Lecture Notes in Mathematics

Catherine Donati-Martin
Michel Émery
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Editors

Catherine Donati-Martin
Laboratoire de Probabilités
et Modèles Aléatoires
Université Pierre et Marie Curie
Boîte courrier 188
4, place Jussieu
75252 Paris cedex 05, France
e-mail: donati@ccr.jussieu.fr

Alain Rouault
Laboratoire de Mathématiques
Bâtiment Fermat
Université Versailles-Saint-Quentin
45, avenue des Etats-Unis
78035 Versailles cedex, France
e-mail: rouault@fermat.math.uvsq.fr

Michel Émery
Institut de Recherche
Mathématique Avancée
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg cedex, France
e-mail: emery@math.u-strasbg.fr

Christophe Stricker
UFR Sciences et techniques
Université de Besançon
16, route de Gray
25030 Besançon cedex, France
e-mail: christophe.stricker@univ-fcomte.fr

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While correcting the proofs of this volume, we received the sad news that Frank Knight had passed away. His deep understanding of stochastic processes, and in particular of their local times, has inspired many an author in the Séminaire de Probabilités; his contributions stand as models for clarity and rigor. The Séminaire has lost a very close friend and contributor.

This volume is dedicated to his memory.

C. Donati-Martin
M. Émery
A. Rouault
C. Stricker
M. Yor

Frank Knight (1933–2007): An appreciation, with respect and admiration

by Marc Yor

In March 2007, Frank passed away after a long illness. He is well known for having extracted some gems in the world of diffusions, e.g., the famous Ray–Knight theorems on local times, to mention one of his celebrated achievements.

Once started on a research topic, he was possessed by a very strong drive to solve the problem in a rigorous way: the reader of these lines should take the opportunity to have a look at his *Impressions of P.A. Meyer as Deus Ex-Machina* [1], where most of the discussion consists in disentangling some flaws in Frank's paper [2], for which P.A. Meyer helped him very generously…

Many exchanges with Frank were of this kind, as he wrote letters about fine points of martingale time changes [3], discussed the Krein theorem in relation with inverse local times [4], was interested in the Brownian spider [5], developed his beloved Prediction Theory ([6], [7]), and so on.

It is a tautology to say that Frank Knight had his own way of looking at things; see e.g., the Foreword to his *Essentials of Brownian Motion and Diffusion* [8] where he explains why no stochastic integrals will be found in the book . . .

Despite his illness, he worked until the end, as shown by his joint paper [9] published in the volume [10] edited by D. Burkholder in memory of J. Doob.

However, a few months after P.A. Meyer's death, when I asked him to be part of the scientific committee for the Memorial Conference in February 2004 in Strasbourg, Frank wrote kindly that I was not being "reasonable". . .

This was typical of Frank's seriousness, often mingled with humor.

I feel, as many of the Séminaire's oldies, that a great probabilist just started off his ultimate random walk.

References

[1] F. Knight: An Impression of P.A. Meyer as Deus Ex-Machina. *Séminaire de Probabilités XXXVII,* LNM 1832. Springer, Berlin Heidelberg New York (2004)

[2] F. Knight: A past predictive view of Gaussian processes. *Ann. Sci. Éc. Norm. Sup.* 4^{e} série, t. 16, 541–566 (1983)

[3] F. Knight: On invertibility of martingale time changes. *Seminar on Stochastic Processes 1987,* 193–221, Progr. Probab. Statist., 15, Birkhäuser, Boston (1988)

[4] F. Knight: Characterization of Lévy measures of inverse local times of gap diffusion. *Seminar on Stochastic Processes 1981,* 53–78, Progr. Probab. Statist., 1, Birkhäuser, Boston (1981)

[5] F. Knight: A remark on Walsh's Brownian motions. *Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993). J. Fourier Anal. Appl.* Special Issue, 317–323 (1995)

[6] F. Knight: Foundations of the prediction process. *Oxford Studies in Probability, 1.* Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1992)

[7] F. Knight: Essays on the prediction process. *Institute of Mathematical Statistics Lecture Notes-Monograph Series, 1.* Institute of Mathematical Statistics, Hayward, Calif. (1981)

[8] F. Knight: Essentials of Brownian motion and diffusion. *Math. Surveys, 18.* American Mathematical Society (1981)

[9] F. Knight and J.L. Steichen: The probability of escaping interference. In [10], 689–699 (2006)

[10] D. Burkholder (ed): Joseph Doob: A Collection of Mathematical Articles in His Memory. Department of Mathematics, University of Illinois at Urbana-Champaign. Originally published as Volume 50 of the *Ill. J. Math.* (2006)

Preface

Who could have predicted that the *Séminaire* de Probabilités would reach the age of 40? This long life is first due to the vitality of the French probabilistic school, for which the *Séminaire* remains one of the most specific media of exchange. Another factor is the amount of enthusiasm, energy and time invested year after year by the Rédacteurs: Michel Ledoux dedicated himself to this task up to Volume XXXVIII, and Marc Yor made his name inseparable from the *Séminaire* by devoting himself to it during a quarter of a century. Browsing among the past volumes can only give a faint glimpse of how much is owed to them; keeping up with the standard they have set is a challenge to the new Rédaction.

In a changing world where the status of paper and ink is questioned and where, alas, pressure for publishing is increasing, in particular among young mathematicians, we shall try and keep the same direction. Although most contributions are anonymously refereed, the *Séminaire* is not a mathematical journal; our first criterion is not mathematical depth, but usefulness to the French and international probabilistic community. We do not insist that everything published in these volumes should have reached its final form or be original, and acceptance–rejection may not be decided on purely scientific grounds. The policy set forth in volume XIII still prevails: "laisser une place aux débutants à côté des mathématiciens déjà connus, publier des articles de mise au point à côté des travaux originaux, et même, de temps en temps, publier un article intéressant, mais faux."

But the *Séminaire* is not gray literature either. Most of its content, from the very beginning, is still interesting; we hope the current volumes will still be read many years from now. The advanced courses, started in volume XXXIII, are continued in this volume with Laure Coutin's account of calculus for fractional Brownian motion. The *Séminaire* also occasionally publishes a series of contributions on some given theme; in this spirit, a few participants to a May 2004 Oberwolfach workshop on local time-space calculus are contributing to the present volume, and the reports of their interventions give an overview on the current state of that subject.

For our 40th anniversary, Mathdoc has made us an invaluable present, for which we are very thankful: in the framework of their NUMDAM programme, the whole collection of *Séminaires de Probabilités* up to volume XXXVI has been digitized. The result is made available on `http://www.numdam.org/`; access to volumes I–XXXV is free, but, for the time being, access to volume XXXVI in only possible for subscribers to the Springer *link*.

C. Donati-Martin
M. Émery
A. Rouault
C. Stricker

Contents

Part III Other Contributions

Part I

Specialized Course

An Introduction to (Stochastic) Calculus with Respect to Fractional Brownian Motion

Laure Coutin

Laboratoire de Statistique et Probabilités, Université Paul Sabatier
118, route de Narbonne, 31062 Toulouse Cedex 4, France
e-mail: coutin@cict.fr

Summary. This survey presents three approaches to (stochastic) integration with respect to fractional Brownian motion. The first, a completely deterministic one, is the Young integral and its extension given by rough path theory; the second one is the extended Stratonovich integral introduced by Russo and Vallois; the third one is the divergence operator. For each type of integral, a change of variable formula or Itô formula is proved. Some existence and uniqueness results for differential equations driven by fractional Brownian motion are available except for the divergence integral. As soon as possible, these integrals are compared.

Key words: Gaussian processes, Fractional Brownian motion, Rough path, Stochastic calculus of variations

1 Introduction

Fractional Brownian motion was originally defined and studied by Kolmogorov, [Kol40] within a Hilbert space framework. Fractional Brownian motion of Hurst index $H \in]0,1[$ is a centered Gaussian process W^H with covariance function

$$\mathbf{E}\left(W^H(t)W^H(s)\right) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]; \quad (s,t \geqslant 0)$$

(for $H = \frac{1}{2}$, $W^{\frac{1}{2}}$ is a Brownian motion). Fractional Brownian motion has stationary increments since

$$\mathbf{E}\left[\left(W^H(t) - W^H(s)\right)^2\right] = |t-s|^{2H} \quad (s,t \geqslant 0)$$

and is H-selfsimilar:

$$\left(\frac{1}{c^H}W^H(ct); \quad t \geqslant 0\right) \stackrel{d}{=} \left(W^H(t); \quad t \geqslant 0\right) \quad \text{for any} \quad c > 0.$$

The Hurst parameter H accounts not only for the sign of the correlation of the increments, but also for the regularity of the sample paths. Indeed, for $H > \frac{1}{2}$, the increments are positively correlated, and for $H < \frac{1}{2}$, they are negatively correlated. Futhermore, for every $\beta \in (0, H)$, the sample paths are almost surely Hölder continuous with index β. Finally, for $H > \frac{1}{2}$, according to Beran's definition [BT99], it is a long memory process; the covariance of the increments at distance u decreases as u^{2H-2}.

These significant properties make fractional Brownian motion a natural candidate as a model for noise in mathematical finance (see Comte and Renault [CR96], Rogers [Rog97], Cheridito, [Che03], and Duncan, [Dun04]); in hydrology (see Hurst, [Hur51]), in communication networks (see, for instance Leland, Taqqu, Wilson, and Willinger [WLW94]). It appears in other fields, for instance, fractional Brownian motion with Hurst parameter $\frac{1}{4}$ is the limit process of the position of a particule in a one-dimensional nearest neighbor model with a convenient renormalization, see Rost and Vares [RV85]. For more applications, the reader should look at the monograph of Doukhan et al., [DOT03].

For $H \neq \frac{1}{2}$, W^H is neither a semimartingale (see, e.g., Example 2 of Section 4.9.13 of Liptser and Shiryaev [LS84]), nor a Markov process, and the usual Itô stochastic calculus does not apply. Our aim is to present different possible definitions of an integral

$$\int_0^t a(s)\, dW^H(s) \tag{1}$$

for a a suitable process and W^H a fractional Brownian motion, such that:

- The link with the Riemann sums is as expected: for a regular enough,

$$\lim_{|\pi|\to 0} \sum_{t_i \in \pi} a(t_i)\left(W^H(t_{i+1}) - W^H(t_i)\right) = \int_0^t a(s)\, dW^H(s)$$

  where $\pi = (t_i)_{i=0}^n$ are subdivisions of $[0, 1]$;
- There exists a change of variable formula, that is, for suitable f,

$$f\left(W^H(t)\right) = f(0) + \int_0^t f'\left(W^H(s)\right)\, dW^H(s);$$

- It allows to define and solve differential equations driven by a d-dimensional fractional Brownian motion $W^H = (W^i)_{i=1,\dots,d}$,

$$y^i(t) = y^i(0) + \int_0^t f_0^i\big(y(s)\big)\, ds + \sum_{j=1}^d \int_0^t f_j^i\big(y(s)\big)\, dW^j(s),$$

where $y^i(0) \in \mathbb{R}$ and the functions f^i_j are smooth enough ($i = 1, \ldots, n$ and $j = 0, \ldots, d$).

The dimension is important. Assume that it is equal to one, $d = 1 = n$, and follow some ideas of Föllmer [Föl81]. Let f be a function, m times continuously differentiable. The sample paths of W^H are Hölder continuous of any index strictly less than H. Then, using a Taylor expansion of order $m > \frac{1}{H}$, the following limit exists

$$\int_0^t f'\left(W^H(s)\right)\, dW^H(s) := \lim_{|\pi|\to 0} \sum_{t_i\in\pi} \sum_{k=1}^m \frac{f^{(k)}\left(W^H(t_i)\right)\left(W^H(t_{i+1}) - W^H(t_i)\right)^k}{k!}$$

and $\int_0^t f'\left(W^H(s)\right)\, dW^H(s) = f\left(W^H(t)\right)$. When $d > 1$, one can also define

$$\int_0^t f\left(W^H(s)\right)\, dW^H(s) := \\ = \lim_{|\pi|\to 0} \sum_{t_i\in\pi} \sum_{k=1}^m \frac{1}{k!} D^k F\left(W^H(t_i)\right) . \left(W^H(t_{i+1}) - W^H(t_i)\right)^{\otimes k}$$

if $f = DF$ with $F : \mathbb{R}^d \to \mathbb{R}$. Moreover, the ideas of Doss, [Dos77], allow to define and solve differential equations driven by a fractional Brownian motion, as proved in [Nou05]. This method extends to the multidimensional case when the Lie algebra generated by $f_1, \ldots, f_d$ is nilpotent, see Yamato [Yam79] for $H = 1/2$. This is pointed out in [BC05b].

These two arguments do not work in the multidimensional case in more general situations. The aim of this survey is to present other integrals which allow to work in dimension greater than one.

Recently, there have been numerous attempts to define a (stochastic) integral with respect to fractional Brownian motion.

- The first kind of attempts are deterministic ones. They rely on the properties of the sample paths. First, the results of Young, [You36], apply to fractional Brownian motion. The sample paths are Hölder continuous of any index strictly less than H. Then, the sequences of Riemann sums converge for any process a with sample paths α Hölder continuous with $\alpha + H > 1$. Secondly, Ciesielski et al. [CKR93] have noticed that the sample paths belong to some Besov–Orlicz space. They define an integral on Besov–Orlicz using wavelet expansions. Third, Zähle, [Zäh98] uses fractional calculus and a generalization of the integration by parts formula.
  For all these integrals, the process $f(W^H)$, for suitable functions f is integrable with respect to itself only if $H > \frac{1}{2}$. For $H > \frac{1}{2}$, there exists

a change of variable formula. Some existence and uniqueness results for stochastic differential equations are available, see for instance, Nualart and Răşcanu, [NR02]. In [Lyo94], Lyons has proved that the Picard iteration scheme converges.
Using the rough path theory of [Lyo98], these results are extended to $H > \frac{1}{4}$ in [CQ02].

- The second kind is related to the integral with generalized covariation of Russo and Vallois, [RV93]. Again, for $H > \frac{1}{2}$, there exists a change of variable formula. In the one-dimensional case, Nourdin, in [Nou05], has obtained existence and uniqueness for stochastic differential equations and an approximation scheme for all $H \in]0, 1[$.
- The third one is related to the divergence operator of a Gaussian process introduced by Gaveau and Trauber in [GT82]. This divergence operator extends the Wiener integral to stochastic processes. It coincides with the Itô integral for Brownian motion. Decreusefond and Üstünel, [DÜ99], have studied the case of fractional Brownian motion. Alos, Mazet, and Nualart, [AMN01] or Cheridito and Nualart, [CN05] and Biaggini, Øksendal, Sulem, and Wallner, [BØSW04] or Carmona, Coutin, and Montseny, [CCM03], Cheridito and Nualart [CN05] and Decreusefond [Dec05] have extended the results of [DÜ99]. Again, the change of variable formula is obtained in the one-dimensional case, for any $H \in]0, 1[$, and only for $H > \frac{1}{4}$ in the multidimensional case. In general, the divergence approach leads to some anticipative differential equations. For $H > \frac{1}{2}$, Kleptsyna et al. have solved the case of linear equations, [KKA98]. Solving nonlinear equations is an open problem in the multidimensional case.

Existence and uniqueness for a differential equation in the case when $H \leqslant \frac{1}{4}$ and $d \geqslant 2$ is an open problem.

All these "stochastic" or "infinitesimal" calculi extend to some Volterra Gaussian processes, see for instance Decreusefond, [Dec05].

2 Fractional Brownian motion

This section is devoted to some properties of fractional Brownian motion and its sample paths. We also give several representations of a fractional Brownian motion.

2.1 First properties

Existence

According to Proposition 2.2 page 8 of [ST94], for all $H \in]0, 1[$ the function

$$R^H(t,s) = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

is definite positive. Then using Proposition 1.3.7 page 35 of [RY99], R^H is the covariance of a centered Gaussian process, denoted by W^H and called fractional Brownian motion with Hurst parameter H.

We now describe a few (geometrical) invariance properties of fractional Brownian motion.

Proposition 1. *Let W^H be a fractional Brownian motion with Hurst parameter $H \in]0,1[$. The following properties hold:*

1. *(Time homogeneity): for any $s > 0$, the process $\{W^H(t+s) - W^H(s),\ t \geqslant 0\}$ is a fractional Brownian motion with Hurst parameter H;*
2. *(Symmetry): the process $\{-W^H(t),\ \ t \geqslant 0\}$ is a fractional Brownian motion with Hurst parameter H;*
3. *(Scaling): for any $c > 0$, the process $\{c^H W^H(\frac{t}{c}),\ t \geqslant 0\}$ is a fractional Brownian motion with Hurst parameter H;*
4. *(Time inversion): the process X defined by $X(0) = 0$ and $X(t) = t^{2H} W^H(1/t)$ for $t > 0$ is also a fractional Brownian motion with Hurst parameter H.*

Remark 1. If W^H is a fractional Brownian motion defined on $[0,1]$, the process $\big\{\tilde{W}(t) = W^H(1-t) - W^H(1),\ \ t \in [0,1]\big\}$ is also a fractional Brownian motion with Hurst parameter H.

Remark 2. The process W^H is not a Markov process since its covariance is not triangular, (see [Nev68]).

Sample paths properties of fractional Brownian motion

From the expression of its covariance, we derive some properties of the sample paths of fractional Brownian motion.

Proposition 2. *For $H \in]0,1[$, the sample paths of fractional Brownian motion W^H are almost surely Hölder continuous of any order α strictly less than H. For all $T \in]0,+\infty[$ and $\alpha < H$, $\sup_{0 \leqslant s < t \leqslant T} \frac{|W^H(t) - W^H(s)|^2}{|t-s|^{2\alpha}}$ has some finite exponential moments.*

Proof. Since the moments of a Gaussian variable are functions of its variance, for any $a > 0$

$$\mathbf{E}\big(|W^H(s) - W^H(t)|^a\big) = C_a |t-s|^{aH}, \text{ where } \mathbf{E}\big(|\mathcal{N}(0,1)|^a\big) = C_a\,.$$

The Kolmogorov criterion implies that the Hölder seminorm of W^H, namely, $\sup_{0 \leqslant s < t \leqslant T} \frac{|W^H(t) - W^H(s)|}{|t-s|^\alpha}$, is almost surely finite for $\alpha < H$. Since W^H is a Gaussian process, the existence of some finite exponential moments follows from Theorem 2.4.6 page 39 of [Fer97]. □

Let π be a subdivision of the interval $[0,t]$ with $0 = t_0 < t_1 < \cdots < t_n = t$. The number $|\pi| = \sup_{i=0,\dots,n-1} |t_{i+1} - t_i|$ is called the mesh of π. For $p \geqslant 1$ and X a real function defined on $\mathbb{R}^+$, we set

$$V^{\pi}_{p,[0,t]}(X)^p = \sum_{i=0}^{n-1} \big|X(t_{i+1}) - X(t_i)\big|^p.$$

Proposition 3. *Let W^H be a fractional Brownian motion with Hurst parameter $H \in]0,1[$. For all $t \geqslant 0$, $V^{\pi}_{1/H,[0,t]}(W^H)^{1/H}$ converges to $t\,\mathbf{E}(|X|^{1/H})$ in L^2, where $X \sim \mathcal{N}(0,1)$, when $|\pi|$ goes to 0.*

Proof. Using the scaling property and the stationarity of the increments, we compute

$$\mathbf{E}\big(V^{\pi}_{1/H,[0,t]}(W^H)^{1/H}\big) = t\,\mathbf{E}\big(|X|^{1/H}\big).$$

To prove the proposition, it suffices to show that

$$\lim_{|\pi|\to 0} \mathbf{E}\big(V^{\pi}_{1/H,[0,t]}(W^H)^{2/H}\big) = t^2\,\mathbf{E}\big(|X|^{1/H}\big)^2. \tag{2}$$

Indeed, put $\Delta_i W^H = W^H(t_{i+1}) - W^H(t_i)$. The linear regression of $\Delta_j W^H$ with respect to $\Delta_i W^H$ yields

$$\begin{aligned}
\Delta_i W^H &= |t_{i+1} - t_i|^H X,\\
\Delta_j W^H &= \frac{\mathbf{E}(\Delta_i W^H \Delta_j W^H)}{|t_{i+1} - t_i|^H}\, X\\
&\quad + \sqrt{\frac{|t_{i+1} - t_i|^{2H}|t_{j+1} - t_j|^{2H} - \mathbf{E}(\Delta_i W^H \Delta_j W^H)^2}{|t_{i+1} - t_i|^{2H}}}\, Y,
\end{aligned}$$

where (X,Y) is a pair of independent standard Gaussian variables. Then one easily computes

$$\mathbf{E}\big(V^{\pi}_{1/H,[0,t]}(W^H)^{2/H}\big) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{-\frac{(x^2+y^2)}{2}}\, F^{\pi}_H(x,y)\,dx\,dy$$

with

$$F^{\pi}_H(x,y) = |x|^{1/H} \sum_i \sum_j (t_{i+1} - t_i)(t_{j+1} - t_j) \left| y\sqrt{1 - d_{i,j}^2} + d_{i,j}\, x \right|^{1/H},$$

where

$$d_{i,j} = \frac{\mathbf{E}(\Delta_i W^H \Delta_j W^H)}{(t_{i+1} - t_i)^H (t_{j+1} - t_j)^H}.$$

Using the Taylor expansion of the function $z \mapsto z^{1/H}$ between y and $y\sqrt{1-d_{i,j}^2}+d_{i,j}x$ and the fact that the $d_{i,j}$ are bounded by 1, one obtains

$$\begin{aligned}&\left|F_H^\pi(x,y)-t^2\,x^{1/H}y^{1/H}\right|\\&\quad\leqslant \frac{2}{H}\,|x|^{1/H}\left[|x|^{1/H}+|y|^{1/H}\right]\sum_{i,j}|t_{i+1}-t_i|\,|t_{j+1}-t_j|\,|d_{i,j}|.\end{aligned}$$

The constant C may vary from one line to the next, but depends only on H. Let $\varepsilon\in\,]0,1]$.

If (i,j) is such that $|t_i-t_j|\geqslant 4\frac{|\pi|}{\varepsilon}$; then $\frac{t_{i+1}-t_i}{|t_i-t_j|}$ and $\frac{t_{j+1}-t_j}{|t_i-t_j|}$ are dominated by $\frac{\varepsilon}{4}$. Lemma 1 below for $s=t_i$, $t=t_j$, $u=t_{i+1}-t_i$ and $v=t_{j+1}-t_j$ yields

$$|d_{i,j}|\leqslant C\,\varepsilon^{2-2H}. \tag{3}$$

Using the Fubini–Tonelli theorem, we compute

$$\sum_{(i,j),|t_i-t_j|\geqslant\frac{4|\pi|}{\varepsilon}}(t_{i+1}-t_i)(t_{j+1}-t_j)\leqslant t^2. \tag{4}$$

If (i,j) is such that $|t_i-t_j|<4\frac{|\pi|}{\varepsilon}$, then $|d_{i,j}|\leqslant 1$ and

$$\sum_{(i,j),|t_i-t_j|\geqslant\frac{4|\pi|}{\varepsilon}}(t_{i+1}-t_i)(t_{j+1}-t_j)\leqslant 16\,t\,\frac{|\pi|}{\varepsilon}. \tag{5}$$

For π such that $|\pi|\leqslant\varepsilon^2$, putting together inequalities (3)–(5) gives

$$\left|F_H^\pi(x,y)-t^2x^{1/H}y^{1/H}\right|\leqslant C\,|x|^{1/H}\left[|x|^{1/H}+|y|^{1/H}\right]\left[\varepsilon^{2-2H}t^2+\varepsilon t\right].$$

Integrating this difference with respect to the Gaussian density yields L^2 convergence. □

Lemma 1. *Let W^H be a fractional Brownian motion with Hurst parameter $H\in\,]0,1[$. There exists a continuous function R on $\left[-\frac14,\frac14\right]^2$ and a constant C^H such that*

$$|R(x,y)|\leqslant C^H\min(|x|,|y|)^3\,|x|^{-H}\,|y|^{-H}$$

and for all $(s,t,u,v)\in\mathbb{R}_+^4$ such that $\max\left(\frac{|u|}{|t-s|},\frac{|v|}{|t-s|}\right)\leqslant\frac14$

$$\begin{aligned}&\frac{\mathbf{E}\left([W^H(s+u)-W^H(s)][W^H(t+v)-W^H(t)]\right)}{u^Hv^H}=\\&\quad=H(1-2H)\left|\frac{u}{|t-s|}\,\frac{v}{|t-s|}\right|^{1-H}+R\Big(\frac{u}{t-s},\frac{v}{t-s}\Big).\end{aligned}$$

Proof. Put

$$F(s,t,u,v) = \frac{\mathbf{E}\left([W^H(s+u) - W^H(s)][W^H(t+v) - W^H(t)]\right)}{u^H v^H}.$$

A tedious computation gives

$$\begin{aligned}
&\mathbf{E}\left([W^H(s+u) - W^H(s)][W^H(t+v) - W^H(t)]\right)\\
&\quad = \frac{1}{2}\left[|s+u-t|^{2H} + |s-v-t|^{2H} - |t-s|^{2H} - |t+v-s-u|^{2H}\right].
\end{aligned}$$

Set $x = \dfrac{u}{t-s}$, $y = \dfrac{v}{t-s}$; then

$$F(s,t,u,v) = \frac{x^{-H}y^{-H}}{2}\left[|1-x|^{2H} + |1+y|^{2H} - 1 - |1+y-x|^{2H}\right].$$

Using the Taylor expansion of $z \mapsto (1+z)^{2H}$ one obtains

$$F(s,t,u,v) = -H(2H-1)\, x^{1-H}y^{1-H} + R(x,y)$$

where

$$R(x,y) = O\Big(\left(|x|^2 + |y|^2\right)^{\frac{3}{2}}\Big).$$

□

Remark 3. If the subdivision π_n is given by the points $t_j = j2^{-n}t$, where j ranges from 0 to 2^n, then the limit in Proposition 3 holds almost surely. (Estimate the variance of $V^{\pi}_{1/H,[0,t]}(W^H)^{1/H} - t\,\mathbf{E}(|X|^{1/H})$ and apply the Borel–Cantelli Lemma.)

Using the same lines as for Brownian motion (see [RY99] Corollaries 1.2.5 and 1.2.6), we derive the following.

Corollary 1. *The fractional Brownian paths a.s. have infinite variation on any interval.*

Corollary 2. *For $\alpha > H$, the fractional Brownian paths are a.s. nowhere locally Hölder continuous of order α.*

Proposition 2 and Corollary 2 leave open the case that $\alpha = H$. The next result shows in particular that the fractional Brownian paths with Hurst parameter H are not Hölder continuous of order $\alpha = H$. Its proof can be derived from Theorem 1.3 in [Ben96].

Theorem 1. *Let W^H be a fractional Brownian motion with Hurst parameter $H \in\,]0,1[$; then for all $t > 0$,*

$$\limsup_{\varepsilon\to 0^+} \frac{W^H(t+\varepsilon) - W^H(t)}{\sqrt{2\varepsilon^{2H}\log\log(1/\varepsilon)}} = 1 \quad a.s.,$$

$$\limsup_{0\leqslant t,t'\leqslant 1,\ |t-t'|=\varepsilon\to 0^+} \frac{W^H(t') - W^H(t)}{\sqrt{2\varepsilon^{2H}\log(1/\varepsilon)}} = 1 \quad a.s.$$

Proposition 4. *Liptser and Shiryaev [LS84]*
A fractional Brownian motion W^H with Hurst parameter $H \in]0,1[$, is a semimartingale if and only if $H = \frac{1}{2}$.

Proof. Assume that W^H is a semimartingale, then there exists a filtration $(\mathcal{F}_t)_{t\in[0,1]}$ fulfilling the usual conditions, a $(\mathcal{F}_t)_{t\in[0,1]}$ continuous local martingale M and an adapted process A of finite variation $(\mathcal{F}_t)_{t\in[0,1]}$ such that

$$W^H(t) = M(t) + A(t), \quad t \in [0,1].$$

According to Theorem 1.8 chapter IV page 111 of [RY99], $V^{\pi}_{2,[0,1]}(M)$ converges uniformly on $[0,1]$ in probability to $\langle M, M\rangle$ when the mesh of the subdivision goes to 0. Since A has finite variation, $\lim_{|\pi|\to 0} V^{\pi}_{2,[0,1]}(W^H)^2 = \langle M, M\rangle$. For $H > \frac{1}{2}$, Proposition 3 says that $\langle M, M\rangle = 0$. Then $M = 0$ and W^H has finite variation, which contradicts Corollary 1.

For $H < \frac{1}{2}$, let τ be a $(\mathcal{F}_t)_{t\in[0,1]}$ stopping time such that $\langle M, M\rangle_\tau < \infty$ almost surely. Then $\langle M, M\rangle_\tau = \lim_{|\pi|\to 0} V^{\pi}_{2,\tau}\left(W^H\right)^2$ and

$$V^{\pi}_{1/H,[0,1]}\left(W^H\right)^{1/H} \leqslant \sup_{s,t\in[0,1],\quad |t-s|<|\pi|} \left|W^H(t) - W^H(s)\right|^{1/H-2} V^{\pi}_{2,[0,1]}\left(W^H\right)^2.$$

According to Proposition 3 $\lim_{|\pi|\to 0} V^{\pi}_{1/H,[0,1]}\left(W^H\right)^{1/H}$ is finite but not null. Since the paths of W^H are continuous,

$$\lim_{|\pi|\to 0} \sup_{s,t\in[0,1]|t-s|<|\pi|} \left|W^H(t) - W^H(s)\right|^{1/H-2} = 0;$$

therefore $\lim_{|\pi|\to 0} V^{\pi}_{2,\tau}\left(W^H\right) = \infty$, which contradicts the hypothesis on τ. □

Proposition 5 (Cheridito 2001 [Che01]). *Let W^H be a fractional Brownian motion with Hurst parameter $H \in]0,1[$, and B be an independent Brownian motion. For $\varepsilon > 0$, $W^H + \varepsilon B$ is a semimartingale if and only if $H > \frac{3}{4}$ or $H = \frac{1}{2}$.*

We close this section with some fractal dimension of the graph of fractional Brownian motion, following [Fal03].

Proposition 6. *With probability 1, the Hausdorff and box dimensions of the graph $\left(t, W^H(t)\right)_{t\in[0,1]}$ equal $2 - H$.*

The law of the supremum of fractional Brownian motion is an open problem. Partial results are available in Duncan et al., [DYY01] and in Lanjri Zadi and Nualart [LZN03].

2.2 Representations of fractional Brownian motion

In [MVN68], Mandelbrot and Van Ness show that fractional Brownian motion is obtained by integrating a deterministic kernel with respect to a Gaussian measure, see Section 2.2. This representation is not unique. In this section, we present several representations of fractional Brownian motion as an integral with respect to a Gaussian measure or a Brownian motion. None of them seems to be universal, and choosing one of them depends on what one wants to do. Each of them leads to the construction of new processes, even not Gaussian by replacing the Gaussian measure with an independently scattered measure or a Lévy measure.

A simple construction of fractional Brownian motion is given by Enriquez, [Enr04]. It is equal to the limit of renormalized correlated random walks on **Z**. It will not be presented in this survey.

Moving average representation

(See the book of Samorodnitsky and Taqqu [ST94] p 321.)
The fractional Brownian motion $\{W^H(t),\ \ t \geqslant 0\}$ has the integral representation

$$\frac{1}{C_1(H)} \int_{-\infty}^{+\infty} \left[(t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}\right] M(dx); \quad t \in \mathbb{R}, \tag{6}$$

where $C_1(H) = \sqrt{\int_0^\infty \left[(1+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right]^2 dx + \frac{1}{2H}}$, M is a Gaussian random measure and $(a)_+ = a\,\mathbf{1}_{[0,\infty[}(a)$.

Remark 4. In (6), the function $(.)_+$ may be replaced by $|.|$ which yields the "well-balanced" representation given in [ST94] page 325 Section 2.7.

Remark 5. The kernel can be modified, see for instance the papers of Benassi, Jaffard, and Roux, [BJR97] or Ayache and Lévy Véhel, [ALV99]. A stochastic integral for these processes is introduced in [Dec05]. The Gaussian measure can be replaced by an independently scattered measure, see for instance Benassi-Roux, [BR03] or by a Lévy measure, see Lacaux, [Lac04].

Harmonizable representation

(See the book of Samorodnitsky and Taqqu [ST94] p 328.)
The fractional Brownian motion $\{W^H(t),\ \ t \geqslant 0\}$ has the integral representation

$$\frac{1}{C_2(H)} \int_{-\infty}^{+\infty} \frac{e^{ixt}-1}{ix} |x|^{\frac{1}{2}-H} M(dx); \quad t \in \mathbb{R},$$

where $C_2(H) = \sqrt{\frac{\pi}{H\Gamma(2H)\sin \pi H}}$.

Remark 6. Heuristically, the harmonizable representation of fractional Brownian motion is deduced from the moving average representation by using the Parseval identity. Indeed, the Fourier transform of a Gaussian random measure is again a Gaussian random measure. Notice that

$$(t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} = \left(H-\frac{1}{2}\right)\int_{-\infty}^{+\infty} \mathbf{1}_{[0,t]}(s)(s-x)_+^{H-\frac{3}{2}}\, ds.$$

Its Fourier transform should be equal to the product of the Fourier transforms of $\mathbf{1}_{[0,t]}$ and $(.)_+^{H-\frac{3}{2}}$. Now the Fourier transform of $\mathbf{1}_{[0,t]}(s)$ is $x \mapsto \frac{1}{\sqrt{2\pi}}\frac{e^{ixt}-1}{ix}$. The Fourier transform of $s_+^{H-\frac{3}{2}}$ does not exist, though, and up to a constant, the Fourier transform of $x \mapsto (-x)_+^{H-\frac{3}{2}}$ is $x \mapsto |x|^{\frac{1}{2}-H}$. Then, the Fourier transform of $x \mapsto (t-x)_+^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}$ is $x \mapsto \frac{e^{ixt}-1}{ix}|x|^{\frac{1}{2}-H}$ up to a suitable constant.

Volterra representations

(See Barton and Poor, [BP98] or Decreusefond and Üstünel [DÜ99].)
The fractional Brownian motion $\{W^H(t),\ \ t \geqslant 0\}$ has the integral representation

$$W^H(t) = \int_0^t K^H(t,s)\, dB(s), \quad t \geqslant 0$$

where

$$K^H(t,s) = \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\, F\left(H-\frac{1}{2};\ \frac{1}{2}-H;\ H+\frac{1}{2};\ 1-\frac{t}{s}\right), \quad s<t, \tag{7}$$

where F denotes the Gauss hypergeometric function (see Lebedev for more details [Leb65]) and $\{B(t),\ t \geqslant 0\}$ is a Brownian motion. According to Lemma 2.2 formula (2.25) page 36 of [SKM93]

$$K^H(t,s) = C_H\left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right)\int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}}\, du\right] \tag{8}$$

where

$$C_H = \sqrt{\frac{\pi H(2H-1)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})^2 \sin\left(\pi(H-\frac{1}{2})\right)}}.$$

It is worth pointing out that Norros, Valkeila, and Virtamo give in [NVV99] a close representation for $H > \frac{1}{2}$. The Brownian motion is replaced by a Gaussian martingale whose variance function is $t \mapsto \lambda_H t^{2-2H}$ for a suitable constant λ_H. The kernel $K^H(t,s)$ is replaced by the kernel $\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}\, dr \mathbf{1}_{]0,t[}(s)$.

As pointed out in Baudoin–Coutin, [BC05a], the Volterra representation of fractional Brownian motion (or more generally a Volterra process X with respect to an underlying Brownian motion B) is unique if and only if W^H (or X) and B have the same filtration.

Aggregation of Ornstein–Uhlenbeck processes

(See [CCM00].) Even if fractional Brownian motion is not a Markov process, it may be obtained from an infinite dimensional Markov process.

Recall that

$$K^H(t,s) = C_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right)\int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}}\,du\right].$$

Observe that $(t-s)^{H-\frac{1}{2}}$ is nearly a Laplace transform,

$$(t-s)^{H-\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{1}{2}-H\right)}\int_0^\infty x^{-H-\frac{1}{2}} e^{-x(t-s)}\,dx, \quad H < \frac{1}{2},$$

$$(t-s)^{H-\frac{1}{2}} = \frac{\left(H-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}\int_0^{+\infty} x^{-H+\frac{1}{2}} \int_s^t e^{-x(u-s)}\,du\,dx, \quad H > \frac{1}{2}.$$

Using Fubini's stochastic theorem (see Protter [Pro04]) we obtain the following representations.

For $H > \frac{1}{2}$, the fractional Brownian motion $(W^H(t), \quad t \geqslant 0)$ has the integral representation

$$W^H(t) = C_H \frac{\left(H-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}\int_0^\infty x^{-H+\frac{1}{2}}\, Y^H(x,t)\,dx, \quad t \geqslant 0,$$

where

$$X^H(t,x) = \int_0^t e^{-x(t-s)} s^{\frac{1}{2}-H}\,dB(s) \quad \text{and} \quad Y^H(t,x) = \int_0^t X^H(u)\, u^{H-\frac{1}{2}}\,du.$$

For $H < \frac{1}{2}$, the fractional Brownian motion $\{W^H(t), \quad t \geqslant 0\}$ has the integral representation

$$W^H(t) = C_H \frac{1}{\Gamma\left(\frac{1}{2}-H\right)}\int_0^\infty x^{-H-\frac{1}{2}}\, Z^H(x,t)\,dx, \quad t \geqslant 0$$

where

$$Z^H(t,x) = \int_0^t \left[t^{H-\frac{1}{2}} e^{-x(t-s)} - \left(H-\frac{1}{2}\right)\int_s^t u^{H-\frac{3}{2}} e^{-x(s-u)}\right] s^{\frac{1}{2}-H}\,dB(s).$$

Remark 7. The processes X^H, Y^H, and Z^H are close to Ornstein–Uhlenbeck processes, and easy to deal with.

Remark 8. Some generalizations are available in [IT99], where Brownian motion is replaced by a Gamma process or in [BNS01], where Brownian motion is replaced by a Lévy process.

3 Deterministic integrals for fractional Brownian motion

In this section, we focus on Young integrals; a few words are said about Besov and fractional integrals at the end. We made this choice, since for $H < \frac{1}{2}$, only the rough paths approach of Lyons, [Lyo98] yields results on existence and uniqueness for solution of differential equations driven by a fractional Brownian motion, in the multidimensional case. Rough path theory can be seen as a generalization of Young integration.

3.1 Young integrals

We briefly recall some facts about Young integrals. The results of this paragraph are contained in the articles by Young, [You36] and Lyons 1994 [Lyo94].

In this paragraph, $x = (x^1, \dots, x^d)$ denotes a continuous function from $[0,1]$ to $\mathbb{R}^d$ endowed with the Euclidean norm $|.|$.

Definition 1. *For $p \geqslant 1$, the trajectory x is said to have finite p variation whenever*

$$\sup_{\pi} \sum_{i=0}^{n-1} |x(t_{i+1}) - x(t_i)|^p < \infty,$$

where the supremum runs over all finite subdivision $\pi = (t_i)_{i=1}^n$ of $[0,1]$, with $0 \leqslant t_0 < \cdots < t_n \leqslant 1$.

Notation 2 *If x has finite p variation, we set*

$$\mathrm{Var}_{p,[0,1]}(x) := \left[\sup_{\pi} \sum_{i} |x(t_{i+1}) - x(t_i)|^p \right]^{\frac{1}{p}}.$$

Proposition 7 (Young 1936, [You36]).
Let p, q in $[1, \infty[$ verify $\frac{1}{p} + \frac{1}{q} > 1$. Assume that x has finite p variation and y finite q variation. The following sequence of Riemann sums converges:

$$\sum_{i=1}^{k_n - 1} x\left(t_i^n\right) \left[y\left(t_{i+1}^n\right) - y\left(t_i^n\right) \right]$$

where $\pi^n = (t_i^n)_{i=1}^{k_n}$ is any sequence of finite subdivisions of $[0,1]$ with mesh going to 0.

The limit, denoted by $\int_0^1 x(s)\, dy(s)$, does not depend upon the choice of the sequence of subdivisions.

This integral has good properties. The most useful one for the sequel is that the function $t \mapsto \int_0^t x(s)\,dy(s) := \int_0^1 \mathbf{1}_{[0,t]}(s)\,x(s)\,dy(s)$ defined on $[0,1]$ has finite q variation and

$$\mathrm{Var}_{q,[0,1]}\left(\int_0^{\cdot} x(s)\,dy(s)\right) \leqslant \|x\|_\infty \,\mathrm{Var}_{q,[0,1]}(y).$$

Corollary 3. *Let x be a path of finite p variation with $1 \leqslant p < 2$, and f be a continuous differentiable function on $\mathbb{R}^d$ with partial derivatives α Hölder continuous for some $\alpha > p-1$. Then*

$$f\big(x(1)\big) = f\left(x(0)\right) + \sum_{i=1}^{d} \int_0^1 \frac{\partial f}{\partial x^i}\big(x(s)\big)\,dx^i(s),$$

where the integral is the one defined in Proposition 7.

Suppose x is a continuous path in $\mathbb{R}^d$ with finite p variation where $1 \leqslant p < 2$. Let $(f^i)_{i=0}^d$ be differentiable vector field on $\mathbb{R}^n$, α Hölder continuous with $\alpha > p-1$ and $a \in \mathbb{R}^n$. For any path z_0 with finite p variation on $\mathbb{R}^n$, the sequence of Picard iterates is $(z_m)_{m\in\mathbb{N}}$ where

$$z_{m+1}(t) = a + \int_0^t f^0\big(z_m(s)\big)\,ds + \sum_{i=1}^{d} \int_0^t f^i\big(z_m(s)\big)\,dx^i(s), \quad t \in [0,1], \quad m \geqslant 0.$$

Theorem 3 (Lyons 1994 [Lyo94]). *Assume that f is differentiable with partial derivatives α Hölder continuous with $\alpha > p-1$. The sequence of Picard iterates converges for the distance in p variation. The limit does not depend on the choice of z_0.*

Definition 2. *The limit, denoted by z, is called the solution of the differential equation*

$$z(t) = a + \int_0^t f^0\big(z(s)\big)\,ds + \sum_{i=1}^{d} \int_0^t f^i\big(z(s)\big)\,dx^i(s), \quad t \in [0,1].$$

Young integrals for fractional Brownian motion with Hurst parameter greater than 1/2

Now, we apply the results of Section 3.1 to fractional Brownian motion with Hurst parameter $H \in]1/2, 1[$. The reader can also see the paper of Ruzmaikina, [Ruz00].

According to Proposition 2, the sample paths of W^H are almost surely α Hölder continuous for all $\alpha < H$. They have almost surely finite p variation for all $p > \frac{1}{H}$.

Let Y be a continuous process with sample paths of finite q variation, with $q < \frac{1}{1-H}$. Let π^n be a sequence of subdivisions of $[0,1]$ whose mesh goes to 0, $\pi^n = (t_i^n)_{i=1}^{k_n}$. The sequence of random variables

$$\sum_{i=1}^{k_n-1} Y(t_i^n)\big[W^H(t_{i+1}^n) - W^H(t_i^n)\big]$$

converges almost surely. The limit, which does not depend on the sequence of subdivisions, is denoted by $\int_0^1 Y(s)\, dW^H(s)$.

Remark 9.

- There is no mesurability or adaptedness hypothesis on the process Y.
- At this point, nothing can be said about the expectation of $\int_0^1 Y(s)\, dW^H(s)$.
- If T is a random time taking its values in $[0,1]$, this approach allows to define $\int_0^T Y(s)\, dW^H(s)$ (see [Ber89]).

A d-dimensional fractional Brownian motion, $W^H = (W^1, \dots, W^d)$, with Hurst parameter H consists of d independent copies of a fractional Brownian motion with Hurst parameter H.

Proposition 8. *Let H be greater than $\frac{1}{2}$ and f be a differentiable function on $\mathbb{R}^d$, with partial derivatives α Hölder continuous with $\alpha > \frac{1}{H} - 1$. Then*

$$f\left(W^H(1)\right) = f(0) + \sum_{i=1}^{d} \int_0^1 \frac{\partial f}{\partial x^i}\left(W^H(s)\right)\, dW^i(s).$$

Let $(f^i)_{i=0}^d$ be differentiable vector field on $\mathbb{R}^n$, α Hölder continuous with $\alpha > \frac{1}{H} - 1$. For any path Z_0 with finite p variation on $\mathbb{R}^n$, $p > \frac{1}{H}$, the Picard iterates $(Z_m)_{m\in\mathbb{N}}$ are given by

$$\begin{aligned} Z_{m+1}(t) = a &+ \int_0^t f^0\big(Z_m(s)\big)\, ds \\ &+ \sum_{i=1}^{d} \int_0^t f^i\big(Z_m(s)\big)\, dW^i(s),\ t \in [0,1], \quad m \geqslant 0. \end{aligned}$$

Corollary 4. *Let $H \in\,]1/2, 1[$. Assume that f is differentiable with partial derivatives α Hölder continuous with $\alpha > \frac{1}{H} - 1$. The sequence of Picard iterates converges, and the limit does not depend on the choice of z_0.*

Definition 3. *The limit, denoted by Z, is called the solution of the differential equation*

$$Z(t) = a + \int_0^t f^0\big(Z(s)\big)\, ds + \sum_{i=1}^{d} \int_0^t f^i\big(Z(s)\big)\, dW^i(s), \quad t \in [0,1] \qquad (9)$$

controlled by W^H.

3.2 Besov and fractional integrals

Two other proofs of Proposition 8 are available.

Besov integral for fractional Brownian motion

A proof of Proposition 8 can be found in the article by Ciesielski, Kerkyacharian and Roynette, [CKR93].

Indeed, let x be in the Besov space $\mathcal{B}^{\alpha}_{p,\infty}$ and y be in the Besov space $\mathcal{B}^{1-\alpha}_{p,1}$ with $\frac{1}{p} < \alpha < 1 - \frac{1}{p}$. The expansion of x in the Schauder basis is

$$x(t) = x_1\varphi_1 + \sum_{j,k} x_{j,k}\,\varphi_{j,k},$$

and the expansion of y in the Haar basis

$$y(t) = y_1\varphi_1 + \sum_{j,k} y_{j,k}\,\xi_{j,k}.$$

Since the derivative of $\varphi_{j,k}$ is $\xi_{j,k}$, the integral $\int_0^1 y(s)\,dx(s)$ is

$$\int_0^t y(s)\,dx(s) = \sum_{j,k,j',k'} x_{j,k}\,y_{j',k'} \int_0^t \xi_{j,k}(u)\,\xi_{j',k'}(u)\,du.$$

This integral belongs to $\mathcal{B}^{\alpha}_{p,\infty}$ and

$$\left\| \int_0^{\cdot} y(s)\,dx(s) \right\|_{\mathcal{B}^{\alpha}_{p,\infty}} \leqslant C_{\alpha,p}\,\|y\|_{\mathcal{B}^{1-\alpha}_{p,1}}\,\|x\|_{\mathcal{B}^{\alpha}_{p,\infty}},$$

for a universal constant $C_{\alpha,p}$. Notice that for $\alpha > \frac{1}{2}$, $\mathcal{B}^{\alpha}_{p,\infty} \subset \mathcal{B}^{1-\alpha}_{p,1}$. Therefore Picard iteration is well defined, and for those f which define a contracting operator on $\mathcal{B}^{\alpha/2}_{p,\infty}$, the differential equation controlled by $x \in \mathcal{B}^{\alpha}_{p,\infty}$

$$z(t) = a + \int_0^t f\big(z(s)\big)\,dx(s)$$

has a unique solution.

