for the velocity potential $\phi(x, y, t)$ of the fluid have to be supplemented by appropriate boundary conditions, by taking into account the fact (see Fig.2) that

(a) the horizontal bed at $y = 0$ is hard and
(b) the upper boundary $y = y(x, t)$ is a free surface .

As a result
(a) the vertical velocity at $y = 0$ vanishes,

$$v(x, 0, t) = 0\,, \tag{11}$$

which implies (using (3) and (5))

$$\phi_y(x, 0, t) = 0\,. \tag{12}$$

(b) As the upper boundary is free, let us specify it by $y = h + \eta(x, t)$ (see Fig.2). Then at the point $x = x_1$, $y = y_1 \equiv y(x, t)$, we can write

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \cdot \frac{\mathrm{d}x_1}{\mathrm{d}t} = \eta_t + \eta_x u_1 = v_1\,. \tag{13}$$

Since $v_1 = \phi_{1y}$, $u_1 = \phi_{1x}$, the last two parts of (13) can be rewritten as

$$\phi_{1y} = \eta_t + \eta_x \phi_{1x}\,. \tag{14}$$

(c) Similarly at $y = y_1$, the pressure $p_1 = 0$. Then from (10), it follows that

$$u_{1t} + u_1 u_{1x} + v_1 v_{1x} + g\eta_x = 0\,. \tag{15}$$

Thus the motion of the surface of water wave is essentially specified by the Laplace equation (8) and (10) along with one fixed boundary condition (12) and two *variable nonlinear* boundary conditions (14) and (15). One has to then solve the Laplace equation subject to these boundary conditions.

### (iv) *Taylor Expansion of $\phi(x,y,t)$ in $y$*

Making use of the fact $\delta = h/l \ll 1$, $h \ll l$, we assume $y(= h+\eta(x,t))$ to be small to introduce the Taylor expansion

$$\phi(x,y,t) = \sum_{n=0}^{\infty} y^n \phi_n(x,t) \; . \tag{16}$$

Substituting the above series for $\phi$ into the Laplace equation (8), solving recursively for $\phi_n(x,t)$'s and making use of the boundary condition (14), $\phi_y(x,0,t) = 0$, one can show that

$$u_1 = \phi_{1x} = f - \frac{1}{2} y_1^2 f_{xx} + \text{ higher order in } \; y_1 \; , \tag{17}$$

$$v_1 = \phi_{1y} = -y_1 f_x + \frac{1}{6} y_1^3 f_{xxx} + \text{ higher order in } \; y_1 \; , \tag{18}$$

where $f = \partial\phi_0/\partial x$. We can then substitute these expressions into the nonlinear boundary conditions (13),(14) and (15) to obtain equations for $f$ and $\eta$.

### (v) *Introduction of Small Parameters $\epsilon$ and $\delta$*

So far the analysis has not taken into account fully the shallow nature of the channel $(a/h = \epsilon \ll 1)$ and the solitary nature of the wave $(a/l = a/h \cdot h/l = \epsilon\delta \ll 1$, $\epsilon \ll 1$, $\delta \ll 1)$, which are essential to realize the Scott Russel phenomenon. For this purpose we stretch the independent and dependent variables in the defining (13)–(15), (17) and (18) through appropriate scale changes, but retaining the overall form of the equations. To realize this we can introduce the natural scale changes

$$x = l x' \; , \quad \eta = a \eta' \tag{19}$$

along with

$$t = \frac{l}{c_0} t' \; , \tag{20}$$

where $c_0$ is a parameter to be determined. Then in order to retain the form of (17), (18) we require

$$u_1 = \epsilon c_0 u_1' \; , \quad v_1 = \epsilon\delta c_0 v_1' \; , \quad f = \epsilon c_0 f' \; . \tag{21}$$

We also have

$$y_1 = h + \eta(x,t) = h\left(1 + \epsilon\eta'\left(x',t'\right)\right) \; . \tag{22}$$

