On Representation and Analysis of Crisp and Fuzzy Information Systems

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Abstract. This paper proposes an approach to representation and analysis of information systems with fuzzy attributes, which combines the variable precision fuzzy rough set (VPFRS) model with the fuzzy flow graph method. An idea of parameterized approximation of crisp and fuzzy sets is presented. A single ε-approximation, which is based on the notion of fuzzy rough inclusion function, can be used to express the crisp approximations in the rough set and variable precision rough set (VPRS) model. A unified form of the ε-approximation is particularly important for defining a consistent VPFRS model. The introduced fuzzy flow graph method enables alternative description of decision tables with fuzzy attributes. The generalized VPFRS model and fuzzy flow graphs, taken together, can be applied to determining a system of fuzzy decision rules from process data.

1 Introduction

Two important paradigms, developed in the recent decades, can be successfully used for modelling and analyzing decision processes performed by a human operator: the rough set theory introduced by Pawlak [19] and the theory of fuzzy sets proposed by Zadeh [34].

The idea of combining fuzzy sets with rough sets was realized by two independent approaches. The method given by Nakamura [18] consists in application of the classical rough set theory to a crisp representation of fuzzy sets. In contrast to that, Dubois and Prade [6] introduced a novel concept of fuzzy rough sets, suitable for expressing vagueness represented in fuzzy sets, and coarseness characteristic of rough sets. The concept of Dubois and Prade has been widely used and developed, see, e.g., [8,12,25].

A significant parameterized extension of the crisp rough set theory is the variable precision rough set (VPRS) model proposed by Ziarko [35]. It bases on the idea of relaxation of strong inclusion requirements. The VPRS model helps to overcome problems caused by errors and noise, which are present in data obtained from real decision processes. More recently, a probabilistic interpretation of the VPRS model was developed, see e.g., [11,29,36]. The original VPRS model and many other extensions of crisp rough sets can be expressed in

J.F. Peters et al. (Eds.): Transactions on Rough Sets VI, LNCS 4374, pp. 191–210, 2007.

the framework of a generalized theory. The rough mereology of Polkowski and Skowron [24] presents an alternative generalized approach to rough sets, which is based on the mereology of Leśniewski. The idea of relaxation of strong inclusion requirements was also applied to fuzzy rough sets [7,33].

Another useful method, introduced and studied by Pawlak [20,21,22], is a hybrid approach to decision algorithms, which combines the idea of flow graphs with the crisp rough set model. It was shown [9] that every decision algorithm can be associated with a flow graph.

We emphasize the problem of obtaining a set of relevant fuzzy decision rules from recorded process data and decision examples. This is a crucial step in applications of fuzzy inference systems [14,32]. The used data can be represented in the form of a decision table with fuzzy attributes. To analyze efficiently this kind of decision table, we adapt and combine all three approaches mentioned above: fuzzy rough sets, variable precision rough set model and flow graphs.

First of all, we present a generalized version of our variable precision fuzzy rough set (VPFRS) model [16,17], which was introduced with the aim to enable analysis of fuzzy decision tables obtained from dynamic processes. There are many ways of performing basic operations on fuzzy sets. In order to get a consistent VPFRS model, we propose a unified parameterized approach to approximation of crisp and fuzzy sets. Basing on the notion of rough and fuzzy rough inclusion function, a definition of a single ε-approximation is given.

Secondly, we propose a fuzzy flow graph approach, which is suitable for representing and analyzing fuzzy decision systems. The connection of the flow graph approach with fuzzy inference systems is discussed, The problem of a correct choice of fuzzy connectives, with the aim to retain the flow conservation equations, is considered. Furthermore, we give new definitions of the path's certainty and strength, by respecting only the relevant part of the flow and disregarding the flow components which come from other paths.

Finally, we show that the VPFRS model can be effectively used for a simpler representation and easier selection of fuzzy decision rules with the help of fuzzy flow graphs.

We start with a formal description of fuzzy information systems.

2 Fuzzy Information Systems

In the classical concept of (crisp) sets with sharp boundaries, any element x of an universe U belongs or does not belong to a given subset of U. In contrast to that, the notion of fuzzy sets admits of partial membership. Any fuzzy set F can be defined by assigning to every element $x \in U$ a membership degree $\mu_F(x) \in [0, 1]$ in the set F. Thus, we get a membership function μ_F which describes the fuzzy set F.

In a crisp information system, a set of attributes Q is used to characterize the elements of an universe U. Each element x of the universe U is described by a combination of attributes values. Only one attribute value of each attribute $q \in Q$ can be assigned to a given element $x \in U$.

In order to generalize the notion of information system, we use a set of fuzzy attributes with linguistic values expressed by membership functions. Several linguistic values of every attribute $q \in Q$ can be assigned to an element $x \in U$. In other words, an element x can belong, to a non-zero membership degree, to many fuzzy sets representing linguistic values of an attribute q. We introduce a formal definition of a fuzzy information system.

Definition 1. *A fuzzy information system is the 4-tuple $S = \langle X, Q, L, f \rangle$, where*

U – is a nonempty set, called the universe,
Q – is a finite set of fuzzy attributes,
L – is a set of fuzzy (linguistic) values of attributes, $L = \bigcup_{q \in Q} L_q$,
L_q is the set of linguistic values of an attribute $q \in Q$,
f – is an information function, $f\colon\ U \times L \to [0, 1]$,
$f(x, l) \in [0, 1]$ for every $l \in L$ and every $x \in U$.

In practice, we use fuzzy decision tables, which constitute a special form of fuzzy information systems with two disjoint groups of condition and decision attributes, respectively.

To give a formal description of decision tables, we assume a finite universe U with N elements: $U = \{x_1, x_2, \dots, x_N\}$. Attributes are divided into a subset of n condition attributes: $C = \{c_1, c_2, \dots, c_n\}$, and a subset of m decision attributes: $D = \{d_1, d_2, \dots, d_m\}$.

Every fuzzy attribute is associated with a set of linguistic values. We denote by $V_i = \{V_{i1}, V_{i2}, \dots, V_{in_i}\}$ the family of linguistic values of a condition attribute c_i, and by $W_j = \{W_{j1}, W_{j2}, \dots, W_{jm_j}\}$ the family of linguistic values of a decision attribute d_j, where n_i and m_j, is the number of the linguistic values of the i-th condition and the j-th decision attribute, respectively, $i = 1, 2, \dots, n$, and $j = 1, 2, \dots, m$.