It remains to prove that the sample paths of W^H belongs to $\mathcal{B}^{1-\alpha}_{p,1}$ for $p > \frac{1}{2}$. Using a wavelet expansion of fractional Brownian motion, the authors have proved that almost surely, the sample paths of W^H belong to the Besov space $\mathcal{B}^{H}_{p,\infty}$ where $\frac{1}{p} < H < 1 - \frac{1}{p}$. Let Y be a process with sample paths in $\mathcal{B}^{1-H/2}_{p,1}$ for $1/p < H/2$. The integral $\int_0^1 Y(s)\,dW^H(s)$ is pathwise well defined. Moreover, Proposition 8 can be recovered using again the Picard iteration scheme.

Fractional integral for fractional Brownian motion

Another proof of Proposition 8 can be found in the article of Nualart–Răşcanu [NR02] which is based on the article of Zähle, [Zäh98]. Indeed, in [Zäh98], Zähle has pointed out that almost surely, the sample paths of W^H belong to the Besov-type space $\mathcal{W}^{\beta}_{2,\infty}$ for $\frac{1}{2} < \beta < H$. Here $\mathcal{W}^{\beta}_{2,\infty}$ denotes the Besov-type space of bounded measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\int_0^1\int_0^1 \frac{\big(f(t)-f(s)\big)^2}{|t-s|^{2\beta+1}}\, ds\, dt < \infty.$$

Moreover, we recover Proposition 8 using again the Picard iteration scheme.

Using the approach of Nualart–Răşcanu, [NR02], Nualart–Saussereau, [NS05], and Nualart–Hu, [HN06], have shown that the solution of

$$X(t) = a + \sum_{i=1}^{n} \int_0^t f_j^i\big(x(s)\big)\, dW^i(s)$$

has a density with respect to the Lebesgue measure, under a suitable non-degeneracy condition (ellipticity).

3.3 Conclusion

These deterministic integrals allow to write an Itô formula and to solve differential equations driven by fractional Brownian motion. Hairer, in [Hai05] has obtained some ergodic properties of the solution. But some stochastic computations seem difficult to perform; for instance, if Z is the solution of (9), we do not known how to deal with

$$\mathbf{E}\big(f(Z_1)\big).$$

This may be possible by using the divergence integral, see Section 7. Now, we will study the case $H < 1/2$.

4 Rough path and fractional Brownian motion

The main theorem of Lyons [Lyo98] can be summarized in the following continuity theorem: The solution of a stochastic differential equation is not a continuous application for the uniform convergence, but is continuous for the p variation distance. This distance is built on all iterated integrals up to order $[p]$, the integer part of p, where $p \geqslant 1$ depends on the roughness of the underlying paths. We refer the reader to [Lyo98], [LQ02], or [Lej03] for a detailed presentation on the theory of rough paths and the objects we introduce now.

4.1 Rough path

In this section, $\mathbb{R}^d$ is endowed with the Euclidean norm, denoted by $|.|$. The tensor product $(\mathbb{R}^d)^{\otimes n}$ is endowed with

$$|\xi| = \sum_{i=1}^{n} |x_i| \quad \text{if} \quad \xi = (x_1, \ldots, x_n).$$

Let x be a continuous path with values in $\mathbb{R}^d$. The path x is said to have finite p variation if its seminorm $\mathrm{Var}_{p,[0,1]}(x)$ is finite where

$$\mathrm{Var}_{p,[0,1]}(x) = \left(\sup_{\pi} \sum_{i=1}^{k-1} \left| x(t_{i+1}) - x(t_i) \right|^p \right)^{1/p},$$

where the supremum over π runs over all subdivisions of $[0,1]$, with the convention that the points of π are $0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant 1$. For a continuous path x with finite variation, the smooth functional of degree $[p]$ over x is $\mathbf{X} = \left(1, \mathbf{X}^1_{s,t}, \ldots, \mathbf{X}^{[p]}_{s,t}\right)_{0 \leqslant s \leqslant t \leqslant 1}$ where

$$\begin{aligned}
\mathbf{X}^1_{s,t} &= x(t) - x(s), \\
\mathbf{X}^2_{s,t} &= \int_{s \leqslant u_1 \leqslant u_2 \leqslant t} dx(u_1) \otimes dx(u_2), \\
\mathbf{X}^i_{s,t} &= \int_{s \leqslant u_1 \leqslant \cdots \leqslant u_i \leqslant t} \otimes_{k=1}^{i} dx(u_k), \quad i = 1, \ldots, [p].
\end{aligned}$$

The set of smooth functionals is endowed with the p variation distance defined by

$$\begin{aligned}
d_p(\mathbf{X}, \mathbf{Y}) &= \sum_{j=1}^{[p]} d_p\left(\mathbf{X}^j, \mathbf{Y}^j\right) + \sup_{t \in [0,1]} |x(t) - y(t)|, \\
d_p\left(\mathbf{X}^j, \mathbf{Y}^j\right) &= \sup_{\pi} \left(\sum_{i=1}^{k-1} \left| \mathbf{X}^j_{t_i, t_{i+1}} - \mathbf{Y}^j_{t_i, t_{i+1}} \right|^{p/j} \right)^{j/p}.
\end{aligned}$$

The closure of the set of the smooth functionals for the p variation distance is called the set of geometric functionals and denoted by $G\Omega^p([0,1], \mathbb{R}^d)$. For x a path with finite p variation, $\mathbf{X}$ is a geometric functional over x if $\mathbf{X} = \left(1, \mathbf{X}^1_{s,t}, \ldots, \mathbf{X}^{[p]}_{s,t}\right)_{0 \leqslant s \leqslant t \leqslant 1}$ belongs to $G\Omega^p\left([0,1], \mathbb{R}^d\right)$ and

$$\forall 0 \leqslant s \leqslant t \leqslant 1, \quad \mathbf{X}^1_{s,t} = x(t) - x(s).$$

Let $f : \mathbb{R}^d \to \mathcal{L}\left(\mathbb{R}^d, \mathbb{R}\right)$ be continuous. Here $\mathcal{L}\left(\mathbb{R}^d, \mathbb{R}\right)$ is the set of linear applications from $\mathbb{R}^d$ to $\mathbb{R}$. For all continuous paths of finite variation x, the path z defined by

$$z(t) := \int_0^t f\big(x(s)\big)\,dx(s), \quad t \in [0,1]$$

has finite variation.

Let $\mathbf{X} = \big(1, \mathbf{X}^1_{s,t}, ..., \mathbf{X}^{[p]}_{s,t}\big)_{0\leqslant s\leqslant t\leqslant 1}$ respectively, $\mathbf{Z} = \big(1, \mathbf{Z}^1_{s,t}, ..., \mathbf{Z}^{[p]}_{s,t}\big)_{0\leqslant s\leqslant t\leqslant 1}$ be the smooth functional built on x (resp. z). In [Lyo98], Lyons proved the following theorem.

Theorem 4. *If f is $[p]+1$ times differentiable and has bounded partial derivatives up to degree $[p]+1$, the application $\mathbf{X} \mapsto \mathbf{Z}$ is continuous for the p variation distance. It admits a unique extension to the set of geometric functionals $G\Omega^p\left([0,1],\mathbb{R}^d\right)$.*

Remark 10. If f is as in Theorem 4 and x be a smooth path, then

$$\begin{aligned}\int_s^t f\big(x(u)\big)\,dx(u) &= f\big(x(s)\big)\,.\,\mathbf{X}^1_{s,t} \\ &\quad + \int_{s\leqslant u_1\leqslant u_2\leqslant t} Df\big(x(u_2)\big)\,.\,\big(dx(u_1)\otimes dx(u_2)\big) \\ &= f\left(x(s)\right).\mathbf{X}^1_{s,t} + \cdots + D^{([p]-1)} f\left(x(s)\right).\mathbf{X}^{[p]}_{s,t} \\ &\quad + \int_{s\leqslant u_1\leqslant\cdots\leqslant u_{[p]+1}\leqslant t} D^{[p]} f\left(x(u_{[p]+1})\right) \\ &\qquad .\,\big(dx(u_1)\otimes\cdots\otimes dx(u_{[p]+1})\big)\,.\end{aligned}$$

Then, one can prove that

$$\begin{aligned}\int_s^t f\big(x(u)\big)\,dx(u) = \lim_{|\pi|\to 0}\sum_{t_i\in\pi}\big[f\big(x(t_i)\big).\mathbf{X}^1_{t_i,t_{i+1}} + \cdots \\ + D^{([p]-1)} f\big(x(t_i)\big).\mathbf{X}^{[p]}_{t_i,t_{i+1}}\big].\end{aligned} \tag{10}$$

Identity (10) remains true when x has finite p variation, $\mathbf{X}$ is the element of $G\Omega^p\left([0,1],\mathbb{R}^d\right)$ over x and

$$\int_s^t f\big(x(u)\big)\,dx(u) = \mathbf{Z}^1_{s,t}.$$

Moreover, let $(x^n)_{n\in\mathbb{N}}$ be a sequence of smooth functions which converges to x. Call $\mathbf{X}^n$ the regular functional built over x^n. Assume that $\mathbf{X}^n_{n\in\mathbb{N}}$ converges to $\mathbf{X}$ in $G\Omega^p\left([0,1],\mathbb{R}^d\right)$. Then exchanging limits is possible in

$$\begin{aligned}\int_s^t f\big(x(u)\big)\,dx(u) = \lim_{n\to\infty}\lim_{|\pi|\to 0}\sum_{t_i\in\pi}\big[f\big(x^n(t_i)\big).(\mathbf{X}^n)^1_{t_i,t_{i+1}} + \cdots \\ + D^{([p]-1)} f\big(x^n(t_i)\big).(\mathbf{X}^n)^{[p]}_{t_i,t_{i+1}}\big].\end{aligned}$$

Consider the following differential equation

$$dy(t) = f\big(y(t)\big)\, dx(t), \quad y_0 \in \mathbb{R}^n, \tag{11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is Lipschitz continuous. For any continuous path x with finite variation, the differential equation (11) has a unique solution y. The application $x \mapsto y$ is called the Itô map associated to the differential equation (11). It is well known that this Itô map is not continuous with respect to the topology of the uniform convergence, see Watanabe [Wat84]. Let $\mathbf{X} = \big(1, \mathbf{X}^1_{s,t}, \ldots, \mathbf{X}^{[p]}_{s,t}\big)_{0\leqslant s\leqslant t\leqslant 1}$ respectively, $\mathbf{Y} = \big(1, \mathbf{Y}^1_{s,t}, \ldots, \mathbf{Y}^{[p]}_{s,t}\big)_{0\leqslant s\leqslant t\leqslant 1}$ be the smooth functional built on x (respectively, y). In [Lyo98], Lyons has proved the following theorem.

Theorem 5. *If f is $[p]+1$ times differentiable with bounded partial derivatives up to order $[p]+1$, then the map $\mathbf{X} \mapsto \mathbf{Y}$ is continuous for the p variation distance. It extends uniquely to the set of geometric functionals $G\Omega^p\left([0,1], \mathbb{R}^d\right)$.*

Remark 11. Let $d = 1$, x be a α Hölder continuous path, $\alpha > 0$. Then the geometric rough path built on x is $\mathbf{X} = \big(1, \mathbf{X}^1_{s,t}, \ldots, \mathbf{X}^{[p]}_{s,t}\big)_{0\leqslant s\leqslant t\leqslant 1}$ where $p \geqslant \frac{1}{\alpha}$ and

$$\mathbf{X}^i_{s,t} = \frac{\big(x(t) - x(s)\big)^i}{i!}, \quad i = 1, \ldots, [p], \quad (s,t) \in [0,1]^2. \tag{12}$$

According to Theorem 5, for $f : \mathbb{R} \to \mathbb{R}$, $[p]+1$ times differentiable with bounded partial derivatives up to order $[p]+1$ and for $p \geqslant \frac{1}{\alpha}$, the differential equation

$$dy(t) = f\big(y(t)\big)\, dx(t), \quad y_0 \in \mathbb{R},$$

has a unique solution y. Moreover, the map $\mathbf{X} \mapsto \mathbf{Y}$ is continuous for the p variation distance, where $\mathbf{Y}^i$ is given by (12).

4.2 Geometric rough path over fractional Brownian motion

For $H \in \,]0,1[$, fractional Brownian motion has a modification with sample paths Hölder continuous of any index α, $\alpha < H$, thus with sample paths of finite p variation for $\frac{1}{H} < p$.

The case when $d = 1$

According to Remark 11, when $d = 1$ one has the following Proposition.

Proposition 9. *Let W^H be a fractional Brownian motion with Hurst parameter $H \in \,]0,1[$; let $f : \mathbb{R} \to \mathbb{R}$ be $[p]+1$ times differentiable with bounded partial derivatives up to order $[p]+1$. For $p > \frac{1}{H}$, the differential equation*

$$dY(t) = f\big(Y(t)\big)\, dW^H(t), \quad y_0 \in \mathbb{R}, \tag{13}$$

has a unique solution.

If $\left(W_n^H\right)_{n\in\mathbb{N}}$ is a sequence which converges to W^H for the $1/p$ norm, then Y is the limit of $(Y_n)_{n\in\mathbb{N}}$ where Y_n is the solution of (13) with W_n^H instead of W^H.

Some applications to waves in random media are given by Marty in [Mar05]. According to the results of [Lyo98], we restrict ourself to order $[p]$ and dimension $d > 1$.

The case when $H > \frac{1}{4}$

In order to build a geometric rough path over $W^H = \left(W^1, \ldots, W^d\right)$ a d-dimensional fractional Brownian motion, we have to consider smooth approximations of W^H. In [CQ02], the authors have chosen the dyadic linear interpolation of W^H; other approximations may give the same result. For $m \in \mathbb{N}^*$, put $t_k^m = k2^{-m}, \quad k = 0, \ldots, 2^m$, and

$$W(m)(t) = W^H(t_{k-1}^m) + \Delta_k^m W \, 2^m \, (t - t_{k-1}^m), \quad t \in [t_{k-1}^m, t_k^m]$$

where $\Delta_k^m W$ is the increment $W(t_k^m) - W(t_{k-1}^m)$ of W^H between t_{k-1}^m and t_k^m. Call $\mathbf{W}(m) = \left(1, \mathbf{W}(m)_{s,t}^1, \mathbf{W}(m)_{s,t}^2, \ldots, \mathbf{W}(m)_{s,t}^{[p]}\right)_{0\leqslant s\leqslant t\leqslant 1}$ the smooth rough path over $W(m)$.

Proposition 10. *Theorem 2 of [CQ02]*

Denote by W^H a d-dimensional fractional Brownian motion with Hurst parameter H and by $\mathbf{W}(m) = \left(1, \mathbf{W}(m)_{s,t}^1, \mathbf{W}(m)_{s,t}^2, \mathbf{W}(m)_{s,t}^3\right)_{0\leqslant s\leqslant t\leqslant 1}$ the smooth rough path built on the dyadic linear interpolation of W.

If $H \in \left]\frac{1}{4}, 1\right[$, then for any $p > 1/H$, $\left(\mathbf{W}(m)\right)_{m\in\mathbb{N}}$ converges almost surely to a geometric rough path over W^H $\mathbf{W} = \left(1, \mathbf{W}_{s,t}^1, \mathbf{W}_{s,t}^2, \mathbf{W}_{s,t}^3\right)_{0\leqslant s\leqslant t\leqslant 1}$ for the p variation distance.

Proof. Since $\Omega G^p\left([0,1], \mathbb{R}^d\right)$ is a complete metric space, we only have to prove that almost surely

$$\sum_{m=1}^{\infty} d_p\big(\mathbf{W}(m), \mathbf{W}(m+1)\big) < \infty.$$

We only give some ideas to prove that almost surely

$$\sum_{m=1}^{\infty} d_p\big(\mathbf{W}(m)^2, \mathbf{W}(m+1)^2\big) < \infty.$$

First the distance in p variation between two geometric functionals, $\mathbf{X}$ and $\mathbf{Y}$, $d_p(\mathbf{X}, \mathbf{Y})$, is controlled by the sum of the increments of $\mathbf{X} - \mathbf{Y}$ along the dyadic subdivision (see Lemmas 8–10 [CQ02]). For instance for the second level path we have Lemma 2 of [LLQ02].

Lemma 2. *For any $p > 2$ and $\gamma > \frac{p}{2} - 1$, there exists a constant C depending only on p and γ such that for all functionals $\mathbf{X}$ and $\mathbf{Y}$ in $G\Omega^3\left([0,1],\mathbb{R}^d\right)$*

$$d_p(\mathbf{X}^2,\mathbf{Y}^2) \leqslant C\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|\mathbf{X}^2_{t^n_{k-1},t^n_k} - \mathbf{Y}^2_{t^n_{k-1},t^n_k}\right|^{\frac{p}{2}}$$

$$+ C\left(\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|\mathbf{X}^1_{t^n_{k-1},t^n_k} - \mathbf{Y}^1_{t^n_{k-1},t^n_k}\right|^{p}\right)^{1/2}$$

$$\times\left(\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n}|\mathbf{X}^1_{t^n_{k-1},t^n_k}|^p + |\mathbf{Y}^1_{t^n_{k-1},t^n_k}|^{p}\right)^{1/2}.$$

Then, we have to prove that almost surely,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n}\left|\mathbf{W}(m)^2_{t^n_{k-1},t^n_k} - \mathbf{W}(m+1)^2_{t^n_{k-1},t^n_k}\right|^{p/2} < \infty.$$

Notice that $W^i(m)$, $i = 1, 2, 3$, is a polynomial of degree i in the variables $\Delta^m_k W,\ \ k = 1,\dots,2^m$. For instance, if $m \leqslant n$, the second level path $\mathbf{W}^2(m)$ is

$$\mathbf{W}^2(m)_{t^n_{k-1},t^n_k} = \frac{1}{2}\,2^{2(m-n)}\,\Delta^m_l W^{\otimes 2}, \tag{14}$$

where l is the unique natural number such that

$$\frac{l-1}{2^m} \leqslant \frac{k-1}{2^n} < \frac{k}{2^n} \leqslant \frac{l}{2^m}.$$

If $m > n$, the second level path $\mathbf{W}^2(m) - \mathbf{W}^2(m+1)$ is

$$\mathbf{W}^2(m+1)_{t^n_{k-1},t^n_k} - \mathbf{W}^2(m)_{t^n_{k-1},t^n_k}$$
$$= \frac{1}{2}\sum_l\left(\Delta^{m+1}_{2l-1}W \otimes \Delta^{m+1}_{2l}W - \Delta^{m+1}_{2l}W \otimes \Delta^{m+1}_{2l-1}W\right)$$

for $k = 1,\dots,2^n$, where the summation over l runs from $2^{m-n}(k-1)+1$ to $2^{m-n}k$. Recall the results of Lemma 1: There exists a constant C such that for all $(s,t,\tau) \in [0,1]^3$, $\tau \neq 0$, verifying $\frac{|t-s|}{\tau} < 1$, one has

$$\mathbf{E}\left(|W^H(t) - W^H(s)|^2\right) = d\,|t-s|^{2H},$$

$$\mathbf{E}\left[\left(W^H(t) - W^H(s)\right)\left(W^H(t+\tau) - W^H(s+\tau)\right)\right] \leqslant C\,\tau^{2H}\,\tfrac{|t-s|^2}{\tau^2}.$$

For $s = r2^{-m}$, $t = l2^{-m}$, and $\tau = 2^{-m}$, we obtain

$$\left|\mathbf{E}\left(\Delta^m_l W^i \Delta^m_r W^j\right)\right| \leqslant (2H-1)\,\delta_{i,j}\,C\,\frac{|k-l|^{2H-2}}{2^{2Hm}}. \tag{15}$$

The following Lemma is a consequence of (15).

Lemma 3. *For $i \neq j$ and $l > r$,*

$$|\mathbf{E}\left(\Delta_{2l-1}^{m+1}W \otimes \Delta_{2l}^{m+1}W - \Delta_{2l}^{m+1}W \otimes \Delta_{2l-1}^{m+1}W\right)$$
$$\left(\Delta_{2r-1}^{m+1}W \otimes \Delta_{2r}^{m+1}W - \Delta_{2r}^{m+1}W \otimes \Delta_{2r-1}^{m+1}W\right)| \leqslant C_1 \left(\frac{l-r}{2^{m+1}}\right)^{4H} \frac{1}{(l-r)^5}.$$

where $C_1(H)$ is a constant depending only on H. If $l = r$, then

$$\mathbf{E}\left(\left[\Delta_{2l-1}^{m+1}W \otimes \Delta_{2l}^{m+1}W - \Delta_{2l}^{m+1}W \otimes \Delta_{2l-1}^{m+1}W\right]^2 \right) = 2(1-2H)2^{-4Hm}.$$

The key estimate for the second level path is given in the following.

Lemma 4. *Let $H > \frac{1}{4}$ and p such that $\max\left(\frac{1}{H}, 3\right) < p \leqslant 4$. There exists a constant C depending only on d, p, and H such that for any n, m and $k = 1, \ldots, 2^n$*

$$\mathbf{E}\left|\mathbf{W}^2(m+1)_{t_{k-1}^n, t_k^n} - \mathbf{W}^2(m)_{t_{k-1}^n, t_k^n}\right|^{p/2} \leqslant C 2^{\frac{p}{4}(m-n)} 2^{-mHp}.$$

Proof. Since $\mathbf{W}^2(m+1)_{t_{k-1}^n, t_k^n} - \mathbf{W}^2(m)_{t_{k-1}^n, t_k^n}$ belongs to the second chaos of the fractional Brownian motion, we only have to prove the Lemma for $p = 4$.

For $m \leqslant n$, it is easily derived from (14) and from $(a+b)^2 \leqslant 2(a^2+b^2)$ that

$$\mathbf{E}\left|\mathbf{W}^2(m+1)_{t_{k-1}^n, t_k^n} - \mathbf{W}^2(m)_{t_{k-1}^n, t_k^n}\right|^2 \leqslant 2^{(m-n)4-4mH-1}.$$

For $m > n$, the diagonal terms of $W^2(m+1)_{t_{k-1}^n, t_k^n} - W^2(m)_{t_{k-1}^n, t_k^n}$ vanish. Using the Hilbert–Schmidt norm on $\mathbb{R}^d \times \mathbb{R}^d$,

$$\mathbf{E}\left|\mathbf{W}^2(m+1)_{t_{k-1}^n, t_k^n} - \mathbf{W}^2(m)_{t_{k-1}^n, t_k^n}\right|^2 = \sum_{i \neq j} \sum_{l,\, r} \mathbf{E}(A_r^{i,j} A_l^{j,i})$$

where l ranges from $2^{m-n}(k-1)+1$ to $2^{m-n}k$ and r from 1 to $l-1$, and

$$A_l^{i,j} = \Delta_{2l-1}^{m+1}W^i\, \Delta_{2l}^{m+1}W^j - \Delta_{2l}^{m+1}W^i\, \Delta_{2l-1}^{m+1}W^j.$$

Using Lemma 3, we obtain for $m > n$,

$$\begin{aligned} \mathbf{E}\left|\mathbf{W}^2(m+1)_{t_{k-1}^n, t_k^n} - \mathbf{W}^2(m)_{t_{k-1}^n, t_k^n}\right|^2 &\leqslant C\, 2^{m-n} 2^{-4Hm} \\ &\quad + C \sum_{l=2}^{2^{m-n}} 2^{-4Hm} \sum_{r=1}^{l-1} (l-r)^{4H-5} \\ &\leqslant C\, 2^{m-n} 2^{-4Hm}. \end{aligned}$$

Since for $H > \frac{1}{4}$,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} 2^{\frac{p}{4}(m-n)} 2^{-mHp} < \infty,$$

one has

$$\mathbf{E}\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{\gamma} \sum_{k=1}^{2^n} \left| \mathbf{W}(m)^2_{t^n_{k-1},t^n_k} - \mathbf{W}(m+1)^2_{t^n_{k-1},t^n_k} \right|^{p/2} \right) < +\infty.$$

Almost surely the following sum is convergent

$$\sum_{m=1}^{\infty} d_p \left(\mathbf{W}(m)^2, \mathbf{W}(m+1)^2 \right) < \infty.$$

The same convergence holds for the other levels. There exists a unique function $\mathbf{W}_{.,.}$ on $\{(s,t) \in [0,1]^2, \quad 0 \leqslant s \leqslant t \leqslant 1\}$, such that $d_p\big(\mathbf{W}, \mathbf{W}(m)\big)$ converges to 0, almost surely when m goes to infinity, in the p variation distance. □

A consequence of Theorem 4 and Proposition 10 is the following Itô formula.

Corollary 5. *Theorem 5 of [CQ02]*
If $H > \frac{1}{4}$, $p > \frac{1}{H}$, and f is $C^{[p]}(\mathbb{R}^d, \mathbb{R})$ then

$$f\left(W^H(1)\right) = f\left(W^H(0)\right) + \int_0^1 Df\left(W^H(s)\right) \, dW^H(s),$$

where

$$\int_0^1 Df\left(W^H(s)\right) \, dW^H(s)$$
$$= \lim_{m\to\infty} \sum_{k=1}^{2^m} \int_{t^m_{k-1}}^{t^m_k} Df\left(W^H(t^m_{k-1}) + (t - t^m_{k-1})\Delta^m_k W\right) \, dt \, 2^m \Delta^m_k W \,.$$

Following [CFV05], $Df(W^H)$ is Stratonovitch integrable with respect to W^j, see [Nua95]. Indeed, for all $t \in [0,1]$ there exists a random variable denoted by $\int_0^t Df\left(W^H(s)\right) \, dW^j(s)$ such that for all sequences $(\pi^n = (t^n_i)_{i=0,\ldots,k_n})_{n\in\mathbb{N}}$ of subdivision of $[0,t]$ such that $|\pi^n| \to_{n\to\infty} 0$, the following convergence holds in probability

$$\lim_{n\to\infty} \sum_{i=0}^{k_n-1} \left(\frac{1}{t^n_{i+1} - t^n_i} \int_{t^n_i}^{t^n_{i+1}} Df\left((W^H(s)\right) \, ds \right) \left(W^j(t^n_{i+1}) - W^j(t^n_i)\right)$$
$$= \int_0^t Df\left(W^H(s)\right) \, dW^j(s).$$

Differential equation in the case when $H > \frac{1}{4}$

Consider the following stochastic differential equation

$$dy^i(t) = f_0^i\,(t, y(t))\; dt + \sum_{j=1}^{d} f_j^i\,(t, y(t))\; dW^j(t), y^i(0) = \xi^j, \quad i = 1, \dots, n, \tag{16}$$

where W^H is a d-dimensional fractional Brownian motion. Assume, for simplicity, that all partial derivatives of f_i^j up to order $[p]+1$ are bounded for $i = 1, \dots, n,\;\; j = 0, \dots, d$. Then one has a Wong–Zakai limit theorem. Namely, if $\big(y(m)^i,\;\; i = 1, \dots, d\big)$ is the unique solution to the ordinary equation

$$dy(m)^i(t) = f_0^i\big(t, y(m)(t)\big)\; dt + \sum_{j=1}^{d} f_j^i\big(t, y(m)(t)\big)\; dW(m)^j(t), \quad y^i(0) = \xi^i,$$

Theorem 5 and Proposition 10 imply that $y(m)$ converges to a continuous sample path y for the p variation distance on $[0,1]$ and $y(0) = \xi$. Of course, the limit path y may be regarded as the strong solution to the stochastic differential (16). In fact, a stronger result holds. Call $\mathbf{Y}(m)$ the smooth functional $\big\{(1, \mathbf{Y}(m)^1_{s,t}, \mathbf{Y}(m)^2_{s,t}, \mathbf{Y}(m)^3_{s,t})\big\}_{0 \leqslant s \leqslant t \leqslant 1}$ built over $y(m)$.

Corollary 6. *Theorem 5 of [CQ02]*
If $H > \frac{1}{4}$ and $p > \frac{1}{H}$, then when m goes to infinity $\mathbf{Y}(m)$ converges in the p variation distance almost surely to some geometric functional $\mathbf{Y} = \big(1, \mathbf{Y}^1_{s,t}, \mathbf{Y}^2_{s,t}, \mathbf{Y}^3_{s,t}\big)_{0 \leqslant s \leqslant t \leqslant 1}$.

Following [CFV05], $\big\{f_j^i(y(0) + \mathbf{Y}^1_{0,t}),\;\; t \in [0,1]\big\}\,,\;\;\; i = 1, \dots, n, \;\;\; j = 1, \dots, d$ is Stratonovich integrable with respect to W^j, see [Nua95]. Indeed, for all $t \in [0,1]$ there exists a random variable denoted by $\int_0^t f_j^i\,\big(y(0) + \mathbf{Y}^1_{0,s}\big)\; dW^j(s)$ such that for all sequences $\big(\pi^n = (t_i^n)_{i=0,\dots,k_n}\big)_{n \in \mathbb{N}}$ of subdivisions of $[0,t]$ such that $|\pi^n| \to_{n \to \infty} 0$, the following convergence holds in probability

$$\begin{aligned} &\lim_{n\to\infty} \sum_{\ell=0}^{k_n - 1} \left(\frac{1}{t^n_{\ell+1} - t^n_\ell} \int_{t^n_\ell}^{t^n_{\ell+1}} f_j^i\,\big(y(0) + \mathbf{Y}^1_{0,s}\big)\; ds \right) \big(W^j(t^n_{\ell+1}) - W^j\,(t^n_\ell)\big) \\ &\quad = \int_0^t f_j^i\,\big(y(0) + \mathbf{Y}^1_{0,s}\big)\; dW^j(s)\,. \end{aligned}$$

Corollary 7. *Under the hypothesis of Corollary 6, the differential equation (16) has a solution in the Stratonovich sense.*

The investigation of the properties of the solution of differential equations driven by fractional Brownian motion is pursued, using rough paths theory. In [MSS06], Millet and Sanz prove a large deviation principle in the space of geometric rough paths, extending classical results on Gaussian processes.

In [BC05b], Baudoin and Coutin, using a Taylor expansion type formula, show how it is possible to associate differential operators with stochastic differential equations driven by a fractional Brownian motion. As an application, they deduce that invariant measures for such SDEs satisfy an infinite dimensional system of partial differential equations.

The case when $\boldsymbol{H} \leqslant \frac{1}{4}$

Proposition 11. *Theorem 2 of [CQ02]*
Let W^H be a d-dimensional fractional Brownian motion with Hurst parameter H and $\mathbf{W}(m) = \left(1, \mathbf{W}(m)^1_{s,t}, \mathbf{W}(m)^2_{s,t}, \mathbf{W}(m)^3_{s,t}\right)_{0\leqslant s\leqslant t\leqslant 1}$ the smooth rough path built on the dyadic linear interpolation of W^H. If $H \leqslant \frac{1}{4}$, the second level path $\mathbf{W}(m)^2$ of its dyadic linear interpolation does not converge in $L^1(\Omega, \mathcal{F}, \mathbf{P})$.

Proof. First, express $\mathbf{W}^2(m)^{i,j}_{0,1}$ as a double Wiener integral for $i \neq j$:

$$\mathbf{W}^2(m)^{i,j}_{0,1} = \iint_{[0,1]^2} f_m(u,v)\, dB^i(u)\, dB^j(v).$$

Second, observe that $(f_m)_{m\in\mathbb{N}}$ converges almost surely to a function f when m goes to 0 but f does not belong to $L^2([0,1],\mathbb{R})$. Then $\mathbf{W}^2(m)^{i,j}_{0,1}$ does not converge in probability nor in $L^p(\Omega, \mathbf{P}, \mathbb{R})$.

The proof continues by extending the Volterra representation of fractional Brownian motion given in Section 2.2 to the multidimensional case. The fractional Brownian motion $(W^H(t),\ \ t \geqslant 0)$ has the integral representation

$$W^i(t) = \int_0^t K^H(t,s)\, dB^i(s), \quad t \geqslant 0,$$

where

$$K^H(t,s) = C_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}}\, du \right]$$

and $B = \left(B^1, \dots, B^d\right)$ is a d-dimensional Brownian motion. For $m \in \mathbb{N}^*$, $W(m)$ is also a Volterra process, that is

$$W(m)^i(t) = \int_0^t K(m)(t,s)\, dB^i(s), \quad t \geqslant 0$$

where if $t^m_k = k2^{-m},\ \ k = 0, \dots, 2^m$, then for $t \in \left[t^m_{k-1}, t^m_k\right]$:

$$K(m)(t,s) = K^H(t^m_{k-1}, s) + 2^m (t - t^m_{k-1}) \left[K^H(t^m_k, s) - K^H(t^m_{k-1}, s)\right],$$

$$\frac{\partial K(m)}{\partial t}(t,s) = 2^m \left[K^H(t^m_k, s) - K^H(t^m_{k-1}, s)\right].$$

The process $W(m)$ is absolutely continuous with respect to Lebesgue measure with derivative given by

$$\frac{dW(m)^j}{dt}(t) = \int_0^1 \frac{\partial K(m)}{\partial t}(t,s)\, dB^j(s), \quad t \geqslant 0.$$

Using Fubini's Theorem, see [Pro04], and the independence of B^i and B^j for $i \neq j$,

$$\begin{aligned}\mathbf{W}^2(m)_{0,1} &= \iint_{[0,1]^2} \left[\int_0^1 K(m)(t,u)\, \frac{\partial K(m)}{\partial t}(t,v)\, dt\right] dB^i(u)\, dB^j(v) \\ &= \iint_{[0,1]^2} f(m)(u,v)\, dB^i(u)\, dB^j(v)\end{aligned}$$

where $f(m)(u,v)$ stands for

$$\sum_{k=1}^{2^m} \frac{K(m)(t_k^m,u) + K(m)(t_{k+1}^m,u)}{2} \left[K(m)(t_{k+1}^m,v) - K(m)(t_k^m,v)\right].$$

Observe that $t \mapsto K^H(t,s)$ is differentiable on $]s,1]$, absolutely continuous on any compact interval of $]s,1]$, and its derivative is

$$\partial_t K^H(t,s) = C_H\, \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}\, (t-s)^{H-\frac{3}{2}}. \tag{17}$$

Then, $du\, dv$ almost surely on $u > v$, one can prove that

$$\lim_{m\to\infty} f(m)(u,v) = \int_u^1 K^H(t,u)\, \partial_t K^H(t,v)\, dt.$$

On $v < u$ an integration by parts yields

$$\lim_{m\to\infty} f(m)(u,v) = K^H(1,u)\, K^H(1,v) - \int_v^1 K^H(t,v)\, \partial_t K^H(t,u)\, dt.$$

Since, for $H < \frac{1}{2}$

$$\partial_t K^H(t,s) \geqslant C_H s^{\frac{1}{2}-H} (t-s)^{H-\frac{3}{2}},$$

and

$$K^H(t,s) \geqslant C_H s^{\frac{1}{2}-H} (t-s)^{H-\frac{1}{2}},$$

then for $H \leqslant \frac{1}{4}$,

$$\int_u^1 K^H\big(t, \max(u,v)\big)\, \partial_t K^H\big(t, \min(u,v)\big)\, dt \notin L^2\left([0,1]^2, du\, dv, \mathbb{R}\right).$$

Indeed, for (u,v) such that $0<u-v<1-u$,

$$\int_u^1 K^H(t,u)\,\partial_t K^H(t,v)\,dt \geqslant C_H^2\,(uv)^{H-\frac{1}{2}}\int_u^1 (t-u)^{H-\frac{1}{2}}(t-v)^{H-\frac{3}{2}}\,dt,$$
$$\geqslant C_H^2\,(uv)^{H-\frac{1}{2}}\int_0^{1-u} x^{H-\frac{1}{2}}(x+u-v)^{H-\frac{3}{2}}\,dx,$$
$$\geqslant C_H^2\,(uv)^{H-\frac{1}{2}}2^{H-\frac{3}{2}}(u-v)^{2H-1}.$$

But $(u-v)^{2H-1}$ does not belong to $L^2([0,1]^2, du\,dv, \mathbb{R})$ for $H \leqslant \frac{1}{4}$. The proof of Proposition 11 is over. □

Conclusion

To study properties of the solution of a differential equation driven by fractional Brownian motion, even for probabilistic ones, the rough path theory is very powerful. To go further, one has to compute the exception of the iterated integrals built on fractional Brownian motion.

To our knowledge, the existence of the density of the solution of the differential equation (16) when $H < 1/2$ is an open problem.

5 Forward, backward integral, and generalized covariation

This section is based on the papers of Russo–Vallois, [RV93], Gradinaru et al., [GRV03]. Let $\mathcal{C}$ be the Fréchet space of continuous processes equipped with the metric topology of uniform convergence in probability (ucp) on each compact interval.

Definition 4. *If X is continuous and if almost all paths of Y belong to $L^1_{loc}(\mathbb{R},\mathbb{R},dx)$, the forward integral is*

$$\int_0^t Y(u)\,d^-X(u) := \lim_{\varepsilon\downarrow 0}\mathrm{ucp}\int_0^t Y(s)\,\frac{X(s+\varepsilon)-X(s)}{\varepsilon}\,ds;$$

the covariation is defined by

$$[X,Y]_t := \lim_{\varepsilon\downarrow 0}\mathrm{ucp}\,\frac{1}{\varepsilon}\int_0^t \big(X(s+\varepsilon)-X(s)\big)\big(Y(s+\varepsilon)-Y(s)\big)\,ds;$$

and the symmetric-Stratonovich integral is

$$\int_0^t Y_s\,d^0X(s) = \int_0^t Y(u)\,d^-X(u) + \frac{1}{2}[X,Y]_t,$$

provided these limits exist.

If X is such that $[X,X]$ exists, X is a finite quadratic variation process. For such a process, if $f \in C^2(\mathbb{R},\mathbb{R})$, then the following Itô formula holds:

$$f(X_t) = f(X_0) + \int_0^t f'\big(X(u)\big)\, d^0X(u), \quad \forall t \in [0,1]. \tag{18}$$

Remark 12.

- If X is a continuous semimartingale and Y a suitable previsible process, then $\int_0^\cdot Y(u)d^-X(u)$ is but the classical Itô integral (see [RV93]).
- If X and Y are continuous semimartingales, then $\int_0^\cdot Y(u)\, d^0X(u)$ is the Fisk–Stratonovich integral and $[X,Y]$ is the ordinary square bracket.
- For $H > \frac{1}{2}$, W^H has a zero quadratic variation process, so the Itô formula (18) holds.

Since the quadratic variation of W^H is infinite when $H < \frac{1}{2}$, the authors of [GRV03] introduce new objects.

Definition 5. *For $\alpha > 0$, the α strong variation of a process X is*

$$[X]_t^{(\alpha)} := \lim_{\varepsilon\downarrow 0} \mathrm{ucp} \int_0^t \frac{\big|X(u+\varepsilon) - X(u)\big|^\alpha}{\varepsilon}\, du, \quad \forall t \in [0,1]$$

provided the limit exists.

Definition 6 (Errami–Russo [ER03]).
Given $n \geqslant 1$, the n-covariation $[X^1,\dots,X^n]$ of a vector $(X^1,\dots,X^n)$ of real continuous processes is for $t \in [0,1]$,

$$[X^1,\dots,X^n]_t := \lim_{\varepsilon\downarrow 0} \mathrm{ucp} \int_0^t \frac{\big(X^1(u+\varepsilon) - X^1(u)\big)\cdots\big(X^n(u+\varepsilon) - X^n(u)\big)}{\varepsilon}\, du,$$

when the limit exists.

For a process X, the vector valued process $(X,\dots,X)$ may have a finite n variation even if the n strong variation of X does not exist.

Proposition 12. *Proposition 3.14 [RV00].*
For $H \in]0,1[$, the fractional Brownian motion with Hurst parameter H has a $1/H$ strong variation and for $2nH > 1$

$$\big[W^H\big]_t^{(2n)} = \mu_{2n}\, t, \quad \forall t \in [0,1],$$

where $\mu_a = \mathbf{E}\left(|W_1^H|^a\right)$.

Then, a natural extension of the symmetric-Stratonovich integral is the following one, introduced by Revuz–Yor [RY99] (see Exercise (2.18) chapter IV). Let ν be a probability measure on $[0,1]$ and $m_k := \int_0^1 \alpha^k \nu(d\alpha)$ its kth moment.

Definition 7. *Fix $m \geqslant 1$. If $g : \mathbb{R} \to \mathbb{R}$ is a locally bounded function, the ν integral of order m of $g(X)$ with respect to X is*

$$\int_0^t g\big(X(u)\big) d^{\nu,m} X(u) \\ := \lim_{\varepsilon \downarrow 0} \mathrm{ucp}\, \frac{1}{\varepsilon} \int_0^t \big(X(s+\varepsilon) - X(s)\big)^m \int_0^1 g\big(X(s) + \alpha[X(s+\varepsilon) - X(s)]\big)\, \nu(d\alpha)\, ds$$

if the limit exists.

This integral with respect to X is defined only for integrands of the form $g(X)$. Nevertheless, $g(X)$ may sometimes be replaced by a general Y.

Example 6

- *If $g=1$, then, for any probability measure ν, the integral $\int_0^\cdot g(X(u))\, d^{\nu,m} X(u)$ is the m variation of X (see Definition 6).*
- *If $\nu = \delta_0$ and $m = 1$, $\int_0^\cdot g\big(X(u)\big)\, d^{\nu,m} X(u)$ is the forward integral defined in Definition 4.*
- *If $\nu = \frac{1}{2}[\delta_0 + \delta_1]$ and $m = 1$, $\int_0^t g\big(X(u)\big)\, d^{\nu,m} X(u)$ is the symmetric integral defined in Definition 4.*

A probability measure ν on $[0,1]$ is called symmetric if it is invariant under the transformation $t \mapsto 1 - t$ of $[0,1]$.

Theorem 7 (see Gradinaru et al. [GRV03]). *Let $n \in \mathbb{N}^*$.*

Let X be a process with strong $(2n)$ variation and $g \in C^{2n}(\mathbb{R}, \mathbb{R})$. Let ν be a symmetric probability measure such that $m_{2j} = \frac{1}{2j+1}$ for $j = 1, \ldots, l-1$. If all integrals involved in the Itô formula (19) but one exist, the last one exists too and for $t \in [0,1]$

$$f\big(X(t)\big) = f(X_0) + \int_0^t f'\big(X(u)\big)\, d^{\nu,1} X(u) \\ + \sum_{j=l}^{n-1} k_{l,j}^{\nu} \int_0^t f^{(2j+1)}\big(X(u)\big)\, d^{\delta_{1/2}, 2j+1} X(u) \tag{19}$$

where the sum is null if $l > n-1$. Here, $k_{l,j}^{\nu}$ are some universal constants.

For fractional Brownian motion, Gradinaru et al., [GNRV05], go further.

Theorem 8.

1. *For $H > \frac{1}{6}$ and $f \in C^6(\mathbb{R}, \mathbb{R})$, the integral $\int_0^\cdot f'\left(W^H(u)\right)\, d^{\nu,1} W^H(u)$ exists for any symmetric probability measure on $[0,1]$, and one has*

$$f\left(W^H(t)\right) = f(0) + \int_0^t f'\left(W^H(u)\right)\, d^{\nu,1} W^H(u), \quad t \in [0,1].$$

2. *Fix $r \geqslant 2$. If $(2r+1)H > \frac{1}{2}$ and if $f \in C^{4r+2}(\mathbb{R},\mathbb{R})$ then the integral $\int_0^{\cdot} f'\left(W^H(u)\right) d^{\nu,1}W^H(u)$ exists for any symmetric probability measure on $[0,1]$ such that $m_{2j} = \frac{1}{2j+1}$ for $j = 1,\dots,r-1$ and*

$$f\left(W^H(t)\right) = f(0) + \int_0^t f'\left(W^H(u)\right) d^{\nu,1}W^H(u), \quad t \in [0,1].$$

Remark 13. In [GNRV05], the authors prove that for $H < \frac{1}{6}$ and for $\nu = \frac{1}{2}[\delta_0 + \delta_1]$, the integral $\int_0^t f'\left(W^H(u)\right) d^{\nu,1}W^H(u)$ does not exist.

Remark 14. In the one-dimensional case, Nourdin [Nou05] shows that this integral gives a meaning to and solves stochastic differential equations, and the Milshtein scheme, see [Tal96], converges. Moreover, Nourdin and Simon, in [NS06] have studied the existence of the density of the solution.

6 The semimartingale approach

In this section, we present some ideas of [CCM03]. The authors of [CCM03] have noticed that fractional Brownian motion is a limit of Gaussian semimartingales using the Volterra representation given in Section 2.2. Then, using stochastic calculus of variation with respect to the underlying Brownian motion B, they obtain a nice representation of integrals with respect to these Gaussian semimartingales. This representation allows to exchange limits and integral signs, and provides an integral with respect to fractional Brownian motion.

Before coming to Volterra representations, we present results on Wiener integrals with respect to fractional Brownian motion and their reproducing kernels. We show that their natural filtrations are Brownian filtrations.

6.1 Wiener integral and reproducing kernel

Following the representations given in Section 2.2, the fractional Brownian motion $\{W^H(t), \quad t \geqslant 0\}$ admits the integral representation

$$W^H(t) = \int_0^t K^H(t,s)\, dB(s), \quad t \geqslant 0$$

where $(s,t) \in \mathbb{R}_+^2$, $K^H(t,s)$ is given by

$$C_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}} (u-s)^{H-\frac{1}{2}}\, du \right] \mathbf{1}_{]0,t[}(s)$$

and B is a Brownian motion. Moreover, for fixed s, the map $t \mapsto K^H(t,s)$ is differentiable on $]s,1]$ with derivative

$$\partial_t K^H(t,s) = C_H\, \frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{3}{2}}.$$

For $0 < s < t \leqslant 1$, one has

$$\begin{aligned} W^H(t) - W^H(s) &= \int_s^t K^H(t,r)\,dB(r) + \int_0^s \left[K^H(t,r) - K^H(s,r)\right]\,dB(r) \\ &= \int_0^1 \Bigg[\mathbf{1}_{[s,t]}(r)\,K^H(1,r) \\ &\qquad + \int_s^1 \left[\mathbf{1}_{[s,t]}(u) - \mathbf{1}_{[s,t]}(r)\right] \partial_1 K^H(u,r)\,du\Bigg]\,dB(r). \quad (20) \end{aligned}$$

Let a be a step function of the form

$$a(s) = \sum_{k=1}^n a_k \mathbf{1}_{[t_{k-1},t_k[}(s)$$

for a subdivision $0 \leqslant t_0 \leqslant \cdots \leqslant t_n$ of $[0,1]$ and $a_k \in \mathbb{R}$, $k = 1, \ldots, n$. From (20) one has

$$\begin{aligned} &\sum_{k=1}^n a_k \left[W^H(t_k) - W^H(t_{k-1})\right] \\ &\quad = \int_0^1 \left[a(r)K^H(1,r) + \int_s^1 \left[a(u) - a(r)\right] \partial_1 K^H(u,r)\,du\right]\,dB(r). \end{aligned}$$

Introduce the following operator on suitable functions

$$I_{1,-}^{K^H}(a)(s) := a(s)K^H(1,s) + \int_s^1 \left[a(u) - a(s)\right] \partial_1 K^H(u,s)\,du, \quad s \in [0,1]. \tag{21}$$

Proposition 13. *Let a be an α Hölder continuous function with $\alpha + H > \frac{1}{2}$. The Wiener integral $\int_0^1 a(s)\,dW^H(s)$ of a with respect to W^H exists and has the following representation*

$$\int_0^1 \left[a(r)K^H(1,r) + \int_s^1 \left[a(u) - a(r)\right] \partial_1 K^H(u,r)\,du\right]\,dB(r).$$

Proof. Let $a(m)$ be the linear interpolation of a along the dyadic subdivision, that is,

$$a(m)(t) = a(m)\left(t_{k-1}^m\right) + 2^m\left(t - t_{k-1}^m\right)\left[a(t_k^m) - a(t_{k-1}^m)\right], \quad t \in \left[t_{k-1}^m, t_k^m\right].$$

Then, for any $\alpha' < \alpha$, $a(m)$ is α' Hölder continuous and converges to a in the α' Hölder norm. If $\alpha' > \frac{1}{2} - H$, then $I_{1,-}^{K^H}\big(a(m)\big)$ converges in $L^2\big([0,1], \mathbb{R}, dr\big)$ to $I_{1,-}^{K^H}(a)$, thus proving the proposition. $\square$

The operator $I_{1,-}^{K^H}$ is close to a Liouville operator, see [SKM93].