Substituting the transformations (19)–(22) into (17), we obtain

$$\begin{aligned} u_1' &= f' - \frac{1}{2}\delta^2\left(1+\epsilon\eta'\right)^2 f'_{x'x'} \\ &= f' - \frac{1}{2}\delta^2 f'_{x'x'} \; , \end{aligned} \tag{23}$$

where we have omitted terms proportional to $\delta^2\epsilon$ as small compared to terms of the order $\delta^2$. Similarly from (18), we obtain

$$v_1' = -\left(1+\epsilon\eta'\right) f'_{x'} + \frac{1}{6}\delta^2 f'_{x'x'x'} \; . \tag{24}$$

Now considering the nonlinear boundary condition (14) in the form

$$v_1 = \eta_t + \eta_x u_1 \ , \tag{25}$$

it can be rewritten, after making use of the transformations (19)–(22) and neglecting terms involving $\epsilon\delta^2$, as

$$\eta'_{t'} + f'_{x'} + \epsilon\eta' f'_{x'} + \epsilon f' \eta'_{x'} - \frac{1}{6}\delta^2 f'_{x'x'x'} = 0 \ . \tag{26}$$

Similarly considering the other boundary condition (16) and making use of the above transformations, it can be rewritten, after neglecting terms of the order $\epsilon^2\delta^2$, as

$$f'_{t'} + \epsilon f' f'_{x'} + \frac{ga}{\epsilon c_0^2}\eta'_{x'} - \frac{1}{2}\delta^2 f'_{x'x't'} = 0 \ . \tag{27}$$

Now choosing the arbitrary parameter $c_0$ as

$$c_0^2 = gh \tag{28}$$

so that $\eta'_x$ term is of order unity, (27) becomes

$$f'_{t'} + \eta'_{x'} + \epsilon f' f'_{x'} - \frac{1}{2}\delta^2 f'_{x'x't'} = 0 \ . \tag{29}$$

For notational convenience we will hereafter omit the prime symbol in all the variables, however remembering that all the variables hereafter correspond to rescaled quantities. Then the evolution equation for the amplitude of the wave and the function related to the velocity potential reads

$$\eta_t + f_x + \epsilon\eta f_x + \epsilon f \eta_x - \frac{1}{6}\delta^2 f_{xxx} = 0 \ , \tag{30}$$

$$f_t + \eta_x + \epsilon f f_x - \frac{1}{2}\delta^2 f_{xxt} = 0 \ . \tag{31}$$

Note that the small parameters $\epsilon$ and $\delta^2$ have occurred in a natural way in (30), (31).

## (vi) ***Perturbation Analysis***

Since the parameters $\epsilon$ and $\delta^2$ are small in (30), (31), we can make a perturbation expansion of $f$ in these parameters:

$$f = f^{(0)} + \epsilon f^{(1)} + \delta^2 f^{(2)} + \text{ higher order terms} \ , \tag{32}$$

where $f^{(i)}$, $i = 0, 1, 2, ...$ are functions of $\eta$ and its spatial derivatives. Substituting this into (30), (31) and regrouping, we obtain

$$\eta_t + f_x^{(0)} + \epsilon\left[f_x^{(1)} + \eta f_x^{(0)} + \eta_x f^{(0)}\right] + \delta^2\left[f_x^{(2)} - \frac{1}{6}f_{xxx}^{(0)}\right] + \text{higher order terms in } (\epsilon, \delta^2) = 0 \ , \tag{33}$$

$$\eta_x + f_t^{(0)} + \epsilon\left[f_t^{(1)} + f^{(0)} f_x^{(0)}\right] + \delta^2\left[f_t^{(2)} - \frac{1}{2}f_{xxt}^{(0)}\right] + \text{higher order terms in } (\epsilon, \delta^2) = 0 \ . \tag{34}$$

