No Multiple Collisions for Mutually Repelling Brownian Particles

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Summary. Although Brownian particles with small mutual electrostatic repulsion may collide, multiple collisions at positive time are always forbidden.

1 Introduction

A three-dimensional Brownian motion $B_t = (B_t^1, B_t^2, B_t^3)$ does not hit the axis $\{x_1 = x_2 = x_3\}$ except possibly at time 0. An easy proof is obtained by applying Ito's formula to $R_t = [(B_t^1 - B_t^2)^2 + (B_t^1 - B_t^3)^2 + (B_t^2 - B_t^3)^2]$ and remarking that up to the multiplicative constant 3 the process R is the square of a two-dimensional Bessel process for which $\{0\}$ is a polar state. This remark will be our guiding line in the sequel.

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ and for $N \geqslant 3$ the following system of stochastic differential equations

$$dX_t^i \;=\; dB_t^i \;+\; \lambda \sum_{1\leqslant j\neq i\leqslant N} \frac{dt}{X_t^i - X_t^j} \;,\; i = 1, 2, \dots, N$$

with boundary conditions

$$X_t^1 \leqslant X_t^2 \leqslant \cdots \leqslant X_t^N \,, \quad 0 \leqslant t < \infty \,,$$

and a random, $\mathcal{F}_0$-measurable, initial value satisfying

$$X_0^1 \leqslant X_0^2 \leqslant \cdots \leqslant X_0^N \,.$$

Here $B_t = (B_t^1, B_t^2, \dots, B_t^N)$ denotes a standard N-dimensional $(\mathcal{F}_t)$-Brownian motion and λ is a positive constant. This system has been extensively studied in [5], [7], [2], [1], [3], [6]. For comments on the relationship between this system and the spectral analysis of Brownian matrices, and also conditioning of Brownian particles, we refer to the introduction and the bibliography in [3].

When $\lambda \geqslant \frac{1}{2}$, establishing strong existence and uniqueness is not difficult, because particles never collide, as proved in [7]. The general case with arbitrary coupling strength is investigated in [2] and it is proved in [3] that collisions occur a.s. if and only if $0 < \lambda < \frac{1}{2}$. As for multiple collisions (three or more particles at the same location), it has been stated without proof in [9] and [4] that they are impossible. The proof we give below, with a Bessel process unexpectedly coming in, is just an exercise on Ito's formula.

2 A remarkable identity in law

We consider for any $t \geqslant 0$

$$S_t = \sum_{j=1}^{N}\sum_{k=1}^{N}(X_t^j - X_t^k)^2 .$$

Theorem 1. *For any $\lambda > 0$, the process S divided by the constant $2N$ is the square of a Bessel process with dimension $(N-1)(\lambda N+1)$.*

Proof. It is purely computational. Ito's formula provides for any $j \neq k$

$$\begin{aligned}(X_t^j - X_t^k)^2 = (X_0^j - X_0^k)^2 &+ 2\int_0^t (X_s^j - X_s^k)\, d(B_s^j - B_s^k) \\ &+ 2\lambda \sum_{1\leqslant l\neq j\leqslant N} \int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l}\, ds \\ &+ 2\lambda \sum_{1\leqslant m\neq k\leqslant N} \int_0^t \frac{X_s^k - X_s^j}{X_s^k - X_s^m}\, ds + 2\,t .\end{aligned}$$

Adding the $N(N-1)$ equalities we get

$$\begin{aligned}S_t = S_0 &+ 2\sum_{j=1}^{N}\sum_{k=1}^{N}\int_0^t (X_s^j - X_s^k)\, d(B_s^j - B_s^k) \\ &+ 4\lambda \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant l\neq j\leqslant N}\int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l}\, ds \;+\; 2N(N-1)t .\end{aligned}$$

But

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant l\neq j\leqslant N}\int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l}\, ds \\ &= \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant j\neq l\leqslant N}\left[\int_0^t \frac{X_s^j - X_s^l}{X_s^j - X_s^l}\, ds + \int_0^t \frac{X_s^l - X_s^k}{X_s^j - X_s^l}\, ds\right] \\ &= N^2(N-1)t - \sum_{l=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant l\neq j\leqslant N}\int_0^t \frac{X_s^l - X_s^k}{X_s^l - X_s^j}\, ds \\ &= \frac{1}{2}N^2(N-1)t .\end{aligned}$$