Lemma 5. *For $H \in]0,1[$, one has the following identification: for suitable a*

$$I_{1,-}^{K^H}(a)(s) = c_H\, s^{\frac{1}{2}-H}\, I_{1,-}^{H-\frac{1}{2}}\big[u^{H-\frac{1}{2}}a(u)\big](s), \quad s \in [0,1].$$

Here according to [SKM93] for $\alpha \in]0,1[$ and $f \in L^p_{loc}(\mathbb{R},\mathbb{R},dx)$, $s<1$,

$$I_{1,-}^{\alpha}(f)(s) = \frac{1}{\Gamma(\alpha)}\int_s^1 f(u)(u-s)^{\alpha-1}du$$

and for suitable f

$$I_{1,-}^{-\alpha}(f)(s) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(s)}{(1-s)^{\alpha}} - \alpha\int_s^1 \frac{f(u)-f(s)}{(u-s)^{\alpha+1}}\,du\right].$$

Proof. In [PT01], Pipiras and Taqqu have pointed out that

$$K^H(t,s) = c_H\, s^{\frac{1}{2}-H}\, I_{1,-}^{H-\frac{1}{2}}\big(u^{H-\frac{1}{2}}\mathbf{1}_{[0,t]}(u)\big)(s), \quad 0 \leqslant s < t \leqslant 1$$

for a suitable constant c_H. So Lemma 5 is true for step functions, and conclusion is reached by a density argument. □

Moreover, $\big(I_{1,-}^{\alpha}\big)_{\alpha\geqslant 0}$ is a semigroup of operators and $I_{1,-}^{\alpha}\circ I_{1,-}^{-\alpha} = Id$. Then for any $t\in[0,1]$, the equation

$$\mathbf{1}_{[0,t]}(s) = I_{1,-}^{K^H}\big(f(t,.)\big)(s), \quad s\in[0,1],$$

with unknown $f(t,.) \in L^2\big([0,1],\mathbb{R},dr\big)$, has a unique solution, namely

$$f^H(t,s) = C_H^{-1}\, s^{\frac{1}{2}-H}\, I_{1,-}^{\frac{1}{2}-H}\big(u^{H-\frac{1}{2}}\mathbf{1}_{[0,t]}(u)\big)(s), \quad s\in[0,1].$$

This result was proved in Lemma 5.1 of [PT01] for $H > \frac{1}{2}$ and is a consequence for $H < \frac{1}{2}$ of the definition of $I_{1,-}^{\alpha}$.

We recover the first part Theorem 4.8 of [DÜ99].

Proposition 14. *The process $B = \{B(t) := \int_0^1 f^H(t,.)\,dW^H(s);\ \ t\in[0,1]\}$ is a Brownian motion whose natural filtration coincides with the natural filtration of W^H. (Here the integral is a Wiener integral.)*

Recall that $\mathcal{H}_H$, the reproducing kernel Hilbert space of fractional Brownian motion is the closure of the linear span $\mathcal{E}$ of the indicator functions $\{\mathbf{1}_{[0,t]};\ \ t\in[0,1]\}$ with respect to the scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle = R_H(t,s)$ (see Appendix A).

We now are in a position to rewrite Theorem 3.3 of [DÜ99].

Proposition 15. *For $H \in]0,1[$, $\big(I_{1,-}^{K^H}\big)^{-1}\big(L^2([0,1],\mathbb{R},dr)\big) = \mathcal{H}_H$ endowed with the scalar product*

$$\langle f,g\rangle_{\mathcal{H}_H} = \Big\langle \big(I_{1,-}^{K^H}\big)^{-1}(f), \big(I_{1,-}^{K^H}\big)^{-1}(g)\Big\rangle_{L^2([0,1],\mathbb{R},du)}.$$

Remark 15. When $H > \frac{1}{2}$, the elements of $\mathcal{H}_H$ may not be functions but distributions of negative order, according to the papers [PT00] or [AN03].

6.2 Approximation by Gaussian semimartingales

For $\varepsilon > 0$, put

$$W^{H,\varepsilon}(t) := \mathbf{E}\left(W^H(t+\varepsilon) \mid \mathcal{F}_t\right) = \int_0^t K^H(t+\varepsilon, s)\, dB(s), \quad t \in [0,1],$$

where $(\mathcal{F}_t, \ \ t \in [0,1])$ is the natural filtration associated to B or W^H. Then $W^{H,\varepsilon}$ converges to W^H in $C([0,1],\mathbb{R})$ almost surely when ε goes to zero. Using Fubini's theorem, [Pro04], $W^{H,\varepsilon}$ is a semimartingale with decomposition given by:

$$W^{H,\varepsilon}(t) = \int_0^t K^H(s+\varepsilon, s)\, dB(s) + \int_0^t du \int_0^u \partial_1 K^H(u+\varepsilon, s)\, dB(s), \quad t \in [0,1].$$

6.3 Construction of the integral

In order to define the integral

$$\int_0^t a(s)\, dW^H(s), \quad t \in [0,1]$$

for suitable processes, we extend the operator $I_{1,-}^{K^H}$ to $I_{t,-}^{K^H}$ where for a regular enough, $I_{t,-}^{K^H}(a)(s)$ is defined as $I_{1,-}^{K^H}\left(a\mathbf{1}_{[0,t]}\right)(s)$ for $0 \leqslant s < t \leqslant 1$, that is,

$$I_{t,-}^{K^H}(a)(s) = K^H(t,s)\, a(s) + \int_s^t \left[a(u) - a(s)\right] \partial_1 K^H(u,s)\, du. \qquad (22)$$

Let a be an adapted process belonging to $\mathbf{D}^{1,2}\left(L^2([0,1],\mathbb{R},dr)\right)$ (see Remark 30 in Appendix A for the definition of this space). Then for $t \in [0,1]$,

$$\int_0^t a(s)\, dW^{H,\varepsilon}(s)$$
$$= \int_0^t a(s)\, K^H(s+\varepsilon, s)\, dB(s) + \int_0^t a(u)\, du \int_0^u \partial_1 K^H(u+\varepsilon, s)\, dB(s).$$

Then using Property P, (32) in Appendix A, we obtain

$$\begin{aligned}\int_0^t a(s)\, dW^{H,\varepsilon}(s) = &\int_0^t a(s)\, K^H(s+\varepsilon, s)\, dB(s) \\ &+ \int_0^t du \int_0^u a(u)\, \partial_1 K^H(u+\varepsilon, s)\, \delta^B B(s) \\ &+ \int_0^t du \int_0^u D_s a(u)\, \partial_1 K^H(u+\varepsilon, s)\, ds.\end{aligned}$$

The second integral in the right-hand side is a divergence since the process $\{a(u)\,\partial_1 K^H(u+\varepsilon, s); \quad s \in [0,u]\}$ is not adapted to $\{B(s), \quad s \leqslant u\}$. The anticipating stochastic Fubini theorem, see Theorem 3.1 [Leó93] yields

$$\begin{aligned}\int_0^t a(s)\,dW^{H,\varepsilon}(s) = &\int_0^t a(s)\,K^H(s+\varepsilon, s)\,dB(s)\\ &+\int_0^t\int_s^t a(u)\,\partial_1 K^H(u+\varepsilon, s)\,du\,\delta^B B(s)\\ &+\int_0^t du\int_0^u D_s a(u)\,\partial_1 K^H(u+\varepsilon, s)\,ds.\end{aligned}$$

The function $t \mapsto K^H(t,s)$ is not absolutely continuous when $H < \frac{1}{2}$, one can set

$$\begin{aligned}\int_0^t a(s)\,dW^{H,\varepsilon}(s) = &\int_0^t a(s)\,K^H(t+\varepsilon, s)\,dB(s)\\ &+\int_0^t\int_s^t [a(u)-a(s)]\;\partial_1 K^H(u+\varepsilon, s)\,du\,\delta^B B(s)\\ &+\int_0^t du\int_0^u D_s a(u)\,\partial_1 K^H(u+\varepsilon, s)\,ds. \qquad (23)\end{aligned}$$

Hypothesis 9 *Assume that a is an adapted process belonging to the space $\mathbf{D}_B^{1,2}\left(L^2([0,1],\mathbb{R},du)\right)$ and that there exists α fulfilling $\alpha+H > \frac{1}{2}$ and $p > 1/H$ such that*

- $\|a\|^2_{\mathbf{L}^{1,2}_{B,\alpha}} := \sup\limits_{0<s<u<1} \dfrac{\mathbf{E}\left[(a(u)-a(s))^2 + \int_0^1 \left(D_r^B a(u) - D_r^B a(s)\right)^2\,dr\right]}{|u-s|^{2\alpha}}$ *is finite,*
- $\sup_{s\in[0,1]} |a(s)|$ *belongs to* $L^p(\Omega,\mathbb{R},\mathbf{P})$.

Proposition 16. *Let a be a process fulfilling Hypothesis 9. For $t \in [0,1]$*

1. *The process*
$$I_{t,-}^{K^H}(a) := \left\{a(s)\,K^H(t,s) + \int_s^t [a(u)-a(s)]\;\partial_1 K^H(u,s)\,du, \quad s\in[0,1]\right\}$$
belongs to $\mathbf{D}^{1,2}\left(L^2([0,1],\mathbb{R},dr)\right)$.
2. *The process $I_{t,-}^{K^H}(a)$ is the limit in $\mathbf{D}_B^{1,2}\left(L^2([0,1],\mathbb{R},dr)\right)$ of the processes*
$$I_{t,-}^{K^H,\varepsilon}(a) := \left\{a(s)\,K^H(t+\varepsilon,s) + \int_s^t [a(u)-a(s)]\;\partial_1 K^H(u+\varepsilon,s)\,du, \; s\in[0,1]\right\}.$$

Proof. The proof relies on the Meyer inequality (31) of Appendix A. □

For such a process a, (see [CCM03] for less strong hypotheses), the family of random variables $\left\{\int_0^t a(s)\,dW^{H,\varepsilon}(s), \varepsilon > 0\right\}$ converges in $L^2(\Omega, \mathbb{R}, \mathbf{P})$ when ε goes to 0. Each term in the right-hand side of (23) converges to the same term where $\varepsilon = 0$. Then, it is "natural" to define

Definition 8. *For a process a fulfilling Hypothesis 9, set*

$$\begin{aligned}\int_0^t a(s)\,dW^H(s) := &\int_0^t a(s)\,K^H(t,s)\,dB(s)\\ &+ \int_0^t\int_s^t \big[a(u) - a(s)\big]\,\partial_1 K^H(u,s)\,du\,\delta^B B(s)\\ &+ \int_0^t du \int_0^u D_s^B a(u)\,\partial_1 K^H(u,s)\,ds.\end{aligned}$$

Remark 16.

- This approach extends to Gaussian Volterra processes, see for instance [Dec05].
- Observe that stochastic calculus of variation appears "naturally."
- It remains to prove that the process $\left\{\int_0^t a(s)\,dW^H(s),\quad t \in [0,1]\right\}$ has a continuous modification, see Theorem 7.1 of [CCM03].
- When the integral using Riemann sums and the integral in Definition 8 exist, they coincide.

Proposition 17. *For a process a fulfilling Hypothesis 9, the process*

$$\left\{\int_0^t a(s)\,dW^H(s),\quad t \in [0,1]\right\}$$

has a continuous modification.

Proof. This integral may be represented as

$$\int_0^1 a(s)\,\mathbf{1}_{[0,t]}(s)\,dW^H(s) = X(t) + Y(t) + Z(t),$$

where

$$\begin{aligned}X(t) &= \int_0^1 K^H(t,s)\,a(s)\,\mathbf{1}_{[0,t]}(s)\,dB(s),\\ Y(t) &= \int_0^1\int_s^t \big[a(u)\,\mathbf{1}_{[0,t]}(u) - a(s)\,\mathbf{1}_{[0,t]}(s)\big]\;\partial K^H(u,s)\,du\,\delta^B B(s),\\ Z(t) &= \int_0^t du \int_0^u D_s^B a(u)\,\partial_1 K^H(u,s)\,ds.\end{aligned}$$

The sample paths of the process Z are absolutely continuous with respect to Lebesgue measure and then they are continuous.

Continuity of X is established via Kolmogorov's continuity criterion. More precisely, for $p > 1/H$, there exists $C = C(T, \alpha)$ such that

$$\mathbf{E}\big[|X(t+\tau) - X(t)|^p\big] \leqslant C\,\tau^{1+(pH-1)}. \tag{24}$$

First write

$$\begin{aligned} X(t+\tau) - X(t) = &\int_0^t \big(K^H(t+\tau,s) - K^H(t,s)\big)\; a(s)\,dB(s) \\ &+ \int_t^{t+\tau} K^H(t+\tau,s)\,a(s)\,dB(s). \end{aligned}$$

Since a is adapted, one can apply the Burkholder–Davis–Gundy inequalities to the martingales

$$\begin{aligned} r &\mapsto \int_0^r \big[K^H(t+\tau,s) - K^H(t,s)\big]\; \mathbf{1}_{[0,t]}(s)\,a(s)\,dB(s), \\ r &\mapsto \int_0^r K^H(t+\tau,s)\,\mathbf{1}_{[t,t+\tau]}(s)\,a(s)\,dB(s) \end{aligned}$$

to obtain for a constant C_p the upper bound

$$\begin{aligned} \mathbf{E}\big[|X(t{+}\tau) - X(t)|^p\big] \leqslant C_p\,\mathbf{E}&\left[\left(\int_0^t \big[K^H(t{+}\tau,s) - K^H(t,s)\big]^2 a(s)^2\,ds\right)^{\frac{p}{2}}\right] \\ &+ C_p\,\mathbf{E}\left[\left(\int_t^{t+\tau} K^H(t{+}\tau,s)^2 a(s)^2\,ds\right)^{\frac{p}{2}}\right]. \end{aligned}$$

The integrability assumptions on a and the fact that

$$\big\|K^H(t{+}\tau,.) - K^H(t,.)\big\|_{L^2([0,1],\mathbb{R},ds)} = \mathbf{E}\left(\big[W^H(t{+}\tau) - W^H(t)\big]^2\right) = \tau^H$$

imply for $p > 2$,

$$\begin{aligned} &\mathbf{E}\left[|X(t{+}\tau) - X(t)|^p\right]^{1/p} \\ &\qquad \leqslant C\,\Big\|\sup_{s\in[0,1]}|a(s)|\Big\|_{L^p(\omega,\mathbb{R},\mathbf{P})}\,\big\|K^H(t{+}\tau,.) - K^H(t,.)\big\|_{L^2([0,1],\mathbb{R},ds)} \\ &\qquad \leqslant C\,\Big\|\sup_{s\in[0,1]}|a(s)|\Big\|_{L^p(\omega,\mathbb{R},\mathbf{P})}\,|\tau|^H \end{aligned}$$

which is exactly (24).

One can prove that for some constant C_α

$$\begin{aligned} &\Big\|\int_.^{t+\tau} (a(u)\; - a(s))\;\partial K^H(u,s)\,du\mathbf{1}_{[0,t+\tau]} \\ &\quad - \int_.^t (a(u) - a(s))\;\partial K^H(u,s)\,du\;\;\mathbf{1}_{[0,t]}\Big\|_{\mathbf{D}_B^{1,2}} \leqslant C_\alpha\,\|a\|_{\mathbf{L}^{1,2}_{B,\alpha}}\,|\tau|^{H+\alpha}. \end{aligned}$$

Then, Meyer's inequality (31) in Appendix A yields

$$\mathbf{E}\left[|Y(t+\tau)-Y(t)|^2\right] \leqslant C\,\tau^{1+(\alpha+H-1)},$$

and Y has a continuous modification. □

Let f belong to $C^2(\mathbb{R},\mathbb{R})$ and $t\in[0,1]$. Notice that

$$D^B(s)f'\left(W^{H,\varepsilon}(u)\right) = f''\left(W^{H,\varepsilon}(u)\right)K^H(u+\varepsilon,s), \quad 0<s<u<1.$$

Writing Itô's formula for $W^{H,\varepsilon}$ in the same spirit as (23), one obtains

$$\begin{aligned} f\left(W^{H,\varepsilon}(t)\right) &= f\left(W^{H,\varepsilon}(0)\right) + \int_0^t f'\left(W^{H,\varepsilon}(s)\right)K^H(t+\varepsilon,s)\,dB(s) \\ &\quad + \int_0^t\int_s^t \left[f'\left(W^{H,\varepsilon}(u)\right) - f'\left(W^{H,\varepsilon}(s)\right)\right]\partial_t K^H(u+\varepsilon,s)\,du\,\delta^B B(s) \\ &\quad + \int_0^t du\, f''\left(W^{H,\varepsilon}(u)\right)\int_0^u K^H(u+\varepsilon,s)\,\partial_1 K^H(u+\varepsilon,s)\,ds. \end{aligned}$$

Observe that

$$\int_0^u K^H(u+\varepsilon,s)\,\partial_1 K^H(u+\varepsilon,s)\,ds = \frac{1}{2}\frac{d\mathbf{E}\left(W^{H,\varepsilon}(u)^2\right)}{du}.$$

Therefore taking the limit when ε goes to 0 yields the following Itô formula.

Proposition 18. *Theorem 8.2 of [CCM03] Let $H>\frac{1}{4}$, $t\in[0,1]$ and f belong to $C^5(\mathbb{R},\mathbb{R})$ then $I_{t,-}^{K^H}\left(f'(W^H)\right)$ belongs to* $\operatorname{Dom}\delta^B$ *and*

$$\begin{aligned} f\left(W^H(t)\right) &= f\left(W^H(0)\right) + \int_0^t f'\left(W^H(s)\right)\,K^H(t,s)\,dB(s) \\ &\quad + \int_0^t\int_s^t \left[f'\left(W^H(u)\right) - f'\left(W^H(s)\right)\right]\partial_1 K^H(u,s)\,du\,\delta^B B \\ &\quad + H\int_0^t f''\left(W^H(s)\right)\,s^{2H-1}\,ds. \end{aligned}$$

Remark 17. For $H>\frac{1}{6}$, a more complicated formula is given in [CCM03].

6.4 Conclusion

Observe that this approach leads to anticipative stochastic differential equations. To our knowledge, solving them is an open problem. Another approach may be to answer the following questions.

Let $W^H = (W^1, \ldots, W^d)$ be a d-dimensional fractional Brownian motion and

$$W^{\varepsilon,i}(t) := \mathbf{E}\left(W^i(t+\varepsilon) \mid \mathcal{F}_t\right) = \int_0^t K^H(t+\varepsilon, s)\, dB^i(s), \quad i = 1, \ldots, d,$$

where $(\mathcal{F}_t, \quad t \in [0,1])$ is the natural filtration generated by W^H. Let $\mathbf{W}^\varepsilon = \left(1, \mathbf{W}^{\varepsilon,1}, \mathbf{W}^{\varepsilon,2}, \mathbf{W}^{\varepsilon,3}\right)$ be the geometric functional over $W^\varepsilon = \left(W^{\varepsilon,1}, \ldots, W^{\varepsilon,d}\right)$ defined by

$$\begin{aligned}
\mathbf{W}^{\varepsilon,1}_{s,t} &= W^{H,\varepsilon}(t) - W^{H,\varepsilon}(s),\\
\mathbf{W}^{\varepsilon,2}_{s,t} &= \int_s^t W^{\varepsilon,1}(s,u) \otimes \circ dW^{H,\varepsilon}(u),\\
\mathbf{W}^{\varepsilon,3}_{s,t} &= \int_s^t W^{\varepsilon,2}(s,u) \otimes \circ dW^{H,\varepsilon}(u),
\end{aligned}$$

there $\circ d$ stands for Stratonovich integral. According to [CL05], $\mathbf{W}^\varepsilon$ is a geometric functional with finite p variation for any $p > 2$.

1. Does $\mathbf{W}^\varepsilon$ converge in the p variation distance for $p > \frac{1}{H}$?
2. If the answer to the previous question is yes, does the limit coincide with the limit obtained in Proposition 10?

If the answer to the first question is positive, Theorem 5 provides a way to solve differential equations driven by fractional Brownian motion. This notion of a solution will coincide with the notion of Corollary 6, if the answer to the second question is positive.

7 Divergence with respect to fractional Brownian motion

This section is devoted to divergence with respect to fractional Brownian motion, as introduced by Decreusefond and Üstünel, [DÜ99]. For Brownian motion, this integral coincides with the Itô integral for adapted integrand. First, we construct the divergence integral with respect to fractional Brownian motion and derive the Girsanov theorem. We notice that when the Hurst parameter is smaller than $\frac{1}{4}$, fractional Brownian motion is not integrable with respect to itself. We present the extensions of the divergence operator given by Cheridito and Nualart, [CN05] and Biagini, Øksendal, Sulem and Wallner, [BØSW04] or Decreusefond [Dec05]. We conclude with the link with other integrals and some Itô and Tanaka formulas.

7.1 Divergence for fractional Brownian motion

In Section 6, we have shown that there always exists a Brownian motion B such that the Volterra representation given in Section 2.2 holds. Then, one can construct two divergence operators: one with respect to the fractional Brownian motion W^H, denoted by δ^{W^H}; the second one with respect to the underlying Brownian motion B, denoted by δ^B.

Recall results from Propositions 15 and 14. For $H \in]0,1[$, $\left(I_{1,-}^{K^H}\right)^{-1}\left(L^2([0,1],\mathbb{R},dr)\right) = \mathcal{H}_H$ endowed with the scalar product

$$\langle f,g\rangle_{\mathcal{H}_H} = \left\langle \left(I_{1,-}^{K^H}\right)^{-1}(f), \left(I_{1,-}^{K^H}\right)^{-1}(g) \right\rangle_{L^2([0,1],\mathbb{R},du)},$$

where $I_{1,-}^{K^H}(a)$ is defined by (21): for a regular enough,

$$I_{1,-}^{K^H}(a)(s) := a(s)\,K^H(1,s) + \int_s^1 [a(u)-a(s)]\;\partial_1 K^H(u,s)\,du, \quad s \in [0,1].$$

We establish the deeper relations between B and W^H, as in Theorem 4.8 of [DÜ99].

Proposition 19.

1. *For all* $t \in [0,1]$, $W^H(t) = \delta^B\big(I_{1,-}^{K^H}(\mathbf{1}_{[0,t]})\big) = \delta^B\left(K^H(t,.)\right)$.
2. $\mathbf{D}^{1,2}_{W^H} = \big(I_{1,-}^{K^H}\big)^{-1}\big(\mathbf{D}^{1,2}_B\big)$ *and for* $F \in \mathbf{D}^{1,2}_{W^H}$, $D^{W^H}F = \big(I_{1,-}^{K^H}\big)^{-1}(D^B F)$.
3. $\operatorname{Dom}\delta^{W^H} = \big(I_{1,-}^{K^H}\big)^{-1}(\operatorname{Dom}\delta^B)$ *and for* $u \in \mathbf{D}^{1,2}_{W^H}(\mathcal{H}^H)$
   $\delta^{W^H}(u) = \delta^B\big(I_{1,-}^{K^H}(u)\big)$.

Proof. Let Λ_B be the isometry between $L^2([0,1],\mathbb{R},dr)$ and the first Wiener chaos of B (see Appendix A).

Let Λ_{W^H} be the isometry between $\mathcal{H}^H$ and the first Wiener chaos of W^H.

Proposition 14 means that for all $t \in [0,1]$, $\Lambda_B\left(\mathbf{1}_{[0,t]}\right) = \Lambda_{W^H} \circ \big(I_{1,-}^{K^H}\big)^{-1}(\mathbf{1}_{[0,t]})$. Since $\mathcal{E}$, the linear span of the indicator functions $\{\mathbf{1}_{[0,t]};\ \ t \in [0,1]\}$, is dense in $L^2([0,1],\mathbb{R},dr)$, one obtains

$$\Lambda_B = \Lambda_{W^H} \circ \big(I_{1,-}^{K^H}\big)^{-1}.$$

Thus, for $f \in \big(I_{1,-}^{K^H}\big)^{-1}\left(L^2([0,1],\mathbb{R},dr)\right) = \mathcal{H}^H$, one has

$$\Lambda_B\big(I_{1,-}^{K^H}(f)\big) = \Lambda_{W^H}(f). \tag{25}$$

Taking $f = \mathbf{1}_{[0,t]}$ yields the first point of Proposition 19.

Let F be a smooth cylindrical random variable given by

$$F = f\left(W^H(\phi_1),\ldots,W^H(\phi_n)\right)$$

where $n \geqslant 1$, $f \in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ (f and all its derivatives with at most polynomial growth, $\phi_i \in \mathcal{H}^H$, $i = 1,\ldots,n$). Recall that

$$D^{W^H}F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\left(W^H(\phi_1),\ldots,W^H(\phi_n)\right)\,\phi_j.$$

Using identity (25),

$$D^{W^H}F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \left[B\big(I_{1,-}^{K^H}(\phi_1)\big), \ldots, B\big(I_{1,-}^{K^H}(\phi_n)\big) \right] \phi_j,$$

we identify

$$D^{W^H}F = \big(I_{1,-}^{K^H}\big)^{-1} \big(D^B F\big).$$

This yields point (2) of Proposition 19 since smooth variables are dense in $\mathbf{D}^{1,2}_{W^H}$.

Moreover, for $u \in \mathbf{D}^{1,2}_{W^H}\left(\mathcal{H}^H\right)$, one has $\big(I_{1,-}^{K^H}\big)(u) \in \mathbf{D}^{1,2}_B$ and

$$\mathbf{E}\left[\delta^{W^H}(u)F\right] = \mathbf{E}\left[\langle u, D^W F\rangle_{\mathcal{H}_H}\right] = \mathbf{E}\left[\langle I_{1,-}^{K^H} u, D^B F\rangle_{L^2([0,1],\mathbb{R},dr)}\right],$$

which is exactly point (3) of Proposition 19. □

At this point we can define the so-called **divergence integral** for fractional Brownian motion:

Definition 9. *For* $u \in \operatorname{Dom} \delta^W$,

$$\int_0^1 u(s)\, \delta^{W^H} W^H(s) := \delta^{W^H}(u).$$

Remark 18.

- This divergence integral is the same as the one defined by Decreusefond–Üstünel [DÜ99], or by Alos–Mazet–Nualart in [AMN01], or Alos–Nualart [AN03] or by Cheredito–Nualart [CN05] or Decreusefond in [Dec05].
- The link with the integral obtained in [CCM03], see Section 6, is the following. If the divergence integral exists in the sense of the Definition 9 and if the integral is defined according to Definition 8, then the following equality holds

$$\int_0^1 a(s)\, dW^H(s) = \int_0^1 a(s)\, \delta^{W^H} W^H(s) + \int_0^1 du \int_0^u D^B(s)\, a(u)\, \partial_1 K^H(s,u)\, ds.$$

7.2 Cameron–Martin and Girsanov Theorems

From Proposition 19, the Cameron–Martin space can easily be identified.

Proposition 20. *For* $f \in C([0,1],\mathbb{R})$, *the law of*

$$W^H + f := \left\{ W^H(t) + f(t), \quad t \in [0,1] \right\}$$

is absolutely continuous with respect to the law of W^H *if and only if there exists an element* $\dot{f} \in L^2([0,1],\mathbb{R},dr)$, *such that*

$$f(t) = \int_0^t K^H(t,s)\,\dot{f}(s)\,ds$$

and in that case

$$\frac{d\mathbf{P}_{W^H+f}}{d\mathbf{P}_{W^H}} = \exp\left[\delta^B(\dot{f}) - \frac{1}{2}\|f\|^2_{\mathcal{H}^H}\right].$$

These results can be found in Theorem 4.1 of [DÜ99]. For $H > \frac{1}{2}$ a simpler proof is given in Theorem 4.1 of Norros–Valkeila–Virtamo [NVV99].

Before stating the Girsanov theorem, we give a characterization of the $\{\mathcal{F}_t^{W^H}, \quad t \in [0,1]\}$ square integrable martingales following Corollary 4.2 of [DÜ99].

Proposition 21. *Every* $\left\{\mathcal{F}_t^{W^H}, \quad t \in [0,1]\right\}$ *square integrable martingale* M *can be written as* $M = \left\{M_0 + \delta^{W^H}\left(u\mathbf{1}_{[0,t]}\right), \quad t \in [0,1]\right\}$ *where*

$$u(t) = \mathbf{E}\left[D^{W^H} M_1 \mid \mathcal{F}_t^{W^H}\right].$$

This proposition can be seen as a consequence of Proposition 19. Now, we are in a position to state the Girsanov theorem, see Theorem 4.9 of Decreusefond–Üstünel [DÜ99]. It is nothing but Girsanov's theorem with respect to B written in terms of W^H.

Theorem 10. *Let* u *be an adapted process in* $L^2\big(\Omega, L^2([0,1],\mathbb{R},dr),\mathbf{P}\big)$ *such that*

$$\mathbf{E}\left[L^u(1)\right] = 1,$$

where

$$L^u(t) = \exp\left(\delta^{W^H}\big((I_{t,-}^{K^H})^{-1}(u)\big) - \frac{1}{2}\left\|(I_{t,-}^{K^H})^{-1}(u)\right\|^2_{\mathcal{H}^H}\right).$$

Define a probability $\mathbf{P}_u$ *by*

$$\left.\frac{d\mathbf{P}_u}{d\mathbf{P}}\right|_{\mathcal{F}_t^{W^H}} = L^u(t), \quad t \in [0,1].$$

Under $\mathbf{P}_u$ *the process*

$$\left\{W^H(t) - \int_0^t K^H(t,s)\,u(s)\,ds, \quad t \in [0,1]\right\}$$

is a fractional Brownian motion with Hurst parameter H.

In order to prove an Itô formula or to study the multidimensional case, we study $\operatorname{Dom}\delta^{W^H}$.

7.3 Is W^H integrable with respect to itself?

First observe that

Proposition 22. *Proposition 3 of [CN05]*
For $0 \leqslant a < b \leqslant 1$, *let* $u = \{\mathbf{1}_{[a,b]}(t) W^H(t);\quad t \in [0,1]\}$; *then*

for $H \in]\frac{1}{4}; 1[$, $\mathbf{P}\left(u \in \mathcal{H}^H\right) = 1$*;*
for $H \in]0, \frac{1}{4}]$, $\mathbf{P}\left(u \in \mathcal{H}^H\right) = 0$.

Note that if u belongs to $\operatorname{Dom} \delta^{W^H}$, then u takes its values in $\mathcal{H}^H$.

Proposition 23. *For* $0 \leqslant a < b \leqslant 1$, *let* $u = \{\mathbf{1}_{[a,b]}(t) W^H(t);\quad t \in [0,1]\}$; *then*

for $H \in]\frac{1}{4}; 1[$, $u \in \operatorname{Dom} \delta^{W^H}$*;*
for $H \in]0, \frac{1}{4}]$, $u \notin \operatorname{Dom} \delta^{W^H}$.

Remark 19. As a consequence, for $H \in]0, \frac{1}{4}]$, W^H is not integrable in the sense of Definition 8, see [CCM03].

Proof. The first point is a consequence of Theorem 8.2 of [CCM03] for the function $f(x) = x^2$. The second point is Proposition 3 of [CN05]. □

According to Proposition 23, we have the following proposition.

Proposition 24. *Let* $d = 2$ *and* $W = \left(W_1^H, W_2^H\right)$ *be a two-dimensional fractional Brownian motion.*

1. *For* $H > \frac{1}{4}$, $u = \left(W_2^H \mathbf{1}_{[0,t]}, 0\right)$ *belongs to* $\operatorname{Dom} \delta^W$,
$$\delta^W(u) = \int_0^t I_{1,-}^{K^H}\left(W_2^H\right)(s)\, dB^1(s).$$
2. *For* $H \leqslant \frac{1}{4}$, $u = \left(W_2^H \mathbf{1}_{[0,t]}, 0\right)$ *does not belong to* $\operatorname{Dom} \delta^W$.

As a consequence, for $H \in]0, \frac{1}{4}]$, u is not integrable in the sense of Definition 8, see [CCM03].

Remark 20. For $H \leqslant 1/4$, since W^H does not belong to $\operatorname{Dom} \delta^{W^H}$, several extensions of $\operatorname{Dom} \delta^{W^H}$ have been proposed.

In [CN05], Cheridito and Nualart weaken the set of smooth cylindrical test functions.

In [BØSW04], Biagini, Øksendal, Sulem, and Wallner extend $\operatorname{Dom} \delta^{W^H}$ to some stochastic distribution process u.

In [BØSW04], the extension of the multidimensional case leads to state

$$\int_0^1 W_1^H(s)\, dW_2^H(s) = W_1^H(1)\, W_2^H(1),$$

where W_1^H and W_2^H are two independent fractional Brownian motions, as pointed out in Example 6.2 of [BØSW04].

7.4 Extension of the domain of the divergence integral

We describe the approach of [CN05].

In [CN05], Cheridito and Nualart work on $\mathbb{R}$, but their approach can be adapted to $[0,1]$.

Let H_n be the nth Hermite polynomial

$$H_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left[\exp\left(\frac{-x^2}{2}\right)\right].$$

Recall that $\mathcal{E}$ is the linear span of the indicator functions $\{\mathbf{1}_{[0,t]};\quad t \in [0,1]\}$. It can be shown as in Theorem 1.1.1 of [Nua95] that for all $p \geqslant 1$,

$$\mathrm{span}\big\{H_n\left(B(\phi)\right) : n \in \mathbb{N},\quad \phi \in \mathcal{E},\quad \|\phi\|_{L^2(\mathbb{R})} = 1\big\}$$

is dense in $L^p(\Omega, \mathbb{R}, \mathbf{P})$. Following Definition 4 of Nualart and Cheridito, [CN05] we set:

Definition 10. *Let $u = \{u(t),\quad t \in [0,1]\}$ be a measurable process. We say that $u \in \mathrm{Dom}^* \delta^{W^H}$ whenever there exists in $\cup_{p>1} L^p(\Omega, \mathbb{R}, \mathbf{P})$ a random variable $\delta^{W^H}(u)$ such that for all $n \in \mathbb{N}^*$ and $\phi \in \mathcal{E}$ verifying $\|\phi\|_{L^2(\mathbb{R})} = 1$, the following conditions are satisfied:*

1. *For almost all $t \in \mathbb{R}$, $u(t)\, H_{n-1}\left(B(\phi)\right) \in L^1(\Omega, \mathbb{R}, \mathbf{P})$,*
2. *$\mathbf{E}\left[u_.\, H_{n-1}\left(B(\phi)\right)\right] (I_{1,-}^{K^H})^{-1,*} \phi(.) \in L^1([0,1])$,*
3. *$C_H^2 \int_0^1 \mathbf{E}\left[u(t)\, H_{n-1}\left(B(\phi)\right)\right] (I_{1,-}^{K^H})^{-1,*}(\phi)(t)\, dt = \mathbf{E}\big[\delta^{W^H}(u)\, H_n\big(B(\phi)\big)\big]$,*

where $\left(I_{1,-}^{K^H}\right)^{-1,*}$ is the adjoint of $\left(I_{1,-}^{K^H}\right)^{-1}$ in $L^2([0,1], \mathbb{R}, dr)$ and

$$C_H^2 = \frac{\Gamma(H+1/2)^2}{\int_0^\infty \left[(1+s)^{H-1/2} - s^{H-1/2}\right]^2 ds + 1/(2H)}.$$

Observe that if $u \in \mathrm{Dom}^* \delta^{W^H}$, then $\delta^{W^H}(u)$ is unique, and the mapping $\delta : \mathrm{Dom}^* \delta^{W^H} \to \cup_{p>1} L^p(\Omega, \mathbb{R}, \mathbf{P})$ is linear.

Remark 21. According to the results of [CN05], $\mathrm{Dom}\, \delta^{W^H} \subset \mathrm{Dom}^* \delta^{W^H}$, and the extended operator δ^{W^H} defined in Definition 10 restricted to $\mathrm{Dom}\, \delta^{W^H}$ coincides with the divergence operator defined in Definition 9.

Remark 22. The extended divergence operator δ^{W^H} is closed in the following sense (point 2 of Remark 5 of [CN05]):

Let $p \in (1, \infty[$ and $q \in (\frac{1}{1/2+H}, \infty]$. Let $u \in L^p\left(\Omega, L^q(\mathbb{R}, \mathbb{R}, dr), \mathbf{P}\right)$ and let $\{u^k\}_{k\in\mathbb{N}}$ be a sequence in $\mathrm{Dom}^* \delta^{W^H} \cap L^p\left(\Omega, L^q(\mathbb{R}, \mathbb{R}, dr), \mathbf{P}\right)$ such that

$$\lim_{k\to\infty} u^k = u \quad \text{in} \quad L^p\big(\Omega, L^q(\mathbb{R}, \mathbb{R}, dr), \mathbf{P}\big).$$

If there exist a $\hat{p} \in (1, \infty]$ and an $X \in L^{\hat{p}}(\Omega, \mathbb{R}, \mathbf{P})$ such that

$$\lim_{k\to\infty} \delta(u^k) = X \quad \text{in} \quad L^{\hat{p}}(\Omega, \mathbb{R}, \mathbf{P}),$$

then $u \in \mathrm{Dom}^* \delta^{W^H}$, and $\delta^{W^H}(u) = X$.

7.5 Link with white noise theory

In order to describe the approach used by Biagini, Øksendal, Sulem, and Wallner in [BØSW04], we present a summary of classical white noise theory. These authors work on $\mathbb{R}$, but their approach can be adapted to $[0,1]$.

Let $(\xi_n)_{n\in\mathbb{N}^*}$ be an orthonormal basis on $L^2\left([0,1],\mathbb{R},dr\right)$. Let $\mathcal{I}$ denote the set of multi-indices $\alpha=(\alpha_1,\ldots,\alpha_{l(\alpha)})$ of finite length, $\alpha_i\in\mathbb{N}$, $\alpha_{l(\alpha)}\neq 0$. The norm of α is $|\alpha|=\sum_{i=1}^{l(\alpha)}|\alpha_i|$, its factorial is $\alpha!=\prod_{i=1}^{l(\alpha)}\alpha_i!$ and the corresponding variable is

$$\mathcal{H}_\alpha=\prod_{i=1}^{l(\alpha)}H_{\alpha_i}\big(B(\xi_i)\big).$$

The unit vectors of $\mathcal{I}$ are $\varepsilon^{(k)}=(0,\ldots,0,1)$ with $l(\varepsilon^{(k)})=k$.

Theorem 11. *Second Wiener-chaos extension theorem (Theorem 2.3 of [BØSW04]) For any $F\in L^2(\Omega,\mathbb{R},\mathbf{P})$ there exists a unique family $(c_\alpha)_{\alpha\in\mathcal{I}}$ in $\mathbb{R}^{\mathcal{I}}$ such that*

$$F(\omega)=\sum_{\alpha\in\mathcal{I}}c_\alpha\,\mathcal{H}_\alpha(\omega)$$

and

$$\mathbf{E}(F^2)=\sum_{\alpha\in\mathcal{I}}c_\alpha^2\,\alpha!\ .$$

Example 12 *For all $t\in[0,1]$,*

$$B(t)=\sum_{k=1}^{\infty}\int_0^t\xi_k(s)\,ds\,\mathcal{H}_{\varepsilon^{(k)}}.$$

In order to give a meaning to $\frac{d}{dt}B(t)$ we introduce the Hida space.

Definition 11. *The Hida space $\mathcal{S}^*$ of stochastic distributions is the set of all formal expansions*

$$\Phi(\omega)=\sum_{\alpha\in\mathcal{I}}c_\alpha\,\mathcal{H}_\alpha(\omega)$$

such that

$$\exists q\in[1,\infty[,\qquad \sum_{\alpha\in\mathcal{I}}c_\alpha^2\,\alpha!\,(2\mathbb{N})^{-q\alpha}<\infty$$

where $(2\mathbb{N})^\gamma$ stands for $\prod_{i=1}^{l(\gamma)}(2i)^{\gamma_i}$.

Remark 23. It is worth observing that $\mathcal{S}^*$ is not included in $L^0(\Omega,\mathbb{R},\mathbf{P})$.

Example 13 *The application B is differentiable in $\mathcal{S}^*$ and its differential, called white noise, is*

$$B^{(0)}(t) = \sum_{k=1}^{\infty} \xi_k(t)\, \mathcal{H}_{\varepsilon^{(k)}}.$$

Definition 12. *If $F^i(\omega) = \sum_{\alpha \in \mathcal{I}} c^i_\alpha \mathcal{H}_\alpha(\omega), \quad i = 1, 2$ are two elements of $\mathcal{S}^*$, then their Wick product $F^1 \diamond F^2$ is the element of $\mathcal{S}^*$*

$$F^1 \diamond F^2(\omega) = \sum_{\gamma \in \mathcal{I}} \sum_{\alpha+\beta=\gamma} c^1_\alpha\, c^2_\beta\, \mathcal{H}_\gamma(\omega).$$

Theorem 14. *Theorem 8 of [BØSW04]*
If $u \in \operatorname{Dom} \delta^B$ then $t \mapsto u(t) \diamond B^{(0)}(t)$ belongs to $L^1([0,1], \mathcal{S}^*, dt)$ and

$$\delta^B(u) = \int_{[0,1]} u(t) \diamond B^{(0)}(t)\, dt.$$

Recall the first point of Proposition 19,

$$W^H(t) = \left\langle I^{K^H}_{1,-}\left(\mathbf{1}_{[0,t]}\right), B \right\rangle_{L^2([0,1],\mathbb{R},dr)}.$$

The process W^H admits the decomposition

$$W^H(t) = \sum_{k=0}^{\infty} \left\langle \mathbf{1}_{[0,t]}, \left(I^{K^H}_{1,-}\right)^* (\xi_k) \right\rangle_{L^2([0,1],\mathbb{R},dr)} \mathcal{H}_{\varepsilon^{(k)}}$$

where $\left(I^{K^H}_{1,-}\right)^*$ is the adjoint of $I^{K^H}_{1,-}$ in $L^2([0,1], \mathbb{R}, dr)$. The process W^H is differentiable in $\mathcal{S}^*$ and its differential is

$$W^{(H)}(t) = \sum_{k=0}^{\infty} \left(I^{K^H}_{1,-}\right)^* \left(\xi_k(t)\right) \mathcal{H}_{\varepsilon^{(k)}}.$$

Definition 13. *Definition 3.3 of [BØSW04]*
Let u be a process taking its values in $\mathcal{S}^*$ such that $t \mapsto u(t) \diamond W^{(H)}(t)$ belongs to $L^1([0,1], \mathcal{S}^*, dt)$. Then u is $\diamond$ integrable with respect to W^H and

$$\int_0^1 u(t)\, d^\diamond W^H(t) = \int_0^1 u(t) \diamond W^{(H)}(t)\, dt.$$

Example 15 *(Example 3.4 of [BØSW04])*
The process $W^H \mathbf{1}_{[0,T]}$ is $\diamond$ integrable with respect to W^H and

$$\int_0^T W^H(t)\, d^\diamond W^H(t) = \frac{1}{2} W^H(T)^2 - \frac{T^{2H}}{2}.$$

Proposition 25. *Proposition 5.2 of [BØSW04]*
If u is $\diamond$ integrable with respect to W^H then $I_{1,-}^{K^H}(u) \diamond B^{(0)}$ is integrable with respect to dt and

$$\int_0^1 u(t)\, d^{\diamond} W^H(t) = \int_0^1 I_{1,-}^{K^H}(u)(t) \diamond B^{(0)}(t)\, dt.$$

The following identification is derived from Theorem 14.

Corollary 8. *Let u be a process such that $I_{1,-}^{K^H}(u) \in \operatorname{Dom} \delta^B$. Then u is $\diamond$ integrable with respect to W^H and*

$$\int_0^1 u(t)\, d^{\diamond} W^H(t) = \delta^B\big(I_{1,-}^{K^H}(u)\big) = \delta^{W^H}(u).$$

7.6 The divergence integral as a process

The divergence integral $\int u(s)\, \delta^{W^H} W^H(s)$ or $\int I_{t,-}^{K^H}(u)(s)\, \delta^B B(s)$ is defined by duality, in a weak sense. We shall exhibit assumptions ensuring that the process $\big\{\int_0^t u(s)\, \delta^{W^H} W^H(s),\ t \in [0,1]\big\}$ or $\big\{\int I_{t,-}^{K^H}(u)(s)\, \delta^B B(s),\ t \in [0,1]\big\}$ has a continuous modification. There exist a lot of such conditions (see Theorem 7.1 of [CCM03], Propositions 1 and 3 of [AMN01], [Dec05]). The existence of a continuous modification follows from a maximal inequality, see [MMV01] and [AN03] for $H > 1/2$, [Dec05] for $H < 1/2$. Here, we only give the simplest conditions, not the optimal ones.

We give more details when $H > 1/2$, since that is the easiest case.

Case when $H > 1/2$

As pointed out by Mémin, Mishura, and Valkeila, [MMV01] for deterministic integrands u, continuity follows from a maximal inequality. This maximal inequality holds for processes $u \in \mathbf{D}_{W^H}^{1,p}(\mathcal{H}^H)$ which live in a subspace of $\mathcal{H}^H$, see Alos and Nualart, [AN03].

Put

$$|\mathcal{H}^H| := \left\{ f \in \mathcal{H}^H;\ \ \|f\|_{|\mathcal{H}^H|} := \int_{[0,1]^2} |f(u)|\,|f(r)|\,|u-r|^{2H-2}\, du\, dr < \infty \right\}.$$

Lemma 6. *For $H > \frac{1}{2}$, the following continuous inclusions hold:*

$$L^{1/H}\big([0,1], \mathbb{R}, dr\big) \subset |\mathcal{H}^H| \subset \mathcal{H}^H.$$

Proof. The covariance function R^H

$$R^H(t,s) = \frac{1}{2}\,\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

is twice differentiable except on the diagonal and its second derivative $\frac{\partial^2 R^H}{\partial t \partial s}$ belongs to $L^1\left([0,1]^2,\mathbb{R},dr\right)$ for $H>\frac{1}{2}$. This means

$$R^H(t,s)=H(2H-1)\int_{[0,1]^2}\mathbf{1}_{[0,t]}(u)\,\mathbf{1}_{[0,s]}(r)\,|u-r|^{2H-2}\,du\,dr.$$

Then, $|\mathcal{H}^H|$ is included in $\mathcal{H}^H$ and for $f,g\in|\mathcal{H}_H|$,

$$\langle f,g\rangle_{\mathcal{H}^H}=H(2H-1)\int_{[0,1]^2}f(u)\,g(r)\,|u-r|^{2H-2}\,du\,dr\,;$$

this inclusion is continuous.