For any element $x \in U$, its membership degrees in all linguistic values of the condition attribute c_i (or decision attribute d_j) should be determined. This is performed in the process called fuzzification, using the recorded crisp value of a particular attribute of the element x. The fuzzy value of an attribute, for a given element x, is a fuzzy set on the domain of all linguistic values of that attribute.

We denote by $V_i(x)$ the fuzzy value of the condition attribute c_i for any $x \in U$, as a fuzzy set on the domain of the linguistic values of c_i

$$V_i(x) = \{\mu_{V_{i1}}(x)/V_{i1},\ \mu_{V_{i2}}(x)/V_{i2},\ \dots,\ \mu_{V_{in_i}}(x)/V_{in_i}\}\,. \tag{1}$$

$W_j(x)$ denotes the fuzzy value of the decision attribute d_j for any $x \in U$, as a fuzzy set on the domain of the linguistic values of d_j

$$W_j(x) = \{\mu_{W_{j1}}(x)/W_{j1},\ \mu_{W_{j2}}(x)/W_{j2},\ \dots,\ \mu_{W_{jm_j}}(x)/W_{jm_j}\}\,. \tag{2}$$

When the linguistic values of all attributes have the form of singletons or disjoint intervals on the original domain of attributes, we get a classical crisp decision table. In such a case, only one linguistic value can be assigned to each condition and decision attribute of an element $x \in U$.

Furthermore, we assume, for any element $x \in U$, that all linguistic values $V_i(x)$ and $W_j(x)$ $(i = 1, 2, \ldots, n\,,\ \ j = 1, 2, \ldots m)$ satisfy the requirements

$$\mathrm{power}(V_i(x)) = \sum_{k=1}^{n_i} \mu_{V_{ik}}(x) = 1\,, \qquad \mathrm{power}(W_j(x)) = \sum_{k=1}^{m_j} \mu_{W_{jk}}(x) = 1\,. \quad (3)$$

The requirements (3) will be used in section 4 for introducing a generalized flow graph approach, which can be applied to analysis of fuzzy information systems.

3 Variable Precision Fuzzy Rough Set Model

3.1 Parameterized Crisp Rough Sets

The rough set theory, proposed by Pawlak [19], is based on the observation that any crisp subset of an universe U can be characterized with respect to an indiscernibility (equivalence) relation $R \subseteq U \times U$. Those classes of indiscernible elements $x \in U$, which are "completely in accordance" with a given set $A \subseteq U$, form the lower approximation of A. Indiscernibility classes, which are "partially in accordance" with A, form the upper approximation of A. A set is called exact, if its lower and upper approximations are equal to each other, otherwise the set is called rough.

The lower approximation $\underline{R}(A)$ and upper approximation $\overline{R}(A)$ of a crisp set A are defined formally as follows

$$\underline{R}(A) = \{x \in U : [x]_R \subseteq A\}\,, \quad (4)$$

$$\overline{R}(A) = \{x \in U : [x]_R \cap A \neq \emptyset\}\,, \quad (5)$$

where $[x]_R$ denotes an indiscernibility class which contains the element $x \in U$.

Observe that the above definitions are constructed using two operations on sets: inclusion and intersection. Let us define the lower and upper approximations, utilizing only the notion of set inclusion.

Definition 2. *Given an indiscernibility relation R, the lower approximation $\underline{R}(A)$ and upper approximation $\overline{R}(A)$ of a crisp set A are defined as follows*

$$\underline{R}(A) = \{x \in U : \forall\, S \subseteq [x]_R \wedge S \neq \emptyset\,,\ \ S \subseteq A\}\,, \quad (6)$$

$$\overline{R}(A) = \{x \in U : \exists\, S \subseteq [x]_R \wedge S \neq \emptyset\,,\ \ S \subseteq A\}\,. \quad (7)$$

The definitions (6) and (7) emphasize two extreme (ideal) cases of approximation obtained by applying the indiscernibility relation R.

The need for defining the lower and upper approximations in a unified way becomes clearer, when we consider the approximation of fuzzy sets. This is because there is no single method of performing basic operations on fuzzy sets. Using only one fuzzy connective is especially important for creating a consistent variable precision fuzzy rough set (VPFRS) model.

Now, let us recall the concept of crisp variable precision rough set (VPRS) model, introduced by Ziarko [35]. In order to cope with inconsistency of information systems, caused by noise and errors in data, it is necessary to admit of some level of misclassification, especially in the case of large information systems. This can be done by relaxing strong inclusion requirements, basing on a modified relation of set inclusion. We explain the VPRS concept using the notion of inclusion degree, $\mathrm{incl}(A, B)$, of a nonempty (crisp) set A in a (crisp) set B, defined as follows

$$\mathrm{incl}(A, B) = \frac{\mathrm{card}(A \cap B)}{\mathrm{card}(A)} . \tag{8}$$

The inclusion degree should be constrained by applying a lower limit l and an upper limit u, introduced in the extended version of VPRS [13], which satisfy the requirement

$$0 \leq l < u \leq 1 . \tag{9}$$

We assume a non-probabilistic interpretation of the VPRS model. The probabilistic rough set approach [27,36], introduced recently, is a generalization of the VPRS model, which bases on conditional probability of inclusion.

Using the limits l and u, which satisfy the constraint (9), one can introduce the notions of u-lower and l-upper approximation of any subset A of the universe U by an indiscernibility relation R.

The u-lower approximation of A by R is a set defined as follows

$$\underline{R}_u(A) = \{x \in U\colon\ \mathrm{incl}([x]_R, A) \geq u\} , \tag{10}$$

where $[x]_R$ denotes an indiscernibility class of R containing the element x.

The l-upper approximation of A by R is a set defined as follows

$$\overline{R}_l(A) = \{x \in U\colon\ \mathrm{incl}([x]_R, A) > l\} . \tag{11}$$

Observe that the definitions (10) and (11) use the same notion of inclusion degree and can be interpreted as a weakened form of (6) and (7).

To extend the crisp VPRS model to a parameterized rough set and fuzzy rough set model, we only apply the degree of set inclusion as the basic notion. For a general treatment of the problem, we adapt the idea of rough inclusion function, given by Skowron and Stepaniuk [26], which is defined on the Cartesian product of the powersets $\mathbb{P}(U)$ of the universe U

$$\nu : \mathbb{P}(U) \times \mathbb{P}(U) \to [0, 1] . \tag{12}$$

We assume that the first parameter represents a nonempty set, and the rough inclusion function should be monotonic with respect to the second parameter

$$\nu(X, Y) \leq \nu(X, Z) \quad \text{for any } Y \subseteq Z, \quad \text{where } X, Y, Z \subseteq U .$$

The inclusion degree (8) is an example of rough inclusion function (12).