In order that (33) and (34) are self consistent as evolution equations for an one-dimensional wave propagating to the right, we can choose

$$f^{(0)} = \eta + O\left(\epsilon, \delta^2\right) \ , \tag{35}$$

where $O(\epsilon, \delta^2)$ stands for terms proportional to $\epsilon$ and $\delta^2$. Then (33), (34) become

$$\eta_t + \eta_x + \epsilon\left[f_x^{(1)} + 2\eta\eta_x\right] + \delta^2\left[f_x^{(2)} - \frac{1}{6}\eta_{xxx}\right] = 0 \ , \tag{36}$$

$$\eta_t + \eta_x + \epsilon\left[f_t^{(1)} + \eta\eta_x\right] + \delta^2\left[f_t^{(2)} - \frac{1}{2}\eta_{xxt}\right] = 0 \ , \tag{37}$$

where higher order terms in $\epsilon$ and $\delta^2$ are neglected. Since $f^{(1)}$ and $f^{(2)}$ are functions of $\eta$ (and its spatial derivatives)

$$f_x^{(1)} = f_\eta^{(1)}\eta_x, \quad f_t^{(1)} = f_\eta^{(1)}\eta_t = -f_\eta^{(1)}\eta_x + O\left(\epsilon, \delta^2\right) = -f_x^{(1)} \ , \tag{38}$$

where in the last relation, (33), (34) have been used for $\eta_t$ and $\eta_x$. Similarly, we can argue that

$$f_x^{(2)} = f_\eta^{(2)}\eta_x, \quad f_t^{(2)} = -f_\eta^{(2)}\eta_x + O\left(\epsilon, \delta^2\right) = -f_x^{(2)} \ , \tag{39}$$

Substituting (38), (39) into (36), (37) , we obtain

$$\eta_t + \eta_x + \epsilon\left[f_x^{(1)} + 2\eta\eta_x\right] + \delta^2\left[f_x^{(2)} - \frac{1}{6}\eta_{xxx}\right] = 0 \ , \tag{40}$$

$$\eta_t + \eta_x + \epsilon[-f_x^{(1)} + \eta\eta_x] + \delta^2\left[-f_x^{(2)} + \frac{1}{2}\eta_{xxx}\right] = 0 \ . \tag{41}$$

Compatibility of these two equations require that

$$f_x^{(1)} = -\frac{1}{2}\eta\eta_x, \qquad f_x^{(2)} = \frac{1}{3}\eta_{xxx} \ . \tag{42}$$

Integrating, we find

$$f^{(1)} = -\frac{1}{4}\eta^2, \quad f^{(2)} = \frac{1}{3}\eta_{xx} \ . \tag{43}$$

Substituting $f^{(1)}$ and $f^{(2)}$ into (40), (41) , we ultimately obtain the KdV equation in the form

$$\eta_t + \eta_x + \frac{3}{2}\epsilon\eta\eta_x + \frac{\delta^2}{6}\eta_{xxx} = 0 \ , \tag{44}$$

describing the unidirectional propagation of shallow water waves.

### (vii) *The Standard (Contemporary) Form of KdV Equation*

Finally, changing to a moving frame of reference,

$$\xi = x - t, \quad \tau = t \tag{45}$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \ , \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} \ , \tag{46}$$

(44) can be rewritten as

$$\eta_\tau + \frac{3}{2}\epsilon\eta\eta_\xi + \frac{1}{6}\delta^2\eta_{\xi\xi\xi} = 0 \,. \quad (47)$$

Then introducing the new variables

$$u = \frac{3\epsilon}{2\delta^2}\eta, \quad \tau' = \frac{6}{\delta^2}\tau \,, \quad (48)$$

(47) can be expressed as

$$u_{\tau'} + 6uu_\xi + u_{\xi\xi\xi} = 0 \,. \quad (49)$$

Redefining the variables $\tau'$ as $t$ and $\xi$ as $x$, again for notational convenience, we finally arrive at the ubiquitous form of the KdV equation as

$$u_t + 6uu_x + u_{xxx} = 0 \,. \quad (50)$$

# 4 Solitary Wave, Solitons and Complete Integrability of KdV equation

The Korteweg-de Vries equation (50) admits cnoidal wave solution and in the limiting case solitary wave solution as well. More importantly, the KdV solitary wave is a soliton : it retains its shape and speed upon collision with another solitary wave of different amplitude, except for a phase shift (11). In fact for an arbitrary initial condition, the solution of the Cauchy initial value problem consists of N-number of solitons of different amplitudes in the background of small amplitude dispersive waves. All these results ultimately lead to the result that the KdV equation is a completely integrable, infinite dimensional, nonlinear Hamiltonian system. It possesses

(i) a Lax pair of linear differential operators and is solvable through the so called inverse scattering transform (IST) method (12),
(ii) infinite number of conservation laws and associated infinite number of involutive integrals of motion
(iii) N-soliton solution
(iv) Hirota bilinear form
(v) Hamiltonian structure

and a host of other interesting properties (see for example (9; 10)). We will very briefly consider some of these properties.