For the martingale term, we compute

$$\begin{aligned}
&\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(X_s^j - X_s^k)\right)^2 \\
&\quad = \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}(X_s^j - X_s^k)(X_s^j - X_s^l) \\
&\quad = \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}(X_s^j - X_s^k)^2 + \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}(X_s^j - X_s^k)(X_s^k - X_s^l) \\
&\quad = \frac{N}{2}S_s\,.
\end{aligned}$$

Let B' be a linear Brownian motion independent of B. The process C defined by:

$$C_t \;=\; \int_0^t \mathbf{I}_{\{S_s>0\}}\,\frac{\displaystyle\sum_{j=1}^{N}\sum_{k=1}^{N}(X_s^j - X_s^k)\,dB_s^j}{\sqrt{\frac{N}{2}S_s}} \;+\; \int_0^t \mathbf{I}_{\{S_s=0\}}dB'_s$$

is a linear Brownian motion and we have

$$S_t \;=\; S_0 \;+\; 2\int_0^t \sqrt{2NS_s}\,dC_s \;+\; 2N(N-1)(\lambda N+1)t\,,$$

which completes the proof. □

3 Multiple collisions are not allowed

Since multiple collisions do not occur for Brownian particles without interaction, we can guess they do not either in case of mutual repulsion. Here is the proof.

Theorem 2. *For any $\lambda > 0$, multiple collisions cannot occur after time* 0.

Proof. i) For $3 \leqslant r \leqslant N$ and $1 \leqslant q \leqslant N - r + 1$, let

$$I = \{q, q+1, \dots, q+r-1\}$$

$$S_t^I = \sum_{j\in I}\sum_{k\in I}(X_t^j - X_t^k)^2$$

$$\tau^I = \inf\{t > 0 : S_t^I = 0\}\;.$$

ii) We first consider the initial condition X_0. From [2], Lemma 3.5, we know that for any $1 \leqslant i < j \leqslant N$ and any $t < \infty$, we have a.s.

$$\int_0^t \frac{du}{X_u^j - X_u^i} \; < \; \infty\,.$$

Therefore for any $u > 0$ there exists $0 < v < u$ such that $X_v^1 < X_v^2 < \cdots < X_v^N$ a.s. In order to prove $\mathbb{P}(\tau^I = \infty) = 1$, we may thus assume $X_0^1 < X_0^2 < \cdots < X_0^N$ a.s., which implies for any I that $S_0^I > 0$ and so $\tau^I > 0$ a.s.
iii) We know ([8], XI, section 1) that $\{0\}$ is polar for the Bessel process $\sqrt{S_t}/\sqrt{2N}$, which means that $\tau^I = \infty$ a.s. for $I = \{1, 2, \ldots, N\}$. We will prove the same result for any I by backward induction on $r = \mathrm{card}(I)$. Assume there are no s-multiple collisions for any $s > r$. Then

$$\begin{aligned} S_t^I = S_0^I \;&+\; 4\sum_{j\in I}\sum_{k\in I}\int_0^t (X_s^j - X_s^k)\, dB_s^j \\ &+ 4\lambda \sum_{j\in I}\sum_{k\in I}\sum_{l\notin I}\int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l}\, ds \;+\; 2r(r-1)(\lambda r + 1)t\,. \end{aligned}$$

We set for $n \in \mathbb{N}^*$, $\tau_n^I \;=\; \inf\{t > 0 : S_t^I \leqslant 1/n\}$. For any $t \geqslant 0$,

$$\begin{aligned} \log S_{t\wedge\tau_n^I}^I = \log S_0^I \;&+\; 4\sum_{j\in I}\sum_{k\in I}\int_0^{t\wedge\tau_n^I} \frac{X_s^j - X_s^k}{S_s^I}\, dB_s^j \\ &+ 2\lambda \sum_{j\in I}\sum_{k\in I}\sum_{l\notin I}\int_0^{t\wedge\tau_n^I} \frac{(X_s^j - X_s^k)}{S_s^I}\left[\frac{1}{X_s^j - X_s^l} - \frac{1}{X_s^k - X_s^l}\right] ds \\ &+ 2r[(r-1)(\lambda r + 1) - 2]\int_0^{t\wedge\tau_n^I} \frac{ds}{S_s^I} > -\infty\;. \end{aligned}$$