Using the rough inclusion function ν, the lower and upper approximations of a crisp set A can be defined by

$$\underline{R}(A) = \{x \in U : \nu([x]_R, A) = 1\} , \tag{13}$$

$$\overline{R}(A) = \{x \in U : \nu([x]_R, A) > 0\} \,. \tag{14}$$

We want to go beyond the standard rough set perspective and introduce a parameterized single form of approximation of crisp sets.

Definition 3. *Given an indiscernibility relation R, the ε-approximation $R_\varepsilon(A)$ of a crisp set A is defined as follows*

$$R_\varepsilon(A) = \{x \in U : \nu([x]_R, A) \geq \varepsilon\} \,, \tag{15}$$

where $\varepsilon \in (0, 1]$.

The ε-approximation R_ε can be used for expressing any kind of approximation, due to the following properties:

$$\begin{array}{lllll}
\text{(P1)} & R_\varepsilon(A) & = & \underline{R}(A) & \text{for } \varepsilon = 1 \,, \\
\text{(P2)} & R_\varepsilon(A) & = & \overline{R}(A) & \text{for } \varepsilon = 0+ \,, \\
\text{(P3)} & R_\varepsilon(A) & = & \underline{R}_u(A) & \text{for } \varepsilon = u \,, \\
\text{(P4)} & R_\varepsilon(A) & = & \overline{R}_l(A) & \text{for } \varepsilon = l+ \;.
\end{array}$$

Although, we have a single notion of ε-approximation, we are still able to determine a pair of approximations, by using a pair of appropriate values of the ε parameter. However, we are mainly interested in determining the consistent part of an analyzed information system. Hence, the lower approximation is the most important notion used for reasoning about data.

The concept of VPRS has turned out to be efficient in applications of the rough set theory to real decision processes [16], e.g. when analyzing the control of dynamic systems, characterized by large decision tables. In such a case the determination of the u-lower approximation (10) should be repeated for different (decreasing) values of the parameter u.

When considering a series of n ε-approximations of a set A, the following property is satisfied due to monotonicity of the inclusion function

$$\text{(P5)} \quad R_{\varepsilon_1}(A) \subseteq R_{\varepsilon_2}(A) \subseteq \ldots \subseteq R_{\varepsilon_n}(A) \quad \text{for} \quad \varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq \varepsilon_n \,.$$

3.2 Parameterized Fuzzy Rough Sets

Our goal is to propose a unified approach to parameterized approximation of fuzzy sets. To this end, we generalize the notion of crisp ε-approximation and adapt the widely used concept of fuzzy rough sets of Dubois and Prade. In consequence, a consistent form of variable precision fuzzy rough set model will be obtained, suitable for analysis of large fuzzy information systems.

Let us recall the definition of fuzzy rough set, introduced by Dubois and Prade [6]. For a given fuzzy set A and a fuzzy partition $\Phi = \{F_1, F_2, \ldots, F_n\}$ on the universe U, the membership functions of the lower and upper approximations of A by Φ are defined as follows

$$\mu_{\underline{\Phi}(A)}(F_i) = \inf_{x \in U} \mathrm{I}(\mu_{F_i}(x), \mu_A(x)) \,, \tag{16}$$

$$\mu_{\overline{\Phi}(A)}(F_i) = \sup_{x \in U} \mathrm{T}(\mu_{F_i}(x), \mu_A(x)) \,, \tag{17}$$

where T and I denote a T-norm operator and an implicator, respectively.

The pair of sets $(\underline{\Phi}F, \overline{\Phi}F)$ is called a fuzzy rough set.

In order to extend the approach given in previous subsection, we need to consider the problem of determining the degree of inclusion of one fuzzy set into another. This problem has been often discussed (see, e.g., [1,2,5,7,15]). Many different measures of fuzzy set inclusion were considered. Among many proposals, implication operators were applied to determination of set inclusion. Sinha-Dougherty [4] proposed an axiomatic approach, which can be formulated using the generalized Łukasiewicz implicators.

We propose a different idea of set inclusion in comparison with various solutions given in the literature. It consists in determination of inclusion with respect to particular elements of sets. This leads to a detailed description of inclusion. In consequence, we get a fuzzy set rather than a number. This method is particulary useful for elaborating an effective variable precision fuzzy rough set model.

A fuzzy set, which describes the inclusion of a fuzzy set A in a fuzzy set B, determined with respect to particular elements of the set A, constitutes the basic notion of our VPFRS model. The obtained fuzzy set will be called the inclusion set of A in B, and denoted by $\mathrm{INCL}(A, B)$.

There are many possibilities of defining the inclusion set. We apply to this end an implication operator denoted by I.

Definition 4. *The implication-based inclusion set* $\mathrm{INCL}(A, B)$ *of a nonempty fuzzy set* A *in a fuzzy set* B *is defined as follows*

$$\mu_{\mathrm{INCL}(A,B)}(x) = \begin{cases} \mathrm{I}(\mu_A(x), \mu_B(x)) & \text{if } \ \mu_A(x) > 0 \,, \\ 0 & \text{otherwise} \,. \end{cases} \tag{18}$$

By assuming that $\mu_{\mathrm{Incl}(A,B)}(x) = 0$, for $\mu_A(x) = 0$, we take into account only the support of the set A. For the sake of simplicity of the computational algorithm, it is not necessary to consider inclusion for all elements of the universe.

Furthermore, we can require that the degree of inclusion with respect to x should be equal to 1, if the inequality $\mu_A(x) \le \mu_B(x)$ for that x is satisfied

$$\mathrm{I}(\mu_A(x), \mu_B(x)) = 1, \quad \text{if } \ \mu_A(x) \le \mu_B(x) \,. \tag{19}$$

The requirement (19) is always satisfied by residual implicators.