The second inclusion was proved by Mémin, Mishura, and Valkeila, [MMV01]. Applying Hölder's inequality with exponent $q=\frac{1}{H}$ yields

$$\begin{aligned}&\|f\|^2_{|\mathcal{H}^H|}\\ &\leqslant H(2H-1)\|f\|^H_{L^{1/H}([0,1],\mathbb{R},dr)}\left[\int_0^1\left(\int_0^1|f(u)|\,|r-u|^{2H-2}\,du\right)^{\frac{1}{1-H}}dr\right]^{1-H}.\end{aligned}$$

Up to a multiplicative constant, the second factor in the above expression is equal to the $\frac{1}{1-H}$ norm of the left sided Liouville integral $I^{2H-1}_{0,+}(|f|)$, where for suitable function g and $\alpha\in\,]0,1]$

$$I^{\alpha}_{0,+}(g)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}g(u)\,du,\quad t\in[0,1].$$

According to Theorem 3.7 page 72 of [SKM93] for $\mu=0$, $p=\frac{1}{H}$, $q=\frac{1}{1-H}$, $\alpha=2H-1$, and $m=0$ the linear operator $I^{2H-1}_{0,+}$ is continuous from $L^{1/H}\left([0,1],\mathbb{R},dr\right)$ to $L^{\frac{1}{1-H}}\left([0,1],\mathbb{R},dr\right)$ and

$$\|f\|^2_{|\mathcal{H}^H|}\leqslant c_H\,\|f\|^{2H}_{L^{1/H}([0,1],\mathbb{R},dr)}\,\|I^{2H-1}_{0,+}\|_{L^{1/H}([0,1],\mathbb{R},dr),L^{\frac{1}{1-H}}([0,1],\mathbb{R},dr)}.\tag{26}$$

□

Now, we are in a position to state the maximal inequality, Theorem 4 of [AN03]

Theorem 16. *Let $p>1/H$. Let $u=\{u(t),\quad t\in[0,1]\}$ be a stochastic process in $\mathbf{D}^{1,p}_{W^H}\left(L^{\frac{1}{H}-\varepsilon}([0,1],\mathbb{R},dr)\right)$ for $0<\varepsilon<H-\frac{1}{p}$. The following inequality holds*

$$\begin{aligned}&\mathbf{E}\left(\sup_{t\in[0,1]}\left|\int_0^t u(s)\delta^{W^H}W^H(s)\right|^p\right)\leqslant C_{H,\varepsilon,p}\\ &\times\left[\left(\int_0^1|\mathbf{E}u(s)|^{\frac{1}{H-\varepsilon}}\,ds\right)^{p(H-\varepsilon)}+\mathbf{E}\left[\int_0^1\left(\int_0^1|D^{W^H}_s u(r)|^{\frac{1}{H}}\,dr\right)^{\frac{H}{H-\varepsilon}}ds\right]^{p(H-\varepsilon)}\right].\end{aligned}$$

Proof. The proof relies on the representation of the divergence integral using the idea of Zähle [Zäh98]. Indeed, let $\alpha = 1-\frac{1}{p}-\varepsilon$. Then for $1-H < \alpha < 1-\frac{1}{p}$, using the identity $c_\alpha = \int_r^t (t-\theta)^{-\alpha}(\theta-r)^{\alpha-1}d\theta$, one has

$$\int_0^t u(s)\delta^{W^H}W^H(s) = \frac{1}{c_\alpha}\int_0^t u(s)\int_s^t (t-r)^{-\alpha}(r-s)^{\alpha-1}\,dr\,\delta^{W^H}W^H(s).$$

Using the Fubini stochastic theorem (see Nualart's book [Nua95]) one has

$$\int_0^t u(s)\delta^{W^H}W^H(s) = \frac{1}{c_\alpha}\int_0^t \left(\int_s^t (t-r)^{-\alpha}(r-s)^{\alpha-1}u(s)\delta^{W^H}W^H(s)\right) dr.$$

Hölder's inequality and the condition $\alpha < 1-\frac{1}{p}$ yield

$$\left|\int_0^t u(s)\delta^{W^H}W^H(s)\right|^p \leqslant \frac{1}{c_\alpha^p(1-\alpha)^{p-1}}\int_0^t \left|\int_0^r u(s)(r-s)^{\alpha-1}\delta^{W^H}W^H(s)\right|^p dr.$$

From Lemma 6, $D^{1,p}_{W^H}\big(L^{1/H}([0,1],\mathbb{R},dr)\big)$ is continuously embedded into $D^{1,p}_{W^H}(\mathcal{H}^H)$ and

$$\begin{aligned}
&\mathbf{E}\sup_{t\in[0,1]}\left|\int_0^t u(s)\delta^{W^H}W^H(s)\right|^p\\
&\leqslant C_{\alpha,H,p}\int_0^1\left(\int_0^r (r-s)^{\frac{\alpha-1}{H}}\mathbf{E}|u(s)|^{\frac{1}{H}}\,ds\right)^{pH} dr\\
&\quad + C_{\alpha,H,p}\,\mathbf{E}\int_0^1\left[\int_0^r\int_0^1 (r-s)^{\frac{\alpha-1}{H}}\left|D_\theta^{W^H}u(s)\right|^{\frac{1}{H}}d\theta\,ds\right]^{pH} dr\\
&=: I_1 + I_2.
\end{aligned}$$

Again the first factor in the right-hand side of the above expression is equal up to a multiplicative constant to $\big\|I_{0,+}^{\frac{\alpha-1+H}{H}}(\mathbf{E}|u_\cdot|^{\frac{1}{H}})\big\|^{pH}_{L^{pH}([0,1],\mathbb{R},dr)}$. According to Theorem 3.7 page 72 of [SKM93] for $\mu = 0$, $p = \frac{H}{H-\varepsilon}$, $q = pH$, $\alpha = \frac{\alpha-1+H}{H}$, and $m = 0$ the linear operator $I_{0,+}^{\frac{\alpha-1+H}{H}}$ is continuous from $L^{1/H}\big([0,1],\mathbb{R},dr\big)$ to $L^{\frac{1}{1-H}}\big([0,1],\mathbb{R},dr\big)$ and

$$I_1 \leqslant C_{\alpha,p,H}\left[\int_0^1 \mathbf{E}\left[|u(r)|^{\frac{1}{H-\varepsilon}}\right] dr\right]^{p(H-\varepsilon)}.$$

A similar trick yields

$$I_2 \leqslant C_{\alpha,p,H}\mathbf{E}\left[\int_0^1\left(\int_0^1 |D_s^{W^H}u(r)|^{\frac{1}{H}}\,ds\right)^{\frac{H}{H-\varepsilon}} dr\right]^{p(H-\varepsilon)},$$

which completes the proof. □

Continuity of the divergence integral is a consequence of the Garsia–Rodemich–Rumsey Lemma [GRRJ71] and the maximal inequality.

Proposition 26. *Theorem 5 of Alos–Nualart [AN03]*
Assume $pH > 1$. Let $u = \{u(t);\ \ t \in [0,1]\}$ be a stochastic process in the space $\mathbf{D}^{1,p}_{W^H}(|\mathcal{H}|$, such that

$$\mathbf{E}\left[\|u\|^p_{L^{1/H}([0,T],\mathbb{R})}\right] + \mathbf{E}\left[\left\|D^{W^H}u\right\|^p_{L^{1/H}([0,T]^2,\mathbb{R})}\right] < \infty,$$
$$\int_0^1 \mathbf{E}\left[|u(r)|^p\right] dr + \int_0^1 \mathbf{E}\left[\int_0^1 \left|D^{W^H}_\theta u(r)\right|^{\frac{1}{H}} d\theta\right]^{pH} dr < \infty.$$

The integral process $X := \left\{X(t) := \int_0^t u(s)\,\delta^{W^H}W^H(s),\quad t \in [0,1]\right\}$ has a modification which is γ-Hölder continuous for all $\gamma < H - \frac{1}{p}$.

Case when $H < 1/2$

Proposition 27. *Suppose that $u := \{u(t);\ t \in [0,1]\}$ is λ-Hölder continuous in the norm of the space $\mathbf{D}^{1,p}_{W^H}$ for some $p \geqslant 2$ and $\lambda > \frac{1}{2} - H$. Then u belongs to the space $\mathbf{D}^{1,p}_{W^H}(\mathcal{H}^H)$ and*

$$\mathbf{E}|X(t) - X(s)|^p \leqslant C\,|t-s|^{pH}$$

where $X(t) = \int_0^t u(s)\,\delta^{W^H}W^H(s)$.
If $p > \frac{1}{H}$, then X has a continuous modification.

Remark 24. This result is proved in Theorem 7.1 of [CCM03] or using Proposition 1 of [AMN01] for $\alpha = \frac{1}{2} - H$.

Proof. The proof is based on the Meyer inequalities (31) and uses the identification given in Proposition 19:

$$X(t) = \int_0^1 I^{K^H}_{1,-}(\mathbf{1}_{[0,t]}u)_s\,\delta^B B(s). \qquad \square$$

7.7 Links with the deterministic and symmetric integrals

This work is done in [Dec03], [Dec05], or [CN05].

Links with the deterministic integrals

Following Decreusefond–Üstünel, [DÜ99], we have the following identity provided both sides exist:

$$\lim_{|\pi_n|\to 0} \sum_{t_i \in \pi_n} u(t_i)\left[W^H(t_{i+1}) - W^H(t_i)\right] = \int_0^1 u(s)\,\delta^{W^H}W^H(s) + \int_0^1 D^{W^H}_s u(s)\,ds.$$

Links with symmetric integrals

The following result proved by Alos–Nualart, [AN03], relates the divergence operator with the symmetric stochastic integral introduced by Russo and Vallois in [RV93] (see definition 4).

Proposition 28. *For $H > \frac{1}{2}$, let $u = \{u(t),\ \ t \in [0,1]\}$ be a stochastic process in the space $\mathbf{D}^{1,2}_{W^H}(\mathcal{H}^H)$. Assume that*

$$\mathbf{E}\left[\|u\|^2_{|\mathcal{H}^H|} + \|D^{W^H} u\|^2_{|\mathcal{H}^H|\otimes|\mathcal{H}^H|}\right] < \infty$$

and

$$\int_0^1\int_0^1 \left|D_s^{W^H} u(t)\right| |t-s|^{2H-2}\, ds\, dt < \infty \quad a.s.$$

Then, the symmetric integral $\int_0^1 u(s)\, d^0 W^H(s)$, defined as the limit in probability as ε goes to zero of

$$\frac{1}{2\varepsilon}\int_0^1 u(s)\ \left[W^H\big((s+\varepsilon)\wedge 1\big) - W^H\big((s-\varepsilon)\vee 0\big)\right]\ ds,$$

exists, and one has

$$\int_0^1 u(t)\, d^0 W^H(t) = \delta^{W^H}(u) + \alpha_H \int_0^1\int_0^1 D_s^{W^H}\ u(t)\, |t-s|^{2H-2}\, ds\, dt.$$

7.8 Itô's and Tanaka's formulas

The divergence integral is well suited to identify the terms of an Itô formula as the sum of a divergence integral and a term with finite variation.

A little Itô formula

This little Itô formula is a change of variable formula for fractional Brownian motion itself.

Proposition 29. *If $H \in\ ]0,1[$ and $f \in C_b^2(\mathbb{R},\mathbb{R})$, then $\left\{f'\big(W^H(s)\big)\,,\ s \in [0,1]\right\}$ belongs to $\mathrm{Dom}^*\,\delta^{W^H}$, and almost surely, for all $t \in [0,1]$,*

$$f\left(W^H(t)\right) = f(0) + \int_0^t f'\left(W^H(s)\right)\,\delta^{W^H} W^H(s) + H\int_0^t f''\left(W^H(s)\right)\, s^{2H-1}\, ds.$$

If $H \in\ ]\frac{1}{4},1[$, then $\left\{f'\left(W^H(s)\right),\ \ s \in [0,1]\right\}$ belongs to $\mathrm{Dom}\,\delta^{W^H}$.

Remark 25.

- This formula was first obtained by Decreusefond and Üstünel for $H > \frac{1}{2}$ in [DÜ99] Theorem 5.1 and extended to all H by Privault (application of Corollary 2) in [Pri98]. This version is proved in Nualart–Cheridito, [CN05].

- In general, for $H \in \,]0, \frac{1}{4}]$, the process $\left\{f'\left(W^H(s)\right), \quad s \in [0,1]\right\}$ does not belong to $\operatorname{Dom} \delta^{W^H}$, see Proposition 23.
- A very elegant proof is given by Biagini, Øksendal, Sulem, and Wallner, [BØSW04]. Indeed the process $\left\{f'\left(W^H(s)\right), \quad s \in [0,1]\right\}$ is $\diamond$ integrable with respect to W^H and $\int_0^1 f'\left(W^H(s)\right) \delta^{W^H} W^H(s) = \int_0^1 f'\left(W^H(s)\right) d^{\diamond} W^H(s)$.
- For $H > \frac{1}{6}$, in [CCM03] Proposition 8.11, Carmona, Coutin, and Montseny have identified the term $\int_0^1 f'\left(W^H(s)\right) \delta^{W^H} W^H(s)$ in terms of the divergence integral δ^B. Unfortunately, this expression does not seem to easily generalize to all H.

Proof. This proposition is proved by writing Itô formulas for a sequence of C^1 or semimartingale Gaussian processes which converges to W^H, and identifying the limit of each term involved in the Itô formula for semimartingales. □

Local time, Tanaka and Itô–Tanaka formula

It can be derived from Theorem 8.1 in Berman [Ber70], that the process $\left\{W^H(t),\ t \in [0,1]\right\}$ has a continuous local time.

Proposition 30. *For all $H \in \,]0,1[$, there exists a two-parameter process*

$$\{l^y(t), \quad t \in [0,1], \quad y \in \mathbb{R}\}$$

such that for every bounded Borel function $g: \mathbb{R} \to \mathbb{R}$,

$$\int_0^t g\left(W^H(s)\right) ds = \int_{\mathbb{R}} g(y)\, l^y(t)\, dy. \tag{27}$$

Moreover, this local time has a version which is jointly continuous in (y,t) almost surely, and which satisfies a Hölder condition in t, uniformly in x: for every $\gamma < 1 - H$, there exist two random variables η and η' which are almost surely positive and finite such that

$$\sup_{y \in \mathbb{R}} \left| l^y_{t+h} - l^y_t \right| \leqslant \eta' \, |h|^{\gamma}$$

for all $t,\ t+h$ in $[0,1]$ and all $|h| < \eta$.

Following the Itô formula given in Proposition 29, we introduce the weighted local time.

Definition 14. *The weighted local time is the two-parameter process, jointly continuous in (y,t), $\left\{L^y(t), \quad t \in [0,1], \quad y \in \mathbb{R}\right\}$ where*

$$L^y(t) := 2H\, t^{2H-1}\, l^y(t) - 2H(2H-1) \int_0^t s^{2H-2}\, l^y(s)\, ds.$$

From Proposition 30, for every continuous function $g:\ \mathbb{R}\to\mathbb{R}$,

$$2H\int_0^t g\left(W^H(s)\right)\, s^{2H-1}\,ds=\int_{\mathbb{R}} g(y)\,L^y(t)\,dy.$$

In Proposition 2 of [CNT01], Coutin, Nualart, and Tudor give the Wiener chaos expansion of this weighted local time.

Remark 26. An extension to the two-dimensional case is given by Nualart, Rovira, and Tindel in [NRT03]. They define vortex filaments based on fractional Brownian motion.

Applying Itô's formula to the function f_k given by

$$f_k(x)=\int_{-\infty}^{x}\int_{-\infty}^{v} p_{1/k}(z-v)\,dz\,dv,\quad x\in\mathbb{R}$$

where $p_k(x)=\frac{k}{\sqrt{2\pi}}\,e^{-\frac{k^2x^2}{2}}$, and taking the limit of each term when k goes to infinity yields the Tanaka formula proved in [CNT01] for $H>\frac{1}{3}$.

Theorem 17. *Theorem 10 of Cheridito–Nualart [CN05].*
For $H\in\,]0,1[$, $t\in[0,1]$, *and* $y\in\mathbb{R}$

$$\left\{\mathbf{1}_{]y,\infty[}\left(W^H(t)\right),\quad t\in[0,1]\right\}\in\mathrm{Dom}^*\,\delta^{W^H}$$

and

$$\delta^{W^H}\left(\mathbf{1}_{]y,\infty[}\left(W^H\right)\mathbf{1}_{[0,t]}\right)=\left(W^H(t)-y\right)_+-\left(W_0^H-y\right)_+-\frac{1}{2}\,L^y(t).\qquad(28)$$

Moreover, for $H\in\,]\frac{1}{3},1[$, $t\in[0,1]$, *and* $y\in\mathbb{R}$,

$$\left\{\mathbf{1}_{]y,\infty[}\left(W^H(t)\right),\quad t\in[0,1]\right\}\in\mathrm{Dom}\,\delta^{W^H}.$$

Remark 27. Using (28) for $H>\frac{1}{3}$, one gets

$$\left|W^H(t)\right|=\int_0^t I_{t,-}^{K^H}\left(\operatorname{sgn}W^H\right)(s)\,\delta^B B(s)+L_t^0$$

where

$$\begin{aligned}I_{t,-}^{K^H}\left(\operatorname{sgn}W^H\right)(s)&=K^H(t,s)\,\operatorname{sgn}W^H(s)\\&\quad+\int_s^t\partial_1K^H(u,s)\left(\operatorname{sgn}W^H(u)-\operatorname{sgn}W^H(s)\right)\,du.\end{aligned}$$

Since the process $\left\{\int_0^t\operatorname{sgn}W^H(s)\,\delta^B B(s),\ t\in[0,1]\right\}$ is a Brownian motion, the process $\tilde{W}^H=\left\{\tilde{W}^H(t):=\int_0^t K^H(t,s)\,\operatorname{sgn}W^H(s)\,\delta^B B(s),\ t\in[0,1]\right\}$ is a fractional Brownian motion with Hurst parameter H, and

$$\left|W^H(t)\right| = \tilde{W}^H(t) + \int_0^t\int_s^t \partial_1 K^H(u,s)\left(\operatorname{sgn} W^H(u) - \operatorname{sgn} W^H(s)\right) du\, \delta^B B(s) + L_t^0.$$

When $H = \frac{1}{2}$

$$\int_0^t\int_s^t \partial_1 K^H(u,s)\left(\operatorname{sgn} W^H(u) - \operatorname{sgn} W^H(s)\right) du\, \delta^B B(s) = 0 \qquad (29)$$

and the processes $\{L_t^0, \quad t \in [0,1]\}$ and $\left\{\sup_{s\in[0,t]} W_s^{1/2}, \quad t\in[0,1]\right\}$ have the same law, (see Revuz–Yor, [RY99] Sect. VI. 2 for details). This is not true for fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$ since the term (29) does not vanish. The law of $\left\{\sup_{s\in[0,t]} W_s^H, \quad t\in[0,1]\right\}$ is still an open problem.

Remark 28. Let $W^H = (W^1, \ldots, W^d)$ be a d-dimensional fractional Brownian motion with Hurst parameter in $]0,1[$. The fractional Bessel process is the process R defined by

$$R(t) = \sqrt{\left(W^1(t)\right)^2 + \cdots + \left(W^d(t)\right)^2}, \quad t \in [0,1].$$

When $H = \frac{1}{2}$, the Bessel process is solution of the integral equation

$$R(t)^2 = 2\int_0^t R(s)\, d\beta(s) + d\, t, \quad t\in[0,1],$$

where $\beta_t = \sum_{i=1}^d \int_0^t \frac{W^i(s)}{R(s)}\, dB^i(s)$ for $d \geqslant 2$ is a Brownian motion.

In [HN05], Hu and Nualart prove that for $H \neq \frac{1}{2}$, β is not a fractional Brownian motion and R does not satisfy the following integral equation

$$R(t)^2 = 2\int_0^t R(s)\, d\beta'(s) + d\, t^{2H}, \quad t\in[0,1],$$

where β' is a fractional Brownian motion with Hurst parameter H.

If f is a convex function, denote by f'_- its left-derivative and by f'' the measure given by $f''\left([y,z[\right) = f'_-(z) - f'_-(y)$ for $\infty < y < z < \infty$.

Theorem 18. *Theorem 12 of Cheridito–Nualart, [CN05] and Proposition 7 of Coutin, Nualart, and Tudor, [CNT01].*

For $H \in]0,1[$, $t\in[0,1]$ and $y \in \mathbb{R}$, let f be a convex function such that

(i) $f\left(W^H(t)\right) \in L^2(\Omega, \mathbb{R}, \mathbf{P})$,
(ii) $f'_-(W^H_\cdot)\mathbf{1}_{[0,t]} \in L^2(\Omega\times[0,1], \mathbb{R}, \mathbf{P}\otimes dr)$.

Then

$$\left\{f'_-\big(W^H(t)\big), \quad t \in [0,1]\right\} \in \operatorname{Dom}^* \delta^{W^H}$$

and

$$\delta\big(f'_-(W^H)\mathbf{1}_{[0,t]}\big) = f\big(W^H(t)\big) - f\big(W^H(0)\big) - \frac{1}{2}\int_{\mathbb{R}} L^y(t)\, f''(dy).$$

Moreover, for $H \in \left]\frac{1}{3}, 1\right[$ *and* $t \in [0,1]$,

$$\left\{f'_-\big(W^H(t)\big), \quad t \in [0,1]\right\} \in \operatorname{Dom} \delta^{W^H}.$$

The proof uses a classical regularization of f.

Itô formula

We present a change of variable formula for the process obtained as a divergence integral.

Theorem 19. *Theorem 3 of Alos-Mazet-Nualart [AMN01]*
Let F *be a function of class* $C_b^2(\mathbb{R}, \mathbb{R})$, *and* $u = \{u(t),\ t \in [0,1]\}$ *an adapted process in the space* $\mathbf{D}^{2,2}_{W^H}(\mathcal{H})$, *satisfying the following conditions:*

- *for* $H > \frac{1}{2}$, *the process* u *is bounded in the norm of the space* $\mathbf{D}^{2,4}_{W^H}(\mathcal{H})$,
- *for* $\frac{1}{2} > H > \frac{1}{3}$, *the process* u *and* $D_r^{W^H} u$ *are* λ*-Hölder continuous in the norm of the space* $\mathbf{D}^{1,4}_{W^H}$ *for some* $\lambda > \frac{1}{2} - H$, *and the function*

$$\gamma(r) = \sup_{s \in [0,1]} \|D_r^{W^H} u(s)\|_{W^H,1,4} + \sup_{0 \leqslant s' < s \leqslant 1} \frac{\|D_r^{W^H} u(s) - D_r^{W^H} u(s')\|_{W^H,1,4}}{|s - s'|^{\lambda}}$$

satisfies $\int_0^1 \gamma(r)^p\, dr < \infty$ *for some* $p > \frac{2}{4H-1}$.

Set $X = \left\{X(t) = \delta^{W^H}(u\,\mathbf{1}_{[0,t]}),\ t \in [0,1]\right\}$. *Then for each* $t \in [0,1]$ *the process* $\left\{F'\big(X(s)\big)\, u(s)\, \mathbf{1}_{[0,t]}(s),\ s \in [0,1]\right\}$ *belongs to* $\operatorname{Dom} \delta^{W^H}$ *and the following formula holds*

$$\begin{aligned} F\big(X(t)\big) &= F(0) + \delta^{W^H}\big(F'(X_.)\, u(.)\, \mathbf{1}_{[0,t]}(.)\big) \\ &\quad + \int_0^t F''\big(X(s)\big)\, u(s) \Big[\int_0^s \partial_1 K^H(s,r) \Big(\int_0^s D_r\big(I^{K^H}_{s,-}(u)\big)(\theta)\, \delta^B B_\theta\Big)\, dr\Big] ds \\ &\quad + \frac{1}{2}\int_0^t F''\big(X(s)\big)\, \frac{\partial}{\partial s}\Big[\int_0^s \big(I^{K^H}_{s,-}(u)(r)\big)^2\, dr\Big] ds. \end{aligned}$$

7.9 Conclusion

The divergence integral is the most powerful one for computing expectations of functionals of fractional Brownian motion. Studying differential equations with the divergence integral seems to be more difficult. Nevertheless, some results are available for linear differential equation in [NT06] and [BC05b]. The case of a nonlinear differential equation is still open.

A Divergence operator of a Gaussian process

We briefly recall some elements of stochastic calculus of variations (see Nualart's book [Nua95] for more details). For the sake of simplicity we work on $[0,1]$.

A.1 Divergence operator with respect to a real Gaussian process

Let $W = \{W(t); \quad t \in [0,1]\}$ be a centered Gaussian process starting from 0 with covariance function

$$R(t,s) = \mathbf{E}\big(W(t)W(s)\big).$$

We assume that W is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where the σ-field $\mathcal{F}$ is generated by W. The natural filtration generated by W is denoted by $\big(\mathcal{F}^W(t), \quad t \in [0,1]\big)$.

The first Wiener chaos, $\mathbf{H}_1$, is the closed subspace of $L^2(\Omega, \mathbb{R}, \mathbf{P})$ generated by W. The closure of $\mathcal{E}$ the linear span of the indicator functions with respect to the scalar product $\big\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \big\rangle = R(t,s)$ is the reproducing kernel Hilbert space, $\mathcal{H}$.

The map $\mathbf{1}_{[0,t]} \mapsto W(t)$ extends to an isometry between $\mathcal{H}$ and $\mathbf{H}_1$. The image of an element $\phi \in \mathcal{H}$ is denoted by $W(\phi)$.

Remark 29. It is but the classical definition of the reproducing kernel given for instance in [Fer97] up to the isomorphism induced by

$$\mathbf{1}_{[0,t]} \mapsto R(t,.).$$

Let $\mathcal{S}$ denote the set of smooth cylindrical random variables of the form

$$F = f\big(W(\phi_1), \ldots, W(\phi_n)\big) \tag{30}$$

where $n \geqslant 1, \quad f \in C_p^\infty(\mathbb{R}^n, \mathbb{R})$ (f and all its derivative have at most polynomial growth), $\phi_i \in \mathcal{H}, \quad i = 1, \ldots, n$. The derivative of a smooth cylindrical random variable F is the $\mathcal{H}$ valued random variable given by

$$D^W F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\big(W(\phi_1), \ldots, W(\phi_n)\big)\, \phi_j.$$

The derivative D^W is a closable unbounded operator from $L^p(\Omega, \mathbb{R}, \mathbf{P})$ to $L^p(\Omega, \mathcal{H}, \mathbf{P})$ for each $p \geqslant 1$. Similarly, the iterated derivative $D^{W,k}$ maps $L^p(\Omega, \mathbb{R}, \mathbf{P})$ to $L^p(\Omega, \mathcal{H}^{\otimes k}, \mathbf{P})$. For any positive integer k and any real $p \geqslant 1$, we denote by $\mathbf{D}_W^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|_{W,k,p}^p = \|F\|_{L^p(\Omega,\mathbb{R},\mathbf{P})}^p + \sum_{j=1}^k \|D^{W,j} F\|_{L^p(\Omega,\mathcal{H}^{\otimes > j})}^p,$$

where $\| \quad \|_{L^p(\Omega,\mathbb{R},\mathbf{P})}$ denotes the norm in $L^p(\Omega, \mathbb{R}, \mathbf{P})$.

The adjoint of the derivative D^W is denoted by δ^W. The domain of δ^W (denoted by $\mathrm{Dom}\,\delta^W$) is the set of all elements $u \in L^2(\Omega, \mathcal{H}, \mathbf{P})$ such that there exists a constant c satisfying

$$\left|\mathbf{E}\left\langle D^W F, u\right\rangle_{\mathcal{H}}\right| \leqslant c\|F\|_{L^2(\Omega,\mathbb{R},\mathbf{P})}$$

for all $F \in \mathcal{S}$. If $u \in \mathrm{Dom}\,\delta^W$, $\delta^W(u)$ is the element in $L^2(\Omega, \mathbb{R}, \mathbf{P})$ defined by the duality relationship

$$\mathbf{E}\left[\delta^W(u)F\right] = \mathbf{E}\langle D^W F, u\rangle_{\mathcal{H}}, \quad F \in \mathbf{D}_W^{1,2}.$$

Furthermore, Meyer's inequalities imply that for all $p > 1$, one has

$$\|\delta^W(u)\|_{L^p(\Omega,\mathbb{R},\mathbf{P})} \leqslant c_p\, \|u\|_{\mathbf{D}_W^{1,p}(\mathcal{H})}, \tag{31}$$

where

$$\|u\|_{\mathbf{D}_W^{1,p}(\mathcal{H})} = \|u\|^p_{L^p(\Omega,\mathbf{H},\mathbf{P})} + \|D^W u\|^p_{L^p(\Omega,\mathcal{H}^{\otimes 2})}.$$

If u is a simple $\mathcal{H}$ valued random variable of the form $u = \sum_{j=1}^n F_j \phi_j$ for some $n \geqslant 1$, $F_j \in \mathbf{D}_W^{1,2}$ and $\phi_j \in \mathcal{H}$, $j = 1, \ldots, n$, then u belongs to the domain of δ^W and

$$\delta^W(u) = \sum_{j=1}^n F_j W(\phi_j) - \left\langle D^W F_j, \phi_j\right\rangle_{\mathcal{H}}.$$

Property P (Integration by parts formula). Suppose that $u \in {\mathbf{D}_W}^{1,2}(\mathcal{H})$. Let F be a random variable belonging to $\mathbf{D}_W^{1,2}$ such that $\mathbf{E}[F^2\|u\|^2_{\mathcal{H}}] < \infty$; then

$$\delta^W(F\,u) = F\,\delta^W(u) - \langle D^W F, u\rangle_{\mathcal{H}} \tag{32}$$

in the sense that $F\,u$ belongs to $\mathrm{Dom}\,\delta^W$ if and only if the right-hand side of (32) belongs to $L^2(\Omega, \mathbb{R}, \mathbf{P})$.

Remark 30. Observe that when W is a Brownian motion, the set $\{u \in L^2(\Omega \times [0,1], \mathbb{R});\ u$ is $\mathcal{F}_t$ progressively measurable$\}$ is included in $\mathrm{Dom}\,\delta^W$, and for such a process u, $\delta^W(u)$ coincides with the usual Itô integral.

Moreover $\mathbf{D}_W^{1,2}(\mathcal{H}) = \mathbf{D}_W^{1,2}\left(L^2([0,1], \mathbb{R}, du)\right)$.

Remark 31. In the general case, the divergence operator δ^W can also be interpreted as a generalized stochastic integral. In fact, for all $\phi \in \mathcal{H}$, $W(\phi) = \delta^W(\phi)$, and in particular for $n \in \mathbb{N}^*$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$,

$$\delta^W\left(\sum_{i=1}^n a_i\, \mathbf{1}_{[t_{i-1},t_i[}\right) = \sum_{i=1}^n a_i\,(W_{t_i} - W_{t_{i-1}}).$$

A.2 Extension to the multidimensional case

This construction extends to the d-dimensional case. Let $\bar{W} = (W^1, \dots, W^d)$ be a centered Gaussian process, with d independent components. The covariance function of W^i is denoted by R^i and its reproducing kernel Hilbert space by $\mathcal{H}^i$.

Introduce $\mathcal{H} = \prod_{i=1}^d \mathcal{H}^i$ which is a Hilbert space for the scalar product

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{i=1}^d \langle f^i, g^i\rangle_{\mathcal{H}^i}$$

if $f = (f^1, \dots, f^d)$ and $g = (g^1, \dots, g^d)$. The smooth and cylindrical random variables are now of the form

$$F = f\left(W^1(\phi_1^1), \dots, W^d(\phi_1^d), \dots, W^1(\phi_n^1), \dots, W^d(\phi_n^d)\right) \tag{33}$$

where $n \geqslant 1$, $f \in C_p^\infty(\mathbb{R}^{dn}, \mathbb{R})$ (f and all its derivative have at most polynomial growth), $\phi_i^j \in \mathcal{H}^j, i = 1, \dots, n,\ j = 1, \dots, d$. The derivative of a smooth cylindrical random variable F is the $\mathcal{H}$ valued random variable given by $D^{\bar{W}}F = \left(D^{W^1}F, \dots, D^{W^d}F\right)$ where

$$D^{W^i}F = \sum_{j=1}^n \frac{\partial f}{\partial x_{i+d(j-1)}}\left(W^1\left(\phi_1^1\right), \dots, W^d(\phi_1^d), \dots, W^1(\phi_n^1), \dots,\right.$$
$$\left. W^d(\phi_n^d)\right) \phi_j^i.$$

The derivative $D^{\bar{W}}$ is a closable unbounded operator from $L^p(\Omega, \mathbb{R}, \mathbf{P})$ to $L^p(\Omega, \mathcal{H}, \mathbf{P})$ for any $p \geqslant 1$. Similary, the iterated derivative $D^{\bar{W},k}$ maps $L^p(\Omega, \mathbb{R}, \mathbf{P})$ to $L^p(\Omega, \mathcal{H}^{\otimes k}, \mathbf{P})$. For any positive integer k and any real $p \geqslant 1$, call $\mathbf{D}_{\bar{W}}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|F\|_{\bar{W},k,p}^p = \|F\|_{L^p(\Omega,\mathbb{R},\mathbf{P})}^p + \sum_{j=1}^k \left\|D^{\bar{W},j}F\right\|_{L^p(\Omega,\mathcal{H}^{\otimes j},\mathbf{P})}^p.$$

Denote by $\delta^{\bar{W}}$ the adjoint of the derivative $D^{\bar{W}}$. The domain of $\delta^{\bar{W}}$ (denoted by $\operatorname{Dom}\delta^{\bar{W}}$) is the set of all $u \in L^2(\Omega, \mathcal{H}, \mathbf{P})$ such that there exists a constant c satisfying

$$\left|\mathbf{E}\left\langle D^{\bar{W}}F, u\right\rangle_{\mathcal{H}}\right| \leqslant c\,\|F\|_{L^2(\Omega,\mathbb{R},\mathbf{P})}$$

for all $F \in \mathcal{S}$. If $u \in \operatorname{Dom}\delta^{\bar{W}}$, $\delta^{\bar{W}}(u)$ is the element in $L^2(\Omega, \mathbb{R}, \mathbf{P})$ defined by the duality relationship

$$\mathbf{E}\left(\delta^{\bar{W}}(u)F\right) = \mathbf{E}\left\langle D^{\bar{W}}F, u\right\rangle_{\mathcal{H}}, \qquad F \in \mathbf{D}_{\bar{W}}^{1,2}. \tag{34}$$

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References

[ALV99] A. Ayache and J. Lévy Véhel. Generalized multifractional Brownian motion: definition and preliminary results. In *Fractals: theory and applications in engineering*, pages 17–32. Springer, Berlin Heidelberg New York, 1999

[AMN01] E. Alòs, O. Mazet, and D. Nualart. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, 29(2):766–801, 2001

[AN03] E. Alòs and D. Nualart. Stochastic integration with respect to the fractional Brownian motion. *Stoch. Stoch. Rep.*, 75(3):129–152, 2003

[BC05a] F. Baudoin and L. Coutin. Bridge over a Volterra process. Preprint, 2005

[BC05b] F. Baudoin and L. Coutin. Operators associated with a stochastic differential equation driven by fractional Brownian motions. Preprint, 2005

[Ben96] A. Benassi. Régularité des processus gaussiens elliptiques et le modèle hiérarchique. *Ann. Math. Blaise Pascal*, 3(1):33–49, 1996

[Ber70] S. M. Berman. Gaussian processes with stationary increments: Local times and sample function properties. *Ann. Math. Statist.*, 41:1260–1272, 1970

[Ber89] J. Bertoin. Sur une intégrale pour les processus à α-variation bornée. *Ann. Probab.*, 17(4):1521–1535, 1989

[BJR97] A. Benassi, S. Jaffard, and D. Roux. Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana*, 13(1):19–90, 1997

[BNS01] O.E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 63(2):167–241, 2001

[BØSW04] F. Biagini, B. Øksendal, A. Sulem, and N. Wallner. An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion. Stochastic analysis with applications to mathematical finance. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2041):347–372, 2004

[BP98] R. J. Barton and V. Poor. Signal detection in fractional Gaussian noise. *IEEE Trans. on Inf. Theor.*, 34(5):943–959, 1998

[BR03] A. Benassi and D. Roux. Elliptic self-similar stochastic processes. *Rev. Mat. Iberoamericana*, 19(3):767–796, 2003

[BT99] J. Beran and N. Terrin. Errata: "Testing for a change of the long-memory parameter" [Biometrika **83** (1996), no. 3, 627–638; MR 98a:62111]. *Biometrika*, 86(1):233, 1999

[CCM00] P. Carmona, L. Coutin, and G. Montseny. Approximation of some Gaussian processes. *Stat. Inference Stoch. Process.*, 3(1–2):161–171, 2000. 19th "Rencontres Franco-Belges de Statisticiens" (Marseille, 1998)

[CCM03] P. Carmona, L. Coutin, and G. Montseny. Stochastic integration with respect to fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(1):27–68, 2003

[CFV05] L. Coutin, P. Friz, and N. Victoir. Good rough path sequences and applications to anticipating and fractional calculus. Preprint. 2005

[Che01] P. Cheridito. Mixed fractional Brownian motion. *Bernoulli*, 7(6):913–934, 2001

[Che03] P. Cheridito. Arbitrage in fractional Brownian motion models. *Finance Stoch.*, 7(4):533–553, 2003

[CKR93] Z. Ciesielski, G. Kerkyacharian, and B. Roynette. Quelques espaces fonctionnels associés à des processus gaussiens. *Studia Math.*, 107(2):171–204, 1993

[CL05] L. Coutin and A. Lejay. Semimartingales and rough paths theory. *Electron. J. Probab.*, 10:no. 23, 761–785 (electronic), 2005

[CN05] P. Cheridito and D. Nualart. Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in \left(0, \frac{1}{2}\right)$. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(6):1049–1081, 2005

[CNT01] L. Coutin, D. Nualart, and C.A. Tudor. Tanaka formula for the fractional Brownian motion. *Stochastic Process. Appl.*, 94(2):301–315, 2001

[CQ02] L. Coutin and Z. Qian. Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theor. Related Fields*, 122(1):108–140, 2002

[CR96] F. Comte and E. Renault. Long memory continuous time models. *J. Econometrics*, 73(1):101–149, 1996

[Dec03] L. Decreusefond. Stochastic integration with respect to fractional Brownian motion. In *Theory and applications of long-range dependence*, pages 203–226. Birkhäuser Boston, Boston, MA, 2003

[Dec05] L. Decreusefond. Stochastic integration with respect to Volterra processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(2):123–149, 2005

[Dos77] H. Doss. Liens entre équations différentielles stochastiques et ordinaires. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 13(2):99–125, 1977

[DOT03] P. Doukhan, G. Oppenheim, and M.S. Taqqu, editors. *Theory and applications of long-range dependence*. Birkhäuser Boston, Boston, MA, 2003

[DÜ99] L. Decreusefond and A. S. Üstünel. Stochastic analysis of the fractional Brownian motion. *Potential Anal.*, 10(2):177–214, 1999

[Dun04] T. E. Duncan. Some processes associated with a fractional Brownian motion. In *Mathematics of finance*, volume 351 of *Contemp. Math.*, pages 93–101. Amer. Math. Soc., Providence, RI, 2004

[DYY01] T.E. Duncan, Y. Yan, and P. Yan. Exact asymptotics for a queue with fractional Brownian input and applications in ATM networks. *J. Appl. Probab.*, 38(4):932–945, 2001

[Enr04] N. Enriquez. A simple construction of the fractional Brownian motion. *Stoch. Process. Appl.*, 109(2):203–223, 2004

[ER03] M. Errami and F. Russo. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stoch. Process. Appl.*, 104(2):259–299, 2003

[Fal03] K. Falconer. *Fractal geometry.* Wiley, Hoboken, NJ, second edition, 2003. Mathematical foundations and applications

[Fer97] X. Fernique. *Fonctions aléatoires gaussiennes, vecteurs aléatoires gaussiens.* Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1997

[Föl81] H. Föllmer. Calcul d'Itô sans probabilités. In *Séminaire de Probabilités, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, pages 143–150. Springer, Berlin Heidelberg New York, 1981

[GNRV05] M. Gradinaru, I. Nourdin, F. Russo, and P. Vallois. m-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(4):781–806, 2005

[GRRJ71] A. M. Garsia, E. Rodemich, and H. Rumsey Jr. A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana Univ. Math. J.*, 20:565–578, 1970/1971

[GRV03] M. Gradinaru, F. Russo, and P. Vallois. Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geqslant \frac{1}{4}$. *Ann. Probab.*, 31(4):1772–1820, 2003

[GT82] B. Gaveau and P. Trauber. L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel. *J. Funct. Anal.*, 46(2):230–238, 1982

[Hai05] M. Hairer. Ergodicity of stochastic differential equations driven by fractional Brownian motion. *Ann. Probab.*, 33(2):703–758, 2005

[HN05] Y. Hu and D. Nualart. Some processes associated with fractional Bessel processes. *J. Theoret. Probab.*, 18(2):377–397, 2005

[HN06] Y. Hu and D. Nualart. Differential equations driven by Hölder continuous functions of order greater than 1/2. *preprint*, 2006

[Hur51] H.E. Hurst. Long term storage capacity of reservoir (with discussion). *Trans. Am. Soc. Civ. Eng.*, 770–808, 1951

[IT99] E. Iglói and G. Terdik. Long-range dependence through Gamma-mixed Ornstein-Uhlenbeck process. *Electron. J. Probab.*, 4, 1999 Paper no 16.

[KKA98] M.L. Kleptsyna, P.E. Kloeden, and V.V. Anh. Existence and uniqueness theorems for fBm stochastic differential equations. *Probl. Inf. Transm.*, 34(4):332–341, 1998. Translation from Probl. Peredachi. Inf. 34, No.4, 51–61 (1998)

[Kol40] A. Kolmogorov. Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *S.R. (Doklady)*, 26(1):115–118, 1940

[Lac04] C. Lacaux. Real harmonizable multifractional Lévy motions. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(3):259–277, 2004

[Leb65] N. N. Lebedev. *Special functions and their applications.* Revised English edition. Translated and edited by Richard A. Silverman. Prentice-Hall, Englewood Cliffs, NJ, 1965

[Lej03] A. Lejay. An introduction to rough paths. In *Séminaire de Probabilités XXXVII*, volume 1832 of *Lecture Notes in Math.*, pages 1–59. Springer, Berlin Heidelberg New York, 2003

[Leó93] J.A. León. Fubini theorem for anticipating stochastic integrals in Hilbert space. *Appl. Math. Optim.*, 27(3):313–327, 1993

[LLQ02] M. Ledoux, T. Lyons, and Z. Qian. Lévy area of Wiener processes in Banach spaces. *Ann. Probab.*, 30(2):546–578, 2002

[LQ02] T. Lyons and Z. Qian. *System control and rough paths.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2002. Oxford Science Publications

[LS84] R.S. Liptser and A.N. Shiryaev. *Theory of martingales.* Springer, Berlin Heidelberg New York, 1984

[Lyo94] T.J. Lyons. Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young. *Math. Res. Lett.*, 1(4):451–464, 1994

[Lyo98] T.J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998

[LZN03] N. Lanjri Zadi and D. Nualart. Smoothness of the law of the supremum of the fractional Brownian motion. *Electron. Comm. Probab.*, 8:102–111 (electronic), 2003

[Mar05] R. Marty. Asymptotic behavior of differential equations driven by periodic and random processes with slowly decaying correlations. *ESAIM Probab. Stat.*, 9:165–184 (electronic), 2005

[MMV01] J. Mémin, Y. Mishura, and E. Valkeila. Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. *Statist. Probab. Lett.*, 51(2):197–206, 2001

[MSS06] A. Millet and M. Sanz-Solé. Large deviations for rough paths of the fractional Brownian motion. *Annales de l'Institut Henri Poincaré (Probabilités et Statistiques*, 2006

[MVN68] B.B. Mandelbrot and J.W. Van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 10:422–437, 1968

[Nev68] J. Neveu. *Processus aléatoires gaussiens*. Séminaire de Mathématiques Supérieures, No. 34 (Été, 1968). Les Presses de l'Université de Montréal, Montreal, Que., 1968

[Nou05] I. Nourdin. Schémas d'approximation associés à une équation différentielle dirigée par une fonction höldérienne; cas du mouvement brownien fractionnaire. *C. R. Math. Acad. Sci. Paris*, 340(8):611–614, 2005

[NR02] D. Nualart and A. Răşcanu. Differential equations driven by fractional Brownian motion. *Collect. Math.*, 53(1):55–81, 2002

[NRT03] D. Nualart, C. Rovira, and S. Tindel. Probabilistic models for vortex filaments based on fractional Brownian motion. *Ann. Probab.*, 31(4):1862–1899, 2003

[NS05] D. Nualart and B. Saussereau. Malliavin calculus for stochastic differential equation driven by a fractional brownian motion. *preprint*, 2005

[NS06] I. Nourdin and T. Simon. On the absolute continuity of one-dimensional sdes driven by a fractional Brownian motion. *Stat. and Probab. Lett.*, 76(9):907–912, 2006

[NT06] I. Nourdin and C. Tudor. Some linear fractional stochastic equations. *Stochastics*, 2006

[Nua95] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer, Berlin Heidelberg New York, 1995

[NVV99] I. Norros, E. Valkeila, and J. Virtamo. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, 5(4):571–587, 1999

[Pri98] N. Privault. Skorohod stochastic integration with respect to nonadapted processes on Wiener space. *Stochastics Stochastics Rep.*, 65(1-2):13–39, 1998

[Pro04] P.E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer, Berlin

Heidelberg New York, second edition, 2004. Stochastic Modelling and Applied Probability

[PT00] V. Pipiras and M.S. Taqqu. Integration questions related to fractional Brownian motion. *Probab. Theor. Related Fields*, 118(2):251–291, 2000

[PT01] V. Pipiras and M.S. Taqqu. Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli*, 7(6):873–897, 2001

[Rog97] L.C.G. Rogers. Arbitrage with fractional Brownian motion. *Math. Finance*, 7(1):95–105, 1997

[Ruz00] A.A. Ruzmaikina. Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion. *J. Statist. Phys.*, 100(5-6):1049–1069, 2000

[RV85] H. Rost and M.E. Vares. Hydrodynamics of a one-dimensional nearest neighbor model. In *Particle systems, random media and large deviations (Brunswick, Maine, 1984)*, volume 41 of *Contemp. Math.*, pages 329–342. Amer. Math. Soc., Providence, RI, 1985

[RV93] F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theor. Related Fields*, 97(3):403–421, 1993

[RV00] F. Russo and P. Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1–2):1–40, 2000

[RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin Heidelberg New York, third edition, 1999

[SKM93] S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications, Edited and with a foreword by S.M. Nikolskii, Translated from the 1987 Russian original, Revised by the authors

[ST94] G. Samorodnitsky and M.S. Taqqu. *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman & Hall, New York, 1994. Stochastic models with infinite variance

[Tal96] D. Talay. Probabilistic numerical methods for partial differential equations: elements of analysis. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, volume 1627 of *Lecture Notes in Math.*, pages 148–196. Springer, Berlin Heidelberg New York, 1996

[Wat84] S. Watanabe. *Lectures on stochastic differential equations and Malliavin calculus*, volume 73 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay, 1984. Notes by M. Gopalan Nair and B. Rajeev

[WLW94] W. Willinger, W. Leland, M. Taqqu and D. Wilson. On self-similar nature of ethernet traffic. *IEEE/ACM Trans. Networking*, 2:1–15, 1994

[Yam79] Y. Yamato. Stochastic differential equations and nilpotent Lie algebras. *Z. Wahrsch. Verw. Gebiete*, 47(2):213–229, 1979

[You36] L.C. Young. An inequality of Hölder type, connected with Stieltjes integration. *Acta. Math.*, 67:251–282, 1936

[Zäh98] M. Zähle. Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theor. Related Fields*, 111(3):333–374, 1998

Part II

Local Time-Space Calculus

A Change-of-Variable Formula with Local Time on Surfaces

Goran Peskir*

Department of Mathematical Sciences, University of Aarhus
Ny Munkegade, 8000 Aarhus, Denmark
e-mail: goran@imf.au.dk

Summary. Let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale and let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale. Setting $C = \{\, (x_1, \ldots, x_n) \in I\!R^n \mid x_n < b(x_1, \ldots, x_{n-1}) \,\}$ and $D = \{ (x_1, \ldots, x_n) \in I\!R^n \mid x_n > b(x_1, \ldots, x_{n-1}) \}$ suppose that a continuous function $F : I\!R^n \to I\!R$ is given such that F is $C^{i_1, \ldots, i_n}$ on $\bar{C}$ and F is $C^{i_1, \ldots, i_n}$ on $\bar{D}$ where each i_k equals 1 or 2 depending on whether X^k is of bounded variation or not. Then the following change-of-variable formula holds:

$$\begin{aligned} F(X_t) = F(X_0) &+ \sum_{i=1}^{n} \int_0^t \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}\big(X_s^1, \ldots, X_s^n +\big) + \frac{\partial F}{\partial x_i}\big(X_s^1, \ldots, X_s^n -\big) \Big)\, dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{1}{2}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}\big(X_s^1, \ldots, X_s^n +\big) + \frac{\partial^2 F}{\partial x_i \partial x_j}\big(X_s^1, \ldots, X_s^n -\big) \Big)\, d\langle X^i, X^j \rangle_s \\ &+ \frac{1}{2} \int_0^t \Big(\frac{\partial F}{\partial x_n}\big(X_s^1, \ldots, X_s^n +\big) - \frac{\partial F}{\partial x_n}\big(X_s^1, \ldots, X_s^n -\big) \Big)\, I\big(X_s^n = b_s^X\big)\, d\ell_s^b(X) \end{aligned}$$

where $\ell_s^b(X)$ is the local time of X on the surface b given by:

$$\ell_s^b(X) = I\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(-\varepsilon < X_r^n - b_r^X < \varepsilon)\; d\langle X^n - b^X, X^n - b^X \rangle_r$$

and $d\ell_s^b(X)$ refers to integration with respect to $s \mapsto \ell_s^b(X)$. The analogous formula extends to general semimartingales X and b^X as well. A version of the same formula under weaker conditions on F is derived for the semimartingale $((t, X_t, S_t))_{t \geqslant 0}$ where $(X_t)_{t \geqslant 0}$ is an Itô diffusion and $(S_t)_{t \geqslant 0}$ is its running maximum.