## 4.1 Korteweg–de Vries Equation and the Solitary Waves and Cnoidal Waves

Let us look for elementary wave solutions of (50) in the form

$$u = 2f(x-ct) \quad (51)$$
$$= 2f(\xi) \,, \quad \xi = x - ct \,. \quad (52)$$

Then the KdV equation reduces to an ordinary differential equation of the form

$$-cf_\xi + 6\left(f^2\right)_\xi + f_{\xi\xi\xi} = 0 \,. \tag{53}$$

Integrating twice the above equation, the solution of can be expressed in terms of Jacobian elliptic function as

$$f(\xi) = f(x-ct) = \alpha_3 - (\alpha_3-\alpha_2)\mathrm{sn}^2\left[\sqrt{\alpha_3-\alpha_1}\,(x-ct)\,,m\right], \tag{54}$$

where

$$(\alpha_1+\alpha_2+\alpha_3) = \frac{c}{4}\,, \quad m^2 = \frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}\,. \tag{55}$$

Here $\alpha_1$, $\alpha_2$ and $\alpha_3$ are related to the three integration constants. Equations (54), (55) represent in fact the so-called *cnoidal wave* for obvious reasons.

## Special Cases:

### (i) $m \approx 0$: *Harmonic wave*

When $m \approx 0$, (54) leads to elementary progressing harmonic wave solutions. This can be verified by taking the limit $m \to 0$ in (54), (55).

### (ii) $m = 1$: *Solitary wave*

When $m = 1$, we have

$$f = \alpha_3 - (\alpha_3-\alpha_2)\tanh^2\left[\sqrt{\alpha_3-\alpha_1}\,(x-ct)\right]\,, \tag{56}$$

that is,

$$f = \alpha_2 + (\alpha_3-\alpha_2)\ \mathrm{sech}^2\left[\sqrt{\alpha_3-\alpha_1}\,(x-ct)\right]\,. \tag{57}$$

Choosing now $\alpha_2 = 0$, $\alpha_1 = 0$, we have

$$f = \frac{\alpha_3}{4}\,\mathrm{sech}^2\left[\sqrt{\alpha_3}\,(x-ct)\right]\,. \tag{58}$$

Using (55), (58) can be written as

$$f = \frac{c}{4}\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}\,(x-ct)\right]\,. \tag{59}$$

Substituting (59) into (51), the solution can be written as

$$u(x,t) = 2f = \frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}\,(x-ct)}{2}\right]\,. \tag{60}$$

This is of course the Scott Russel solitary wave (Fig.4),

The characteristic feature of the above solitary wave is that the velocity of the wave ($v = c$) is directly proportional to the amplitude ($a = c/2$): the larger the wave, the faster it moves. Unlike the progressing wave, it is fully localized, decaying exponentially fast as $x \to \pm\infty$ (see Fig.4).

### 4.2 Lax pair and linearization

The KdV equation is well known to posses the Lax pair (12)

$$L = -\frac{\partial^2}{\partial x^2} + u(x,t) \tag{61}$$

and

$$B = -4\frac{\partial^3}{\partial x^3} + 3\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right) \tag{62}$$

so that the Lax equation

$$L_t = [B, L] \tag{63}$$

is equivalent to the KdV equation. Or in other words the KdV equation is linearizable in the sense that it is the compatibility condition corresponding to a linear eigenvalue problem (the Schrödinger spectral problem) and a linear time evolution equation for the eigen function

$$L\psi = \lambda\psi \ , \tag{64}$$

$$\psi_t = B\psi \ . \tag{65}$$

Consequently a nonlinear generalization of the Fourier transform method, namely the inverse scattering transform (IST) technique can be formulated to solve the Cauchy initial value problem of the KdV equation. Schematically it is shown in Fig.5, which is self explanatory. For more details, see for example refs. [9,10]. The final result is that the general solution of the KdV equation can be written as

$$u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x, x+0, t) \ . \tag{66}$$

where $K(x,y,t)$ is the solution of the Gelfand-Levitan-Marchenko linear integral equation

$$\begin{aligned} K(x,y,t) &+ F(x+y,t) \\ &+ \int_x^\infty F(y+z,t)K(x,z,t)\mathrm{d}z = 0 \ , \quad y > x \end{aligned} \tag{67}$$