In order to define a suitable fuzzy counterpart of the rough inclusion function (12), we apply the notions of α-cut, power (cardinality) and support of a fuzzy set. Given a fuzzy subset A of the universe U, the α-cut of A, denoted by A_α, is a crisp set defined as follows

$$A_\alpha = \{x \in U : \ \mu_A(x) \ge \alpha\} \quad \text{for} \quad \alpha \in [0, 1] \,. \tag{20}$$

For a finite fuzzy set A with n elements, power(A) and support(A) are given by

$$\mathrm{power}(A) = \sum_{i=1}^{n} \mu_A(x_i) \,, \qquad \mathrm{support}(A) = \{x : \mu_A(x_i) > 0\} \,. \tag{21}$$

Using the above notions, we define the fuzzy rough inclusion function on the Cartesian product of the families $\mathbb{F}(U)$ of all fuzzy subsets of the universe U

$$\nu_\alpha : \mathbb{F}(U) \times \mathbb{F}(U) \to [0,1] \,. \tag{22}$$

Definition 5. *The fuzzy rough α-inclusion function $\nu_\alpha(A,B)$ of any nonempty fuzzy set A in a fuzzy set B is defined as follows, for any $\alpha \in (0,1]$*

$$\nu_\alpha(A,B) = \frac{\mathrm{power}(A \cap \mathrm{INCL}(A,B)_\alpha)}{\mathrm{power}(A)} \,, \tag{23}$$

The value $\nu_\alpha(A,B)$ expresses how many elements of the nonempty fuzzy set A belong, at least to the degree α, to the fuzzy set B.

First, we prove monotonicity of the proposed fuzzy rough inclusion function.

Theorem 1. *Implication-based fuzzy rough inclusion function ν_α is monotonic with respect to the second parameter, for any $\alpha \in (0,1]$*

$$\nu_\alpha(X,Y) \le \nu_\alpha(X,Z) \quad \text{for any } Y \subseteq Z, \quad \text{where } X,Y,Z \subseteq \mathbb{F}(U) \,.$$

Proof. According to the definition of a fuzzy subset [14], for $Y \subseteq Z$, we have $\mu_Y(x) \le \mu_Z(x), \forall\, x \in U$. Since every R-implicator, S-implicator and QL-implicator is right monotonic [25], it holds that: $\mu_{\mathrm{I}(X,Y)}(x) \le \mu_{\mathrm{I}(X,Z)}(x), \forall\, x \in U$. Thus, using the definition (18), we get

$$\mu_{\mathrm{INCL}(X,Y)}(x) \le \mu_{\mathrm{INCL}(X,Z)}(x), \quad \forall\, x \in U \,.$$

Finally, for any $\alpha \in (0,1]$, we can easy show that

$$\frac{\mathrm{power}(X \cap \mathrm{INCL}(X,Y)_\alpha)}{\mathrm{power}(X)} \le \frac{\mathrm{power}(X \cap \mathrm{INCL}(X,Z)_\alpha)}{\mathrm{power}(X)} \,.$$

Hence $\nu_\alpha(X,Y) \le \nu_\alpha(X,Z)$. □

Furthermore, we can show that the rough inclusion function used in formulae (10) and (11) is a special case of the fuzzy rough inclusion function (23), when we use the implication-based inclusion set.

Theorem 2. *For any nonempty crisp set A, any crisp set B, and for $\alpha \in (0,1]$, the implication-based inclusion function $\nu_\alpha(A,B)$ is equal to the inclusion degree* $\mathrm{incl}(A,B)$.

Proof. We show that for any crisp sets A and B, the inclusion set $\mathrm{Incl}(A,B)$ is equal to the crisp intersection $A \cap B$. The membership function of any crisp set X is given by

$$\mu_X(x) = \begin{cases} 1 & \text{for } x \in X \\ 0 & \text{for } x \notin X \,. \end{cases} \tag{24}$$

Every implicator I satisfies the conditions: $\mathrm{I}(1,1) = \mathrm{I}(0,1) = \mathrm{I}(0,0) = 1$, and $\mathrm{I}(1,0) = 0$.

Thus, applying the definition (18), we get

$$\mu_{\mathrm{Incl}(A,B)}(x) = \mu_{A\cap B}(x) = \begin{cases} 1 & \text{if } x \in A \text{ and } x \in B \\ 0 & \text{otherwise}. \end{cases} \tag{25}$$

For any finite crisp set X, and any $\alpha \in (0,1]$, by formulae and (20), (21) and (24) we get: $\mathrm{power}(X) = \mathrm{card}(X)$, and $X_\alpha = X$.

Consequently, applying (25), we finally have

$$\frac{\mathrm{power}(A \cap \mathrm{Incl}(A,B)_\alpha)}{\mathrm{power}(A)} = \frac{\mathrm{card}(A \cap B)}{\mathrm{card}(A)} .$$

Hence, we proved that $\nu_\alpha(A,B) = \mathrm{incl}(A,B)$, for any $\alpha \in (0,1]$. □

We want to formulate the fuzzy rough approximation in a general way. Therefore, we introduce a function called `res`, defined on the Cartesian product $\mathbb{P}(U)\times\mathbb{F}(U)$, where $\mathbb{P}(U)$ denotes the powerset of the universe U, and $\mathbb{F}(U)$ the family of all fuzzy subsets of the universe U, respectively

$$\mathrm{res} : \mathbb{P}(U) \times \mathbb{F}(U) \to [0,1] . \tag{26}$$

We require that

$$\begin{aligned}
&\mathrm{res}(\emptyset, Y) = 0 , \\
&\mathrm{res}(X,Y) \in \{0,1\}, \quad \text{if } Y \text{ is a crisp set} , \\
&\mathrm{res}(X,Y) \le \mathrm{res}(X,Z) \quad \text{for any } Y \subseteq Z, \quad \text{where } X \in \mathbb{P}(U), \text{ and } Y, Z \in \mathbb{F}(U) .
\end{aligned}$$

The form of the function `res` can be chosen depending on requirements of a considered application. For a given crisp set X and fuzzy set Y, the value of function $\mathrm{res}(X,Y)$ should express the resulting membership degree in the set Y, taking into account not all elements of the universe, but only the elements of the set X. When we apply the limit-based approach, according to Dubois and Prade, we obtain the following form of the function `res`

$$\mathrm{res}(X,Y) = \inf_{x \in X} \mu_Y(x) . \tag{27}$$

In the definition (27) of the function `res`, only one (limit) value of membership degree of elements in the set Y is taken into account. However, this means that we disregard the character (shape) of the membership function. Basing on a single value of membership degree is not always acceptable, especially in the case of large information systems. Hence, we can use the opportunity of giving another definitions of `res`, in which many values of membership degree are considered.

Now, we introduce the notion of generalized fuzzy rough ε-approximation.