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Key words: Local time-space calculus, Itô's formula, Tanaka's formula, Local time, Curve, Surface, Brownian motion, Diffusion, Semimartingale, Weak convergence, Signed measure, Free-boundary problems, Optimal stopping

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1 Introduction

Let $(X_t)_{t \geqslant 0}$ be a continuous semimartingale (see, e.g., [13]) and let $b : I\!R_+ \to I\!R$ be a continuous function of bounded variation. Setting $C = \{\, (t,x) \in I\!R_+ \times I\!R \mid x < b(t) \,\}$ and $D = \{\, (t,x) \in I\!R_+ \times I\!R \mid x > b(t) \,\}$ suppose that a continuous function $F : I\!R_+ \times I\!R \to I\!R$ is given such that F is $C^{1,2}$ on $\bar{C}$ and F is $C^{1,2}$ on $\bar{D}$.

Then the following change-of-variable formula is known to be valid (cf. [11]):

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{1}{2}\Big(F_t(s,X_s+)+F_t(s,X_s-)\Big)\, ds \\ &+ \int_0^t \frac{1}{2}\Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\, dX_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\; I(X_s \neq b(s))\; d\langle X,X\rangle_s \\ &+ \frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big)\, I(X_s = b(s))\; d\ell_s^b(X) \end{aligned} \tag{1.1}$$

where $\ell_s^b(X)$ is the local time of X on the curve b given by:

$$\ell_s^b(X) = I\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(b(r)-\varepsilon < X_r < b(r)+\varepsilon)\; d\langle X,X\rangle_r \tag{1.2}$$

and $d\ell_s^b(X)$ refers to integration with respect to the continuous increasing function $s \mapsto \ell_s^b(X)$. A version of the same formula for an Itô diffusion X derived under weaker conditions on F has found applications in free-boundary problems of optimal stopping (cf. [11]).

The main aim of the present paper is to extend the change-of-variable formula (1.1) to a multidimensional setting of continuous functions F which are smooth above and below surfaces. Continuous semimartingales are considered in Section 2, and semimartingales with jumps are considered in Section 3. A version of the same formula under weaker conditions on F is derived in Section 4 for the continuous semimartingale $((t,X_t,S_t))_{t \geqslant 0}$ where $(X_t)_{t \geqslant 0}$ is an Itô diffusion and $(S_t)_{t \geqslant 0}$ is its running maximum. This version is useful in the study of free-boundary problems for optimal stopping of the maximum process when the horizon is finite (for the infinite horizon case see [10] with references).

The study of Section 4 serves as an example of what generally needs to be done in order to relax the smoothness conditions on F from $\bar{C}$ and $\bar{D}$ to $C \cup D$. These relaxed versions of the formula are important for applications. It is thus hoped that the programme started in Section 3 of [11] and in Section 4 of the present paper will be continued.

For related results on the local time-space calculus see [1], [5], [3], [2], [8]. Older references on the topic include [7], [14], [9], [15], [4].

2 Continuous semimartingales

Let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale and let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale. [Note that the sufficient condition $b \in C^2$ is by no means necessary.] Setting:

$$C = \{ \, (x_1, \ldots, x_n) \in I\!R^n \mid x_n < b(x_1, \ldots, x_{n-1}) \, \} \tag{2.1}$$

$$D = \{ \, (x_1, \ldots, x_n) \in I\!R^n \mid x_n > b(x_1, \ldots, x_{n-1}) \, \} \tag{2.2}$$

suppose that a continuous function $F : I\!R^n \to I\!R$ is given such that:

$$F \;\text{ is }\; C^{i_1, \ldots, i_n} \;\text{ on }\; \bar{C} \tag{2.3}$$

$$F \;\text{ is }\; C^{i_1, \ldots, i_n} \;\text{ on }\; \bar{D} \tag{2.4}$$

where each i_j equals 1 or 2 depending on whether X^j is of bounded variation or not. More explicitly, it means that F restricted to C coincides with a function F_1 which is $C^{i_1, \ldots, i_n}$ on $I\!R^n$, and F restricted to D coincides with a function F_2 which is $C^{i_1, \ldots, i_n}$ on $I\!R^n$. [We recall that a continuous function $F_k : I\!R^n \to I\!R$ is $C^{i_1, \ldots, i_n}$ on $I\!R^n$ if the partial derivatives $\partial F_k / \partial x_j$ when $i_j = 1$ as well as $\partial^2 F_k / \partial x_j^2$ when $i_j = 2$ exist and are continuous as functions from $I\!R^n$ to $I\!R$ for all $1 \leqslant j \leqslant n$ where k equals 1 or 2.]

Then the natural desire arising in free-boundary problems of optimal stopping (and other problems where the hitting time of D by the process X plays a role) is to apply a change-of-variable formula to $F(X_t)$ so to account for possible jumps of $(\partial F / \partial x_n)(x_1, \ldots, x_n)$ at $x_n = b(x_1, \ldots, x_{n-1})$ being measured by:

$$\ell_s^b(X) = I\!P\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I\big(-\varepsilon < X_r^n - b_r^X < \varepsilon\big) \; d\big\langle X^n - b^X, X^n - b^X \big\rangle_r \tag{2.5}$$

which represents the local time of X on the surface b for $s \in [0, t]$. Note that the limit in (2.5) exists (as a limit in probability) since $X^n - b^X$ is a continuous semimartingale.

In the special case when the semimartingale equals (t, X_t) it is evident that the previous setting reduces to the setting leading to the change-of-variable formula (1.1) above. Further particular cases of the formula (1.1) are reviewed in [11]. The following theorem provides a general formula of this kind for continuous semimartingales (see also Section 3 below).

Theorem 2.1. *Let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale, let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale, and let $F : I\!R^n \to I\!R$ be a continuous function satisfying (2.3) and (2.4) above.*

Then the following change-of-variable formula holds:

$$
\begin{aligned}
F(X_t) &= F(X_0) \\
&\quad + \sum_{i=1}^{n} \int_0^t \frac{1}{2}\bigg(\frac{\partial F}{\partial x_i}\,(X_s^1,\dots,X_s^n+) + \frac{\partial F}{\partial x_i}\,(X_s^1,\dots,X_s^n-)\bigg)\, dX_s^i \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n} \int_0^t \frac{1}{2}\bigg(\frac{\partial^2 F}{\partial x_i \partial x_j}\,(X_s^1,\dots,X_s^n+) + \frac{\partial^2 F}{\partial x_i \partial x_j}\,(X_s^1,\dots,X_s^n-)\bigg) \\
&\qquad\qquad \times d\langle X^i, X^j\rangle_s \\
&\quad + \frac{1}{2}\int_0^t \bigg(\frac{\partial F}{\partial x_n}\,(X_s^1,\dots,X_s^n+) - \frac{\partial F}{\partial x_n}\,(X_s^1,\dots,X_s^n-)\bigg) \\
&\qquad\qquad \times I(X_s^n = b_s^X)\; d\ell_s^b(X) \qquad (2.6)
\end{aligned}
$$

where $\ell_s^b(X)$ is the local time of X on the surface b given in (2.5) above, and $d\ell_s^b(X)$ refers to integration with respect to the continuous increasing function $s \mapsto \ell_s^b(X)$.

Proof. 1. Set $Z_t^1 = X_t^n \wedge b_t^X$ and $Z_t^2 = X_t^n \vee b_t^X$ for $t > 0$ given and fixed. Denoting $\hat{X}_t = (X_t^1,\dots,X_t^{n-1}, Z_t^1)$, $\check{X}_t = (X_t^1,\dots,X_t^{n-1}, Z_t^2)$ and $\tilde{X}_t = (X_t^1,\dots,X_t^{n-1}, b_t^X)$, we see that the following identity holds:

$$
F(X_t) = F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t) \qquad (2.7)
$$

where we use that $F(x) = F_1(x) = F_2(x)$ for $x = (x_1,\dots,x_{n-1}, b(x_1,\dots,x_{n-1}))$. The processes $(Z_t^1)_{t\geqslant 0}$ and $(Z_t^2)_{t\geqslant 0}$ are continuous semimartingales admitting the following representations:

$$
Z_t^1 = \frac{1}{2}\,\big(X_t^n + b_t^X - \big|X_t^n - b_t^X\big|\big) \qquad (2.8)
$$

$$
Z_t^2 = \frac{1}{2}\,\big(X_t^n + b_t^X + \big|X_t^n - b_t^X\big|\big)\,. \qquad (2.9)
$$

Recalling the Tanaka formula:

$$
\big|X_t^n - b_t^X\big| = \big|X_0^n - b_0^X\big| + \int_0^t \operatorname{sign}\big(X_s^n - b_s^X\big)\; d\big(X_s^n - b_s^X\big) + \ell_t^b(X) \qquad (2.10)
$$

where $\operatorname{sign}(0) = 0$, we find that:

$$
\begin{aligned}
dZ_t^1 &= \frac{1}{2}\Big(d\big(X_t^n + b_t^X\big) - \operatorname{sign}\big(X_t^n - b_t^X\big)\; d\big(X_t^n - b_t^X\big) - d\ell_t^b(X)\Big) \\
&= \frac{1}{2}\Big(\big(1 - \operatorname{sign}\big(X_t^n - b_t^X\big)\big)\, dX_t^n + \big(1 + \operatorname{sign}\big(X_t^n - b_t^X\big)\big)\, db_t^X - d\ell_t^b(X)\Big)
\end{aligned}
\qquad (2.11)
$$

$$dZ_t^2 = \frac{1}{2}\Big(d\big(X_t^n + b_t^X\big) + \operatorname{sign}\big(X_t^n - b_t^X\big)\ d\big(X_t^n - b_t^X\big) + d\ell_t^b(X)\Big)$$
$$= \frac{1}{2}\Big(\big(1 + \operatorname{sign}(X_t^n - b_t^X)\big)\ dX_t^n + \big(1 - \operatorname{sign}(X_t^n - b_t^X)\big)\ db_t^X + d\ell_t^b(X)\Big). \tag{2.12}$$

In the sequel we set $D_i = \partial/\partial x_i$ and $D_{ij} = \partial^2/\partial x_i \partial x_j$ as well as $D_i^2 = \partial^2/\partial x_i^2$.

2. Applying the Itô formula to $F_1(\hat{X}_t)$ and using (2.11) we get:

$$\begin{aligned}
F_1(\hat{X}_t) &= F_1(\hat{X}_0) + \sum_{i=1}^{n} \int_0^t D_i F_1(\hat{X}_s)\, d\hat{X}_s^i \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\hat{X}_s)\, d\langle \hat{X}^i, \hat{X}^j \rangle_s \\
&= F_1(\hat{X}_0) + \sum_{i=1}^{n-1} \int_0^t D_i F_1(\hat{X}_s)\, dX_s^i \\
&\quad + \frac{1}{2} \int_0^t \Big(1 - \operatorname{sign}\big(X_s^n - b_s^X\big)\Big) D_n F_1(\hat{X}_s)\, dX_s^n \\
&\quad + \frac{1}{2} \int_0^t \Big(1 + \operatorname{sign}\big(X_s^n - b_s^X\big)\Big) D_n F_1(\hat{X}_s)\, db_s^X \\
&\quad - \frac{1}{2} \int_0^t D_n F_1(\hat{X}_s)\, d\ell_s^b(X) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I\big(X_s^n < b_s^X\big)\ d\langle X^i, X^j \rangle_s \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I\big(X_s^n = b_s^X\big)\ d\langle X^i, X^j \rangle_s \\
&\quad + \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s
\end{aligned} \tag{2.13}$$

where in the last four integrals we make use of the general fact:

$$I\big(Y_s^1 = Y_s^2\big)\ d\langle Y^1, Y^3 \rangle_s = I\big(Y_s^1 = Y_s^2\big)\ d\langle Y^2, Y^3 \rangle_s \tag{2.14}$$

whenever Y^1, Y^2, and Y^3 are continuous (one-dimensional) semimartingales. The identity (2.14) can easily be verified using the Kunita–Watanabe inequality and the occupation times formula (for more details see the proof following (3.11) below).

The right-hand side of (2.13) can further be expressed in terms of $\tilde{X}$ using (2.14) as follows:

$$
\begin{aligned}
F_1(\hat{X}_t) = F_1(\hat{X}_0) &+ \sum_{i=1}^{n-1} \int_0^t D_i F_1(X_s)\, I\big(X_s^n < b_s^X\big)\; dX_s^i \\
&+ \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_1(X_s)\, I\big(X_s^n = b_s^X\big)\; dX_s^i \\
&+ \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\tilde{X}_s^i \\
&+ \sum_{i=1}^{n-1} \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\; d\tilde{X}_s^i \\
&+ \int_0^t D_n F_1(X_s)\, I\big(X_s^n < b_s^X\big)\; dX_s^n \\
&+ \frac{1}{2} \int_0^t D_n F_1(X_s)\, I\big(X_s^n = b_s^X\big)\; dX_s^n \\
&+ \int_0^t D_n F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\; d\tilde{X}_s^n \\
&+ \frac{1}{2} \int_0^t D_n F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\tilde{X}_s^n \\
&- \frac{1}{2} \int_0^t D_n F_1(X_s)\, I\big(X_s^n = b_s^X\big)\; d\ell_s^b(X) \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I\big(X_s^n < b_s^X\big)\; d\langle X^i, X^j\rangle_s \\
&+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I\big(X_s^n = b_s^X\big)\; d\langle X^i, X^j\rangle_s \\
&+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\langle \tilde{X}^i, \tilde{X}^j\rangle_s \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\; d\langle \tilde{X}^i, \tilde{X}^j\rangle_s . \qquad (2.15)
\end{aligned}
$$

By grouping the corresponding terms in (2.15) we obtain:

$$\begin{aligned}
F_1(\hat{X}_t) = F_1(\hat{X}_0) &+ \sum_{i=1}^{n} \int_0^t D_i F_1(X_s)\, I\big(X_s^n < b_s^X\big)\ dX_s^i \\
&+ \frac{1}{2} \sum_{i=1}^{n} \int_0^t D_i F_1(X_s)\, I\big(X_s^n = b_s^X\big)\ dX_s^i \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I\big(X_s^n < b_s^X\big)\ d\langle X^i, X^j\rangle_s \\
&+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(X_s)\, I\big(X_s^n = b_s^X\big)\ d\langle X^i, X^j\rangle_s \\
&- \frac{1}{2} \int_0^t D_n F_1(X_s)\, I\big(X_s^n = b_s^X\big)\ d\ell_s^b(X) \\
&+ \sum_{i=1}^{n} \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\tilde{X}_s^i \\
&+ \frac{1}{2} \sum_{i=1}^{n} \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\tilde{X}_s^i \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j\rangle_s \\
&+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j\rangle_s. \qquad (2.16)
\end{aligned}$$

3. Applying the Itô formula to $F_2(\hat{X}_t)$ and using (2.12) we get:

$$\begin{aligned}
F_2(\check{X}_t) &= F_2(\check{X}_0) + \sum_{i=1}^{n} \int_0^t D_i F_2(\check{X}_s)\, d\check{X}_s^i \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\check{X}_s)\, d\langle \check{X}^i, \check{X}^j\rangle_s \\
&= F_2(\check{X}_0) + \sum_{i=1}^{n-1} \int_0^t D_i F_2(\check{X}_s)\, dX_s^i \\
&\quad + \frac{1}{2} \int_0^t \Big(1+\text{sign}\big(X_s^n - b_s^X\big)\Big) D_n F_2(\check{X}_s)\, dX_s^n \\
&\quad + \frac{1}{2} \int_0^t \Big(1-\text{sign}\big(X_s^n - b_s^X\big)\Big) D_n F_2(\check{X}_s)\, db_s^X \\
&\quad + \frac{1}{2} \int_0^t D_n F_2(\check{X}_s)\, d\ell_s^b(X)
\end{aligned}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I\big(X_s^n > b_s^X\big)\; d\langle X^i, X^j \rangle_s$$
$$+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I\big(X_s^n = b_s^X\big)\; d\langle X^i, X^j \rangle_s$$
$$+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\langle \tilde{X}^i, \tilde{X}^j \rangle_s$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n < b_s^X\big)\; d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \tag{2.17}$$

where in the last four integrals we make use of the general fact (2.14).

The right-hand side of (2.17) can further be expressed in terms of $\tilde{X}$ using (2.14) as follows:

$$F_2(\check{X}_t) = F_2(\check{X}_0) + \sum_{i=1}^{n-1} \int_0^t D_i F_2(X_s)\, I\big(X_s^n > b_s^X\big)\; dX_s^i$$
$$+ \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_2(X_s)\, I\big(X_s^n = b_s^X\big)\; dX_s^i$$
$$+ \frac{1}{2} \sum_{i=1}^{n-1} \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\tilde{X}_s^i$$
$$+ \sum_{i=1}^{n-1} \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n < b_s^X\big)\; d\tilde{X}_s^i$$
$$+ \int_0^t D_n F_2(X_s)\, I\big(X_s^n > b_s^X\big)\; dX_s^n$$
$$+ \frac{1}{2} \int_0^t D_n F_2(X_s)\, I\big(X_s^n = b_s^X\big)\; dX_s^n$$
$$+ \int_0^t D_n F_2(\tilde{X}_s)\, I\big(X_s^n < b_s^X\big)\; d\tilde{X}_s^n$$
$$+ \frac{1}{2} \int_0^t D_n F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\tilde{X}_s^n$$
$$+ \frac{1}{2} \int_0^t D_n F_2(X_s)\, I\big(X_s^n = b_s^X\big)\; d\ell_s^b(X)$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I\big(X_s^n > b_s^X\big)\; d\langle X^i, X^j \rangle_s$$
$$+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I\big(X_s^n = b_s^X\big)\; d\langle X^i, X^j \rangle_s$$

$$+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n < b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s. \tag{2.18}$$

By grouping the corresponding terms in (2.18) we obtain:

$$\begin{aligned}
F_2(\check{X}_t) = F_2(\check{X}_0) &+ \sum_{i=1}^{n} \int_0^t D_i F_2(X_s)\, I\big(X_s^n < b_s^X\big)\ dX_s^i \\
&+ \frac{1}{2} \sum_{i=1}^{n} \int_0^t D_i F_2(X_s)\, I\big(X_s^n = b_s^X\big)\ dX_s^i \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I\big(X_s^n < b_s^X\big)\ d\langle X^i, X^j \rangle_s \\
&+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(X_s)\, I\big(X_s^n = b_s^X\big)\ d\langle X^i, X^j \rangle_s \\
&+ \frac{1}{2} \int_0^t D_n F_2(X_s)\, I\big(X_s^n = b_s^X\big)\ d\ell_s^b(X) \\
&+ \sum_{i=1}^{n} \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\tilde{X}_s^i \\
&+ \frac{1}{2} \sum_{i=1}^{n} \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\tilde{X}_s^i \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\
&+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s.
\end{aligned} \tag{2.19}$$

4. Combining the right-hand sides of (2.16) and (2.19) we conclude:

$$\begin{aligned}
&F(X_t) = F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t) = F(X_0) \\
&\quad + \sum_{i=1}^{n} \int_0^t \frac{1}{2}\Big(D_i F\big(X_s^1, \ldots, X_s^n +\big) + D_i F\big(X_s^1, \ldots, X_s^n -\big)\Big)\, dX_s^i \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{1}{2}\Big(D_{ij} F\big(X_s^1, \ldots, X_s^n +\big) + D_{ij} F\big(X_s^1, \ldots, X_s^n -\big)\Big) d\langle X^i, X^j \rangle_s \\
&\quad + \frac{1}{2} \int_0^t \Big(D_n F\big(X_s^1, \ldots, X_s^n +\big) - D_n F\big(X_s^1, \ldots, X_s^n -\big)\Big) \\
&\qquad\qquad \times I\big(X_s^n = b_s^X\big)\ d\ell_s^b(X) + R_t
\end{aligned} \tag{2.20}$$

where the final term is given by:

$$\begin{aligned}R_t = F(\tilde{X}_0) &+ \sum_{i=1}^n \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\tilde{X}_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\ &+ \frac{1}{2} \sum_{i=1}^n \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\tilde{X}_s^i \\ &+ \frac{1}{4} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\ &+ \sum_{i=1}^n \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n < b_s^X\big)\ d\tilde{X}_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n < b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\ &+ \frac{1}{2} \sum_{i=1}^n \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\tilde{X}_s^i \\ &+ \frac{1}{4} \sum_{i,j=1}^n \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s - F(\tilde{X}_t). \qquad (2.21)\end{aligned}$$

Hence we see that (2.6) will be proved if we show that $R_t = 0$. Note that if $F_1 = F_2$ then the identity $R_t = 0$ reduces to the Itô formula applied to $F(\tilde{X}_t)$. In the general case we may proceed as follows.

5. Since $F_1(x) = F_2(x)$ for $x = \big(x_1, \ldots, x_{n-1}, b(x_1, \ldots, x_{n-1})\big)$, we see that the two semimartingales $F_1(\tilde{X})$ and $F_2(\tilde{X})$ coincide, so that:

$$\int_0^t I\big(X_s^n > b_s^X\big)\ d\big(F_1(\tilde{X}_s)\big) = \int_0^t I\big(X_s^n > b_s^X\big)\ d\big(F_2(\tilde{X}_s)\big) \qquad (2.22)$$

$$\int_0^t I\big(X_s^n = b_s^X\big)\ d\big(F_1(\tilde{X}_s)\big) = \int_0^t I\big(X_s^n = b_s^X\big)\ d\big(F_2(\tilde{X}_s)\big). \qquad (2.23)$$

Applying the Itô formula to $F_1(\tilde{X}_s)$ and $F_2(\tilde{X}_s)$ we see that (2.22) and (2.23) become:

$$\begin{aligned}&\sum_{i=1}^n \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\tilde{X}_s^i \\ &\qquad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \\ &\quad = \sum_{i=1}^n \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\tilde{X}_s^i \\ &\qquad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n > b_s^X\big)\ d\langle \tilde{X}^i, \tilde{X}^j \rangle_s \qquad (2.24)\end{aligned}$$

$$\sum_{i=1}^{n} \int_0^t D_i F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\tilde{X}_s^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\langle \tilde{X}^i, \tilde{X}^j \rangle_s$$
$$= \sum_{i=1}^{n} \int_0^t D_i F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\tilde{X}_s^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_s)\, I\big(X_s^n = b_s^X\big)\; d\langle \tilde{X}^i, \tilde{X}^j \rangle_s. \tag{2.25}$$

Making use of (2.24) and (2.25) we see that F_1 in the first four integrals on the right-hand side of (2.21) can be replaced by F_2. This combined with the remaining terms shows that the identity $R_t = 0$ reduces to the Itô formula applied to $F_2(\tilde{X}_t)$. This completes the proof of the theorem. □

Remark 2.2. The change-of-variable formula (2.6) can obviously be extended to the case when instead of one function b we are given finitely many functions $b_1, b_2, \ldots, b_m$ which do not intersect.

More precisely, let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale and let us assume that the following conditions are satisfied:

$b_k : \mathbb{R}^{n-1} \to \mathbb{R}$ is continuous such that $b^{k,X} = b_k(X^1, \ldots, X^{n-1})$ is a semimartingale for $1 \leqslant k \leqslant m$ (2.26)

$F_k : \mathbb{R}^n \to \mathbb{R}$ is $C^{i_1,\ldots,i_n}$ for $1 \leqslant k \leqslant m+1$ where each i_j equals 1 or 2 depending on whether X^j is of bounded variation or not (2.27)

$$\begin{aligned} F(x) &= F_1(x) \quad \text{if} \quad x_n < b_1(x_1, \ldots, x_{n-1}) \\ &= F_k(x) \quad \text{if} \quad b_k(x_1, \ldots, x_{n-1}) < x_n < b_{k+1}(x_1, \ldots, x_{n-1}) \\ &\qquad \text{for} \quad 2 \leqslant k \leqslant m \\ &= F_{m+1}(x) \quad \text{if} \quad x_n > b_{m+1}(x_1, \ldots, x_{n-1}) \end{aligned} \tag{2.28}$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is continuous and $x = (x_1, \ldots, x_n)$ belongs to $\mathbb{R}^n$.

Then the change-of-variable formula (2.6) extends as follows:

$$F(X_t) = F(X_0) + \sum_{i=1}^{n} \int_0^t \frac{1}{2} \bigg(\frac{\partial F}{\partial x_i}\, \big(X_s^1, \ldots, X_s^n +\big) + \frac{\partial F}{\partial x_i}\, \big(X_s^1, \ldots, X_s^n -\big) \bigg)\, dX_s^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{1}{2} \bigg(\frac{\partial^2 F}{\partial x_i \partial x_j}\, \big(X_s^1, \ldots, X_s^n +\big) + \frac{\partial^2 F}{\partial x_i \partial x_j}\, \big(X_s^1, \ldots, X_s^n -\big) \bigg) d\langle X^i, X^j \rangle_s$$
$$+ \frac{1}{2} \sum_{k=1}^{m} \int_0^t \bigg(\frac{\partial F}{\partial x_n}\, \big(X_s^1, \ldots, X_s^n +\big) - \frac{\partial F}{\partial x_n}\, \big(X_s^1, \ldots, X_s^n -\big) \bigg) I\big(X_s^n = b_s^{k,X}\big)\, d\ell_s^{b_k}(X) \tag{2.29}$$

where $\ell_s^{b_k}(X)$ is the local time of X on the surface b_k given in (2.5) above, and $d\ell_s^{b_k}(X)$ refers to integration with respect to $s \mapsto \ell_s^{b_k}(X)$.

Note in particular that an open set C in $\mathbb{R}^n$ (such as a ball) can often be described in terms of functions $b_1, b_2, \ldots, b_m$ so that (2.29) becomes applicable. Perhaps the most interesting example of a function F is obtained by looking at $\tau_D = \inf\{\, t > 0 \mid X_t \in D \,\}$ and setting $F(x) = E_x(G(X_{\tau_D}))$ where G is an admissible function and $X_0 = x$ under P_x for $x \in \mathbb{R}^n$. One such example will be studied in Section 4 below.

Remark 2.3. The change-of-variable formula (2.6) is expressed in terms of the *symmetric* local time (2.5). It is evident from the proof above that one could also use the *one-sided* local times defined by:

$$\ell_s^{b+}(X) = \mathbb{P}\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s I\big(0 \leqslant X_r^n - b_r^X < \varepsilon\big)\; d\langle X^n - b^X, X^n - b^X \rangle_r \tag{2.30}$$

$$\ell_s^{b-}(X) = \mathbb{P}\!-\!\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s I\big(-\varepsilon < X_r^n - b_r^X \leqslant 0\big)\; d\langle X^n - b^X, X^n - b^X \rangle_r. \tag{2.31}$$

Then under the same conditions as in Theorem 2.1 we find that the following two equivalent formulations of (2.6) are valid:

$$\begin{aligned} F(X_t) = F(X_0) &+ \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}\big(X_s^1, \ldots, X_s^n \mp\big)\; dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}\big(X_s^1, \ldots, X_s^n \mp\big)\; d\langle X^i, X^j \rangle_s \\ &+ \frac{1}{2} \int_0^t \left(\frac{\partial F}{\partial x_n}\big(X_s^1, \ldots, X_s^n +\big) - \frac{\partial F}{\partial x_n}\big(X_s^1, \ldots, X_s^n -\big) \right) \\ &\qquad \times I\big(X_s^n = b_s^X\big)\; d\ell_s^{b\pm}(X). \end{aligned} \tag{2.32}$$

Clearly (2.29) above can also be expressed in terms of one-sided local times. Note finally that if $X^n - b^X$ is a continuous local martingale, then the three definitions (2.5), (2.30), and (2.31) coincide.

3 Semimartingales with jumps

In this section we will extend the change-of-variable formula (2.6) first to semimartingales with jumps of bounded variation (Theorem 3.1) and then to general semimartingales (Theorem 3.2).

1. Let $X = (X^1, \ldots, X^n)$ be a semimartingale (see, e.g., [12]). Recall that each sample path $t \mapsto X_t^i$ is right continuous and has left limits for $1 \leqslant i \leqslant n$.

In Theorem 3.1 below we will assume that each semimartingale X^i has jumps of bounded variation in the sense that:

$$\sum_{0<s\leqslant t} \big|\Delta X_s^i\big| < \infty \tag{3.1}$$

where $\Delta X_s^i = X_s^i - X_{s-}^i$ for $1 \leqslant i \leqslant n$. In this case each X^i can be uniquely decomposed into:

$$X_t^i = X_0^i + X_t^{i,c} + X_t^{i,d} \tag{3.2}$$

where $X^{i,c} = M^{i,c} + A^{i,c}$ is a continuous semimartingale and $X^{i,d}$ is a discrete semimartingale (of bounded variation) given by:

$$X_t^{i,d} = \sum_{0<s\leqslant t} \Delta X_s^i. \tag{3.3}$$

Moreover, if $F : I\!R^n \to I\!R$ is C^2 then Itô's formula takes any of the two equivalent forms:

$$\begin{aligned} F(X_t) &= F(X_0) \\ &\quad + \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(X_{s-})\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_{s-})\, d[X^{i,c}, X^{j,c}]_s \\ &\quad + \sum_{0<s\leqslant t} \Big(F(X_s) - F(X_{s-}) - \sum_{i=1}^n \frac{\partial F}{\partial x_i}(X_{s-})\, \Delta X_s^i \Big) \\ &= F(X_0) + \sum_{i-1}^n \int_0^t \frac{\partial F}{\partial x_i}(X_{s-})\, dX_s^{i,c} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_{s-}) d[X^{i,c}, X^{j,c}]_s + \sum_{0<s\leqslant t} \Big(F(X_s) - F(X_{s-}) \Big). \end{aligned} \tag{3.4}$$

Both of these forms will be used freely below without further mentioning.

Let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale with jumps of bounded variation. Then $X^n - b^X$ is a semimartingale with jumps of bounded variation and the local time of X on the surface b is well defined as follows:

$$\ell_s^b(X) = I\!P\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^s I\big(-\varepsilon < X_r^n - b_r^X < \varepsilon\big)\ d[X^n - b^X, X^n - b^X]_r^c \tag{3.5}$$

where $[X^n - b^X,\ X^n - b^X]^c$ is the continuous (path by path) component of $[X^n - b^X,\ X^n - b^X]$. Recalling that $X^{n,c}$ and $b^{X,c}$ are continuous semimartingales associated with X^n and b^X as in (3.2) above, we know that $[X^n - b^X,\ X^n - b^X]^c = [X^{n,c} - b^{X,c},\ X^{n,c} - b^{X,c}]$. The following theorem extends the change-of-variable formula (2.6) to semimartingales with jumps of bounded variation.

Theorem 3.1. *Let $X = (X^1, \ldots, X^n)$ be a semimartingale where each X^i has jumps of bounded variation, let $b : \mathbb{R}^{n-1} \to \mathbb{R}$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale with jumps of bounded variation, and let $F : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying (2.3) and (2.4) above.*

Then the following change-of-variable formula holds:

$$
\begin{aligned}
F(X_t) = F(X_0) &+ \sum_{i=1}^{n} \int_0^t \frac{1}{2} \bigg(\frac{\partial F}{\partial x_i} \, (X^1_{s-}, \ldots, X^n_{s-}+) \\
&\qquad + \frac{\partial F}{\partial x_i} \, (X^1_{s-}, \ldots, X^n_{s-}-) \bigg) \, dX^{i,c}_s \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{1}{2} \bigg(\frac{\partial^2 F}{\partial x_i \partial x_j} \, (X^1_{s-}, \ldots, X^n_{s-}+) \\
&\qquad + \frac{\partial^2 F}{\partial x_i \partial x_j} \, (X^1_{s-}, \ldots, X^n_{s-}-) \bigg) \, d[X^{i,c}, X^{j,c}]_s \\
&+ \sum_{0<s\leqslant t} \Big(F(X_s) - F(X_{s-}) \Big) \\
&+ \frac{1}{2} \int_0^t \bigg(\frac{\partial F}{\partial x_n} \, (X^1_{s-}, \ldots, X^n_{s-}+) - \frac{\partial F}{\partial x_n} \, (X^1_{s-}, \ldots, X^n_{s-}-) \bigg) \\
&\qquad \times I\big(X^n_{s-} = b^X_{s-}, \, X^n_s = b^X_s \big) \; d\ell^b_s(X)
\end{aligned}
\tag{3.6}
$$

where $\ell^b_s(X)$ is the local time of X on the surface b given in (3.5) above, and $d\ell^b_s(X)$ refers to integration with respect to the continuous increasing function $s \mapsto \ell^b_s(X)$.

Proof. The proof can be carried out similarly to the proof of Theorem 2.1 and we will only highlight a few novel points appearing due to the existence of jumps. The remaining details are the same as in the proof of Theorem 2.1.

1. We begin as in the proof of Theorem 2.1 by introducing the processes Z^1, Z^2, $\hat{X}$, $\check{X}$, $\tilde{X}$ and observing that (2.7)–(2.9) carries over unchanged. Since X^n and b^X both have jumps of bounded variation, it is easily seen that so do Z^1 and Z^2 as well. Thus the analogue of (2.10) which is obtained by applying the Tanaka formula reads:

$$
\begin{aligned}
\big| X^n_t - b^X_t \big| = \big| X^n_0 - b^X_0 \big| &+ \int_0^t \operatorname{sign}\big(X^n_{s-} - b^X_{s-} \big) \; d\big(X^{n,c}_s - b^{X,c}_s \big) + \ell^b_t(X) \\
&+ \sum_{0<s\leqslant t} \Big(\big| X^n_s - b^X_s \big| - \big| X^n_{s-} - b^X_{s-} \big| \Big)
\end{aligned}
\tag{3.7}
$$

where $\operatorname{sign}(0) = 0$. Similarly to (2.11) and (2.12) we find that:

$$dZ_t^{1,c} = \frac{1}{2}\Big(\big(1-\operatorname{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\, dX_t^{n,c} + \big(1+\operatorname{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\, db_t^{X,c} - d\ell_t^b(X)\Big) \tag{3.8}$$

$$dZ_t^{2,c} = \frac{1}{2}\Big(\big(1+\operatorname{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\, dX_t^{n,c} + \big(1-\operatorname{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\, db_t^{X,c} + d\ell_t^b(X)\Big). \tag{3.9}$$

2. Applying the Itô formula to $F_1(\hat{X}_t)$ we get:

$$F_1(\hat{X}_t) = F_1(\hat{X}_0) + \sum_{i=1}^n \int_0^t D_i F_1(\hat{X}_{s-})\, d\hat{X}_s^{i,c} + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij} F_1(\hat{X}_{s-})\, d[\hat{X}^{i,c}, \hat{X}^{j,c}]_s + \sum_{0<s\leqslant t} \big(F_1(\hat{X}_s) - F_1(\hat{X}_{s-})\big). \tag{3.10}$$

Hence using (3.8) and proceeding in the same way as in (2.13) and (2.15) we obtain the analogue of the identity (2.16) where all X^i and $\tilde{X}^i$ in the integrators (including those with the angle brackets) are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ (now written as the square brackets).

It may be noted (as in the proof of Theorem 2.1) that in the preceding derivation (and in the derivation following (3.12) below) we need to make use of the general fact:

$$I\big(Y_{s-}^1 = Y_{s-}^2\big)\, d[Y^{1,c}, Y^{3,c}]_s = I\big(Y_{s-}^1 = Y_{s-}^2\big)\, d[Y^{2,c}, Y^{3,c}]_s \tag{3.11}$$

whenever Y^1, Y^2, and Y^3 are (one-dimensional) semimartingales. To verify (3.11) note that the claim is equivalent to the fact that for two (one-dimensional) semimartingales Y^1 and Y^2 we have $I(Y_{s-}^1 = 0)\, d[Y^{1,c}, Y^{2,c}] = 0$. To derive the latter we may invoke the Kunita–Watanabe inequality (cf. [12, p. 61]) according to which it is enough to show that $I(Y_{s-}^1 = 0)\, d[Y^{1,c}, Y^{1,c}] = 0$. This identity however follows by the occupation times formula (cf. [12, p. 168]) since $g = 1_{\{0\}}$ equals zero almost everywhere with respect to Lebesgue measure on $\mathbb{R}$. This proves (3.11) in the general case (recall also (2.14) above).

3. Applying the Itô formula to $F_2(\check{X}_t)$ we get:

$$F_2(\check{X}_t) = F_2(\check{X}_0) + \sum_{i=1}^n \int_0^t D_i F_2(\check{X}_{s-})\, d\check{X}_s^{i,c} + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij} F_2(\check{X}_{s-})\, d[\check{X}^{i,c}, \check{X}^{j,c}]_s + \sum_{0<s\leqslant t} \big(F_2(\check{X}_s) - F_2(\check{X}_{s-})\big). \tag{3.12}$$

Hence using (3.9) and proceeding in the same way as in (2.17) and (2.18) we obtain the analogue of the identity (2.19) where all X^i and $\tilde{X}^i$ in the integrators (including those with the angle brackets) are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ (now written as the square brackets).

4. Combining the right-hand sides of the resulting identities we find the analogue of (2.20) to be:

$$
\begin{aligned}
&F(X_t) = F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t)\\
&= F(X_0) + \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(D_iF(X_{s-}^1,\dots,X_{s-}^n+) + D_iF(X_{s-}^1,\dots,X_{s-}^n-)\Big)\, dX_s^{i,c}\\
&+\frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(D_{ij}F(X_{s-}^1,\dots,X_{s-}^n+) + D_{ij}F(X_{s-}^1,\dots,X_{s-}^n-)\Big) d[X^{i,c},X^{j,c}]_s\\
&+\frac{1}{2}\int_0^t \Big(D_nF(X_{s-}^1,\dots,X_{s-}^n+) - D_nF(X_{s-}^1,\dots,X_{s-}^n-)\Big) d\ell_s^b(X)\\
&+\sum_{0<s\leqslant t}\big(F(X_s)-F(X_{s-})\big) + R_t
\end{aligned}
\tag{3.13}
$$

where we use that:

$$
\begin{aligned}
&\sum_{0<s\leqslant t}\big(F_1(\hat{X}_s)-F_1(\hat{X}_{s-})\big) + \sum_{0<s\leqslant t}\big(F_2(\check{X}_s)-F_2(\check{X}_{s-})\big) - \sum_{0<s\leqslant t}\big(F(\tilde{X}_s)-F(\tilde{X}_{s-})\big)\\
&\qquad = \sum_{0<s\leqslant t}\big(F(X_s)-F(X_{s-})\big)
\end{aligned}
\tag{3.14}
$$

and the final term in (3.13) is given by:

$$
\begin{aligned}
R_t = F(\tilde{X}_0) &+ \sum_{i=1}^n \int_0^t D_iF_1(\tilde{X}_{s-})\, I\big(X_{s-}^n > b_{s-}^X\big)\; d\tilde{X}_s^{i,c}\\
&+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_1(\tilde{X}_{s-})\, I\big(X_{s-}^n > b_{s-}^X\big)\; d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s\\
&+ \frac{1}{2}\sum_{i=1}^n \int_0^t D_iF_1(\tilde{X}_{s-})\, I\big(X_{s-}^n = b_{s-}^X\big)\; d\tilde{X}_s^{i,c}\\
&+ \frac{1}{4}\sum_{i,j=1}^n \int_0^t D_{ij}F_1(\tilde{X}_{s-})\, I\big(X_{s-}^n = b_{s-}^X\big)\; d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s\\
&+ \sum_{i=1}^n \int_0^t D_iF_2(\tilde{X}_{s-})\, I\big(X_{s-}^n < b_{s-}^X\big)\; d\tilde{X}_s^{i,c}\\
&+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(\tilde{X}_{s-})\, I\big(X_{s-}^n < b_{s-}^X\big)\; d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s
\end{aligned}
$$

$$+\ \frac{1}{2}\sum_{i=1}^{n}\int_0^t D_iF_2(\tilde{X}_{s-})\,I\big(X^n_{s-}=b^X_{s-}\big)\ d\tilde{X}^{i,c}_s$$
$$+\ \frac{1}{4}\sum_{i,j=1}^{n}\int_0^t D_{ij}F_2(\tilde{X}_{s-})\,I\big(X^n_{s-}=b^X_{s-}\big)\ d[\tilde{X}^{i,c},\tilde{X}^{j,c}]_s - F(\tilde{X}_t)^c \qquad (3.15)$$

where $F(\tilde{X})^c$ is the continuous semimartingale part of $F(\tilde{X})$. From (3.13) and (3.15) we see that (3.6) will be proved if we show that $R_t = 0$.

5. The same arguments as those given in (2.22)–(2.25) show again that F_1 in the first four integrals on the right-hand side of (3.15) can be replaced by F_2. This combined with the remaining terms shows that the identity $R_t = 0$ reduces to applying the Itô formula to $F_2(\tilde{X}_t)$ and identifying the continuous part of the resulting semimartingale. This completes the proof of the theorem. □

2. The condition (3.1) applied to the semimartingale $X^n - b^X$ is the best known sufficient condition for the local time of X on the surface b to be given by means of the explicit expression (3.5) above. In the case of general semimartingales X and b^X, however, the local time of X on the surface b (i.e., the local time of the semimartingale $X^n - b^X$ at zero) can still be defined by means of the Tanaka formula (3.17) retaining its role as the occupation density relative to the random clock $[X^n - b^X, X^n - b^X]^c$ (see [12, p. 168]) but we do not have the explicit representation (3.5) anymore and the use of the local time is somewhat less transparent.

If $X = (X^1, \ldots, X^n)$ is a general semimartingale (not necessarily satisfying (3.1) above) then each X^i can still be decomposed into (3.2) with $X^{i,c} = M^{i,c} + A^{i,c}$ and $X^{i,d} = M^{i,d} + A^{i,d}$ where $M^{i,c}$ is a continuous local martingale, $A^{i,c}$ is a continuous process of bounded variation, $M^{i,d}$ is a purely discontinuous local martingale, and $A^{i,d}$ is a pure jump process of bounded variation. Since the condition (3.1) may fail (due to the existence of many small jumps) we know that Itô's formula takes only the first form in (3.4) above. It is well known (and easily verified by localization using Taylor's theorem) that the first series over $0 < s \leqslant t$ in (3.4) is absolutely convergent (even if (3.1) fails to hold).

The following theorem extends the change-of-variable formula (2.6) to general semimartingales. Note that (1.1), (2.6), and (3.6) above are special cases of the general formula (3.16) below.

Theorem 3.2. *Let $X = (X^1, \ldots, X^n)$ be a semimartingale, let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \ldots, X^{n-1})$ is a semimartingale, and let $F : I\!R^n \to I\!R$ be a continuous function satisfying (2.3) and (2.4) above.*

Then the following change-of-variable formula holds:

$$
\begin{aligned}
F(X_t) = F(X_0) &+ \sum_{i=1}^{n} \int_0^t \frac{1}{2}\Big(\frac{\partial F}{\partial x_i}\, (X_{s-}^1, \ldots, X_{s-}^n +) \\
&\qquad + \frac{\partial F}{\partial x_i}\, (X_{s-}^1, \ldots, X_{s-}^n -)\Big)\, dX_s^i \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{1}{2}\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}\, (X_{s-}^1, \ldots, X_{s-}^n +) \\
&\qquad + \frac{\partial^2 F}{\partial x_i \partial x_j}\, (X_{s-}^1, \ldots, X_{s-}^n -)\Big)\, d[X^{i,c}, X^{j,c}]_s \\
&+ \sum_{0<s\leqslant t} \Bigg(F(X_s) - F(X_{s-}) - \sum_{i=1}^{n} \frac{1}{2}\Big(\frac{\partial F}{\partial x_i} (X_{s-}^1, \ldots, X_{s-}^n +) \\
&\qquad + \frac{\partial F}{\partial x_i} (X_{s-}^1, \ldots, X_{s-}^n -)\Big)\, \Delta X_s^i \Bigg) \\
&+ \frac{1}{2} \int_0^t \Big(\frac{\partial F}{\partial x_n}\, (X_{s-}^1, \ldots, X_{s-}^n +) - \frac{\partial F}{\partial x_n}\, (X_{s-}^1, \ldots, X_{s-}^n -)\Big) \\
&\qquad \times I\big(X_{s-}^n = b_{s-}^X,\, X_s^n = b_s^X\big)\; d\ell_s^b(X) \qquad (3.16)
\end{aligned}
$$

where $\ell_s^b(X)$ is the local time of X on the surface b given by means of (3.17) below and $d\ell_s^b(X)$ refers to integration with respect to the continuous increasing function $s \mapsto \ell_s^b(X)$.