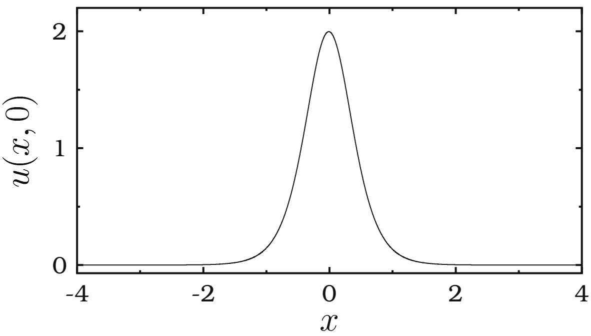


**Fig. 4.** Solitary wave solution (60) of the KdV equation.

and

$$F(x+y,t)=\sum_{n=1}^{N} C_n^2(0)\mathrm{e}^{8\kappa_n^3 t}\mathrm{e}^{-\kappa_n(x+y)}+\frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,0)\mathrm{e}^{-8\mathrm{i}k^3 t}\mathrm{e}^{\mathrm{i}k(x+y)}\mathrm{d}k\;. \quad (68)$$

Here $C_n$, $\kappa_n$ and $R(k,0)$ are the spectral data associated with the Schrödinger spectral problem (64) for $t=0$. Then the discrete states in eq.(68) essentially lead to the soliton picture: N discrete states correspond to N-soliton solutions.

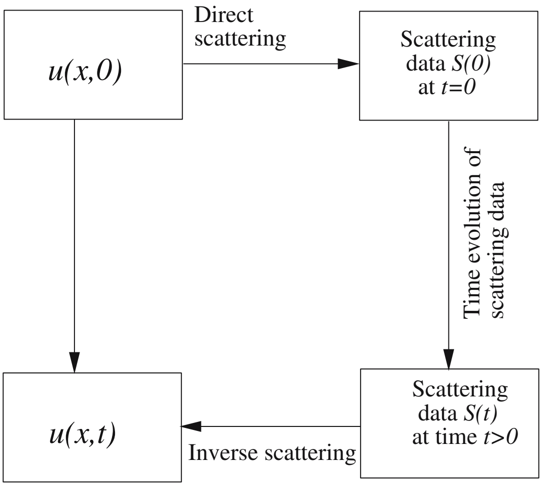


**Fig. 5.** Schematic diagram of the inverse scattering transform method

For example, the two soliton solution can be written as

$$u(x,t)=-2\left(\kappa_2^2-\kappa_1^2\right)\frac{\kappa_2^2\;\mathrm{cosech}^2\gamma_2+\kappa_1^2\;\mathrm{sech}^2\gamma_1}{\left(\kappa_2\;\coth\gamma_2-\kappa_1\;\tanh\gamma_1\right)^2}\;, \quad (69)$$

where $\gamma_i=\kappa_i x-4\kappa_i^3 t-\delta_i$, $\delta_i=\frac{1}{2}\log\left(\frac{C_{i0}^2}{2\kappa_i}\frac{(\kappa_2-\kappa_1)}{(\kappa_2+\kappa_1)}\right)$, $i=1,2$. When the solution (69) is plotted as in Fig.6, it clearly demonstrates the basic soliton property of elastic collision.