Definition 6. *For $\varepsilon \in (0,1]$, the ε-approximation $\Phi_\varepsilon(A)$ of a fuzzy set A, by a fuzzy partition $\Phi = \{F_1, F_2, \ldots, F_n\}$, is a fuzzy set on the domain Φ with membership function expressed as*

$$\mu_{\Phi_\varepsilon(A)}(F_i) = \mathrm{res}(S_\varepsilon(F_i, A), \mathrm{INCL}(F_i, A)) , \tag{28}$$

where

$$S_\varepsilon(F_i, A) = \text{support}(F_i \cap \text{INCL}(F_i, A)_{\alpha_\varepsilon}) ,$$

$$\alpha_\varepsilon = \sup\{\alpha \in [0,1] : \ \nu_\alpha(F_i, A) \geq \varepsilon\} .$$

The set $S_\varepsilon(F_i, A)$ is equal to support of the intersection of the class F_i with the part of $\text{INCL}(F_i, A)$, which contains those elements of the approximating class F_i which are included in A at least to the degree α_ε. The resulting membership $\mu_{\Phi_\varepsilon(A)}(F_i)$ is determined using only the elements from $S_\varepsilon(F_i, A)$ instead of the whole class F_i. This is accomplished by applying the function `res`.

It can be easy shown that applying the definition (27) of the function `res` leads to a simple form of the ε-approximation (28)

$$\mu_{\Phi_\varepsilon(A)}(F_i) = \sup\{\alpha \in [0,1] : \ \nu_\alpha(F_i, A) \geq \varepsilon\} . \tag{29}$$

In contrast to the approximations (16) and (17), which use two different fuzzy connectives, we have a single unified definition of fuzzy rough approximation. In this way we obtain a consistent variable precision fuzzy rough set model. Thus, we are able to compare approximations determined for various values of the parameter ε.

3.3 Analysis of Fuzzy Decision Tables

In the analysis of fuzzy decision tables, two fuzzy partitions are generated with the help of a suitable similarity relation. The partition obtained with respect to condition attributes is used for approximation of fuzzy similarity classes obtained with respect to decision attributes. It is necessary to address the problem of comparing objects described by fuzzy sets. This issue has been widely studied in the literature, see, for example, [3,7,8]. In our considerations, we apply a symmetric T-transitive fuzzy similarity relation [3], which is defined by means of the distance between the compared elements. In the following, we only give formulae for condition attributes. We apply the notation given in section 2.

If we need to compare any two elements x and y of the universe U with respect to the condition attribute c_i, $i = 1, 2, \ldots, n$, then the similarity between x and y could be expressed using a T-similarity relation based on the Łukasiewicz T-norm [7].

$$S_{c_i}(x, y) = 1 - \max_{k=1,n_i} |\mu_{V_{ik}}(x) - \mu_{V_{ik}}(y)| . \tag{30}$$

In order to evaluate the similarity $S_C(x, y)$, with respect to all condition attributes C, we must aggregate the results obtained for particular attributes c_i, $i = 1, 2, \ldots, n$. This can be done by using the T-norm operator `min` as follows

$$S_C(x, y) = \min_{i=1,n} S_{c_i}(x, y) = \min_{i=1,n} (1 - \max_{k=1,n_i} |\mu_{V_{ik}}(x) - \mu_{V_{ik}}(y)|) . \tag{31}$$

By calculating the similarity for all pairs of elements of the universe U, we obtain a symmetric similarity matrix. If the value of similarity between the elements x and y is equal to 1, they belong to the same similarity class. In that case two rows of the similarity matrix should be merged into one fuzzy set with membership degrees equal to 1 for x and y. In consequence, we get a family of fuzzy similarity classes $\widetilde{C} = \{C_1, C_2, \ldots, C_{\widetilde{n}}\}$, for condition attributes C and a family of fuzzy similarity classes $\widetilde{D} = \{D_1, D_2, \ldots, D_{\widetilde{m}}\}$, for decision attributes D.

In the next step, we determine fuzzy rough approximations of elements of the family $\widetilde{D}$ by the family $\widetilde{C}$, using the parameterized fuzzy rough set model.

To determine the consistency of fuzzy decision tables and significance of attributes, we apply a generalized measure of ε-approximation quality [17]. For the family $\widetilde{D} = \{D_1, D_2, \ldots, D_{\widetilde{m}}\}$ and the family $\widetilde{C} = \{C_1, C_2, \ldots, C_{\widetilde{n}}\}$ the ε-approximation quality of $\widetilde{D}$ by $\widetilde{C}$ is defined as follows

$$\gamma_{\widetilde{C}_\varepsilon}(\widetilde{D}) = \frac{\mathrm{power}(\mathrm{Pos}_{\widetilde{C}_\varepsilon}(\widetilde{D}))}{\mathrm{card}(U)}\,, \tag{32}$$

where

$$\mathrm{Pos}_{\widetilde{C}_\varepsilon}(\widetilde{D}) = \bigcup_{D_j \in \widetilde{D}} \omega(\widetilde{C}_\varepsilon(D_j)) \cap D_j\,.$$

The fuzzy extension ω denotes a mapping from the domain $\widetilde{C}$ into the domain of the universe U, which is expressed for any fuzzy set X by

$$\mu_{\omega(X)}(x) = \mu_X(C_i), \quad \text{if } \ \mu_{C_i}(x) = 1\,. \tag{33}$$

The definition (32) is based on the generalized notion of positive region. For any fuzzy set X and a similarity relation R, the positive region of X is defined as follows

$$\mathrm{Pos}_{R_\varepsilon}(X) = X \cap \omega(R_\varepsilon(X))\,. \tag{34}$$

In the definition of the positive region (34), we take into account only those elements of the ε-approximation, for which there is no contradiction between the set X and the approximating similarity classes.

4 Fuzzy Flow Graphs

In addition to the VPFRS model, we want to introduce fuzzy flow graphs as a second tool for analysis of fuzzy information systems. The idea of applying flow graphs in the framework of crisp rough sets, for discovering the statistical properties of decision algorithms, was proposed by Pawlak [20,21,22]. We should start with recalling the basic notions of the crisp flow graph approach.

A flow graph is given in the form of directed acyclic final graph $G = (\mathcal{N}, \mathcal{B}, \varphi)$, where $\mathcal{N}$ is a set of nodes, $\mathcal{B} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of directed branches, φ: $\mathcal{B} \to \mathrm{R}^+$ is a flow function with values in the set of non-negative reals R^+.

For any $(X, Y) \in \mathcal{B}$, X is an input of Y and Y is an output of X. The quantity $\varphi(X, Y)$ is called the throughflow from X to Y.