Proof. The proof can be carried out similarly to the proof of Theorem 2.1 and Theorem 3.1 and we will only highlight a few novel points appearing due to the absence of the condition (3.1). The remaining details are the same as in the proof of Theorem 2.1 and Theorem 3.1.

1. We begin as in the proof of Theorem 2.1 by introducing the processes Z^1, Z^2, $\hat{X}$, $\check{X}$, $\tilde{X}$ and observing that (2.7)–(2.9) carries over unchanged. The analogue of (2.10) which is obtained by applying the Tanaka formula now reads:

$$
\begin{aligned}
\big|X_t^n - b_t^X\big| = \big|X_0^n - b_0^X\big| &+ \int_0^t \operatorname{sign}\big(X_{s-}^n - b_{s-}^X\big)\; d\big(X_s^n - b_s^X\big) + \ell_t^b(X) \\
&+ \sum_{0<s\leqslant t} \Big(\big|X_s^n - b_s^X\big| - \big|X_{s-}^n - b_{s-}^X\big| - \operatorname{sign}\big(X_{s-}^n - b_{s-}^X\big)\, \Delta(X^n - b^X)_s \Big)
\end{aligned}
\qquad (3.17)
$$

where $\operatorname{sign}(0) = 0$. Similarly to (2.11) and (2.12) we now find that:

$$dZ_t^1 = \frac{1}{2}\Big(\big(1-\text{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\,dX_t^n + \big(1+\text{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\,db_t^X - d\ell_t^b(X) - dJ_t(X)\Big) \tag{3.18}$$

$$dZ_t^2 = \frac{1}{2}\Big(\big(1+\text{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\,dX_t^n + \big(1-\text{sign}\big(X_{t-}^n - b_{t-}^X\big)\big)\,db_t^X + d\ell_t^b(X) + dJ_t(X)\Big) \tag{3.19}$$

where we denote:

$$J_t(X) = \sum_{0<s\leqslant t}\Big(\big|X_s^n - b_s^X\big| - \big|X_{s-}^n - b_{s-}^X\big| - \text{sign}\big(X_{s-}^n - b_{s-}^X\big)\,\Delta\big(X^n - b^X\big)_s\Big). \tag{3.20}$$

2. Applying the Itô formula to $F_1(\hat{X}_t)$ we get:

$$F_1(\hat{X}_t) = F_1(\hat{X}_0) + \sum_{i=1}^n \int_0^t D_i F_1(\hat{X}_{s-})\,d\hat{X}_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_1(\hat{X}_{s-})\,d[\hat{X}^{i,c},\hat{X}^{j,c}]_s + \sum_{0<s\leqslant t}\Big(F_1(\hat{X}_s) - F_1(\hat{X}_{s-}) - \sum_{i=1}^n D_i F_1(\hat{X}_{s-})\,\Delta X_s^i\Big). \tag{3.21}$$

Hence using (3.18) and proceeding in the same way as in (2.13) and (2.15), making use of the general fact (3.11), we obtain the analogue of the identity (2.16) where all X^i and $\tilde{X}^i$ in the integrators with the angle brackets are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ now written as the square brackets, and the right-hand side of the identity contains a new term given by:

$$-\frac{1}{2}\int_0^t D_n F_1(\hat{X}_{s-})\,dJ_s(X) \tag{3.22}$$

due to the appearance of $-dJ_t(X)$ in (3.18).

3. Applying the Itô formula to $F_2(\check{X}_t)$ we get:

$$F_2(\check{X}_t) = F_2(\check{X}_0) + \sum_{i=1}^n \int_0^t D_i F_2(\check{X}_{s-})\,d\check{X}_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij}F_2(\check{X}_{s-})\,d[\check{X}^{i,c},\check{X}^{j,c}]_s + \sum_{0<s\leqslant t}\Big(F_2(\check{X}_s) - F_2(\check{X}_{s-}) - \sum_{i=1}^n D_i F_2(\check{X}_{s-})\,\Delta X_s^i\Big). \tag{3.23}$$

Hence using (3.19) and proceeding in the same way as in (2.17) and (2.18), making use of the general fact (3.11), we obtain the analogue of the identity (2.19) where all X^i and $\tilde{X}^i$ in the integrators with the angle brackets are replaced by $X^{i,c}$ and $\tilde{X}^{i,c}$ now written as the square brackets, and the right-hand side of the identity contains a new term given by:

$$\frac{1}{2}\int_0^t D_n F_2(\check{X}_{s-})\, dJ_s(X) \tag{3.24}$$

due to the appearance of $dJ_t(X)$ in (3.19).

4. Combining the right-hand sides of the resulting identities we find the analogue of (2.20) to be:

$$\begin{aligned}
F(X_t) &= F_1(\hat{X}_t) + F_2(\check{X}_t) - F(\tilde{X}_t) \\
&= F(X_0) + \sum_{i=1}^n \int_0^t \frac{1}{2}\Big(D_i F\big(X_{s-}^1, \ldots, X_{s-}^n +\big) + D_i F\big(X_{s-}^1, \ldots, X_{s-}^n -\big)\Big)\, dX_s^i \\
&\quad + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \frac{1}{2}\Big(D_{ij} F\big(X_{s-}^1, \ldots, X_{s-}^n +\big) \\
&\qquad\qquad + D_{ij} F\big(X_{s-}^1, \ldots, X_{s-}^n -\big)\Big) d[X^{i,c}, X^{j,c}]_s \\
&\quad + \frac{1}{2}\int_0^t \Big(D_n F\big(X_{s-}^1, \ldots, X_{s-}^n +\big) - D_n F\big(X_{s-}^1, \ldots, X_{s-}^n -\big)\Big)\, d\ell_s^b(X) + R_t
\end{aligned} \tag{3.25}$$

where the final term is given by:

$$\begin{aligned}
R_t = F(\tilde{X}_0) &+ \sum_{i=1}^n \int_0^t D_i F_1(\tilde{X}_{s-})\, I\big(X_{s-}^n > b_{s-}^X\big)\, d\tilde{X}_s^i \\
&+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij} F_1(\tilde{X}_{s-})\, I\big(X_{s-}^n > b_{s-}^X\big)\, d[\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s \\
&+ \frac{1}{2}\sum_{i=1}^n \int_0^t D_i F_1(\tilde{X}_{s-})\, I\big(X_{s-}^n = b_{s-}^X\big)\, d\tilde{X}_s^i \\
&+ \frac{1}{4}\sum_{i,j=1}^n \int_0^t D_{ij} F_1(\tilde{X}_{s-})\, I\big(X_{s-}^n = b_{s-}^X\big)\, d[\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s \\
&+ \sum_{i=1}^n \int_0^t D_i F_2(\tilde{X}_{s-})\, I\big(X_{s-}^n < b_{s-}^X\big)\, d\tilde{X}_s^i \\
&+ \frac{1}{2}\sum_{i,j=1}^n \int_0^t D_{ij} F_2(\tilde{X}_{s-})\, I\big(X_{s-}^n < b_{s-}^X\big)\, d[\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2} \sum_{i=1}^{n} \int_0^t D_i F_2(\tilde{X}_{s-})\, I\big(X^n_{s-} = b^X_{s-}\big)\; d\tilde{X}^i_s \\
&+ \frac{1}{4} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_{s-})\, I\big(X^n_{s-} = b^X_{s-}\big)\; d[\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s - F(\tilde{X}_t) \\
&+ \sum_{0<s\leqslant t} \Big(F_1(\hat{X}_s) - F_1(\hat{X}_{s-}) - \sum_{i=1}^{n} D_i F_1(\hat{X}_{s-})\, \Delta \hat{X}^i_s \Big) \\
&+ \sum_{0<s\leqslant t} \Big(F_2(\check{X}_s) - F_2(\check{X}_{s-}) - \sum_{i=1}^{n} D_i F_2(\check{X}_{s-})\, \Delta \check{X}^i_s \Big) \\
&+ \frac{1}{2} \int_0^t D_n F_2(\check{X}_{s-})\, dJ_s(X) - \frac{1}{2} \int_0^t D_n F_1(\hat{X}_{s-})\, dJ_s(X). \qquad (3.26)
\end{aligned}
$$

5. The same arguments as those given in (2.22) and (2.23) now lead to the following analogues of (2.24) and (2.25), respectively:

$$
\begin{aligned}
&\sum_{i=1}^{n} \int_0^t D_i F_1(\tilde{X}_{s-})\, I\big(X^n_{s-} > b^X_{s-}\big)\; d\tilde{X}^i_s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_{s-})\, I\big(X^n_{s-} > b^X_{s-}\big)\; [\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s \\
&\quad + \sum_{0<s\leqslant t} I\big(X^n_{s-} > b^X_{s-}\big)\; \Big(F_1(\tilde{X}_s) - F_1(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_1(\tilde{X}_{s-})\, \Delta \tilde{X}^i_s \Big) \\
&\ = \sum_{i=1}^{n} \int_0^t D_i F_2(\tilde{X}_{s-})\, I\big(X^n_{s-} > b^X_{s-}\big)\; d\tilde{X}^i_s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_{s-})\, I\big(X^n_{s-} > b^X_{s-}\big)\; [\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s \\
&\quad + \sum_{0<s\leqslant t} I\big(X^n_{s-} > b^X_{s-}\big)\; \Big(F_2(\tilde{X}_s) - F_2(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_2(\tilde{X}_{s-})\, \Delta \tilde{X}^i_s \Big) \\
&\qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n} \int_0^t D_i F_1(\tilde{X}_{s-})\, I\big(X^n_{s-} = b^X_{s-}\big)\; d\tilde{X}^i_s \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_1(\tilde{X}_{s-})\, I\big(X^n_{s-} = b^X_{s-}\big)\; [\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s \\
&\quad + \sum_{0<s\leqslant t} I\big(X^n_{s-} = b^X_{s-}\big)\; \Big(F_1(\tilde{X}_s) - F_1(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_1(\tilde{X}_{s-})\, \Delta \tilde{X}^i_s \Big)
\end{aligned}
$$

$$= \sum_{i=1}^{n} \int_0^t D_i F_2(\tilde{X}_{s-})\, I\big(X^n_{s-} = b^X_{s-}\big)\; d\tilde{X}^i_s$$
$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t D_{ij} F_2(\tilde{X}_{s-})\, I\big(X^n_{s-} = b^X_{s-}\big)\; d[\tilde{X}^{i,c}, \tilde{X}^{j,c}]_s$$
$$+ \sum_{0<s\leqslant t} I\big(X^n_{s-} = b^X_{s-}\big)\; \Big(F_2(\tilde{X}_s) - F_2(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_2(\tilde{X}_{s-})\, \Delta\tilde{X}^i_s\Big). \tag{3.28}$$

Making use of (3.27) and (3.28) we see that F_1 in the first four integrals in (3.26) can be replaced by F_2 upon taking into account the four series over $0 < s \leqslant t$ appearing in (3.27) and (3.28). Adding and subtracting the same series over $0 < s \leqslant t$ we see that the first nine terms on the right-hand side of (3.26), together with the series added, assemble exactly the expression obtained by applying the Itô formula to $F_2(\tilde{X}_t)$. Since $F(\tilde{X}_t) = F_2(\tilde{X}_t)$ hence we see that the first ten terms obtained on the right-hand side of (3.26) equals the eleventh term which is the series subtracted. Recalling also the four series from (3.27) and (3.28) this shows that:

$$R_t = \sum_{0<s\leqslant t} I\big(X^n_{s-} > b^X_{s-}\big)\; \Big(F_2(\tilde{X}_s) - F_2(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_2(\tilde{X}_{s-})\, \Delta\tilde{X}^i_s\Big)$$
$$- \sum_{0<s\leqslant t} I\big(X^n_{s-} > b^X_{s-}\big)\; \Big(F_1(\tilde{X}_s) - F_1(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_1(\tilde{X}_{s-})\, \Delta\tilde{X}^i_s\Big)$$
$$+ \frac{1}{2} \sum_{0<s\leqslant t} I\big(X^n_{s-} = b^X_{s-}\big)\; \Big(F_2(\tilde{X}_s) - F_2(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_2(\tilde{X}_{s-})\, \Delta\tilde{X}^i_s\Big)$$
$$- \frac{1}{2} \sum_{0<s\leqslant t} I\big(X^n_{s-} = b^X_{s-}\big)\; \Big(F_1(\tilde{X}_s) - F_1(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_1(\tilde{X}_{s-})\, \Delta\tilde{X}^i_s\Big)$$
$$- \sum_{0<s\leqslant t} \Big(F_2(\tilde{X}_s) - F_2(\tilde{X}_{s-}) - \sum_{i=1}^{n} D_i F_2(\tilde{X}_{s-})\, \Delta\tilde{X}^i_s\Big)$$
$$+ \sum_{0<s\leqslant t} \Big(F_1(\hat{X}_s) - F_1(\hat{X}_{s-}) - \sum_{i=1}^{n} D_i F_1(\hat{X}_{s-})\, \Delta\hat{X}^i_s\Big)$$
$$+ \sum_{0<s\leqslant t} \Big(F_1(\check{X}_s) - F_1(\check{X}_{s-}) - \sum_{i=1}^{n} D_i F_1(\check{X}_{s-})\, \Delta\check{X}^i_s\Big)$$
$$+ \frac{1}{2} \int_0^t D_n F_2(\check{X}_{s-})\, dJ_s(X) - \frac{1}{2} \int_0^t D_n F_1(\hat{X}_{s-})\, dJ_s(X). \tag{3.29}$$

From (3.25) we thus see that the proof of (3.16) reduces to verify the following identity:

$$R_t = \sum_{0<s\leqslant t} \Big(F(X_s) - F(X_{s-}) - \sum_{i=1}^{n} \Big(I\big(X_{s-}^n < b_{s-}^X\big)\ D_i F_1(X_{s-})\, \Delta X_s^i + I\big(X_{s-}^n = b_{s-}^X\big)\ \frac{1}{2}\Big(D_i F_1(\tilde{X}_{s-}) + D_i F_2(\tilde{X}_{s-})\Big)\, \Delta X_s^i + I\big(X_{s-}^n > b_{s-}^X\big)\ D_i F_2(X_{s-})\, \Delta X_s^i \Big)\Big). \tag{3.30}$$

To this end it is helpful to note that:

$$\begin{aligned}
&\frac{1}{2}\int_0^t D_n F_2(\check{X}_{s-})\, dJ_s(X) - \frac{1}{2}\int_0^t D_n F_1(\hat{X}_{s-})\, dJ_s(X) \\
&= \frac{1}{2}\sum_{0<s\leqslant t} I\big(X_s^n > b_s^X,\ X_{s-}^n = b_{s-}^X\big)\ \Big(D_n F_2(\tilde{X}_{s-}) - D_n F_1(\tilde{X}_{s-})\Big)(X_s^n - b_s^X) \\
&\quad + \sum_{0<s\leqslant t} I(X_s^n > b_s^X,\ X_{s-}^n < b_{s-}^X)\ \Big(D_n F_2(\tilde{X}_{s-}) - D_n F_1(X_{s-})\Big)(X_s^n - b_s^X) \\
&\quad - \sum_{0<s\leqslant t} I(X_s^n < b_s^X,\ X_{s-}^n > b_{s-}^X)\ \Big(D_n F_2(X_{s-}) - D_n F_1(\tilde{X}_{s-})\Big)(X_s^n - b_s^X) \\
&\quad - \frac{1}{2}\sum_{0<s\leqslant t} I(X_s^n < b_s^X,\ X_{s-}^n = b_{s-}^X)\ \Big(D_n F_2(\tilde{X}_{s-}) - D_n F_1(\tilde{X}_{s-})\Big)(X_s^n - b_s^X).
\end{aligned} \tag{3.31}$$

A lengthy but straightforward verification shows that the two sides in (3.30) coincide, i.e., that the right-hand side of (3.29) equals the right-hand side of (3.30). This can be done by recalling that each series over $0 < s \leqslant t$ in (3.29) and (3.31) is absolutely convergent so that all eleven of them appearing on the right-hand side of (3.29) can be combined into a single series of the finite sum of the eleven individual terms. Multiplying the sum by each of the indicators

$$I(X_s^n > b_s^X,\ X_{s-}^n = b_{s-}^X),\ I(X_s^n = b_s^X,\ X_{s-}^n = b_{s-}^X),\ I(X_s^n < b_s^X,\ X_{s-}^n = b_{s-}^X),$$
$$I(X_s^n \geqslant b_s^X,\ X_{s-}^n > b_{s-}^X),\ I(X_s^n < b_s^X,\ X_{s-}^n > b_{s-}^X),$$
$$I(X_s^n > b_s^X,\ X_{s-}^n < b_{s-}^X),\ I(X_s^n = b_s^X,\ X_{s-}^n < b_{s-}^X),\ I(X_s^n < b_s^X,\ X_{s-}^n < b_{s-}^X)$$

and comparing the result with the corresponding expression on the right-hand side of (3.30) it is seen that all eight of them coincide. This establishes the identity (3.30) and completes the proof of the theorem. □

Remark 3.3. It is evident that the contents of Remark 2.2 and Remark 2.3 carry over to the setting of Theorem 3.2 (or Theorem 3.1) without major change. By adding the corresponding jump terms to (2.29) and (2.32) one obtains their extension to general semimartingales (or semimartingales with jumps of bounded variation). We will omit the explicit expressions of these formulas.

4 The time-space maximum process

In this section we first apply the change-of-variable formula (2.6) to a three-dimensional continuous semimartingale and then derive a version of the same formula under weaker conditions on the function. This version is useful in the study of free-boundary problems.

1. Let X be a diffusion process solving:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \tag{4.1}$$

in Itô's sense. The latter more precisely means that X satisfies:

$$X_t = X_0 + \int_0^t \mu(r, X_r)\, dr + \int_0^t \sigma(r, X_r)\, dB_r \tag{4.2}$$

for all $t \geqslant 0$ where μ and σ are locally bounded (continuous) functions for which the integrals in (4.2) are well defined (the second being Itô's) so that X itself is a continuous semimartingale (the process B is a standard Brownian motion). To ensure that X is nondegenerate we will assume that $\sigma > 0$.

Associated with X we consider the maximum process S defined by:

$$S_t = \Big(\max_{0 \leqslant r \leqslant t} S_r \Big) \vee S_0. \tag{4.3}$$

Then $((t, X_t, S_t))_{t \geqslant 0}$ is a continuous semimartingale taking values in $I\!R_+ \times E$ where we set

$$E = \{ (x, s) \in I\!R^2 \mid x \leqslant s \}.$$

2. Let $b : I\!R_+ \times I\!R \to I\!R$ be a continuous function such that the process b^X defined by $b^X_t = b(t, S_t)$ is a semimartingale. Setting:

$$C = \{ (t, x, s) \in I\!R_+ \times E \mid x > b(t, s) \} \tag{4.4}$$

$$D = \{ (t, x, s) \in I\!R_+ \times E \mid x < b(t, s) \} \tag{4.5}$$

suppose that a continuous function $F : I\!R_+ \times E \to I\!R$ is given such that:

$$F \quad \text{is} \quad C^{1,2,1} \quad \text{on} \quad \bar{C} \tag{4.6}$$

$$F \quad \text{is} \quad C^{1,2,1} \quad \text{on} \quad \bar{D} \tag{4.7}$$

in the sense explained following (2.3) and (2.4) above. [A slight notational change in the definition of the process $((t, X_t, S_t))_{t \geqslant 0}$ and the sets C and D in comparison with those given in Section 2 above is made to meet the notation used in [10] and related papers.]

Moreover, since $\sigma > 0$ it follows that:

$$P\left(X_r = b_r^X\right) = 0 \quad \text{for} \quad r \in \langle 0, t] \tag{4.8}$$

so that under (4.6) and (4.7) the change-of-variable formula (2.6) takes the simpler form:

$$\begin{aligned} F(t, X_t, S_t) = F(0, X_0, S_0) &+ \int_0^t F_t(r, X_r, S_r)\, I\big(X_r \neq b_r^X\big)\; dr \\ &+ \int_0^t F_x(r, X_r, S_r)\, I\big(X_r \neq b_r^X\big)\; dX_r \\ &+ \int_0^t F_s(r, X_r, S_r)\, I\big(X_r \neq b_r^X\big)\; dS_r \\ &+ \int_0^t F_{xx}(r, X_r, S_r)\, I\big(X_r \neq b_r^X\big)\; d\langle X, X\rangle_r \\ &+ \frac{1}{2}\int_0^t \Big(F_x(r, X_r+, S_r) - F_x(r, X_r-, S_r)\Big)\, d\ell_r^b(X) \end{aligned} \tag{4.9}$$

where $\ell_r^b(X)$ is the local time of X on the surface b given by:

$$\ell_r^b(X) = I\!P\text{-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^r I\big(-\varepsilon < X_u - b_u^X < \varepsilon\big)\;\; d\big\langle X - b^X, X - b^X\big\rangle_u \tag{4.10}$$

and $d\ell_r^b(X)$ in (4.9) refers to integration with respect to the continuous increasing function $r \mapsto \ell_r^b(X)$. [The appearance of X in $d\ell_r^b(X)$ is motivated by the fact that S_t is a functional of X.] Note also that using (4.1) the formula (4.9) can be rewritten as (4.22) below.

3. It turns out, however, that similarly to the case studied in Section 3 of [11] the conditions (4.6) and (4.7) are not always readily verified. The main example we have in mind (arising from the free-boundary problems mentioned above) is:

$$F(t, x, s) = E_{t,x,s}\big(G(t+\tau_D, X_{t+\tau_D}, S_{t+\tau_D})\big) \tag{4.11}$$

where $(X_t, S_t) = (x, s)$ under $P_{t,x,s}$, an admissible function G is given and fixed, and:

$$\tau_D = \inf\{\, r > 0 \mid (t+r, X_{t+r}, S_{t+r}) \in D \,\}. \tag{4.12}$$

Then one directly obtains the "interior condition" (4.13) by standard means while the "closure condition" (4.6) is harder to verify at b since (unless we know a priori that $r \mapsto b(r, s)$ is Lipschitz continuous or even differentiable) both F_t and F_{xx} may in principle diverge when b is approached from the interior of C.

Motivated by applications in free-boundary problems we will now present a version of the formula (4.9) where (4.6) and (4.7) are replaced by the conditions:

$$F \quad \text{is} \quad C^{1,2,1} \quad \text{on} \quad C \tag{4.13}$$

$$F \quad \text{is} \quad C^{1,2,1} \quad \text{on} \quad D. \tag{4.14}$$

The rationale behind this version is the same as in [11]. Given that one has some basic control over F_x at b (in free-boundary problems mentioned above such a control is provided by the principle of smooth fit) even if F_t is formally to diverge when the boundary b is approached from the interior of C, this deficiency is counterbalanced by a similar behaviour of F_{xx} through the infinitesimal generator of X, and consequently the first integral in (4.22) below is still well defined and finite.

4. Given a subset A of $I\!R_+ \times E$ and a function $f : A \to I\!R$ we say that f is *locally bounded* on A (in $I\!R_+ \times E$) if for each a in $\bar{A}$ there is an open set U in $I\!R_+ \times E$ containing a such that f restricted to $A \cap U$ is bounded. Note that f is locally bounded on A if and only if for each compact set K in $I\!R_+ \times E$ the restriction of f to $A \cap K \neq \emptyset$ is bounded. Given a function $g : [0,t] \to I\!R$ of bounded variation we let $V(g)(t)$ denote the total variation of g on $[0,t]$.

To grasp the meaning of the condition (4.19) below in the case of F from (4.11) above, letting $I\!L_X = \partial/\partial t + \mu\, \partial/\partial x + \big(\sigma^2/2\big)\, \partial^2/\partial x^2$ denote the infinitesimal generator of X, recall that the infinitesimal generator $I\!L$ of $((t, X_t, S_t))_{t \geqslant 0}$ can formally be described as follows (cf. [10]):

$$I\!L = I\!L_X \quad \text{in} \quad x < s \tag{4.15}$$

$$\frac{\partial}{\partial s} = 0 \quad \text{at} \quad x = s. \tag{4.16}$$

Denoting $C_s = \{\, (t,x) \mid (t,x,s) \in C \,\}$ and $D_s = \{\, (t,x) \mid (t,x,s) \in D \,\}$ hence we see that:

$$I\!L F = 0 \quad \text{in} \quad C_s \tag{4.17}$$

$$I\!L F = I\!L G \quad \text{in} \quad D_s. \tag{4.18}$$

This shows that $I\!L F$ is locally bounded on $C_s \cup D_s$ as soon as $I\!L G$ is so on D_s. The latter condition (in free-boundary problems) is easily verified since G is given explicitly.

The main result of the present section may now be stated as follows (see also Remark 4.2 below for further sufficient conditions).

Theorem 4.1. *Let X be a diffusion process solving (4.1) in Itô's sense, let $b : I\!R_+ \times I\!R \to I\!R$ be a continuous function such that the process b^X defined by $b^X_t = b(t, S_t)$ is a semimartingale, and let $F : I\!R_+ \times E \to I\!R$ be a continuous function satisfying (4.13) and (4.14) above.*

If the following conditions are satisfied:

$$\big(F_t + \mu F_x + (\sigma^2/2) F_{xx}\big)\,(\,\cdot\,,\,\cdot\,,s) \quad \textit{is locally bounded on} \quad C_s \cup D_s \tag{4.19}$$

$$F_x(\,\cdot\,,b(\,\cdot\,,s)\pm\varepsilon,s) \rightarrow F_x(\,\cdot\,,b(\,\cdot\,,s)\pm,s) \quad \textit{uniformly on} \quad [0,t] \quad \textit{as} \quad \varepsilon \downarrow 0 \tag{4.20}$$

$$\sup_{0<\varepsilon<\delta} V(F(\,\cdot\,,b(\,\cdot\,,s)\pm\varepsilon,s))(t) < \infty \quad \textit{for some} \quad \delta > 0 \tag{4.21}$$

for each s given and fixed, then the following change-of-variable formula holds:

$$\begin{aligned} F(t,X_t,S_t) &= F(0,X_0,S_0) \\ &\quad + \int_0^t \big(F_t + \mu F_x + (\sigma^2/2)F_{xx}\big)\,(r,X_r,S_r)\; I\big(X_r \neq b_r^X\big)\; dr \\ &\quad + \int_0^t (\sigma F_x)(r,X_r,S_r)\; I\big(X_r \neq b_r^X\big)\; dB_r \\ &\quad + \int_0^t F_s(r,X_r,S_r)\; I\big(X_r \neq b_r^X, X_r = S_r\big)\; dS_r \\ &\quad + \frac{1}{2}\int_0^t (F_x(r,X_r+,S_r) - F_x(r,X_r-,S_r))\; I\big(X_r = b_r^X\big)\; d\ell_r^b(X) \end{aligned} \tag{4.22}$$

where $\ell_r^b(X)$ is the local time of X at the surface b given by (4.10) above, and $d\ell_r^b(X)$ refers to integration with respect to the continuous increasing function $r \mapsto \ell_r^b(X)$.

Proof. The key observation is that off the diagonal $x = s$ in E the process (t,X_t,S_t) can be identified with a process (t,X_t) and the surface process $b(t,S_t)$ can be identified with a curve $b(t)$. By slightly extending the "two-map argument" given in Remark 4.2 of [6] the previous observation can be embedded rigorously in a well-defined mathematical setting. In this setting the problem becomes equivalent to the problem treated in Theorem 3.1 of [11]. Applying the same method of proof, upon making use of (2.16) and (2.19) above, and relying upon the properties of the local time and Helly's selection theorem, it is seen that the conditions (3.26)–(3.28) in Theorem 3.1 of [11] become the conditions (4.19)–(4.21) above. As this verification is lengthy, but in principle the same, further details will be omitted (for more details see [11]). □

Remark 4.2. It is evident that all of the number of *sufficient conditions* discussed in [11], which are either to imply (4.19)–(4.21) or could be used instead, can easily be translated into the present setting. We will state explicitly only one set of these conditions. *Assume that F satisfies (4.13) and (4.14) above. If (4.19) is satisfied and for each s given and fixed we have*:

$$x \mapsto F(r,x,s) \text{ is convex or concave on } [b(r,s)-\delta, b(r,s)] \text{ and convex or concave on } [b(r,s), b(r,s)+\delta] \text{ for each } r \in [0,t] \text{ with some } \delta > 0 \tag{4.23}$$

$$r \mapsto F_x(r, b(r,s)\pm, s) \text{ is continuous on } [0,t] \text{ with values in } I\!R \tag{4.24}$$

then both (4.20) and (4.21) hold. This shows that *(4.23) and (4.24) imply (4.22) when (4.19) holds.* The condition (4.23) can further be relaxed to the form where:

$$F_{xx}(\,\cdot\,,\,\cdot\,,s) = G_1(\,\cdot\,,\,\cdot\,,s) + G_2(\,\cdot\,,\,\cdot\,,s) \quad \text{on} \quad C_s \cup D_s \tag{4.25}$$

where $G_1(\,\cdot\,,\,\cdot\,,s)$ is non-negative (nonpositive) and $G_2(\,\cdot\,,\,\cdot\,,s)$ is continuous on $\bar{C}_s$ and $\bar{D}_s$ for each s given and fixed. Thus, if (4.24) and (4.25) hold, then both (4.20) and (4.21) hold implying also (4.22) when (4.19) holds.

References

[1] Eisenbaum, N.: Integration with respect to local time. Potential Anal., **13**, 303–328, (2000)

[2] Eisenbaum, N.: Local time-space stochastic calculus for Lévy processes. Stochastic Processes and their Appl., **116**, 757–778 (2006)

[3] Elworthy, K.D., Truman, A., Zhao, H.Z.: A generalized Itô formula and asymptotics of heat equations with caustics, in one-dimension. Preprint, (2003)

[4] Föllmer, H., Protter, P., Shiryaev, A.N.: Quadratic covariation and an extension of Itô's formula. Bernoulli, **1**, 149–169, (1995)

[5] Ghomrasni, R., Peskir, G.: Local time-space calculus and extensions of Itô's formula. Proc. High Dim. Probab. III (Sandbjerg 2002), Progr. Probab., **55**, Birkhäuser Basel, 177–192, (2003)

[6] Graversen, S.E., Peskir, G.: Optimal stopping and maximal inequalities for geometric Brownian motion. J. Appl. Probab., **35**, 856–872, (1998)

[7] Itô, K.: Stochastic integral. Proc. Imp. Acad. Tokyo, **20**, 519–524, (1944)

[8] Kyprianou, A.E., Surya, B.A.: A change-of-variable formula with local time on curves for Lévy processes of bounded variation. Preprint, (2004)

[9] Meyer, P.A.: Un cours sur les intégrales stochastiques. Sém. Probab. 10, Lect. Notes in Math., **511**, 245–400, (1976)

[10] Peskir, G.: Optimal stopping of the maximum process: The maximality principle. Ann. Probab., **26**, 1614–1640, (1998)

[11] Peskir, G.: A change-of-variable formula with local time on curves. Research Report No. 428, Dept. Theoret. Statist. Aarhus, (30 pp), (2002). To appear in *J. Theoret. Probab.*

[12] Protter, P.: Stochastic Integration and Differential Equations. Springer, Berlin Heidelberg New York (2004)

[13] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. Springer, Berlin Heidelberg New York (1999)

[14] Tanaka, H.: Note on continuous additive functionals of the 1-dimensional Brownian path. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **1**, 251–257, (1963)

[15] Wang, A.T.: Generalized Itô's formula and additive functionals of Brownian motion. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **41**, 153–159, (1977)

A Note on a Change of Variable Formula with Local Time-Space for Lévy Processes of Bounded Variation

Andreas E. Kyprianou[1,2] and Budhi A. Surya[1]

[1] Mathematical Institute, Utrecht University
P.O. Box 80.010, 3508 TA Utrecht, The Netherlands
e-mail: a.e.kyprianou@ma.hw.ac.uk

[2] School of Mathematics, Heriot Watt University
Edinburgh, EH14 4AS Scotland, United Kingdom
e-mail: surya@math.uu.nl

Summary. We establish a mild variant of the change of variable formula with local time-space for "ripped" functions and Lévy processes of bounded variation.

1 Lévy processes of bounded variation and local time-space

In this short note we shall establish a change of variable formula for "ripped" time-space functions of Lévy processes of bounded variation at the cost of an additional integral with respect to local time-space in the formula. Roughly speaking, by a ripped function, we mean here a time-space function which is $C^{1,1}$ on both sides of a time dependent barrier and which may exhibit a discontinuity along the barrier itself. Such functions have appeared in the theory of optimal stopping problems for Markov processes of bounded variation (cf. [1, 3, 10, 11]). Our starting point is to give a brief review of the relevant features of Lévy processes of bounded variation and what is meant by local time-space for these processes.

Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geqslant 0\}$ satisfying the usual conditions of right continuity and completion. In this text, we take as our definition of a Lévy process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the strong Markov, $\mathbb{F}$-adapted process $X = \{X(t) : t \geqslant 0\}$ with paths that are right continuous with left limits (càdlàg) having the properties that $P(X(0) = 0) = 1$ and for each $0 \leqslant s \leqslant t$, the increment $X(t) - X(s)$ is independent of $\mathcal{F}_s$ and has the same distribution as $X(t - s)$. On each finite time interval, X has paths of bounded variation (or just X has bounded variation for short) if and only if for each $t \geqslant 0$,

$$X(t) = \mathrm{d}t + \sum_{0<s\leqslant t} \Delta_s \tag{1}$$

where $\mathrm{d} \in \mathbb{R}$ and $\{\Delta_s : s \geqslant 0\}$ is a Poisson point process on $[0,\infty) \times (\mathbb{R}\backslash\{0\})$ with (time-space) intensity measure $dt \times \Pi\,(dx)$ satisfying

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge |x|)\, \Pi\,(dx) < \infty.$$

Note that the latter integrability condition is necessary and sufficient for the convergence of $\sum_{0<s\leqslant t} |\Delta_s|$. The process X is further a compound Poisson process with drift if and only if $\Pi\,(\mathbb{R}\backslash\{0\}) < \infty$.

For any such Lévy process we say that 0 *is irregular for itself* if

$$P(T = 0) = 0$$

where T is the first visit of X to the origin,

$$T = \inf\{t > 0 : X_t = 0\}$$

with the usual definition $\inf \emptyset = \infty$ being understood in the present context as corresponding to the case that Y never visits the origin over the time interval $(0,\infty)$. Standard theory allows us to deduce that T is a stopping time. With the exception of a compound Poisson process, 0 is always irregular for itself within the class of Lévy processes of bounded variation. Further, again excluding the case of a compound Poisson process,

$$P(T < \infty) > 0 \Longleftrightarrow \mathrm{d} \neq 0. \tag{2}$$

We refer to [2] for a much deeper account of regularity properties of Lévy processes. For the purpose of this text we need to extend the idea of irregularity for points to irregularity of time-space curves.

Definition 1. Given a Lévy process X with finite variation, a measurable time-space curve $b : [0,\infty) \to \mathbb{R}$ is said to be *irregular for itself for X* if all $\infty > T \geqslant s \geqslant 0$,

$$P_{(s,b(s))}(\#\{t \in (s,T] : X(t) = b(t)\} < \infty) = 1$$

and $t \in \{s \geqslant 0 : X(s) = b(s)\}$ if and only if $\lim_{s\uparrow t} |X(s) - b(s)| = 0$.

A curve b which is irregular for itself for X allows for the construction of the almost surely finite counting measure

$$L^b : \mathcal{B}[0,\infty) \to \mathbb{N}$$

defined by

$$L^b[0,t] = 1 + \sum_{0<s\leqslant t} \delta_{(X(s)=b(s))}(s) \tag{3}$$

where $\delta_{\cdot}(s)$ is the Dirac unit mass at time s. Further, $L^b[0,\infty)$ is almost surely 1 if $\mathrm{d} = 0$. We call the right continuous process

$$L^b = \left\{L^b_t := L[0,t] : t \geqslant 0\right\}$$

local time-space for the curve b. Our choice of terminology here is motivated by [9] who gave the name *local time-space* for an analogous object defined for continuous semi-martingales.

Little seems to be known about local times of Lévy processes of bounded variation (see however [7]) and hence a full classification of all such curves b which are irregular for themselves for X remains an open question. The definition as given is not empty however as we shall now show with the following simple examples.

Example 2. Suppose simply that $b(t) = x$ for all $t \geqslant 0$ and some $x \in \mathbb{R}$ and that X is not a compound Poisson process. In this case, the local time process is nothing more than the number of visits to x plus one which is a similar definition to the one given in [7]. As can be deduced from the above introduction to Lévy processes of bounded variation, if $\mathrm{d} = 0$ then $L_t = 1$ for all $t > 0$. If on the other hand $\mathrm{d} \neq 0$ then since X has the property that $\{0\}$ is irregular for itself for X then the number of times X hits x in each finite time interval is almost surely finite. Further, X hits x by either creeping upwards over it or creeping downwards below according to the respective sign of d. (Creeping both upwards and downwards is not possible for Lévy processes which do not possess a Gaussian component). Creeping upwards above x occurs at first passage time T if and only if $\lim_{s\uparrow T} X(s) = x$. Since the same statement is true of downward creeping and X may only creep in at most one direction, it follows with the help of the Strong Markov Property that $t \in \{s > 0 : X(s) = x\}$ if and only if $\lim_{s\uparrow t} |X(s) - x| = 0$.

Example 3. More generally, if $\Pi(\mathbb{R}\backslash\{0\}) = \infty$ then an argument similar to the above shows that if b, satisfying $b(0+) = b(0)$ and $|b'(0+)| < \infty$, belongs to the class $C^1(0,\infty)$, then it is also irregular for itself for X. One needs to take advantage in this case of the fact that b has locally linear behaviour. Furthermore, one sees that points t for which $b'(t) = \mathrm{d}$ cannot be hit. We have excluded $\Pi(\mathbb{R}\backslash\{0\}) < \infty$ in order to avoid simple pathological examples such as the case of the compound Poisson process and $b(t) = 0$ for all $t \geqslant 0$.

2 A generalization of the change of variable formula

In this section we state our result. The idea is to take the change of variable formula and to weaken the assumption on the class of functions to which it applies. For clarity, let us first state the change of variable formula in the special form that it takes for bounded variation Lévy processes. See [13] or [12] for details of its proof.

Theorem 4. *Suppose that the time-space function f belongs to the class $C^{1,1}([0,\infty)\times\mathbb{R})$. Then for any Lévy process X of bounded variation,*

$$f(t,X(t))-f(0,X(0))=\int_0^t\frac{\partial f}{\partial t}(s,X(s-))\,ds+\mathrm{d}\int_0^t\frac{\partial f}{\partial x}(s,X(s))ds+\sum_{0<s\leqslant t}\{f(s,X(s))-f(s,X(s-))\}$$

almost surely.

Remark 5. By inspection of the proof of the change of variable formula it is also clear that if for some random time T, $X_t\in D$ for all $t<T$ where D is an open set, then the change of variable formula as given above still holds on the event $\{t\leqslant T\}$ for functions $f\in C^{1,1}([0,\infty),D)$.

The generalization we are interested in consists in weakening the class $C^{1,1}([0,\infty)\times\mathbb{R})$ in the Change of Variable formula to the following class.

Definition 6. Suppose that $b:[0,\infty)\to\mathbb{R}$ is a measurable function. A function f is said to be $C^{1,1}([0,\infty)\times\mathbb{R})$ *ripped along* b if

$$f(t,x)=\begin{cases}f^{(1)}(t,x)\ x>b(t),\ t\geqslant 0\\ f^{(2)}(t,x)\ x<b(t),\ t\geqslant 0\end{cases}$$

where $f^{(1)}$ and $f^{(2)}$ each belong to the class $C^{1,1}([0,\infty)\times\mathbb{R})$.

We shall prove the following theorem.

Theorem 7. *Suppose that b is a measurable function which is irregular for itself for X and f is $C^{1,1}([0,\infty)\times\mathbb{R})$ ripped along b. Then for any Lévy process of bounded variation, X,*

$$\begin{aligned}f(t,X(t))-f(0,X(0+))&=\int_0^t\frac{\partial f}{\partial t}(s,X(s-))ds+\mathrm{d}\int_0^t\frac{\partial f}{\partial x}(s,X(s-))ds\\&+\sum_{0<s\leqslant t}\{f(s,X(s))-f(s,X(s-))\}\\&+\int_0^t\{f(s,X(s+))-f(s,X(s-))\}\,dL_s^b\end{aligned}$$

almost surely.

Note, the term $f(0,X(0+))$ is deliberate in place of $f(0,X(0))$ as, in the case that $X(0)=b(0)$, it is possible that the process $f(\cdot,X(\cdot))$ starts with a jump.

This result complements the recent results of [9] which concern an extension of Itô's formula for continuous semi-martingales. Peskir accommodates for the case that the time-space function, f, to which Itô's formula is applied has a disruption in its smoothness along a *continuous* space time barrier of *bounded variation.* In particular, on either side of the barrier, the function is equal to a $C^{1,2}(\mathbb{R} \times [0, \infty))$ time-space function but, unlike the case here, it is assumed that there is continuity in f across the barrier. The formula that Peskir obtained has an additional integral with respect to the semi-martingale local time at zero of the distance of the underlying semi-martingale from the boundary (this is again a semi-martingale) which he calls *local time-space.* As mentioned above, we have chosen for obvious reasons to refer to the integrator in the additional term obtained in Theorem 7 as local time-space also. Peskir's results build further on those of [8] and [4] for Brownian motion and in this sense our results now bring the discussion into the particular and somewhat simpler class of bounded variation semi-martingales that we study here. [5,6,9] all have further results for general and special types of semi-martingales. However, the present study is currently the only one which considers discontinuous functions. We have introduced local time-space as a counting measure rather than an occupation density at zero of the semi-martingale $X - b$ as one normally sees. In the current context, the latter is in fact identically zero (cf. [12]). Other definitions of local time-space may be possible in order to work with more general classes of curves than those given in Definition 1 and hence the current presentation merely scratches the surface of the problem considered.

3 Proofs

Proof (of Theorem 7). The essence of the proof is based around a telescopic sum which we shall now describe. Define the inverse local time process $\tau = \{\tau_t : t \geqslant 0\}$ where

$$\tau_t = \inf\{s > 0 : L^b_s > t\}$$

for each $t \geqslant 0$. Note the second strict inequality in the definition ensures that τ is a càdlàg process and since $L^b_0 = 1$ by definition, it follows that $\tau_0 = 0$. The process τ is nothing more than a step function which increases on the integers $k = 1, 2, 3, \ldots$ by an amount corresponding to the length of the excursion of X from b whose right end point corresponds to the k-th crossing of b by X. Note that even when $X_0 \neq b(0)$ we count the section of the path of X until it first meets b as an (incomplete) excursion.

The increment in $\{f(s, X(s)) : s \geqslant 0\}$ between $s = 0+$ and $s = t$ can be seen as the accumulation of the increments incurred by X crossing the boundary b, the excursions of X from b and the final increment between the last time of contact of X with b and time t. We have

$$\begin{aligned} f(t,X(t))-f(0,X(0+)) &= \int_0^t \{f(s,X(s+))-f(s,X(s-))\}\, dL_s^b \\ &\quad + \sum_{s\leqslant L_t^b} \left\{f(\tau_s,X_{\tau_s})-f(\tau_{s-},X_{\tau_{s-}})\right\}\mathbf{1}_{(|\Delta\tau_s|>0)} \\ &\quad + \left\{f(t,X(t))-f\left(\tau_{L_t^b},X_{\tau_{L_t^b}-}\right)\right\} \end{aligned} \quad (4)$$

The proof is then completed once we know that the increments in the curly brackets of the second and third term on the right hand side of (4) observe the same development as the change of variable formula. Indeed, taking account of the Strong Markov Property, it would suffice to prove that under the given assumptions on f we have that for all $t\in(0,\infty)$

$$\begin{aligned} & f(t\wedge\eta,X(t\wedge\eta))-f(0,X(0+)) \\ &= \int_0^{t\wedge\eta}\frac{\partial f}{\partial t}(s,X(s-))ds + \mathrm{d}\int_0^{t\wedge\eta}\frac{\partial f}{\partial x}(s,X(s-))ds \\ &\quad + \sum_{0<s\leqslant t\wedge\eta}\{f(s,X(s))-f(s,X(s-))\}\,. \end{aligned} \quad (5)$$

Note that η is the first strictly positive time that $X-b=0$.