From the induction hypothesis we deduce that for $j, k \in I$ and $l \notin I$, a.s. on $\{\tau^I < \infty\}$, $(X_{\tau^I}^j - X_{\tau^I}^l)(X_{\tau^I}^k - X_{\tau^I}^l) \;>\; 0$ and so

$$\begin{aligned} &\int_0^{t\wedge\tau^I} \frac{(X_s^j - X_s^k)}{S_s^I}\left[\frac{1}{X_s^j - X_s^l} - \frac{1}{X_s^k - X_s^l}\right] ds \\ &\quad = \; -\int_0^{t\wedge\tau^I} \frac{(X_s^j - X_s^k)^2}{S_s^I}\frac{ds}{(X_s^j - X_s^l)(X_s^k - X_s^l)} > -\infty\,. \end{aligned}$$

The martingale $(M_n, \mathcal{F}_{t\wedge\tau_n^I})_{n\geqslant 1}$ defined by

$$M_n \;=\; 4\sum_{j\in I}\sum_{k\in I}\int_0^{t\wedge\tau_n^I} \frac{X_s^j - X_s^k}{S_s^I}\, dB_s^j$$

has associated increasing process $A_n = 8r \int_0^{t\wedge\tau_n^I} \frac{ds}{S_s^I}$. It follows that $M_n + \frac{1}{4}[(r-1)(\lambda r+1)-2]A_n$ either tends to a finite limit or to $+\infty$ as n tends to $+\infty$. Then for any $t \geqslant 0$, $\log S^I_{t\wedge\tau^I} > -\infty$ and so $\mathbb{P}(\tau^I = \infty) = 1$, which completes the proof. $\square$

4 Brownian particles on the circle

We now turn to the popular model of interacting Brownian particles on the circle ([9], [3]). Consider the system of stochastic differential equations

$$dX_t^i \ = \ dB_t^i \ + \ \frac{\lambda}{2} \sum_{1\leqslant j\neq i\leqslant N} \cot\left(\frac{X_t^i - X_t^j}{2}\right) dt\,, \ i = 1, 2, \ldots, N$$

with the boundary conditions

$$X_t^1 \leqslant X_t^2 \leqslant \cdots \leqslant X_t^N \leqslant X_t^1 \ + \ 2\pi\,, \qquad 0 \leqslant t < \infty\,.$$

As expected we can prove there are no multiple collisions for the particles $Z_t^j \ = \ e^{i\,X_t^j}$ that live on the unit circle. The proof is more involved and will be deduced by approximation from the previous one.

Theorem 3. *Multiple collisions for the particles on the circle do not occur after time* 0 *for any* $\lambda > 0$.

Sketch of the proof. For the sake of simplicity, we only deal with the N-collisions. Let

$$R_t = \sum_{j=1}^{N}\sum_{k=1}^{N} \sin^2\left(\frac{X_t^j - X_t^k}{2}\right)$$

$$\sigma_n = \inf\{t > 0 \ : \ R_t \leqslant \frac{1}{n}\}\ .$$

We apply Ito's formula to $\log\, R_t$ and get

$$\log\, R_{t\wedge\sigma_n} \ = \ \log\, R_0 \ + \sum_{j=1}^{N} \int_0^{t\wedge\sigma_n} H_s^j\, dB_s^j \ + \ \int_0^{t\wedge\sigma_n} L_s\, ds$$

for some continuous processes H^j and L. We divide each integral into an integral over $\{R_s \geqslant \frac{1}{2}\}$ and an integral over $\{R_s < \frac{1}{2}\}$. The first type integrals do not pose any problem. When $R_s < \frac{1}{2}$, we replace X_s^j with

$$Y_s^j \ = \ X_s^j \quad \text{ or } Y_s^j \ = \ X_s^j \ - \ 2\pi$$

in such a way that for any j, k we have $|Y_s^j - Y_s^k| < \pi/3$. The processes H^j and L have the same expressions in terms of X or Y. With this change of variables we may approximate $\sin x$ by x, $\cos x$ by 1 and replace the trigonometric functions by approximations of the linear ones which we have met in the previous sections. We obtain that

$$\log R_{t\wedge\sigma_n} = \log R_0 + M_n + \frac{1}{4}[(N-1)(\lambda N+1)-2]A_n + \int_0^{t\wedge\sigma_n} D_s\,ds$$

where M_n is a martingale with associated increasing process A_n and D is a.s. a locally integrable process. Details are left to the reader as well as the case of an arbitrary subset I like those in Section 3.

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