$I(X)$ and $O(X)$ denote an input and an output of X, respectively. The input $I(G)$ and output $O(G)$ of a graph G are defined by

$$I(G) = \{X \in \mathcal{N}\colon \ I(X) = \emptyset\}\,, \qquad O(G) = \{X \in \mathcal{N}\colon \ O(X) = \emptyset\}\,. \tag{35}$$

Every node $X \in \mathcal{N}$ of a flow graph G is characterized by its inflow

$$\varphi_{+}(X) = \sum_{Y \in I(X)} \varphi(Y, X)\,, \tag{36}$$

and by its outflow

$$\varphi_{-}(X) = \sum_{Y \in O(X)} \varphi(X, Y)\,. \tag{37}$$

For any internal node X, the equality $\varphi_{+}(X) = \varphi_{-}(X) = \varphi(X)$ is satisfied. The quantity $\varphi(X)$ is called the flow of the node X.

The flow for the whole graph G is defined by

$$\varphi(G) = \sum_{x \in I(G)} \varphi_{-}(X) = \sum_{x \in O(G)} \varphi_{+}(X)\,. \tag{38}$$

By using the flow $\varphi(G)$, the normalized throughflow $\sigma(X, Y)$ and the normalized flow $\sigma(X)$ are determined as follows

$$\sigma(X, Y) = \frac{\varphi(X, Y)}{\varphi(G)}\,, \qquad \sigma(X) = \frac{\varphi(X)}{\varphi(G)}\,. \tag{39}$$

For every branch of a flow graph G the certainty factor is defined by

$$\mathrm{cer}(X, Y) = \frac{\sigma(X, Y)}{\sigma(X)}\,. \tag{40}$$

The coverage factor for every branch of a flow graph G is defined by

$$\mathrm{cov}(X, Y) = \frac{\sigma(X, Y)}{\sigma(Y)}\,. \tag{41}$$

The certainty and coverage factors satisfy the following properties

$$\sum_{Y \in O(X)} \mathrm{cer}(X, Y) = 1\,, \qquad \sum_{X \in I(Y)} \mathrm{cov}(X, Y) = 1\,. \tag{42}$$

The measures of certainty (40) and coverage (41) are useful for analysis of decision algorithms [10].

Now, we consider the issue of applying flow graphs to representation and analysis of fuzzy decision algorithms. We use decision tables with fuzzy values of attributes, presented in section 2. All possible decision rules, generated by the Cartesian product of sets of linguistic values of the attributes, will be examined. According to notation used in section 2, we obtain $r = \prod_{i=1}^{n} n_i \prod_{j=1}^{m} m_j$ possible rules. The k-th decision rule, denoted by R_k, is expressed as follows

$$\begin{array}{ll} R_k\colon & \text{IF } c_1 \text{ is } V_1^k \text{ AND } c_2 \text{ is } V_2^k \ \ldots \text{ AND } c_n \text{ is } V_n^k \\ & \text{THEN } d_1 \text{ is } W_1^k \text{ AND } d_2 \text{ is } W_2^k \ \ldots \text{ AND } d_m \text{ is } W_m^k \end{array} \tag{43}$$

where $k = 1, 2, \ldots, r\,,\ V_i^k \in V_i\,,\ i = 1, 2, \ldots n\,,\ W_j^k \in W_j\,,\ j = 1, 2, \ldots, m.$

When we use the fuzzy Cartesian products $C^k = V_1^k \times V_2^k \times \ldots \times V_n^k$ and $D^k = W_1^k \times W_2^k \times \ldots \times W_m^k$, the k-th decision rule can be expressed in the form of a fuzzy implication, denoted here by $C^k \rightarrow D^k$.

It is necessary to select a subset of decision rules which are relevant to the considered decision process. This can be done by determining to what degree any element $x \in U$, corresponding to a single row of the decision table, confirms particular decision rules. We calculate the truth value of the decision rule's antecedent and the truth value of the decision rule's consequent, by determining the conjunction of the respective membership degrees of x in the linguistic values of attributes.

If we take a decision table with crisp attributes, a decision rule can be confirmed for some x, if the result of conjunction is equal to 1, both for the rule's premise and the rule's conclusion. Otherwise, the element x does not confirm the considered decision rule. The set of elements $x \in U$, which confirm a decision rule, is called the support of the decision rule.

To determine the confirmation degree of fuzzy decision rules, a T-norm operator need to be applied. By $\mathrm{cd}(x,k)$, we denote the confirmation degree of the k-th decision rule by the element $x \in U$

$$\mathrm{cd}(x,k) = \mathrm{T}(\mathrm{cda}(x,k), \mathrm{cdc}(x,k))\,, \tag{44}$$

where $\mathrm{cda}(x,k)$ denotes the confirmation degree of the decision rule's antecedent

$$\mathrm{cda}(x,k) = \mathrm{T}(\mu_{V_1^k}(x), \mu_{V_2^k}(x), \ldots, \mu_{V_n^k}(x))\,, \tag{45}$$

and $\mathrm{cdc}(x,k)$ the confirmation degree of the decision rule's consequent

$$\mathrm{cdc}(x,k) = \mathrm{T}(\mu_{W_1^k}(x), \mu_{W_2^k}(x), \ldots, \mu_{W_m^k}(x))\,. \tag{46}$$

Through determining the confirmation degrees (45), (46) and (44), we generate the following fuzzy sets on the domain U:

the support of the decision rule's antecedent

$$\mathrm{support}(\mathrm{cda}(x,k)) = \{\mathrm{cda}(x_1,k)/x_1,\ \mathrm{cda}(x_2,k)/x_2,\ \ldots,\ \mathrm{cda}(x_N,k)/x_N\}, \tag{47}$$

the support of the decision rule's consequent

$$\mathrm{support}(\mathrm{cdc}(x,k)) = \{\mathrm{cdc}(x_1,k)/x_1,\ \mathrm{cdc}(x_2,k)/x_2,\ \ldots,\ \mathrm{cda}(x_N,k)/x_N\}\,, \tag{48}$$

and the support of the decision rule R_k, respectively

$$\mathrm{support}(R_k) = \{\mathrm{cd}(x_1,k)/x_1,\ \mathrm{cd}(x_2,k)/x_2,\ \ldots,\ \mathrm{cd}(x_N,k)/x_N\}\,. \tag{49}$$

The introduced notions (47), (48) and (49) will be used for defining strength, certainty, and coverage factors of a decision rule.

Now, let us explain the way of constructing fuzzy flow graphs on the basis of a decision table with fuzzy attributes.

Every fuzzy attribute is represented by a layer of nodes. The nodes of a layer correspond to linguistic values of a given attribute.

We denote by $\widetilde{X}$ a fuzzy set on the universe U, which describes membership degree of particular elements $x \in U$ in the linguistic value represented by X. The membership degrees of all x in the set $\widetilde{X}$ can be found in a respective column of the considered decision table.