The statement in (5) is intuitively appealing since up to the stopping time η the process X does not intersect with the boundary b and hence the discontinuity in f should not appear in a development of the function $f(\cdot,X(\cdot))$. The result is proved in the lemma below and thus concludes the proof of the main result. □

Lemma 8. *Under the assumptions of Theorem 7, the identity (5) holds for all $t\in(0,\infty)$.*

Proof. First fix some $\kappa>0$, define

$$\sigma_{\kappa,0}=\inf\{t\geqslant 0:|X(t)-b(t)|>\kappa\}.$$

and $\Omega_\kappa=\{\omega\in\Omega:\sigma_{\kappa,0}<\eta\}$. Next define for each $j\geqslant 1$ the stopping times

$$\sigma_{\kappa,j}=\inf\left\{t>\sigma_{\kappa,j-1}:|X(t)-b(t)|<\frac{1}{2}|X\left(\sigma_{\kappa,j-1}\right)-b(\sigma_{\kappa,j-1})|\right\}$$

where we again work with the usual definition $\inf\emptyset=\infty$. On the set $\Omega_\kappa\cap\{\eta<\infty\}$ we have that

$$\limsup_{j\uparrow\infty}|X(\sigma_{\kappa,j})-b(\sigma_{\kappa,j})|\leqslant\lim_{j\uparrow\infty}\left(\frac{1}{2}\right)^j|X_{\sigma_{\kappa,0}}|=0$$

and hence by the definition of irregularity of b for itself for X,

$$\lim_{j\uparrow\infty}\sigma_{\kappa,j}=\eta \quad (6)$$

where the limit is interpreted to be infinite on the set $\{\eta = \infty\}$. It is also clear that, since X has right continuous paths,

$$\lim_{\kappa \downarrow 0} P\left(\Omega_\kappa\right) = 1. \tag{7}$$

Over the time interval $[\sigma_{\kappa,j-1}, \sigma_{\kappa,j})$ the process X does not enter a tube of positive, $\mathcal{F}_{\sigma_{\kappa,j-1}}$-measurable radius around the curve b, we may appeal to then the standard Change of Variable Formula to deduce that on Ω_κ

$$\begin{aligned} &f\left(\sigma_{\kappa,j} \wedge t, X_{\sigma_{\kappa,j}\wedge t}\right) - f\left(\sigma_{\kappa,j-1} \wedge t, X_{\sigma_{\kappa,j-1}\wedge t}\right) \\ &\quad = \int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t} \frac{\partial f}{\partial t}(s, X(s-))ds + \mathrm{d}\int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t} \frac{\partial f}{\partial x}(s, X(s-))ds \\ &\qquad + \sum_{\sigma_{\kappa,j-1}\wedge t < s \leqslant \sigma_{\kappa,j}\wedge t} \left\{ f(s, X(s)) - f(s, X(s-)) \right\}. \end{aligned} \tag{8}$$

Hence on Ω_κ we have

$$\begin{aligned} &f(\eta \wedge t, X(\eta \wedge t)) - f(\sigma_{\kappa,0}, X(\sigma_{\kappa,0})) \\ &\quad = \sum_{j\geqslant 1} \left\{ f\left(\sigma_{\kappa,j} \wedge t, X(\sigma_{\kappa,j} \wedge t)\right) - f\left(\sigma_{\kappa,j-1} \wedge t, X(\sigma_{\kappa,j-1} \wedge t)\right) \right\} \\ &\quad = \sum_{j\geqslant 1} \int_0^{\eta\wedge t} \left\{ \frac{\partial f}{\partial t}(s, X(s-))ds + \mathrm{d}\frac{\partial f}{\partial x}(s, X(s-)) \right\} \mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t < s \leqslant \sigma_{\kappa,j}\wedge t)} ds \\ &\qquad + \sum_{j\geqslant 1} \sum_{0<s\leqslant \eta\wedge t} \left\{ f(s, X(s)) - f(s, X(s-)) \right\} \mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t < s \leqslant \sigma_{\kappa,j}\wedge t)} \\ &\quad = \int_0^{\eta\wedge t} \frac{\partial f}{\partial t}(s, X(s-))ds + \mathrm{d}\int_0^{\eta\wedge t} \frac{\partial f}{\partial x}(s, X(s-))ds \\ &\qquad + \sum_{0<s\leqslant \eta\wedge t} \left\{ f(s, X(s)) - f(s, X(s-)) \right\}. \end{aligned}$$

where the final equality follows from (an almost sure version of) Fubini's theorem which in turn appeals to the assumption that the limits of f, $\partial f/\partial t$ and $\partial f/\partial x$ all exist and are finite when approaching any point on the curve b. In particular, to deal with the final term in the second equality, note that an almost sure uniform bound of the form

$$|f(s, X(s)) - f(s, X(s-))| \leqslant C|\triangle X(s)|$$

holds (for random C) because of the assumptions on $\partial f/\partial x$ and hence the double sum converges (as X is a process of bounded variation). Since κ may be chosen arbitrarily small, (7) shows that (5) is true almost surely on Ω. □

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References

1. Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004) Exit problems for spectrally negative Lévy processes and applications to Russian, American and Canadized options. *Ann. Appl. Probab.* (2004) **14**, 215–238
2. Bertoin, J. (1996) *Lévy Processes*, Cambridge Univerity Press
3. Chan, T. (2005) Pricing perpetual American options driven by spectrally one sided Lévy processes. In *Exotic option pricing and advanced Lévy models* (eds. A.E. Kyprianou, W. Schoutens and P. Wilmott), 195–216. Wiley.
4. Eisenbaum, N. (2000) Integration with respect to local time. *Potential Anal.* **13**, 303–228
5. Eisenbaum, N. (2006) Local time-space stochastic calculus for Lévy processes. Stochastic Processes and their Appl. **116**, 757–778
6. Elworthy, K.D., Truman, A. and Zhou, H.Z. (2003) A generalized Itô formula and asymptotics of heat equations with caustics in one-dimension. *Preprint*
7. Fitzsimmons, P.J. and Port, S.C. (1991) Local times, occupation times, and the Lebesgue measure of the range of a Lévy process. *Seminar on Stochastic Processes* 1989, pp. 59–73. Birkhäuser, Boston, Mass
8. Föllmer, H., Protter, P. and Shiryaev, A.N. (1995) Quadratic covariation and an extension of Itô's formula. *Bernoulli* **1**, 149–169
9. Peskir, G. (2005). A change-of-variable formula with local time on curves. *J. Theor. Probab.* **18**, 499–535
10. Peskir, G., Shiryaev, A.N. (2000), Sequential testing problems for Poisson processes. Ann. Statist. **28**, 837–859
11. Peskir, G., Shiryaev, A.N. (2002) Solving the Poisson disorder problem. *Advances in finance and stochastics*, 295–312, Springer, Berlin Heidelberg New York
12. Protter, P. (2004) *Stochastic Integration and Differential Equations 2nd Edition*, Springer, Berlin Heidelberg New York
13. Revuz, D. and Yor, M. (1994) *Continuous martingales and Brownian motion.* Second edition. Springer, Berlin Heidelberg New York

Integration with Respect to Self-Intersection Local Time of a One-Dimensional Brownian Motion

Joseph Najnudel

Laboratoire de Probabilités et Modèles Aléatoires, Université Paris VI
175 rue du Chevaleret, 75013 Paris, France

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Introduction

For any locally square-integrable function f, it is possible to define the expression $\int f(a) d_a L_t^a$ by the following equality:

$$\int f(a)\, d_a L_t^a = 2\left[\int_0^t f(B_s)\, dB_s - \big(F(B_t) - F(B_0)\big)\right]$$

where F is a primitive of f, and L_t^a is the local time at a of a one-dimensional Brownian motion on $[0, t]$, denoted by $(B_s)_{0 \leqslant s \leqslant t}$.

More precisely, it has been proven (see Bouleau and Yor [3], Eisenbaum [4]) that if the previous definition is taken,

$$\Phi : f \to \int f(a)\, d_a L_t^a$$

is the unique linear and continuous application from $L^2(\mathbf{R})$ to L^2, such that $\Phi\left(\mathbf{1}_{]a,b]}\right) = L_t^b - L_t^a$ for all $a, b \in \mathbf{R}$.

Therefore, this definition of $\int f(a)\, d_a L_t^a$ is compatible with the natural definition for step functions f.

Moreover, if f is the primitive of a locally integrable function f':

$$\int f(a)\, d_a L_t^a = -\int_0^t f'(B_s)\, ds$$

This equality allows us to define $\int_0^t g(B_s)\, ds$ if g is the derivative of a locally square-integrable function, in the sense of distribution theory.

For example, we can define the previous integral if g is the principal value of $1/x$, the finite part of $1/x_+^\alpha$ for $\alpha < 3/2$, etc. (see Biane and Yor [1], Yamada

[10]). In this paper, we will prove that it is possible to do approximately the same thing with the self-intersection local time.

To define this self-intersection local time, we consider a one-dimensional Brownian motion on $[0,t]$ (denoted by $(B_s)_{0\leqslant s\leqslant t}$); there exists a.s. a continuous function $a \to \alpha_t^a$ such that:

$$\int_0^t du \int_0^u ds\, f(B_s - B_u) = \int f(a)\alpha_t^a\, da$$

for any locally integrable function f. By definition, the self-intersection local time at a of $(B_s)_{0\leqslant s\leqslant t}$ is equal to α_t^a (here, as in Rosen [8], we consider only the one-dimensional self-intersection local time; there are a lot of papers about intersection local times in dimensions 2 and 3, for example, see Bertoin [2], Le Gall [5], Westwater [9], Yor [11]).

In Section 1, we show that $a \to \gamma_t^a = \alpha_t^a + 2ta^-$ is derivable, and that if δ_t^a is its derivative, it is possible to give a meaning to the expression:

$$\int f(a)\, d_a\delta_t^a$$

for any locally square-integrable function.

In other words, it is possible to do the same integration for the derivative of self-intersection local time as for the local time.

This integration will allow us to extend the definition of

$$\int_0^t du \int_0^u ds\, g(B_s - B_u)$$

to distributions g which are not necessary integrable functions.

Finally, in Section 2, we will use the results of Section 1 to study the behaviour of:

$$\int_0^t du \int_0^u ds\, h(B_s - B_u)\mathbf{1}_{|B_s - B_u|>\epsilon}$$

where h is an odd function having some good properties.

1 Construction of the integration with respect to self-intersection local time

In this section, we study the behaviour of the self-intersection local time. To do that, we use a version of Fubini's theorem, which is available for stochastic integrals. More precisely, we have the following proposition (for $t \in \mathbf{R}_+$, $a, b \in \mathbf{R}$, $a < b$):

Proposition 1.1. *Let $(B_s)_{0\leqslant s\leqslant t}$ be a Brownian motion on a probability space $(\Omega, \mathcal{A}, \mu)$, $\mathcal{P}$ the predictable σ-algebra on $[0,t] \times \Omega$, and A a $\mathcal{B}([a,b]) \otimes \mathcal{P}$-measurable function from $[a,b] \times [0,t] \times \Omega$ to $\mathbf{R}$, such that:*

$$\int_a^b dx \int_0^t du \, \mathbf{E}\big[A(x,u)^2\big] < \infty$$

If $(x,\omega) \to Z(x)(\omega)$ *is a* $\mathcal{B}([a,b]) \otimes \mathcal{A}$*-measurable function on* $[a,b] \times \Omega$*, such that for any* x*,* $Z(x) = \int_0^t A(x,u)\, dB_u$*, then* $\int_a^b Z(x)\, dx$ *and* $\int_0^t \big(\int_a^b A(x,u)\, dx\big) dB_u$ *exist and are a.s. equal.*

The proof of this proposition is essentially given in Protter ([6], Theorem 46, p. 160), so we omit it.

Now, let $(B_s)_{s \geqslant 0}$ be a one-dimensional Brownian motion, and f a locally integrable function. The following equality holds:

$$\int_0^t ds \int_s^t du |f(B_s - B_u)| = \int_0^t ds \int |f(a)| L_t^{-a}(\tilde{B}^{(s)})\, da$$

where $L_t^{-a}\big(\tilde{B}^{(s)}\big)$ is the local time at $-a$ of the process $\big(\tilde{B}_u^{(s)}\big)_{s \leqslant u \leqslant t}$, defined by $\tilde{B}_u^{(s)} = B_u - B_s$.

We observe that

$$L_t^{-a}\big(\tilde{B}^{(s)}\big) = L_t^{B_s - a}(B) - L_s^{B_s - a}(B)$$

(with continuous local time), therefore:

$$\int_0^t ds \int |f(a)| L_t^{-a}\big(\tilde{B}^{(s)}\big)\, da \leqslant \int_0^t ds \int |f(a)| L_t^{B_s - a}\, da$$

where $L_t^{B_s - a} = 0$ if $|a| > 2 \sup_{u \leqslant t} |B_u|$. Consequently:

$$\int_0^t ds \int |f(a)|\, L_t^{-a}\big(\tilde{B}^{(s)}\big)\, da \leqslant t \sup_{b \in \mathbf{R}} L_t^b \int_{-2 \sup |B|}^{2 \sup |B|} |f(x)|\, dx < \infty$$

Hence, we can apply Fubini's theorem:

$$\begin{aligned}
\int_0^t du \int_0^u ds\, f(B_s - B_u) &= \int_0^t ds \int_s^t du f(B_s - B_u) \\
&= \int_0^t ds \int f(a) L_t^{-a}\big(\tilde{B}^{(s)}\big)\, da \\
&= \int da\, f(a) \int_0^t ds\, L_t^{-a}\big(\tilde{B}^{(s)}\big) = \int f(a) \alpha_t^a\, da
\end{aligned}$$

where

$$\alpha_t^a = \int_0^t ds\, L_t^{-a}\big(\tilde{B}^{(s)}\big).$$

Because of the previous equality, α_t^a is the self-intersection local time at a of $(B_s)_{0\leqslant s\leqslant t}$. Now, we have a.s.:

$$L_t^{-a}\big(\tilde{B}^{(s)}\big) = 2\left[(B_t - B_s + a)^- - a^- + \int_s^t \mathbf{1}_{B_u - B_s < -a}\, dB_u\right]$$

Therefore, by applying Proposition 1.1, we prove that for all a, the following equality holds a.s.:

$$\alpha_t^a = 2\int_0^t \big((B_s - B_t - a)^+ - (-a)^+\big)\, ds + 2\int_0^t dB_u \int_0^u \mathbf{1}_{B_s - B_u > a}\, ds$$

If we define γ by $\gamma_t^a = \alpha_t^a + 2ta^-$, we obtain:

$$\gamma_t^a = 2\left[\int_0^t (B_s - B_t - a)^+\, ds + \int_0^t dB_u \int_0^u \mathbf{1}_{B_s - B_u > a}\, ds\right]$$

Hence, for every a, b $(a < b)$, we have a.s.:

$$\gamma_t^b - \gamma_t^a = 2\left[\int_0^t \left(\int_a^b (-\mathbf{1}_{B_s - B_t > x})\, dx\right) ds + \int_0^t dB_u \left(-\int_a^b L_u^{x+B_u}\, dx\right)\right]$$

For all x, u, $L_u^{x+B_u}$ is $\mathcal{F}_u$-measurable and continuous with respect to (x, u).

On the other hand, there exists a continuous version of $x \to \int_0^t L_u^{x+B_u}\, dB_u$ (this is a consequence of Burkholder's inequality and Kolmogorov's criteria).

Moreover, for all $u \in [0, t]$, $x \in [a, b]$,

$$\mathbf{E}\left[\big(L_u^{x+B_u}\big)^2\right] = \mathbf{E}\left[\big(L_u^{-x}(B^{(u)})\big)^2\right] \leqslant Cu$$

where $B^{(u)} : v \to B_u - B_{u-v}$ is a standard Brownian motion on $[0, u]$ and $C > 0$ is a constant.

Consequently:

$$\int_a^b \int_0^t \mathbf{E}\big[(L_u^{x+B_u})^2\big]\, du\, dx < \infty$$

and we can apply Proposition 1.1 to prove that for any a, b, we have a.s:

$$\gamma_t^b - \gamma_t^a = -2\int_a^b dx \left[\int_0^t \mathbf{1}_{B_s - B_t > x}\, ds + \int_0^t L_u^{x+B_u}\, dB_u\right] = \int_a^b \delta_t^x\, dx$$

where

$$\delta_t^x = -2\left[\int_0^t \mathbf{1}_{B_s - B_t > x}\, ds + \int_0^t L_u^{x+B_u}\, dB_u\right]$$

We observe that $a \to \gamma_t^a$ is continuous (because $\alpha_t^a = \int_0^t ds\, L_t^{-a}(\tilde{B}))$.

Consequently, it is almost sure that the previous equality (about $\gamma_t^b - \gamma_t^a$) is true for all a, b.

Therefore, $a \to \gamma_t^a$ is a.s. derivable on $\mathbf{R}$, and its derivative is $a \to \delta_t^a$.

Consequently, $a \to \alpha_t^a$ is derivable on $\mathbf{R}^*$ (with derivative: $a \to \beta_t^a = \delta_t^a + 2t\mathbf{1}_{a<0}$), and we can resume our results by the following proposition:

Proposition 1.2. *Let $(B_s)_{0\leqslant s\leqslant t}$ be a one-dimensional Brownian motion. Its self-intersection local time $(\alpha_t^a)_{a\in\mathbf{R}}$ is given by:*

$$\alpha_t^a = 2\left[\int_0^t \left((B_s - B_t - a)^+ - (-a)^+\right) ds + \int_0^t dB_u \int_0^u \mathbf{1}_{B_s - B_u > a}\, ds\right]$$

Moreover, $a \to \alpha_t^a + 2ta^-$ is derivable on $\mathbf{R}$, and its derivative is given by:

$$\delta_t^a = -2\left[\int_0^t \mathbf{1}_{B_s - B_t > a}\, ds + \int_0^t L_u^{a+B_u}\, dB_u\right]$$

Remark. The derivability of $a \to \gamma_t^a$ is a particular case of a more general study about derivability of self-intersection local time for stable processes (see Rosen [8]).

We can also remark that:

$$\delta_t^0 = -2\left[\int_0^t \mathbf{1}_{B_s > B_t}\, ds + \int_0^t L_u^{B_u}\, dB_u\right]$$

so δ_t^0 is strongly related to the quantity $A(t, B_t) = \int_0^t \mathbf{1}_{B_s < B_t}\, ds$.

In fact, it is possible to prove that $t \to \delta_t^0$ has a 4/3-variation which is finite and different from zero, so $A(t, B_t)$ is not a semimartingale (see Rogers and Walsh [7]).

Now, let us prove the following proposition:

Proposition 1.3. *If $f = \sum_i \lambda_i \mathbf{1}_{]a_i, b_i]}$ is a step function, let $\int f(a)\, d_a\delta_t^a$ be defined by $\sum_i \lambda_i(\delta_t^{b_i} - \delta_t^{a_i})$.*

In these conditions, there exists a unique linear and continuous application from $L^2(\mathbf{R})$ to L^2 which coincides with $\int f(a)\, d_a\delta_t^a$ if f is a step function.

Proof. For all $u \in [0, t]$, $s \to B_s^{(u)} = B_u - B_{u-s}$ is a Brownian motion (from $[0, u]$ to $\mathbf{R}$), and the following equality holds a.s. :

$$L_u^{A+B_u} = L_u^{-A}(B^{(u)}) = 2\left[(B_u^{(u)} + A)^- - A^- + \int_0^u \mathbf{1}_{B_s^{(u)} + A < 0}\, dB_s^{(u)})\right]$$

Hence, we have:

$$-\delta_t^A = 2\int_0^t \mathbf{1}_{B_s-B_t>A}\,ds$$
$$+4\int_0^t \left[(B_0-B_u-A)^+ - (-A)^+ + \int_0^u \mathbf{1}_{B_s^{(u)}<-A}\,dB_s^{(u)}\right] dB_u$$
$$= 2\int_0^t f(B_s-B_t)\,ds + 4\int_0^t \left[\int_0^{B_0-B_u} f + \int_0^u f\big(-B_s^{(u)}\big)\,dB_s^{(u)}\right] dB_u$$

where $f = \mathbf{1}_{[A,\infty[}$. By linearity, it is not difficult to prove that:

$$\int f(a)d_a\delta_t^a = 2\int_0^t f(B_s-B_t)\,ds$$
$$+4\int_0^t \left[\int_0^{B_0-B_u} f + \int_0^u f\big(-B_s^{(u)}\big)\,dB_s^{(u)}\right] dB_u$$

for any step function f, if we take a representation of the stochastic integrals such that

$$u \to \int_0^u f\big(-B_s^{(u)}\big)\,dB_s^{(u)}$$

is continuous.

On the other hand, for all $f \in L^2(\mathbf{R})$:

$$\mathbf{E}\left[\left(\int_0^t f(B_s-B_t)\,ds\right)^2\right] \leqslant t\int_0^t \mathbf{E}\left[f(B_s-B_t)^2\right] ds$$
$$= t\int_0^t \frac{du}{\sqrt{2\pi u}}\int f(a)^2 e^{-a^2/2u}\,da = Ct^{3/2}||f||_{L^2}^2$$
$$\mathbf{E}\left[\left(\int_0^{B_0-B_u} f\right)^2\right] \leqslant \mathbf{E}[|B_0-B_u|]||f||_{L^2}^2 \leqslant C\sqrt{u}||f||_{L^2}^2 \leqslant C\sqrt{t}||f||_{L^2}^2$$
$$\mathbf{E}\left[\left(\int_0^u f\big(-B_s^{(u)}\big)\,dB_s^{(u)}\right)^2\right] \leqslant \int_0^u \mathbf{E}\left[f\big(-B_s^{(u)}\big)^2\right] ds \leqslant C\sqrt{t}||f||_{L^2}^2$$

If f is a step function, we know that it is possible to define the double stochastic integral:

$$\int_0^t \left(\int_0^u f\big(-B_s^{(u)}\big)\,dB_s^{(u)}\right) dB_u$$

Now, in the general case, it is possible to take (for all $n \in \mathbf{N}$) step functions f_n such that $||f-f_n||_{L^2} \leqslant 2^{-n}$. In these conditions, for all $u \in [0,t]$, $\int_0^u f\big(-B_s^{(u)}\big)\,dB_s^{(u)}$ is a.s. the limit of $\int_0^u f_n\big(-B_s^{(u)}\big)\,dB_s^{(u)}$.

Consequently, the conditions of measurability which are needed for the existence of the double stochastic integral are true in the case of f, since there are true for f_n.

The L^2-integrability is not difficult to check, so the previous double stochastic integral is well defined for all $f \in L^2$, and we can write:

$$\mathbf{E}\left[\left(\int_0^t \left[\int_0^{B_0-B_u} f + \int_0^u f\big(-B_s^{(u)}\big)\, dB_s^{(u)}\right] dB_u\right)^2\right]$$
$$\leqslant \int_0^t du\, \mathbf{E}\left[\left(\int_0^{B_0-B_u} f + \int_0^u f\big(-B_s^{(u)}\big)\, dB_s^{(u)}\right)^2\right] \leqslant Ct^{3/2}||f||_{L^2}^2$$

By density of step functions in $L^2(\mathbf{R})$, $f \to \int f(a) d_a\delta_t^a$ can be extended to a unique linear and continuous application from $L^2(\mathbf{R})$ to L^2, such that:

$$\int f(a)\, d_a\delta_t^a = 2\int_0^t f(B_s - B_t)\, ds + 4\int_0^t \left[\int_0^{B_0-B_u} f + \int_0^u f\big(-B_s^{(u)}\big)\, dB_s^{(u)}\right] dB_u$$

Proposition 1.3 is proven. □

We observe that if f and g are in $L^2(\mathbf{R})$ and coincide on the interval $[A, A']$, then $\int f(a)\, d_a\delta_t^a = \int g(a)\, d_a\delta_t^a$ when $B_s - B_u \in [A, A']$ for all $s, u \in [0, t]$ (it is easy to prove this for step functions and we conclude by density).

This remark can be used to extend the definition of $\int f(a)\, d_a\delta_t^a$ to locally square integrable functions f (we replace f by $f\mathbf{1}_{[-A,A]}$ when $\sup|B| \leqslant A/2$).

In good cases, it is also possible to integrate with respect to $d_a\beta_t^a$: formally, $d_a\beta_t^a = d_a\delta_t^a + 2t\, d(\mathbf{1}_{a<0})$, so if $f(0)$ is well defined, we will take the following definition:

$$\int f(a)\, d_a\beta_t^a = \int f(a)\, d_a\delta_t^a - 2tf(0)$$

Therefore, we have:

$$\int f(a)\, d_a\beta_t^a = 2\int_0^t \big(f(B_s - B_t) - f(0)\big)\, ds$$
$$+ 4\int_0^t \left[\int_0^{B_0-B_u} f + \int_0^u f\big(-B_s^{(u)}\big)\, dB_s^{(u)}\right] dB_u$$

if this expression has a meaning, and it is only possible if we know the value of $f(0)$ (which for example is naturally defined if a version of f is continuous at 0).

Now, we have the following proposition:

Proposition 1.4. *If f is the second primitive of a locally integrable function (f''), the following equalities hold:*

$$\int f(a)\, d_a\beta_t^a = \int f''(a)\alpha_t^a\, da = \int_0^t du \int_0^u ds\, f''(B_s - B_u).$$

Proof. If $f \in C^2$ and f'' has a compact support (which implies that f' is bounded), we define h by $h(x) = f(-x)$, and we have:

$$\begin{aligned}\int_0^{B_0-B_u} f + \int_0^u f\big(-B_s^{(u)}\big)\,dB_s^{(u)} &= -\int_0^{B_u^{(u)}} h + \int_0^u h\big(B_s^{(u)}\big)\,dB_s^{(u)}\\ &= -\frac{1}{2}\int_0^u h'\big(B_s^{(u)}\big)\,ds\\ &= \frac{1}{2}\int_0^u f'(B_s - B_u)\,ds\end{aligned}$$

Consequently, it is possible to write:

$$\int f(a)\,d_a\beta_t^a = 2\left[\int_0^t \big(f(B_s - B_t) - f(0)\big)\,ds + \int_0^t \left(\int_0^u f'(B_s - B_u)\,ds\right) dB_u\right]$$

It is not difficult to check that if there is a measurable version of $s \to \int_s^t f'(B_s - B_u)\,dB_u$, the measurability and integrability conditions of Proposition 1.1 are satisfied.

Now, for all s:

$$f(B_s - B_t) - f(0) + \int_s^t f'(B_s - B_u)\,dB_u = \int_s^t f''(B_s - B_u)\,du$$

almost surely, so a measurable version of the previous family of stochastic integrals exists and we have:

$$\begin{aligned}\int f(a)\,d_a\beta_t^a &= 2\int_0^t ds\left[f(B_s - B_t) - f(0) + \int_s^t f'(B_s - B_u)\,dB_u\right]\\ &= \int_0^t ds\int_s^t f''(B_s - B_u)\,du = \int_0^t du\int_0^u f''(B_s - B_u)\,ds\end{aligned}$$

Therefore, $\int f(a)\,d_a\beta_t^a = \int f''(a)\alpha_t^a\,da$ for all $f \in C^2$ such that f'' has compact support; if f is affine, $\int f(a)\,d_a\beta_t^a = 0$.

This remark shows that if f'' is an integrable function and f a second primitive of f'', $\int f(a)\,d_a\beta_t^a$ depends only on f''.

So we have a linear application from $L^1(\mathbf{R})$ to a set of random variables:

$$f'' \to \int f(a)\,d_a\beta_t^a - \int f''(a)\alpha_t^a\,da$$

For the second term of this expression, we have:

$$\mathbf{E}\left[\left|\int f''(a)\alpha_t^a\,da\right|\right] \leqslant \int |f''| \sup_{a\in\mathbf{R}} \mathbf{E}[\alpha_t^a] = Ct^{3/2}||f''||_{L^1}$$

We can suppose that $f(0) = f'(0) = 0$, hence $|f(x)| \leqslant |x|||f''||_{L^1}$ and:

$$\mathbf{E}\left[\left|\int_0^t f(B_s - B_t)\,ds\right|\right] \leqslant \left(\int_0^t \mathbf{E}[|B_s - B_t|]\,ds\right) ||f''||_{L^1} \leqslant Ct^{3/2}||f''||_{L^1}$$

$$\mathbf{E}\left[\left(\int_0^{B_0 - B_u} f\right)^2\right] \leqslant \mathbf{E}[(B_0 - B_u)^4/4]\;||f''||^2_{L^1} = Cu^2||f''||^2_{L^1} \leqslant Ct^2||f''||^2_{L^1}$$

$$\begin{aligned}\mathbf{E}\left[\left(\int_0^u f\left(-B_s^{(u)}\right)\,dB_s^{(u)}\right)^2\right] &\leqslant \int_0^u ds\,\mathbf{E}\left[f\left(-B_s^{(u)}\right)^2\right] ds\\ &\leqslant ||f''||^2_{L^1}\int_0^u ds\,\mathbf{E}[(B_s - B_u)^2] \leqslant Ct^2||f''||^2_{L^1}\end{aligned}$$

On the other hand, we have:

$$\int_0^{B_0 - B_u} f + \int_0^u f\left(-B_s^{(u)}\right)\,dB_s^{(u)} = \frac{1}{2}\int_0^u ds\,f'(B_s - B_u)$$

so the double stochastic integral is well defined. Therefore, the previous application is continuous from $L^1(\mathbf{R})$ to L^1.

This application is equal to zero if f'' is continuous; by density:

$$\int f(a)\,d_a\beta_t^a = \int f''(a)\alpha_t^a\,da = \int_0^t du \int_0^u ds\,f''(B_s - B_u)$$

for all $f \in L^1$.

This equality can be extended to locally integrable functions without any difficulty; consequently, we have proven Proposition 1.4. □

Remark. By integration with respect to the derivative of self-intersection local time, we can give a meaning to the expression $\int_0^t du \int_0^u ds\,g(B_s - B_u)$ when g is the second derivative (in the sense of distribution theory) of a locally square-integrable function f, if $f(0)$ is well-defined.

For example, f can be defined by $f(x) = \frac{sgn(x)}{|x|^\alpha}$ where $\alpha < 1/2$.

In this case, g can be considered as the principal value of $k\frac{sgn(x)}{|x|^\beta}$, where $\beta < 5/2$ (k is a multiplicative constant).

f is an odd function, so it is natural to take $f(0) = 0$.

2 An application

Let f be a locally square-integrable function, which is odd and C^2 on $\mathbf{R}^*$.

We will consider the following integral (for $\epsilon > 0$):

$$I_\epsilon = \int_0^t du \int_0^u ds\,h(B_s - B_u)\mathbf{1}_{|B_s - B_u| > \epsilon} = \int h_\epsilon(a)\alpha_t^a\,da$$

where $h = f''$ and $h_\epsilon(a) = h(a)\mathbf{1}_{|a|>\epsilon}$. A second primitive of h_ϵ is the function f_ϵ, such that $f_\epsilon(a) = ag(\epsilon)$ if $|a| \leqslant \epsilon$ and

$$f_\epsilon(a) = f(a) + (\epsilon g(\epsilon) - f(\epsilon))sgn(a)$$

if $|a| > \epsilon$ (g is the derivative of f).

This function is equal to zero at 0. On the other hand, we have $f_\epsilon(a) = k_\epsilon(a) + l_\epsilon(a)$ where:

$$k_\epsilon(a) = (ag(\epsilon) + (f(\epsilon) - \epsilon g(\epsilon))sgn(a))\mathbf{1}_{|a|\leqslant\epsilon} + f(a)\mathbf{1}_{|a|>\epsilon}$$

$$l_\epsilon(a) = (\epsilon g(\epsilon) - f(\epsilon))sgn(a)$$

$f_\epsilon(0) = 0$ and k_ϵ, l_ϵ are locally square-integrable, so we can write:

$$I_\epsilon = \int f_\epsilon(a)\, d_a\delta_t^a = \int k_\epsilon(a)\, d_a\delta_t^a + \int l_\epsilon(a)\, d_a\delta_t^a$$

When ϵ tends to zero, $k_\epsilon(a)$ tends to $f(a)$ almost everywhere.

Now, we will suppose that there exists a function $\phi \in L^2$ such that, if $c > 0$ is small enough: $|f(c)| + c|g(c)| \leqslant \inf_{|b|\leqslant c} \phi(b)$.

In these conditions, k_ϵ is dominated by $|f| + \phi$ if ϵ is small enough. Now, if we suppose that $\epsilon < 1$, $k_\epsilon(a) = f(a)$ for any a out of $[-1, 1]$.

Therefore, $k_\epsilon - f$ is dominated by $(|f| + \phi)\mathbf{1}_{[-1,1]}$, which is in L^2: it tends to zero in L^2.

Consequently, if $J = \int f(a)\, d_a\delta_t^a$, $\int k_\epsilon(a)\, d_a\delta_t^a - J$ tends to zero in L^2.

On the other hand, if A is large enough:

$$\begin{aligned}\int l_\epsilon(a)\, d_a\delta_t^a &= (\epsilon g(\epsilon) - f(\epsilon))\left(\delta_t^A - 2\delta_t^0 + \delta_t^{-A}\right)\\ &= (\epsilon g(\epsilon) - f(\epsilon))\left(\beta_t^A - \beta_t^{0+} - \beta_t^{0-} + \beta_t^{-A}\right)\\ &= (f(\epsilon) - \epsilon g(\epsilon))\left(\beta_t^{0+} + \beta_t^{0-}\right)\end{aligned}$$

(the last equality is true because $a \to \beta_t^a$ has a compact support).

Consequently, we have the following proposition:

Proposition 2.1. *Let f be a locally square-integrable function, which is odd and C^2 on $\mathbf{R}^*$.*

We suppose that there exists a function $\phi \in L^2$, and a number $d > 0$ such that $|f(c)| + c|f'(c)| \leqslant \phi(b)$ if $0 < |b| \leqslant c < d$.

We denote by β^+ and β^- the two derivatives at 0 (right and left) of the self-intersection local time of a Brownian motion $(B_s)_{0\leqslant s\leqslant t}$.

Then, there exists a random variable J, such that:

$$\int_0^t du \int_0^u ds f''(B_s - B_u)\mathbf{1}_{|B_s-B_u|>\epsilon} - (f(\epsilon) - \epsilon f'(\epsilon))(\beta^+ + \beta^-) - J \underset{\epsilon\to 0}{\to} 0$$

in the sense of L^2.

This proposition can be applied to study the behaviour of the quantity:

$$I_\epsilon^\alpha = \int_0^t du \int_0^u ds |B_s - B_u|^\alpha sgn(B_s - B_u) \mathbf{1}_{|B_s - B_u| > \epsilon}$$

If $\alpha > -2$, we can take $f(a) = C sgn(a)|a|^{\alpha+2}$ and $\phi = \mathbf{1}_{[-1,1]}$.

The quantity $f(\epsilon) - \epsilon f'(\epsilon)$ tends to 0 when ϵ tends to 0. Since $\beta^+ + \beta^-$ is in L^2, we can deduce that $I_\epsilon^\alpha - J_\alpha$ tends to zero in L^2 (for a random variable J_α).

If $\alpha = -2$, we can take $f(a) = -\log|a| sgn(a)$, $\quad \phi = f\mathbf{1}_{[-1,1]}$, We can check that we obtain:

$$I_\epsilon^{-2} - (1 - \log\epsilon)(\beta^+ + \beta^-) - J_{-2} \underset{\epsilon \to 0}{\to} 0$$

If $-5/2 < \alpha < -2$, we check that:

$$I_\epsilon^\alpha - \frac{1}{(-2-\alpha)\epsilon^{-2-\alpha}}(\beta^+ + \beta^-) - J_\alpha \underset{\epsilon \to 0}{\to} 0$$

(If $\alpha \leqslant -5/2$, proposition 2.1 does not apply because f is not locally square-integrable)

References

[1] P. Biane, M. Yor : Valeurs principales associées aux temps locaux browniens, Bull. Sci. Math. **111** (1987), no. 1, 23–101

[2] J. Bertoin : Applications des processus de Dirichlet aux temps locaux et temps locaux d'intersection d'un mouvement brownien, Probab. Theor. and Related Fields **80** (1989), 433–460

[3] N. Bouleau, M. Yor : Sur la variation quadratique des temps locaux de certaines semimartingales, C.R. Acad. Sc. Paris **292** (1981), 491–494

[4] N. Eisenbaum : Integration with respect to local time, Potential Anal. **13** (2000), 303–328

[5] J. F. Le Gall : Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan, Sém. Probab. XIX, Lect. Notes Math. **1123**, Springer, Berlin Heidelberg New York (1985), 314–331

[6] P. Protter : Stochastic integration and stochastic differential equations, Springer, Berlin Heidelberg New York (1990), 157–162

[7] L.C.G. Rogers, J.B. Walsh : $A(t, B_t)$ is not a semimartingale, Seminar on Stochastic Process (1990), 275–283

[8] J. Rosen : Derivatives of self-intersection local time, Séminaire de Probabilités XXXVIII, Lect. Notes Math. **1857**, Springer, Berlin Heidelberg New york (2005), 263–281

[9] M.J. Westwater : On Edwards' model for long polymer chains, Comm. Math. Phys. **72** (1980), 131–174

[10] T. Yamada : Principal values of Brownian local times and their related topics, Itô's stochastic calculus and probability theory, Springer, Tokyo (1996), 413–422

[11] M. Yor : Précisions sur l'existence et la continuité des temps locaux d'intersection du mouvement brownien dans $\mathbf{R}^2$, Sém. Probab. XX, Lect. Notes Math. **1204**, Springer, Berlin Heidelberg New York (1986), 532–542

Generalized Itô Formulae and Space-Time Lebesgue–Stieltjes Integrals of Local Times

K. David Elworthy[1], Aubrey Truman[2] and Huaizhong Zhao[3]

[1] Mathematics Institute, University of Warwick
Coventry CV4 7AL, United Kingdom
e-mail: kde@maths.warwick.ac.uk
[2] Department of Mathematics, University of Wales Swansea
Singleton Park, Swansea SA2 8PP, United Kingdom
e-mail: A.Truman@swansea.ac.uk
[3] Department of Mathematical Sciences, Loughborough University
LE11 3TU, United Kingdom
e-mail: H.Zhao@lboro.ac.uk

Summary. Generalized Itô formulae are proved for time dependent functions of continuous real valued semi-martingales. The conditions involve left space and time first derivatives, with the left space derivative required to have locally bounded two-dimensional variation. In particular a class of functions with discontinuous first derivative is included. An estimate of Krylov allows further weakening of these conditions when the semi-martingale is a diffusion.

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1 Introduction

Extensions of Itô formula to less smooth functions are useful in studying many problems such as partial differential equations with some singularities, see below, and in the mathematics of finance. The first extension was obtained for $|X(t)|$ by Tanaka [24] with a beautiful use of local time. The generalized Itô formula in one-dimension for time independent convex functions was developed in [20] and for superharmonic functions in multidimensions in [5] and for distance functions in [15]. Extensions of Itô's formula have also been studied by [16], [12], [21], [11]. In [11], Itô's formula for $W_{loc}^{1,2}$ functions was studied using Lyons–Zheng's backward and forward stochastic integrals [18]. In [4], Itô's formula was extended to absolutely continuous functions with locally bounded derivative using the integral $\int_{-\infty}^{\infty} \nabla f(x)\, \mathrm{d}_x L_s(x)$.

This integral was defined through the existence of the expression $f(X(t)) - f(X(0)) - \int_0^t \frac{\partial}{\partial x} f(X(s))\,\mathrm{d}X(s)$; it was extended to $\int_0^t \int_{-\infty}^{\infty} \nabla f(s,x)\,\mathrm{d}_{s,x} L_s(x)$ for a time dependent function $f(s,x)$ using forward and backward integrals for Brownian motion in [6]. Recent activities in this direction have been to look for minimal assumptions on f to make this integral well defined for semi-martingales other than Brownian motion [7]. However, our motivation in establishing generalized Itô formulae was to use them to describe the asymptotics of the solution of heat equations in the presence of a caustic. Due to the appearance of caustics, the solution of the Hamilton–Jacobi equation, the leading term in the asymptotics, is no longer differentiable, but has a jump in the gradient across the shock wave front of the associated Burgers' equation. Therefore, the local time of continuous semi-martingales in a neighbourhood of the shock wave front of the Burgers equation and the jump of the derivatives of the Hamilton–Jacobi function (or equivalently the jump in the Burgers' velocity) appear naturally in the semi-classical representation of the corresponding solution to the heat equation [8]. None of the earlier versions of Itô's formula apply directly to this situation.

In this paper, we first generalize Itô's formula to the case of a continuous semi-martingale and a left continuous and locally bounded function $f(t,x)$ which satisfies (1) its left derivative $\frac{\partial^-}{\partial t} f(t,x)$ exists and is left continuous, (2) $f(t,x) = f_h(t,x) + f_v(t,x)$ with $f_h(t,x)$ being C^1 in x and $\nabla f_h(t,x)$ having left continuous and locally bounded left derivative $\Delta^- f_h(t,x)$, and f_v having left derivative $\nabla^- f_v(t,x)$ which is left continuous and of locally bounded variation in (t,x). Here we use the two-dimensional Lebesgue–Stieltjes integral of local time with respect to $\nabla^- f(t,x)$. The main result of this paper is formula (2.24). Formula (2.26) follows from (2.24) easily as a special case. These formulae appear to be new and in a good form for extensions to two dimensions [9]. Moreover, in [10], Feng and Zhao observed that the local time $L_t(x)$ can be considered as a rough path in x of finite 2-variation and therefore defined $\int_0^t \int_{-\infty}^{\infty} \nabla^- f(s,x)\,\mathrm{d}_{\mathrm{s,x}} L_s(x)$ pathwise by extending Young and Lyons' profound idea of rough path integration ([17], [25]) to two parameters. When this paper was nearly completed, we received two preprints concerning a generalized Itô's formula for a continuous function $f(t,x)$ with jump derivative $\nabla^- f(t,x)$, ([22], [13]). We remark that formula (2.26) was also observed by [22] independently.

In Section 3, we consider diffusion processes $X(t)$. We prove the generalized Itô formula for a function f with generalized derivative $\frac{\partial}{\partial t} f$ in $L^2_{loc}(dtdx)$ and generalized derivative $\nabla f(t,x)$ being of locally bounded variation in (t,x). We use an inequality from Krylov [16].

2 The continuous semimartingale case

We need the following definitions (see, e.g. [2], [19]): A two-variable function $f(s,x)$ is called monotonically increasing if whenever $s_2 \geqslant s_1$, $x_2 \geqslant x_1$,

$$f(s_2,x_2) - f(s_2,x_1) - f(s_1,x_2) + f(s_1,x_1) \geqslant 0.$$

It is called monotonically decreasing if $-f$ is monotonically increasing. The function f is called left continuous iff it is left continuous in both variables together, in other words, for any sequence $(s_1,x_1) \leqslant (s_2,x_2) \leqslant \cdots \leqslant (s_k,x_k) \to (s,x)$, we have $f(s_k,x_k) \to f(s,x)$ as $k \to \infty$. Here $(s,x) \leqslant (t,y)$ means $s \leqslant t$ and $x \leqslant y$. For a monotonically increasing and left continuous function $f(s,x)$, we can define a Lebesgue–Stieltjes measure by setting

$$\mu([s_1,s_2) \times [x_1,x_2)) = f(s_2,x_2) - f(s_2,x_1) - f(s_1,x_2) + f(s_1,x_1),$$

for $s_2 > s_1$ and $x_2 > x_1$. So for a measurable function $g(s,x)$, we can define the Lebesgue–Stieltjes integral by

$$\int_{t_1}^{t_2} \int_a^b g(s,x)\, \mathrm{d}_{s,x} f(s,x) = \int_{t_1}^{t_2} \int_a^b g(s,x)\, \mathrm{d}\mu.$$

Denote a partition $\mathcal{P}$ of $[t,s] \times [a,x]$ by $t = s_1 < s_2 < \cdots < s_m = s$, $a = x_1 < x_2 < \cdots < x_n = x$ and the variation of f associated with $\mathcal{P}$ by

$$\begin{aligned} &V_{\mathcal{P}}(f,[t,s] \times [a,x]) \\ &\quad = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} |f(s_{i+1},x_{j+1}) - f(s_{i+1},x_j) - f(s_i,x_{j+1}) + f(s_i,x_j)| \end{aligned}$$

and the variation of f on $[t,s] \times [a,x]$ by

$$V_f([t,s] \times [a,x]) = \sup_{\mathcal{P}} V_{\mathcal{P}}(f,[t,s] \times [a,x]).$$

One can find Proposition 2.2, its proof and definition of the multi-dimensional Lebesgue–Stieltjes integral with respect to measures generated by functions of bounded variation in [19]. For the convenience of the reader, we include them here briefly.

Proposition 2.1 (Additivity of variation) *For $s_2 \geqslant s_1 \geqslant t$, and $a_2 \geqslant a_1 \geqslant a$,*

$$\begin{aligned} V_f([t,s_2] \times [a,a_2]) &= V_f([t,s_1] \times [a,a_2]) + V_f([t,s_2] \times [a,a_1]) \\ &\quad + V_f([s_1,s_2] \times [a_1,a_2]) - V_f([t,s_1] \times [a,a_1]). \quad (2.1) \end{aligned}$$

Proof. We only need to prove that for $a \leqslant a_1 < a_2$ and $t \leqslant s_1$,

$$V_f([t,s_1] \times [a,a_2]) = V_f([t,s_1] \times [a,a_1]) + V_f([t,s_1] \times [a_1,a_2]). \qquad (2.2)$$

Our proof is similar to the case of one-dimension. We can always refine a partition $\mathcal{P}$ of $[t, s_1] \times [a, a_2]$ to include a_1. The refined partition is denoted by $\mathcal{P}'$. Then

$$V_{\mathcal{P}}(f, [t, s_1] \times [a, a_2]) \leqslant V_{\mathcal{P}'}(f, [t, s_1] \times [a, a_2]).$$

Then (2.2) follows easily. □

Proposition 2.2 *A function $f(s, x)$ of locally bounded variation can be decomposed as the difference of two increasing functions $f_1(s, x)$ and $f_2(s, x)$, in any quarter space $s \geqslant t, x \geqslant a$. Moreover, if f is also left continuous, then f_1 and f_2 can be taken left continuous.*

Proof. For any $(t, x) \in R^2$, define for $s \geqslant t$ and $x \geqslant a$,

$$2\tilde{f}_1(s, x) = V_f([t, s] \times [a, x]) + f(s, x),$$
$$2\tilde{f}_2(s, x) = V_f([t, s] \times [a, x]) - f(s, x).$$

Then $f(s, x) = \tilde{f}_1(s, x) - \tilde{f}_2(s, x)$. We need to prove that $\tilde{f}_1$ and $\tilde{f}_2$ are increasing functions. For this, let $s_2 \geqslant s_1 \geqslant t$, $a_2 \geqslant a_1 \geqslant a$, then use Proposition 2.1,

$$\begin{aligned}
&2(\tilde{f}_1(s_2, a_2) - \tilde{f}_1(s_1, a_2) - \tilde{f}_1(s_2, a_1) + \tilde{f}_1(s_1, a_1)) \\
&\quad = V_f([t, s_2] \times [a, a_2]) - V_f([t, s_1] \times [a, a_2]) - V_f([t, s_2] \times [a, a_1]) \\
&\qquad + V_f([t, s_1] \times [a, a_1]) + f(s_2, a_2) - f(s_1, a_2) - f(s_2, a_1) + f(s_1, a_1) \\
&\quad = V_f([s_1, s_2] \times [a_1, a_2]) + f(s_2, a_2) - f(s_1, a_2) - f(s_2, a_1) + f(s_1, a_1) \\
&\quad \geqslant 0.
\end{aligned}$$

So $\tilde{f}_1(s, x)$ is an increasing function. Similarly one can prove that $\tilde{f}_2(s, x)$ is an increasing function.

Define

$$f_i(s, x) = \lim_{t \uparrow s, y \uparrow x} \tilde{f}_i(t, y), \quad i = 1, 2.$$

Then since f is left continuous, so

$$f(s, x) = f_1(s, x) - f_2(s, x), \tag{2.3}$$

and f_1 and f_2 are as required. □

From Proposition 2.2, the two-dimensional Lebesgue–Stieltjes integral of a measurable function g with respect to the left continuous function f of bounded variation can be defined by

$$\begin{aligned}
\int_{t_1}^{t_2} \int_a^b g(s, x) \, \mathrm{d}_{s,x} f(s, x) = &\int_{t_1}^{t_2} \int_a^b g(s, x) \, \mathrm{d}_{s,x} f_1(s, x) \\
&- \int_{t_1}^{t_2} \int_a^b g(s, x) \, \mathrm{d}_{s,x} f_2(s, x) \text{ for } t_2 \geqslant t_1, b \geqslant a.
\end{aligned}$$

Here f_1 and f_2 are taken to be left continuous.

It is worth pointing out that it is possible that a function $f(s,x)$ is of locally bounded variation in (s,x) but not of locally bounded variation in x for fixed s. For instance consider $f(s,x)=b(x)$, where $b(x)$ is not of locally bounded variation, then $V_f=0$. However it is easy to see that when a function $f(s,x)$ is of locally bounded variation in (s,x) and of locally bounded variation in x for a fixed $s=s_0$, then it is of locally bounded variation in x for all s. We denote by $V_{f(s)}[a,b]$ the variation of $f(s,x)$ on $[a,b]$ as a function of x for a fixed s.