## 4.3 KdV as a Hamiltonian system

Defining the Hamiltonian density

$$\mathcal{H}=\frac{1}{2}u_x^2+u^3 \quad (70)$$

so that the Hamiltonian becomes

$$\widehat{\mathcal{H}}=\int\left(\frac{1}{2}u_x^2+u^3\right)\mathrm{d}x\;. \quad (71)$$

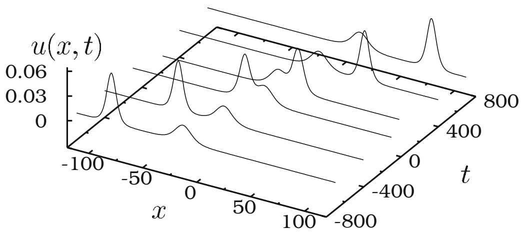


**Fig. 6.** Two soliton interaction of the KdV equation

KdV equation can be written as

$$u_t = \frac{\partial}{\partial x} \frac{\delta \widehat{\mathcal{H}}}{\delta u} \ . \tag{72}$$

Thus, KdV equation has a Hamiltonian structure.

Now, defining the Poisson bracket between two functionals $U$ and $V$ as

$$\{U, V\} = \int \mathrm{d}x \frac{\delta U}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta V}{\delta u(x)} \ . \tag{73}$$

it has been shown (13) that the inverse scattering transform discussed earlier allows one to identify appropriate set of (infinite number of) action-angle variables. Further one can show that in terms of these new variables, the Hamiltonian (71) can be expressed purely in terms of action variables alone. Consequently the corresponding canonical equations can be trivially integrated. In this sense KdV equation has been proved to be a complete integrable infinite dimensional Hamiltonian system.

### 4.4 Bilinearization of KdV

The KdV equation is not only linearizable but also can be bilinearized (14) under the transformation

$$u = 2 \frac{\partial^2}{\partial x^2} \log F \ . \tag{74}$$

Eq. (52) can be rewritten in the bilinear form as

$$F_{xt} F - F_x F_t + F_{xxxx} F - 4 F_{xxx} F_x + 3 F_{xx}^2 = 0 \ . \tag{75}$$

Then it is fairly straightforward to obtain the soliton solutions by expanding $F$ formally as a power series in terms of a small parameter $\epsilon$ so that eq. (75) is written as a system of linear partial differential equations. Restricting to a finite number of terms in the power series and solving the resultant system of linear partial differential equation recursively, one can obtain the N-soliton solution explicitly and the soliton property can be analysed.

Besides the above properties, KdV equation possesses many other characteristic features of integrable systems: (i) Existence of infinite number of conservation laws and constants of motion (ii) Bäcklund transformations (iii) Lie-Bäcklund symmetries, (iv) Painlevé property and so on. Again for details see refs. (9; 10)

## 5 KdV related Integrable and Nonintegrable Equations

Depending on the actual physical situation, the derivation of the shallow water wave equation can be suitably modified to obtain other forms of nonlinear dispersive wave equations in (1+1) dimensions as well as in (2+1) dimensions without going into the actual derivations. Some of the important equations are listed below (11).

- Boussinesq equation

$$u_t + uu_x + g\eta_x - \frac{1}{3}h^2 u_{txx} = 0 \tag{76}$$

$$\eta_t + [u(h+\eta)]_x = 0 \tag{77}$$

- Benjamin-Bona-Mahoney (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{78}$$

- Camassa-Holm equation (15)

$$u_t + 2\kappa u_x + 3uu_x - u_{xxt} = 3u_x u_{xx} + uu_{xxx} \tag{79}$$

- Kadomtsev-Petviashville (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \tag{80}$$

$\sigma^2$=-1: KP-I, $\sigma^2$=+1: KP-II.

There also exist some interesting nonlinar dispersive wave equations to describe deep water wave propagation. These include the following.

- Nonlinear Schrödinger (NLS) equation

$$\mathrm{i}q_t + q_{xx} + |q|^2 q = 0, \tag{81}$$

- Davey-Stewartson equation

$$\begin{aligned} \mathrm{i}q_t + q_{xx} + q_{yy} + 2|q|^2 q + qu = 0 \ , \\ u_{xx} - u_{yy} = 4\left(|q|^2\right)_{xx} \ . \end{aligned} \tag{82}$$

In the derivation of the above equations, generally the bottom of water column or fluid bed is assumed to be flat. However in realistic situations the water depth varies as a function of the horizontal coordinates. In this situation, one often encounters inhomogeneous forms of the above wave equations. Typical example is the variable coefficient KdV equation (16):

$$u_t + f(x,t)uu_x + g(x,t)u_{xxx} = 0, \tag{83}$$

where $f$ and $g$ are functions of $x$, $t$. More general forms can also be deduced depending upon the actual situations, see for example ref. (17).