Let us pick out such two attributes, which are represented by two consecutive layers of the flow graph. We denote by X a linguistic value of the first attribute, and by Y a linguistic value of the second attribute. In the case of crisp flow graphs, the flow between nodes X and Y is equal to the number of elements of the universe U, which are characterized by the combination of attribute values X and Y. In consequence, a particular element $x \in U$ can only be assigned to a unique path in the flow graph. In a fuzzy information system, however, every element of the universe can belong to several linguistic values, and it can be assigned to several paths in the flow graph.

It is possible to determine the flow distribution in the crisp flow graph by using the operations of set intersection and set cardinality. To obtain the flow $\varphi(X, Y)$ for the branch (X, Y) of a fuzzy flow graph, we have to calculate power of the intersection of fuzzy sets $\widetilde{X}$ and $\widetilde{Y}$. Many definitions of fuzzy intersection (T-norm operator) are known. In order to satisfy the flow conservation equations, it is necessary to use the T-norm operator `prod` for determining the intersection of sets. Furthermore, we should assume that the linguistic values of attributes satisfy the requirement (3). We conclude the above discussion with the following theorem.

Theorem 3. *Let* S *be a fuzzy information systems with the linguistic values of attributes satisfying the requirement* (*3*)*, and let* $\cap$ *denote a fuzzy intersection operator based on the* T*-norm* prod*. The following properties are satisfied for the flow graph, which represents the information system* S :

(G1) *the inflow for any output or internal layer node* X *is given by*

$$\varphi_{+}(X) = \mathrm{power}(\widetilde{X}) = \sum_{Y \in I(X)} \varphi(Y, X) = \sum_{Y \in I(X)} \mathrm{power}(\widetilde{X} \cap \widetilde{Y})\,, \tag{50}$$

(G2) *the outflow for any input or internal layer node* X *is given by*

$$\varphi_{-}(X) = \mathrm{power}(\widetilde{X}) = \sum_{Y \in O(X)} \varphi(X, Y) = \sum_{Y \in O(X)} \mathrm{power}(\widetilde{X} \cap \widetilde{Y})\,, \tag{51}$$

(G3) *for any internal layer node* X*, it holds that*

$$\varphi_{+}(X) = \varphi_{-}(X)\,. \tag{52}$$

The properties (G1), (G2) and (G3) do not hold in general, if we use another T-norm operator, e.g. `min`. In the special case of crisp decision tables, the formulae (50) and (51) become equivalent to (36) and (37).

The layers corresponding to condition attributes can be merged into a single layer, which contains nodes representing all possible combinations of linguistic values of the condition attributes. We can also merge all the layers corresponding to decision attributes. Let us denote by X^*, a node of the resulting layer obtained for condition attributes and by Y^*, a node of the resulting layer obtained for decision attributes. The node X^* corresponds to antecedent of some decision rule R_k. Support of the antecedent of the decision rule R_k is determined with the help of formula (47).

The decision rule R_k is represented by the branch (X^*, Y^*). Power of the support of the rule R_k is equal to the flow between the nodes X^* and Y^*, which is obtained using formula (49)

$$\varphi(X^*, Y^*) = \text{power}(\text{support}(R_k)) \,. \tag{53}$$

By applying the formulae (47), (48) and (49), we can determine, for every decision rule R_k, the certainty factor $\text{cer}(X^*, Y^*)$, the coverage factor $\text{cov}(X^*, Y^*)$, and the strength of the rule $\sigma(X^*, Y^*)$

$$\text{cer}(X^*, Y^*) = \text{cer}(R_k) = \frac{\text{power}(\text{support}(R_k))}{\text{power}(\text{support}(\text{cda}(x,k)))} \,, \tag{54}$$

$$\text{cov}(X^*, Y^*) = \text{cov}(R_k) = \frac{\text{power}(\text{support}(R_k))}{\text{power}(\text{support}(\text{cdc}(x,k)))} \,, \tag{55}$$

$$\sigma(X^*, Y^*) = \text{strength}(R_k) = \frac{\text{power}(\text{support}(R_k))}{\text{card}(U)} \,. \tag{56}$$

It is possible to represent any decision rule by a sequence of nodes $[X_1 \ldots X_n]$, namely by a path from the 1-th to the n-th layer of the flow graph G. For a given path $[X_1 \ldots X_n]$, the resulting certainty and strength can be defined. In contrast to the definitions presented in [20,21,22], in which the statistical properties of flow are taken into account, we propose a different form of the path's certainty and strength

$$\text{cer}[X_1 \ldots X_n] = \prod_{i=1}^{n-1} \text{cer}(X_1 \ldots X_i, X_{i+1}) \,, \tag{57}$$

$$\sigma[X_1 \ldots X_n] = \sigma(X_1)\, \text{cer}[X_1 \ldots X_n] \,, \tag{58}$$

where

$$\text{cer}(X_1 \ldots X_i, X_{i+1}) = \frac{\text{power}(\widetilde{X}_1 \cap \widetilde{X}_2 \cap \ldots \cap \widetilde{X}_{i+1})}{\text{power}(\widetilde{X}_1 \cap \widetilde{X}_2 \cap \ldots \cap \widetilde{X}_i)} \,. \tag{59}$$

The resulting certainty (57) of the path $[X_1 \ldots X_n]$, expresses what part of the flow of the starting node X_1 reaches the final node X_n, passing through all nodes of the path.

5 Examples

Let us analyze a fuzzy decision table (Table 1) with condition attributes c_1 and c_2 and one decision attribute d. All attributes have three linguistic values.

Table 1. Decision table with fuzzy attributes

	c_1			c_2			d		
	V_{11}	V_{12}	V_{13}	V_{21}	V_{22}	V_{23}	W_{11}	W_{12}	W_{13}
x_1	0.1	0.9	0.0	0.0	0.9	0.1	0.0	1.0	0.0
x_2	0.8	0.2	0.0	1.0	0.0	0.0	0.0	0.1	0.9
x_3	0.0	0.2	0.8	0.0	0.2	0.8	0.9	0.1	0.0
x_4	0.1	0.9	0.0	0.0	0.9	0.1	0.0	1.0	0.0
x_5	0.0	0.8	0.2	0.8	0.2	0.0	0.0	0.1	0.9
x_6	0.8	0.2	0.0	0.0	0.2	0.8	1.0	0.0	0.0
x_7	0.1	0.9	0.0	0.0	0.9	0.1	0.1	0.9	0.0
x_8	0.0	0.1	0.9	0.8	0.2	0.0	0.0	0.0	1.0
x_9	0.0	0.2	0.8	0.0	0.2	0.8	0.9	0.1	0.0
x_{10}	0.1	0.9	0.0	0.1	0.9	0.0	0.0	0.9	0.1

First, we apply the variable precision fuzzy rough set approach. Using similarity relation in the form (31), we determine similarity matrices with respect to condition and decision attributes. By merging identical rows of the similarity matrix, we get 9 condition similarity classes and and 6 decision similarity classes. We calculate ε-approximation quality using the Łukasiewicz implication operator. The results are presented in table 2.