All these equations can be helpful to deal with tsunami wave propagation at different situations. Which one will suit which situation requires detailed analysis depending upon the experimental observations. Many of the above equations are integrable such as the Boussinesq, Camassa-Holm, KP, nonlinear Schrödinger and Davey-Stewartson equations and certain forms of inhomogeneous KdV equations, while BBM equation and general forms of inhomogeneous KdV equations are non-integrable but may possess solitary wave solutions and are amenable to perturbation analysis. Integrable equations in the above list admit interesting new types of solutions. For example, the Camassa-Holm equation admits peakon solution (see Fig. 7), while the KP equation and Davey-Stewartson equation can admit lump (algebraically decaying) solutions and line soliton solutions. The latter one also admits dromion (exponentially localized) solutions (see Figs. 8−10). These solutions can also be used for possible description of tsunami wave propagation in the appropriate situations.

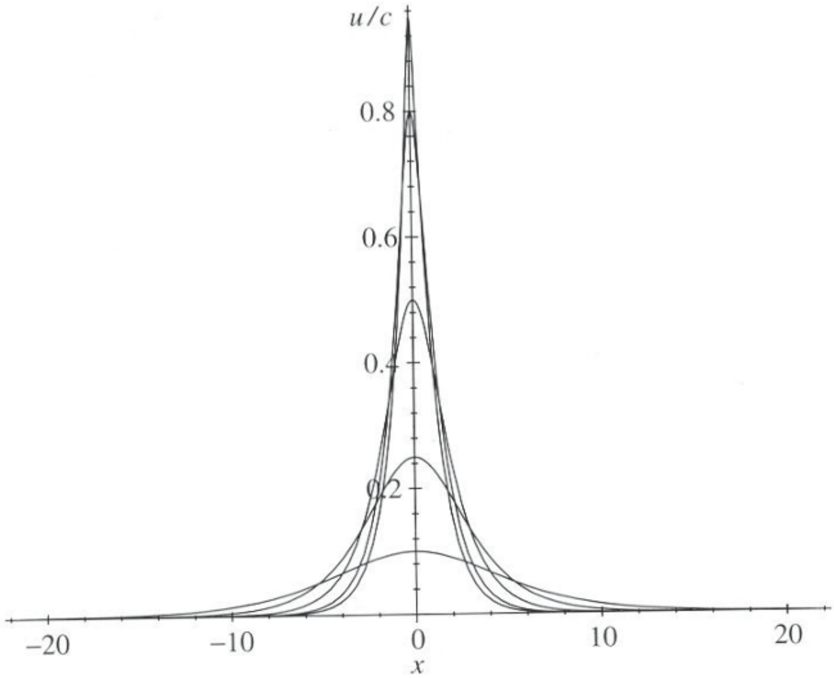


**Fig. 7.** Solitary wave (including peakon ) solution of Camassa-Holm equation

## 6 Summary and Conclusions

In this article, we have discussed briefly the possibility of describing tsunami waves of the type which occurred in the Indian Ocean 2004 earthquake in terms of nonlinear shallow water wave equation of dispersive type like the Korteweg-de Vries equation. In particular, we have pointed how the KdV wave equation can be derived to describe the Scott-Russel phenomenon of unidirectional shallow water wave propagation. Existence of solitary waves, solitons and complete integrability properties of the KdV

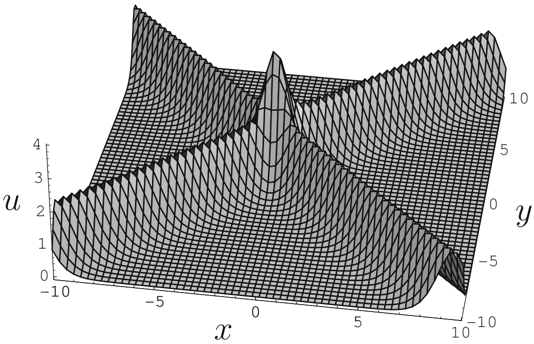


**Fig. 8.** Two line soliton solution of KP-II equation

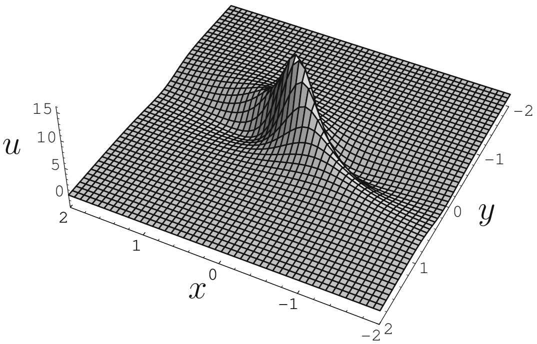


**Fig. 9.** Lump soliton solution of KP-I equation

equation was briefly explained. Other related equations which can be of some use in tsunami dynamics were also briefly touched upon. The generation and propagation of tsunami waves is an extremely complex process. Yet nonlinear dispersive waves of shallow water may be of considerable importance in describing the tsunami dynamics and much work remains to be done in this direction.

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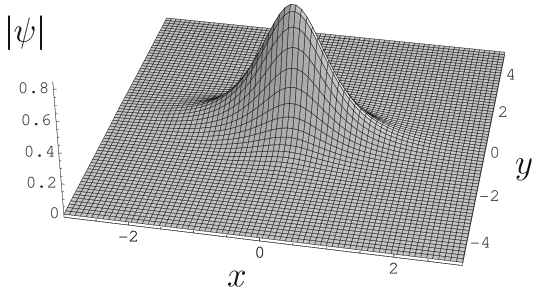


**Fig. 10.** Dromion solution of Davey-Stewartson equation

## References

[1] Lay T etal (2005) The Great Sumatra-Andaman earthquake of 26 December 2004. Science 308:1127–1133

[2] Ram Mohan V etal (2006) Impact of South Asian Tsunami on 26 December 2004 on south east coast of India−A field report (preprint)

[3] Dudley WC and Min L (1988) Tsunami!. Honolulu, Hawaii: Univeristy of Hawaii press

[4] Okada Y (1985) Surface deformation due to sheer and tensile fault in a half space. Bull Seism Soc Am 82:1135–54

[5] Gower J (2005) Jason 1 detects the December 2004 tsunami. EOS 86:37-38

[6] Constantin A and Johnson RS (2006) Modelling tsunami. J Phys A: Math Gen 39:L215-L217

[7] See for example, Bullough RK (1988) The wave par excellence: the solitary progressive great wave of equilibrium of the fluid-an early history of solitary wave. In: Lakshmanan M (Ed.) Solitons: Introduction and applications. Springer, Berlin

[8] Kortweg DJ and de Vries D (1895) On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Phil Mag 39:422-443

[9] Lakshmanan M and Rajasekar S (2003) Nonlinear Dynamics: Integrability, Chaos and Patterns. Springer, Berlin

[10] Ablowitz MJ and Clarkson PA (1991) Solitons, Nonlinear evolution equations and inverse scattering. Cambridge University Press, Cambridge

[11] Zabusky NJ and Kruskal MD (1965) Interaction of “solitons” in a collisionless plasma and the recurrence of initial states. Phys Rev Lett 15:240-43

[12] Gardner CS, Greene JM, Kruskal MD and Miura RM (1967) Method for solving the Kortweg-de Vries equation, Phys Rev Lett 19:1095-97

[13] Zakharov VE and Faddeev LD (1971) The Kortweg-de Vries equation is a fully integrable Hamiltonian system. Funct Anal Appl 5:280-87

[14] Hirota R (1971) Exact solution of the Kortweg-de Vries equation for multiple collisions of solitons. Phys Rev Lett 27:1192-94

[15] Camassa R and Holm D (1993) An integrable shallow water equation with peaked solitons. Phys Rev Lett 71:1661-64

[16] Calogero F and Degasperis A (1982) Spectral transform and solitons. North-Holland, Amsterdam

[17] Caputo JG and Stepanyants YA (2003) Bore formation, evolution and disintegration into solitons in shallow inhomogeneous channels. Nonlinear Processes in Geophysics 10:407-24