Table 2. ε-approximation quality for different values of parameter ε

Method	Removed attribute	$\gamma_{\widetilde{C}_\varepsilon}(\widetilde{D})$			
		$\varepsilon = 1$	$\varepsilon = 0.9$	$\varepsilon = 0.85$	$\varepsilon = 0.8$
Ł-inf	none	0.830	0.900	0.900	0.910
	c_1	0.820	0.880	0.880	0.910
	c_2	0.250	0.250	0.410	0.450

We can state that the considered information system has a high consistency. The condition attribute c_1 can be omitted from the decision table without a significant decrease of the ε-approximation quality.

In the next step, the flow graph method will be applied. We use the same labels for both the linguistic values of the attributes and the corresponding nodes of the flow graph. As stated in previous section, the T-norm operator `prod` should be used in our calculations. The obtained fuzzy flow graph has a very simple form, because there is only one condition attribute c_2 and one decision attribute d. Values of the normalized flow between nodes of the condition layer and nodes of the decision layer are given in Table 3.

Table 3. Normalized flow between nodes of condition and decision layers

	$\sigma(V_{2i}, W_{1j})$			
	W_{11}	W_{12}	W_{13}	Σ
V_{21}	0.000	0.027	0.243	0.270
V_{22}	0.065	0.348	0.047	0.460
V_{23}	0.225	0.045	0.000	0.270
Σ	0.290	0.420	0.290	1.000

We see that the flow conservation equations (50) and (51), are satisfied, for example,

$$\sigma_-(V_{21}) = \frac{\mathrm{power}(\widetilde{V}_{21})}{\mathrm{card}(U)} = \sum_{i=1}^{3} \sigma(V_{21}, W_{1i}) = 0.270\,,$$

$$\sigma_+(W_{11}) = \frac{\mathrm{power}(\widetilde{W}_{11})}{\mathrm{card}(U)} = \sum_{i=1}^{3} \sigma(V_{2i}, W_{11}) = 0.290\,.$$

Let us determine the certainty and coverage factors for branches between the layers according to formulae (54), (55). The results are given in Tables 4 and 5.

Table 4. Certainty factor for branches between condition and decision layers

	$\mathrm{cer}(V_{2i}, W_{1j})$			
	W_{11}	W_{12}	W_{13}	Σ
V_{21}	0.0000	0.1000	0.9000	1.0000
V_{22}	0.1413	0.7565	0.1022	1.0000
V_{23}	0.8333	0.1667	0.0000	1.0000

Table 5. Coverage factor for branches between condition and decision layers

	$\mathrm{cov}(V_{2i}, W_{1j})$		
	W_{11}	W_{12}	W_{13}
V_{21}	0.0000	0.0643	0.8379
V_{22}	0.2241	0.8286	0.1621
V_{23}	0.7759	0.1071	0.0000
Σ	1.0000	1.0000	1.0000

Fuzzy decision rules with the largest values of certainty factor (Table 6) can be included in the final fuzzy inference system. The respective values of coverage factor are useful for explaining the selected decision rules. Only 3 decision rules

Table 6. Decision rules with the largest value of certainty factor

decision rule	certainty	coverage	strength [%]
$V_{21} \to W_{13}$	0.9000	0.8379	24.30
$V_{22} \to W_{12}$	0.7565	0.8286	34.80
$V_{23} \to W_{11}$	0.8333	0.7759	22.50

could be generated from our decision table. Owing to the application of the VPFRS approach, we got a simple fuzzy flow graph.

Let us construct a flow graph without a prior reduction of attributes. We merge the layers corresponding to condition attributes c_1 and c_2 to a resulting layer, which represents all possible linguistic values in the antecedences of decision rules.

We determine the degrees of satisfaction of the rules' antecedences for particular elements $x \in U$. For the antecedence represented by $V_{12}V_{22}$, we get:

$$\widetilde{V_{12}V_{22}} = \widetilde{V}_{12} \cap \widetilde{V}_{22} = \{\ 0.81/x_1,\ \ 0.00/x_2,\ \ 0.04/x_3,\ \ 0.81/x_4,\ \ 0.16/x_5,\ \ 0.04/x_6,\ \ 0.81/x_7,\ \ 0.02/x_8,\ \ 0.04/x_9,\ \ 0.81/x_{10}\},$$

$$\varphi(V_{12}, V_{22}) = \mathrm{power}(\widetilde{V_{12}V_{22}}) = 3.54, \quad \sigma(V_{12}, V_{22}) = \frac{\varphi(V_{12}, V_{22})}{\mathrm{card}U} = 0.354.$$

Table 7. Decision rules with the largest certainty factor (full information system)

decision rule	certainty	coverage	strength [%]
$V_{11}V_{21} \to W_{13}$	0.8901	0.2486	7.21
$V_{11}V_{23} \to W_{11}$	0.9567	0.2210	6.41
$V_{12}V_{21} \to W_{13}$	0.8366	0.2914	8.45
$V_{12}V_{22} \to W_{12}$	0.8763	0.7386	31.02
$V_{13}V_{21} \to W_{13}$	0.9818	0.2979	8.64
$V_{13}V_{23} \to W_{11}$	0.9000	0.3972	11.52

Finally, we determine the normalized throughflow, certainty and coverage factors for branches between of the resulting condition and decision layers. Decision rules with the largest value of certainty factor are given in Table 7. We can observe that the attribute c_1 is superfluous in the obtained decision rules.

6 Conclusions

Information systems with crisp and fuzzy attributes can be effectively analyzed by a hybrid approach which combines the variable precision fuzzy rough set (VPFRS) model with fuzzy flow graphs. The VPFRS model can be defined in a unified way with the help of a single notion of ε-approximation. This allows to avoid the inconsistency of the VPFRS model caused by different forms of fuzzy connectives. The proposed fuzzy flow graph method is suitable for representing and analyzing decision tables with fuzzy attributes. Every fuzzy attribute can be represented by a layer of a flow graph. All nodes of a layer correspond to linguistic values of an attribute. A fuzzy decision table can be reduced by applying the VPFRS approach prior to using the fuzzy flow graph method for determining a system of fuzzy decision rules.

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