Now we recall some well-known results of local time which will be used later in this paper. Let $X(s)$ be a continuous semi-martingale $X(s)=X(0)+M_s+V_s$ on a probability space $\{\Omega,\mathcal{F},P\}$. Here M_s is a continuous local martingale and V_s is a continuous process of bounded variation. Let $L_t(a)$ be the local time introduced by P. Lévy

$$L_t(a)=\lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t 1_{[a,a+\epsilon)}(X(s))\,\mathrm{d}\langle M,M\rangle_s \quad a.s., \tag{2.4}$$

for each t and a. Then it is well known that for each fixed $a\in R$, $L_t(a,\omega)$ is continuous, and nondecreasing in t and right continuous with left limit (cadlag) with respect to a ([14], [23]). Therefore we can consider the Lebesgue–Stieltjes integral $\int_0^\infty \phi(s)\,\mathrm{d}L_s(a,\omega)$ for each a for any Borel-measurable function ϕ. In particular

$$\int_0^\infty 1_{R-\{a\}}(X(s))dL_s(a,\omega)=0 \quad a.s. \tag{2.5}$$

Furthermore if ϕ is in $L^{1,1}_{loc}(ds)$, i.e. ϕ has locally integrable generalized derivative, then we have the following integration by parts formula

$$\int_0^t \phi(s)dL_s(a,\omega)=\phi(t)L_t(a,\omega)-\int_0^t \phi'(s)L_s(a,\omega)ds \quad a.s. \tag{2.6}$$

Moreover, if $g(s,x)$ is Borel measurable in s and x and bounded, by the occupation times formula (e.g. see [14], [23]),

$$\int_0^t g(s,X(s))d\langle M,M\rangle_s=2\int_{-\infty}^{\infty}\int_0^t g(s,a)dL_s(a,\omega)da \quad a.s.$$

If further $g(s,x)$ is in $L^{1,1}_{loc}(ds)$ for almost all x, then using the integration by parts formula, we have

$$\begin{aligned}\int_0^t g(s,X(s))d\langle M,M\rangle_s &= 2\int_{-\infty}^{\infty}\int_0^t g(s,a)dL_s(a,\omega)da\\ &= 2\int_{-\infty}^{\infty} g(t,a)L_t(a,\omega)da\\ &\quad -2\int_{-\infty}^{\infty}\int_0^t \frac{\partial}{\partial s}g(s,a)L_s(a,\omega)dsda \quad a.s.\end{aligned}$$

We first prove a theorem with $f_h = 0$. The result with a term f_h is a trivial generalization of Theorem 2.1.

Theorem 2.1 *Assume $f : [0,\infty) \times R \to R$ satisfies*

(i) f is left continuous and locally bounded, with $f(t,x)$ jointly continuous from the right in t and left in x at each point $(0,x)$,

(ii) the left derivatives $\frac{\partial^-}{\partial t} f$ and $\nabla^- f$ exist at all points of $(0,\infty)\times R$ and $[0,\infty)\times R$, respectively,

(iii) $\frac{\partial^-}{\partial t} f$ and $\nabla^- f$ are left continuous and locally bounded,

(iv) $\nabla^- f$ is of locally bounded variation in (t,x) and $\nabla^- f(0,x)$ is of locally bounded variation in x.

Then for any continuous semi-martingale $\{X(t), t \geqslant 0\}$

$$f(t,X(t)) = f(0,X(0)) + \int_0^t \frac{\partial^-}{\partial s} f(s,X(s))ds + \int_0^t \nabla^- f(s,X(s))dX(s)$$
$$+ \int_{-\infty}^{\infty} L_t(x)\,\mathrm{d}_x \nabla^- f(t,x) - \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla^- f(s,x) \; a.s. \tag{2.7}$$

Proof. By a standard localization argument we can assume that X and its quadratic variation are bounded processes and that f, $\frac{\partial^-}{\partial t} f$, $\nabla^- f$, $V_{\nabla^- f(t)}$ and $V_{\nabla^- f}$ are bounded (note here $V_{\nabla^- f(0)} < \infty$ and $V_{\nabla^- f} < \infty$ imply $V_{\nabla^- f(t)} < \infty$ for all $t \geqslant 0$). Note first that from (i) to (iii) the left partial derivatives of f agree with the distributional derivatives and so (iii) implies that f is absolutely continuous in each variable. We use standard regularizing mollifiers (e.g. see [14]). Define

$$\rho(x) = \begin{cases} c\mathrm{e}^{\frac{1}{(x-1)^2-1}}, & \text{if } x \in (0,2), \\ 0, & \text{otherwise.} \end{cases}$$

Here c is chosen such that $\int_0^2 \rho(x)dx = 1$. Take $\rho_n(x) = n\rho(nx)$ as mollifiers. Define

$$f_n(s,x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho_n(x-y)\rho_n(s-\tau)f(\tau,y)d\tau dy, \quad n \geqslant 1, \tag{2.8}$$

where we set $f(\tau,y) = f(-\tau,y)$ if $\tau < 0$. Then $f_n(s,x)$ are smooth and

$$f_n(s,x) = \int_0^2\int_0^2 \rho(\tau)\rho(z) f\left(s - \frac{\tau}{n}, x - \frac{z}{n}\right) d\tau dz, \quad n \geqslant 1. \tag{2.9}$$

Because of the absolutely continuity mentioned above, we can differentiate under the integral in (2.9) to see that $\frac{\partial}{\partial t} f_n(t,x)$, $\nabla f_n(t,x)$, $V_{\nabla f_n(t)}$ and $V_{\nabla f_n}$ are uniformly bounded. Moreover using Lebesgue's dominated convergence theorem, one can prove that as $n \to \infty$, for each (t,x) with $t \geqslant 0$,

$$f_n(t,x) \to f(t,x). \tag{2.10}$$

Also

$$\frac{\partial}{\partial t}f_n(t,x) \to \frac{\partial^-}{\partial t}f(t,x), \quad t>0 \tag{2.11}$$

$$\nabla f_n(t,x) \to \nabla^- f(t,x), \quad t \geqslant 0. \tag{2.12}$$

Note the convergence in (2.10), (2.11), (2.12) is also in L^p_{loc}, $1 \leqslant p < \infty$.

It turns out for any $g(t,x)$ being continuous in t and C^1 in x and having a compact support, using the integration by parts formula and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{n\to+\infty}\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx &= -\lim_{n\to+\infty}\int_{-\infty}^{\infty}\nabla g(t,x)\nabla f_n(t,x)dx \\ &= -\int_{-\infty}^{\infty}\nabla g(t,x)\nabla^- f(t,x)dx. \end{aligned}\tag{2.13}$$

Note $\nabla^- f(t,x)$ is of bounded variation in x and $\nabla g(t,x)$ has a compact support, so

$$-\int_{-\infty}^{+\infty}\nabla g(t,x)\nabla^- f(t,x)dx = \int_{-\infty}^{+\infty} g(t,x)\,\mathrm{d}_x\nabla^- f(t,x). \tag{2.14}$$

Thus

$$\lim_{n\to+\infty}\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx = \int_{-\infty}^{\infty} g(t,x)\,\mathrm{d}_x\nabla^- f(t,x). \tag{2.15}$$

Similarly, one can easily see from the integration by parts formula and Lebesgue's dominated convergence theorem, if $g(s,x)$ is C^1 in x with $\frac{\partial}{\partial s}\nabla g(s,x)$ being continuous and has a compact support in x,

$$\begin{aligned}&\lim_{n\to+\infty}\int_0^t\int_{-\infty}^{+\infty} g(s,x)\Delta\frac{\partial}{\partial s}f_n(s,x)dxds \\ &\quad= -\lim_{n\to+\infty}\int_0^t\int_{-\infty}^{+\infty}\nabla g(s,x)\nabla\frac{\partial}{\partial s}f_n(s,x)dxds \\ &\quad= -\lim_{n\to+\infty}\int_{-\infty}^{\infty}\left[\nabla g(s,x)\nabla f_n(s,x)\right]_0^t dx \\ &\qquad+\lim_{n\to+\infty}\int_0^t\int_{-\infty}^{+\infty}\frac{\partial}{\partial s}\nabla g(s,x)\nabla f_n(s,x)dxds \\ &\quad= -\int_{-\infty}^{+\infty}\left[\nabla g(s,x)\nabla^- f(s,x)\right]_0^t dx \\ &\qquad+\int_0^t\int_{-\infty}^{\infty}\frac{\partial}{\partial s}\nabla g(s,x)\nabla^- f(s,x)dxds. \end{aligned}\tag{2.16}$$

Thus

$$\lim_{n\to+\infty}\int_0^t\int_{-\infty}^{+\infty} g(s,x)\Delta\frac{\partial}{\partial s}f_n(s,x)dxds$$
$$=\int_0^t\int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}\nabla^- f(s,x). \qquad (2.17)$$

Now suppose $g(s,x)$ is continuous in s and cadlag in x jointly, and has compact support. (In particular, g is bounded). We claim (2.15) and (2.17) still valid. For this define

$$g_m(t,x)=\int_{-\infty}^{\infty}\rho_m(y-x)g(t,y)dy=\int_0^2\rho(z)g\left(x+\frac{z}{m}\right)dz.$$

To see (2.15), using Lebesgue's dominated convergence theorem, note that there is a compact set $G_t\subset R^1$ such that

$$\max_{x\in G_t}|g_m(t,x)-g(t,x)|\to 0 \text{ as } m\to+\infty,$$
$$g_m(t,x)=g(t,x)=0 \qquad \text{for } x\notin G_t.$$

Note

$$\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx = \int_{-\infty}^{+\infty} g_m(t,x)\Delta f_n(t,x)dx$$
$$+\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\Delta f_n(t,x)dx. \qquad (2.18)$$

It is easy to see from (2.15) and using Lebesgue's dominated convergence theorem, that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{-\infty}^{+\infty} g_m(t,x)\Delta f_n(t,x)dx=\lim_{m\to\infty}\int_{-\infty}^{\infty} g_m(t,x)\,\mathrm{d}_x\nabla^- f(t,x)$$
$$=\int_{-\infty}^{\infty} g(t,x)\,\mathrm{d}_x\nabla^- f(t,x). \qquad (2.19)$$

Moreover,

$$\left|\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\Delta f_n(t,x)dx\right|$$
$$\leqslant\left|\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\,\mathrm{d}_x\nabla f_n(t,x)\right|$$
$$\leqslant\max_{x\in G}|g(t,x)-g_m(t,x)|V_{\nabla f_n(t)}(G). \qquad (2.20)$$

This leads easily to

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\Delta f_n(t,x)dx\right|=0. \qquad (2.21)$$

Now we use (2.18), (2.19), (2.21)

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx \\
&\quad = \lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty} g_m(t,x)\Delta f_n(t,x)dx \\
&\qquad + \lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty} (g(t,x)-g_m(t,x))\Delta f_n(t,x)dx \\
&\quad = \int_{-\infty}^{\infty} g(t,x)\mathrm{d}_x\nabla^- f(t,x).
\end{aligned}$$

Similarly we also have

$$\liminf_{n\to\infty}\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx = \int_{-\infty}^{\infty} g(t,x)\,\mathrm{d}_x\nabla^- f(t,x).$$

So (2.15) holds for a cadlag function g with a compact support.

Similarly we can prove (2.17) holds for a cadlag function g with a compact support. That is there exists a compact $G\subset R^1$ such that $g(s,x)=0$ for $x\notin G$ and $s\in[0,t]$.

To complete the proof of (2.7), use Itô's formula for the smooth function $f_n(s,X(s))$, then a.s.

$$\begin{aligned}
f_n(t,X(t)) - f_n(0,X(0)) &= \int_0^t \frac{\partial}{\partial s} f_n(s,X(s))ds + \int_0^t \nabla f_n(s,X(s))dX(s) \\
&\quad + \frac{1}{2}\int_0^t \Delta f_n(s,X(s))d\langle M,M\rangle_s. \qquad (2.22)
\end{aligned}$$

As $n\to\infty$, for all $t\geqslant 0$,

$$f_n(t,X(t)) - f_n(0,X(0)) \to f(t,X(t)) - f(0,X(0)) \quad a.s.,$$

and

$$\int_0^t \frac{\partial}{\partial s} f_n(s,X(s))ds \to \int_0^t \frac{\partial^-}{\partial s} f(s,X(s))ds \quad a.s.,$$

$$\int_0^t \nabla f_n(s,X(s))dV_s \to \int_0^t \nabla^- f(s,X(s))dV_s \quad a.s.$$

and

$$E\int_0^t \big(\nabla f_n(s,X(s))\big)^2 d\langle M,M\rangle_s \to E\int_0^t \big(\nabla^- f(s,X(s))\big)^2 d\langle M,M\rangle_s.$$

Therefore in $L^2(\Omega, P)$,

$$\int_0^t \nabla f_n(s, X(s))dM_s \to \int_0^t \nabla^- f(s, X(s))dM_s.$$

To see the convergence of the last term, we recall the well-known result that the local time $L_s(x)$ is jointly continuous in s and cadlag with respect to x and has a compact support in space x for each s. As $L_s(x)$ is an increasing function of s for each x , so if $G \subset R^1$ is the support of L_t, then $L_s(x) = 0$ for all $x \notin G$ and $s \leqslant t$. Now we use the occupation times formula, the integration by parts formula and (2.15), (2.17) for the case when g is cadlag with compact support in x,

$$\begin{aligned}
&\frac{1}{2}\int_0^t \Delta f_n(s, X(s))d\langle M, M\rangle_s \\
&\quad = \int_{-\infty}^{+\infty}\int_0^t \Delta f_n(s, x)d_s L_s(x)dx \\
&\quad = \int_{-\infty}^{+\infty} \Delta f_n(t, x)L_t(x)dx - \int_{-\infty}^{+\infty}\int_0^t \frac{d}{ds}\Delta f_n(s, x)L_s(x)dsdx \\
&\quad \to \int_{-\infty}^{\infty} L_t(x)\,\mathrm{d}_x\nabla^- f(t, x) - \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla^- f(s, x) \ \ a.s.,
\end{aligned}$$

as $n \to \infty$. This proves the desired formula. □

The smoothing procedure can easily be modified to prove that if $f : R^+ \times R \to R$ satisfies (i), (ii) and (iii) of Theorem 2.1, is also C^1 in x and the left derivative $\Delta^- f(t, x)$ exists at all points of $[0, \infty) \times R$ and is jointly left continuous and locally bounded, then $\Delta f_n(t, x) \to \Delta^- f(t, x)$ as $n \to \infty$, $t > 0$. Thus

$$\begin{aligned}
f(t, X(t)) = f(0, X(0)) + \int_0^t \frac{\partial^-}{\partial s} f(s, X(s))ds + \int_0^t \nabla f(s, X(s))dX(s) \\
+ \frac{1}{2}\int_0^t \Delta^- f(s, X(s))d\langle X\rangle_s \ \ a.s. \qquad (2.23)
\end{aligned}$$

The next theorem is an easy extension of Theorem 2.1 and formula (2.23).

Theorem 2.2 *Assume $f : R^+ \times R \to R$ satisfies conditions (i), (ii) and (iii) of Theorem 2.1. Further suppose*

$$f(t, x) = f_h(t, x) + f_v(t, x)$$

where

(i) $f_h(t, x)$ is C^1 in x with $\nabla f_h(t, x)$ having left partial derivative $\Delta^- f_h(t, x)$, (with respect to x), which is left continuous and locally bounded,

(ii) $f_v(t,x)$ has a left continuous derivative $\nabla^- f_v(t,x)$ at all points (t,x) $[0,\infty)\times R$, which is of locally bounded variation in (t,x) and of locally bounded in x for $t=0$.
Then for any continuous semi-martingale $\{X(t), t \geqslant 0\}$,

$$\begin{aligned} f(t,X(t)) = f(0,X(0)) &+ \int_0^t \frac{\partial^-}{\partial s} f(s,X(s))ds + \int_0^t \nabla^- f(s,X(s))dX(s) \\ &+\frac{1}{2}\int_0^t \Delta^- f_h(s,X(s))d\langle X\rangle_s + \int_{-\infty}^{\infty} L_t(x)\,\mathrm{d}_x \nabla^- f_v(t,x) \\ &-\int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla^- f_v(s,x) \quad a.s. \end{aligned} \tag{2.24}$$

Proof. Mollify f_h and f_v, and so f, as in the proof of Theorem 2.1. Apply Itô's formula to the mollification of f and take the limits as in the proofs of Theorem 2.1 and (2.23). □

If f has discontinuity of first and second order derivatives across a curve $x = l(t)$, where $l(t)$ is a continuous function of locally bounded variation, it will be convenient to consider the continuous semi-martingale

$$X^*(s) = X(s) - l(s),$$

and let $L^*_s(a)$ be its local time. We can prove the following version of our main results:

Theorem 2.3 *Assume $f : R^+ \times R \to R$ satisfies conditions (i), (ii) and (iii) of Theorem 2.1. Moreover, suppose $f(t,x) = f_h(t,x) + f_v(t,x)$, where $f_h(t,x)$ is C^1 in x and $\nabla f_h(t,x)$ has left derivative $\Delta^- f_h(t,x)$ which is left continuous and locally bounded, and there exists a curve $x = l(t)$, $t \geqslant 0$, a continuous function of locally bounded variation such that $\nabla^- f_v(t, x+l(t))$ as a function of (t,x) is of locally bounded variation in (t,x) and of locally bounded in x for $t=0$. Then*

$$\begin{aligned} f(t,X(t)) = {} & f(0,z) + \int_0^t \frac{\partial}{\partial s} f(s,X(s))ds + \int_0^t \nabla^- f(s,X(s))dX(s) \\ &+ \frac{1}{2}\int_0^t \Delta^- f_h(s,X(s))d\langle X\rangle_s + \int_{-\infty}^{\infty} L^*_t(x)\,\mathrm{d}_x \nabla^- f_v(t, x+l(t)) \\ &- \int_{-\infty}^{+\infty}\int_0^t L^*_s(x)\mathrm{d}_{s,x}\nabla^- f_v(s, x+l(s)) \quad a.s. \end{aligned} \tag{2.25}$$

Proof. We only need to consider the case when $f_h = 0$ as the general case will follow easily. We basically follow the proof of Theorem 2.1 and apply Itô's formula to f_n and $X(s)$. We still have (2.22). But by the occupation times formula, a.s.

$$
\begin{aligned}
&\frac{1}{2}\int_0^t \Delta f_n(s,X(s))d\langle M,M\rangle_s\\
&=\frac{1}{2}\int_0^t \Delta f_n(s,X^*(s)+l(s))d\langle M,M\rangle_s\\
&=\int_{-\infty}^{+\infty}\int_0^t \Delta f_n(s,x+l(s))d_sL_s^*(x)dx\\
&=\int_{-\infty}^{+\infty}\Delta f_n(t,x+l(t))L_t^*(x)dx-\int_{-\infty}^{+\infty}\int_0^t \frac{d}{ds}\Delta f_n(s,x+l(s))L_s^*(x)dsdx\\
&\to\int_{-\infty}^{\infty}L_t^*(x)\,\mathrm{d}_x\nabla^- f(t,x+l(t))-\int_{-\infty}^{+\infty}\int_0^t L_s^*(x)\mathrm{d}_{s,x}\nabla^- f(s,x+l(s)),
\end{aligned}
$$

as $n\to\infty$ as in the proof of Theorem 2.1. This proves the desired formula. □

Corollary 2.1 *Assume $f: R^+\times R\to R$ satisfies condition (i) of Theorem 2.1 and its left derivative $\frac{\partial^-}{\partial t}f$ exists on $(0,\infty)\times R$ and is left continuous. Further suppose that there exists a curve $x=l(t)$ of locally bounded variation such that f is C^1 in x off the curve with ∇f having left and right limits in x at each point (t,x) and a left continuous and locally bounded left derivative $\Delta^- f$ on x not equal to $l(t)$. Also assume $\nabla f(t,l(t)+y-)$ as a function of t and y is locally bounded and jointly left continuous if $y\leqslant 0$, and $\nabla f(t,l(t)+y+)$ is locally bounded and jointly left continuous in t and right continuous in y if $y\geqslant 0$. Then for any continuous semi-martingale $\{X(t),t\geqslant 0\}$,*

$$
\begin{aligned}
f(t,X(t))&=f(0,X(0))+\int_0^t\frac{\partial^-}{\partial s}f(s,X(s))ds\\
&\quad+\int_0^t\nabla^- f(s,X(s))dX(s)+\frac{1}{2}\int_0^t\Delta^- f(s,X(s))d\langle X,X\rangle_s\\
&\quad+\int_0^t(\nabla f(s,l(s)+)-\nabla f(s,l(s)-))d_sL_s^*(0)\quad a.s.
\end{aligned}
\tag{2.26}
$$

Proof. At first we assume temporarily that $(\nabla f(t,l(t)+)-\nabla f(t,l(t)-))$ is of bounded variation. This condition will be dropped later. Formula (2.26) can be read from (2.25) by considering

$$
\begin{aligned}
f_h(t,x)&=f(t,x)+(\nabla f(t,l(t)-)-\nabla f(t,l(t)+))(x-l(t))^+,\\
f_v(t,x)&=(\nabla f(t,l(t)+)-\nabla f(t,l(t)-))(x-l(t))^+,
\end{aligned}
$$

and integration by parts formula and noticing $\nabla^- f_v(t,x+l(t))$ is of locally bounded variation in (t,x). Let $g(t,y)=f(t,y+l(t))$. In terms of X^*, (2.26) can be rewritten as

$$\begin{aligned} g(t, X^*(t)) &= g(0, X^*(0)) + \int_0^t g(ds, X^*(s)) + \int_0^t \nabla^- g(s, X^*(s)) dX^*(s) \\ &\quad + \frac{1}{2} \int_0^t \Delta^- g(s, X^*(s)) d\langle X^*, X^* \rangle_s \\ &\quad + \int_0^t (\nabla g(s, 0+) - \nabla g(s, 0-)) d_s L_s^*(0) \quad a.s. \end{aligned} \tag{2.27}$$

Here

$$g(ds, y) = d_s g(s, y) = \frac{\partial^-}{\partial s} f(s, y + l(s)) ds + \nabla^- f(s, y + l(s)) dl(s).$$

Now without assuming that $(\nabla f(t, l(t)+) - \nabla f(t, l(t)-))$ is of bounded variation, we can prove the formula by a smoothing procedure in the variable t. To see this, let

$$g_n(t, y) = \int_0^2 \rho(\tau) g\left(t - \frac{\tau}{n}, y\right) d\tau = \int_0^2 \rho(\tau) f\left(t - \frac{\tau}{n}, y + l\left(t - \frac{\tau}{n}\right)\right) d\tau,$$

with $l(s) = l(0)$ if $s < 0$ and $f(s, x) = f(-s, x)$ for $s < 0$ as usual. Then as $n \to \infty$,

$$\begin{aligned} \int_0^t g_n(ds, X^*(s)) &= \int_0^t \int_0^2 \rho(\tau) \frac{\partial^-}{\partial s} f\left(s - \frac{\tau}{n}, X^*(s) + l\left(s - \frac{\tau}{n}\right)\right) d\tau ds \\ &\quad + \int_0^t \int_0^2 \rho(\tau) \nabla^- f\left(s - \frac{\tau}{n}, X^*(s) + l\left(s - \frac{\tau}{n}\right)\right) \\ &\qquad \times dl\left(s - \frac{\tau}{n}\right) d\tau \\ &\to \int_0^t \frac{\partial^-}{\partial s} f\left(s, X^*(s) + l(s)\right) ds + \nabla^- f(s, X^*(s) + l(s)) dl(s) \\ &= \int_0^t g(ds, X^*(s)) \quad a.s. \end{aligned} \tag{2.28}$$

It is easy to see that for all (t, y)

$$g_n(t, y) \to g(t, y) \tag{2.29}$$

and for all $y \neq 0$,

$$\nabla g_n(t, y) \to \nabla g(t, y), \qquad \Delta^- g_n(t, y) \to \Delta^- g(t, y), \tag{2.30}$$

with uniform local bounds. Moreover, we can see that as $y \to 0\pm$ and $n \to \infty$,

$$\nabla^\pm g_n(t, y) = \int_0^2 \rho(\tau) \nabla^\pm g\left(t - \frac{\tau}{n}, y\right) d\tau \to \nabla g(t, 0\pm). \tag{2.31}$$

Since $\nabla g_n(t, 0\pm)$ are smooth in t they are of locally bounded variation. From (2.27),

$$g_n(t, X^*(t)) = g_n(0, X^*(0)) + \int_0^t g_n(ds, X^*(s)) + \int_0^t \nabla^- g_n(s, X^*(s))dX^*(s) \\ + \frac{1}{2}\int_0^t \Delta^- g_n(s, X^*(s))d\langle X^*, X^*\rangle_s \\ + \int_0^t (\nabla g_n(s, 0+) - \nabla g_n(s, 0-))d_s L_s^*(0) \quad a.s. \tag{2.32}$$

We obtain the desired formula by passing to the limits using (2.28), (2.29), (2.30) and (2.31). □

Remark 2.1 *(i) Formula (2.26) was also observed by Peskir in [22] and [13] independently.*

(ii) From the proof of Theorem 2.1, one can take different mollifications, e.g. one can take (2.9) as

$$f_n(s, x) = \int_0^2 \int_0^2 \rho(\tau)\rho(z) f\left(s + \frac{\tau}{n}, x + \frac{z}{n}\right) d\tau dz, \quad n \geqslant 1.$$

This will lead to as $n \to \infty$,

$$\frac{\partial}{\partial s} f_n(s, x) \to \frac{\partial^+}{\partial s} f(s, x)$$

instead of (2.11), if $\frac{\partial^+}{\partial s} f(s, x)$ is jointly right continuous. Therefore we have the following more general Itô's formula

$$f(t, X(t)) = f(0, z) + \int_0^t \frac{\partial^{s_1}}{\partial s} f(s, X(s))ds + \int_0^t \nabla^{s_2} f(s, X(s))dX(s) \\ + \frac{1}{2}\int_0^t \Delta^{s_2} f_h(s, X(s))d\langle X\rangle_s \\ + \int_{-\infty}^{\infty} L_t(x)\, \mathrm{d}_x \nabla^{s_2} f_v(t, x) - \int_{-\infty}^{+\infty} \int_0^t L_s(x) \mathrm{d}_{s,x} \nabla^{s_2} f_v(s, x) \ a.s.,$$

where $s_1 = \pm$ and $s_2 = \pm$.

Formula (2.24) is in a very general form. It includes the classical Itô formula, Tanaka's formula, Meyer's formula for convex functions, the formula given by Azéma, Jeulin, Knight and Yor [3] and formula (2.26). In the following we will give some examples for which (2.26) and some known generalized Itô formulae do not immediately apply, but formula (2.24) can be applied. These examples can be presented in different forms to include local times on curves.

Example 2.1 *Consider the function*

$$f(t,x) = (\sin \pi x \sin \pi t)^+.$$

Then

$$\nabla^- f(t,x) = \pi \cos \pi x \sin \pi t \; 1_{\sin \pi x \sin \pi t > 0}.$$

One can verify that $\nabla^- f(t,x)$ is of locally bounded variation in (t,x). This can be easily seen from Proposition 2.1 and the simple fact that

$$\cos \pi x \sin \pi t \; 1_{\sin \pi x \sin \pi t > 0} = \begin{cases} \cos \pi x \sin \pi t, & \text{if } i \leqslant t < i+1, j \leqslant x < j+1, i+j \text{ is even} \\ 0, & \text{otherwise}. \end{cases}$$

Therefore

$$\begin{aligned}(\sin \pi X(t) \sin \pi t)^+ = {} & \pi \int_0^t \cos \pi s \sin \pi X(s) \; 1_{\sin \pi X(s) \sin \pi s > 0} ds \\ & + \pi \int_0^t \cos \pi X(s) \sin \pi s \; 1_{\sin \pi X(s) \sin \pi s > 0} dX(s) \\ & + \pi \sin \pi t \int_{-\infty}^{\infty} L_t(a) \, \mathrm{d_a} (\cos \pi a \; 1_{\sin \pi a \sin \pi t > 0}) \\ & - \pi \int_0^t \int_{-\infty}^{\infty} L_s(a) \mathrm{d_{s,a}} (\cos \pi a \sin \pi s \; 1_{\sin \pi a \sin \pi s > 0}).\end{aligned}$$

One can expand the last two integrals to see the jump of

$$\cos \pi a \, \sin \pi s \; 1_{\sin \pi a \, \sin \pi s > 0}.$$

Note in Example 2.1, $\nabla^- f(t,x)$ has jump on the boundary of each interval $i \leqslant t < i+1$, $j \leqslant x < j+1$. One can use this example as a prototype to construct many other examples with other types of derivative jumps.

Example 2.2 *Consider the function*

$$f(t,x) = (\sin \pi x)^{\frac{1}{3}} (\sin \pi x \sin \pi t)^+.$$

Then

$$\begin{aligned}\nabla^- f(t,x) = {} & \frac{1}{3} \pi \cos \pi x (\sin \pi x)^{-\frac{2}{3}} (\sin \pi x \sin \pi t)^+ \\ & + \pi (\sin \pi x)^{\frac{1}{3}} \cos \pi x \sin \pi t \; 1_{\sin \pi x \sin \pi t > 0}.\end{aligned}$$

One can verify that $\nabla^- f(t,x)$ is of locally bounded variation in (t,x) and continuous. In fact,

$$\cos \pi x(\sin \pi x)^{-\frac{2}{3}}(\sin \pi x \sin \pi t)^+ = \begin{cases} \cos \pi x(\sin \pi x)^{\frac{1}{3}} \sin \pi t, & \text{if } i \leqslant t < i+1, \\ & j \leqslant x < j+1, i+j \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

then it is easy to see that $\cos \pi x(\sin \pi x)^{-\frac{2}{3}}(\sin \pi x \sin \pi t)^+$ *is of locally bounded variation in* (t,x) *using Proposition 2.1. Similarly one can see that* $(\sin \pi x)^{\frac{1}{3}} \cos \pi x \sin \pi t \, 1_{\sin \pi x \sin \pi t > 0}$ *is of locally bounded variation in* (t,x) *as well.*

Note $\Delta^- f(t,x)$ *blows up when* x *is near an integer value, and their left and right limits also blow up. However one can apply our generalized Itô's formula (2.24) to this function so that*

$$\begin{aligned} &(\sin \pi X(t))^{\frac{1}{3}}(\sin \pi X(t) \sin \pi t)^+ \\ &= \pi \int_0^t (\sin \pi X(s))^{\frac{4}{3}} \cos \pi s \, 1_{\sin \pi X(s) \sin \pi s > 0} \mathrm{ds} \\ &\quad + \int_{-\infty}^{\infty} L_t(a) \, \mathrm{d_a} \Big(\frac{1}{3} \pi \cos \pi a (\sin \pi a)^{-\frac{2}{3}} (\sin \pi a \sin \pi t)^+ \\ &\qquad + \pi (\sin \pi a)^{\frac{1}{3}} \cos \pi a \sin \pi t \, 1_{\sin \pi a \sin \pi t > 0} \Big) \\ &\quad - \int_0^t \int_{-\infty}^{\infty} L_s(a) \, \mathrm{d_{s,a}} \Big(\frac{1}{3} \pi \cos \pi a (\sin \pi a)^{-\frac{2}{3}} (\sin \pi a \sin \pi s)^+ \\ &\qquad + \pi (\sin \pi a)^{\frac{1}{3}} \cos \pi a \sin \pi s \, 1_{\sin \pi a \sin \pi s > 0} \Big). \end{aligned}$$

3 The case for Itô processes

For Itô processes, we can allow some of the generalized derivatives of f to be only in $L^2_{loc}(dtdx)$. Consider

$$X(t) = X(0) + \int_0^t \sigma_r dW_r + \int_0^t b_r dr. \tag{3.1}$$

Here W_r is a one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_r\}_{r \geqslant 0}, P)$ and σ_r and b_r are progressively measurable with respect to $\{\mathcal{F}_r\}$ and satisfy the following conditions: for all $t > 0$

$$\int_0^t |\sigma_r|^2 dr < \infty, \quad \int_0^t |b_r| dr < \infty \quad a.s. \tag{3.2}$$

Under condition (3.2), the process (3.1) is well defined. For any $N > 0$, define $\tau_N = \inf\{s : |X(s)| \geqslant N\}$. Assume there exist constants $\delta > 0$ and $K > 0$ such that

$$\sigma_t(\omega) \geqslant \delta > 0, \quad |\sigma_t(\omega)| + |b_t(\omega)| \leqslant K, \quad \text{for all } (t,\omega) \text{ with } t \leqslant \tau_N. \tag{3.3}$$

The following inequality due to Krylov [16] plays an important role.

Lemma 3.1 *Assume condition (3.2) and (3.3). Then there exists a constant $M>0$, depending only on δ and K such that*

$$E\int_0^{t\wedge\tau_N}|f(r,X(r))|dr \leqslant M\left(\int_0^t\int_{-N}^{+N}(f(r,x))^2 drdx\right)^{\frac{1}{2}}. \tag{3.4}$$

Denote again by $L_t(x)$ the local time of the diffusion process $X(t)$ at level x. We can prove the following theorem.

Theorem 3.1 *Assume $f(t,x)$ is continuous with generalized derivative $\frac{\partial}{\partial t}f$ in $L^2_{loc}(dtdx)$ and generalized derivative ∇f of locally bounded variation in (t,x) and of locally bounded variation in x for $t=0$. Consider an Itô process $X(t)$ given by (3.1) with σ and b satisfying (3.2) and (3.3). Then a.s.*

$$\begin{aligned} f(t,X(t)) = f(0,X(0)) &+ \int_0^t \frac{\partial}{\partial s}f(s,X(s))ds + \int_0^t \nabla f(s,X(s))dX(s) \\ &+ \int_{-\infty}^{\infty} L_t(x)\,\mathrm{d}_x\nabla f(t,x) - \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla f(s,x). \end{aligned} \tag{3.5}$$

Proof. Define f_n by (2.8). From a well-known result on Sobolev spaces (see Theorem 3.16, p.52 in [1]), we know that as $n\to\infty$,

$$f_n(t,x)\to f(t,x),$$

for all (t,x) and for any $N>0$

$$\begin{aligned} &\frac{\partial}{\partial t}f_n \to \frac{\partial}{\partial t}f, \text{ in } L^2([0,t]\times[-N,N]) \\ &\nabla f_n \to \nabla f, \text{ in } L^4([0,t]\times[-N,N]). \end{aligned}$$

As in the proof of Theorem 2.1, we have the Itô formula (2.22) for $f_n(t\wedge\tau_N, X(t\wedge\tau_N))$. The convergence of the terms $f_n(t\wedge\tau_N, X(t\wedge\tau_N))$, and $\frac{1}{2}\int_0^{t\wedge\tau_N}\sigma_s^2\Delta f_n(s,X(s))ds$ is the same as before. Now by using Lemma 3.1,

$$\begin{aligned} &E\left|\int_0^{t\wedge\tau_N}\frac{\partial}{\partial s}f_n(s,X(s))ds - \int_0^{t\wedge\tau_N}\frac{\partial}{\partial s}f(s,X(s))ds\right| \\ &\leqslant E\int_0^{t\wedge\tau_N}\left|\frac{\partial}{\partial s}f_n(s,X(s)) - \frac{\partial}{\partial s}f(s,X(s))\right|ds \\ &\leqslant M\left(\int_0^t\int_{-N}^{N}\left(\frac{\partial}{\partial s}f_n(s,x)-\frac{\partial}{\partial s}f(s,x)\right)^2 dsdx\right)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $n \to \infty$. Similarly one can prove

$$\int_0^{t\wedge\tau_N} b_s \nabla f_n(s, X(s))ds \to \int_0^{t\wedge\tau_N} b_s \nabla f(s, X(s))ds \quad in \quad L^1(dP).$$

Moreover, there exists a constant $M > 0$ such that

$$\begin{aligned} E&\left(\int_0^{t\wedge\tau_N} \sigma_s \nabla f_n(s, X(s))dW_s - \int_0^t \sigma_s \nabla f(s, X(s))dW_s\right)^2 \\ &= E\left(\int_0^{t\wedge\tau_N} \sigma_s^2 (\nabla f_n(s, X(s)) - \nabla f(s, X(s))\right)^2 ds \\ &\leqslant M\left(\int_0^t \int_{-N}^N (\nabla f_n(s, x) - \nabla f(s, x)\right)^4 dsdx \to 0 \end{aligned}$$

as $n \to \infty$. Therefore we have proved that

$$\begin{aligned} &f(t\wedge\tau_N, X(t\wedge\tau_N)) \\ &\quad = f(0, X(0)) + \int_0^{t\wedge\tau_N} \frac{\partial}{\partial s} f(s, X(s))ds + \int_0^{t\wedge\tau_N} \nabla f(s, X(s))dX(s) \\ &\quad + \int_{-\infty}^{\infty} L_{t\wedge\tau_N}(x)\, \mathrm{d}_x \nabla f(t\wedge\tau_N, x) - \int_{-\infty}^{+\infty} \int_0^{t\wedge\tau_N} L_s(x) \mathrm{d}_{s,x} \nabla f(s, x). \end{aligned}$$

The desired formula follows. □

Recall the following extension of Itô's formula due to Krylov [16]: if $f : R^+ \times R$ is C^1 in x and ∇f is absolutely continuous with respect to x for each t and the generalized derivatives $\frac{\partial}{\partial s} f(s, x)$ and Δf are in $L^2_{loc}(dtdx)$, then

$$\begin{aligned} f(t, X(t)) = f(0, z) &+ \int_0^t \frac{\partial}{\partial s} f(s, X(s))ds + \int_0^t \nabla f(s, X(s))dX(s) \\ &+ \frac{1}{2} \int_0^t \sigma_s^2 \Delta f(s, X(s))ds \quad a.s. \end{aligned} \tag{3.6}$$

The next theorem is an easy consequence of the method of proof of Theorem 3.1 and of formula (3.6).

Theorem 3.2 *Assume $f(t, x)$ is continuous and its generalized derivative $\frac{\partial}{\partial t} f$ is in $L^2_{loc}(dtdx)$. Moreover $f(t, x) = f_h(t, x) + f_v(t, x)$ with $f_h(t, x)$ being C^1 in x and $\nabla f_h(t, x)$ having generalized derivative $\Delta f_h(t, x)$ in $L^2_{loc}(dtdx)$, and f_v having generalized derivative $\nabla f_v(t, x)$ being of locally bounded variation in (t, x) and of locally bounded variation in x for $t = 0$. Suppose $X(t)$ is an Itô process given by (3.1) with σ and b satisfying (3.2) and (3.3). Then,*

$$f(t,X(t)) = f(0,X(0)) + \int_0^t \frac{\partial}{\partial s} f(s,X(s))ds + \int_0^t \nabla f(s,X(s))dX(s)$$
$$+\frac{1}{2}\int_0^t \Delta f_h(s,X(s))d\langle X\rangle_s + \int_{-\infty}^{\infty} L_t(x)\, \mathrm{d}_x \nabla f_v(t,x)$$
$$-\int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla f_v(s,x) \quad a.s. \tag{3.7}$$

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References

1. Adams, R.A.: Sobolev Spaces. Academic Press, San Diego (1978)
2. Ash, R.B., Doléans-Dade, C.A.: Probability and Measure Theory. Second Edition, Academic Press, San Diego (2000)
3. Azéma, J., Jeulin, T., Knight, F., Yor, M.: Quelques calculs de compensateurs impliquant l'injectivité de certains processus croissants. In: Séminaire de Probabilités XXXII, LNM1686, 316–327, Springer, Berlin Heidelberg New York (1998)
4. Bouleau, N., Yor, M.: Sur la variation quadratique des temps locaux de certaines semi-martingales. C.R.Acad, Sci. Paris, Ser.I Math **292**, 491–494 (1981)
5. Brosamler, G.A.: Quadratic variation of potentials and harmonic functions. Transactions of the American Mathematical Society **149**, 243–257 (1970)
6. Eisenbaum, N.: Integration with respect to local time. Potential analysis **13**, 303–328 (2000)
7. Eisenbaum, N.: Local time-space calculus for reversible semi-martingales, Séminaire de Probabilités, Vol. 40, Lecture Notes in Mathematics, Springer-Verlag, in this volume
8. Elworthy, K.D., Truman, A., Zhao, H.Z.: The asymptotics of heat equations in the presence of caustics. To be submitted for publication
9. Feng, C.R., Zhao, H.Z.: A generalized Itô's formula in Two-Dimensions and Stochastic Lebesgue–Stieltjes Integrals. Submitted for publication (2004)
10. Feng, C.R., Zhao, H.Z.: Two-parameter p,q-variation paths and integrations of local times. Potential Anal. **25**, 165–204 (2006)

11. Flandoli, F., Russo, F., Wolf, J.: Some stochastic differential equations with distributional drift. Osaka J. Math., **40**, 493–542 (2003)
12. Föllmer, H., Protter, P.: On Itô's formula for multidimensional Brownian motion. Probab. Theory Relat. Fields, **116**, 1–20 (2000)
13. Ghomrasni, R., Peskir, G.: Local time-space calculus and extensions of Itô's formula. University of Aarhus research report (2003)
14. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Second Edition, Springer, Berlin Heidelberg New York (1998)
15. Kendall, W.: The radial part of a Γ-martingale and non-implosion theorem. The Annals of Probability, **23**, 479–500 (1995)
16. Krylov, N.V.: Controlled Diffusion Processes. Springer, Berlin Heidelberg New York (1980)
17. Lyons, T., Qian, Z.: System Control and Rough Paths. Oxford Science Publications, Oxford (2002)
18. Lyons, T.J., Zheng, W.A.: A crossing estimate for the canonical process on a Dirichlet space and a tightness result, Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987). Astérisque, **157-158**, 249–271 (1988)
19. McShane, E.J.: Integration. Princeton University Press, Princeton (1944)
20. Meyer, P.A.: Un cours sur les intégrales stochastiques. In: Lecture Notes in Mathematics, **511**, 245-398, Springer, Berlin Heidelberg New York (1976)
21. Moret, S., Nualart, D.: Generalization of Itô's formula for smooth nondegenerate martingales. Stochastic Process. Appl., **91**, 115–149 (2001)
22. Peskir, G.: A change-of-variable formula and local time on curves. J. Theoret Probab. **18**, 499–535 (2005)
23. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. Second Edition, Springer, Berlin Heidelberg New York (1994)
24. Tanaka, H.: Note on continuous additive functionals of the 1-dimensional Brownian path. Z. Wahrscheinlichkeitstheorie **1**, 251–257(1963)
25. Young, L.C.: General inequality for Stieltjes integrals and the convergence of Fourier series. Mathematische Annalen **115**, 581–612 (1938)

Local Time-Space Calculus for Reversible Semimartingales

Nathalie Eisenbaum

Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6
4 pl. Jussieu, Case 188, 75252 Paris Cedex 05, France
e-mail: nae@ccr.jussieu.fr

Summary. In a recent work [4], we have constructed a stochastic calculus over the plane with respect to the local times of Lévy processes; this led to a general Itô formula requiring only the existence of locally bounded first-order derivatives. We extend this construction to reversible semimartingales and show the part that it can play for extended Itô formulas.

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Key words: Reversible Semimartingale, Stochastic calculus, Local time, Itô formula

1 Introduction

We consider here a semimartingale X satisfying the following two conditions

(H1) $\sum_{0\leqslant t\leqslant 1} |\Delta X_t| < \infty$

(H2) the process $\hat{X}$ is a semimartingale too, where $\hat{X}$ is given by the following definition.

Definition 1.1. Let $(Y_t, t \geqslant 0)$ be a random process. The process $\hat{Y}$ is defined by $\hat{Y}_t = Y_{(1-t)-}$ if $t \in [0,1)$, and $\hat{Y}_1 = Y_0$.

Under these assumptions we construct (Section 2) a stochastic integration over the plane of deterministic functions with respect to $(L_t^x, x \in \mathbb{R}, 0 \leqslant t \leqslant 1)$ the local time process of X. This construction is available for locally bounded deterministic functions over the plane. This allows in particular to define properly a local time process on any measurable curve b for X (Section 3).

Let F be a deterministic function from $\mathbb{R}\times[0,\infty)$ to $\mathbb{R}$. To expand $F(X_t, t)$ according to the classical Itô formula, we would need to assume that F is in $\mathcal{C}^2$ (see Meyer [8]). But Errami, Russo and Vallois [6] have recently established an Itô formula for F in $\mathcal{C}^1$, which, thanks to the construction of Section 2, can be rewritten under the following form (Section 4):

$$\begin{aligned} F(X_t,t) = F(X_0,0) &+ \int_0^t \frac{\partial F}{\partial t}(X_s,s)\,ds + \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)\,dX_s \\ &+ \sum_{0<s\leqslant t}\left\{F(X_s,s) - F(X_{s-},s) - \frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\right\} \\ &- \frac{1}{2}\int_0^t\int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)\,dL_s^x\,. \end{aligned} \tag{1}$$

Note that each term in (1) is still well defined for a function F admitting locally bounded first-order partial derivatives. In view of the recent works of Elworthy, Truman and Zhao [5], Ghomrasni and Peskir [7], and Peskir [9], it hence seems natural to conjecture that this extended Itô formula involving local time-space stochastic integrals should remain true without assuming continuity of the first derivatives.

2 Integration with respect to local times for reversible semimartingales

The semimartingale X admits a local time process that we denote $(L_t^x, x \in \mathbb{R}, t \geqslant 0)$. For the existence and basic properties of this process, one can consult [1].

Consider now an elementary function f_Δ i.e. there exist a finite sequence $(x_i)_{1\leqslant i\leqslant n}$ of real numbers, a subdivision $(s_j)_{1\leqslant j\leqslant m}$ of $[0,1]$ and a family of real numbers $\{f_{ij}, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ such that

$$f_\Delta(x,s) = \sum_{(x_i,s_j)\in\Delta} f_{ij} 1_{(x_i,x_{i+1}]}(x) 1_{(s_j,s_{j+1}]}(s)$$

where $\Delta = \{(x_i,s_j), 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$.

For such a function, integration with respect to L is defined by

$$\int_0^1 \int_{\mathbb{R}} f_\Delta(x,s)\,dL_s^x = \sum_{(x_i,s_j)\in\Delta} f_{ij}\left(L_{s_{j+1}}^{x_{i+1}} - L_{s_j}^{x_{i+1}} - L_{s_{j+1}}^{x_i} + L_{s_j}^{x_i}\right)$$

The problem is to find the set of functions to which this integration could be extended.

The semimartingale $\hat{X}$ admits a local time process on the time interval $[0,1]$ that we denote by $(\hat{L}_t^x, x \in \mathbb{R})$. The occupation time formula gives the following respective characterization of the local times. For any bounded Borel function f

$$\int_0^t f(X_s)\, d[X]_s^c = \int_{\mathbb{R}} f(x) L_t^x \, dx$$
$$\int_0^t f(\hat{X}_s)\, d[\hat{X}]_s^c = \int_{\mathbb{R}} f(x) \hat{L}_t^x \, dx$$

where $[Y]^c$ denotes the continuous part of the quadratic variation process of a process Y. Note that:

$$[\hat{X}]_t^c = [X]_1^c - [X]_{1-t}^c \, .$$

Consequently, we obtain:

$$L_t^x = \hat{L}_1^x - \hat{L}_{1-t}^x \tag{2}$$

Consider now a bounded Borel function f and set $F(x) = \int_0^x f(u)du$. Bouleau and Yor [2] have established the following formula

$$\begin{aligned} F(X_t) = F(X_0) &+ \int_0^t f(X_{s-})\, dX_s - \frac{1}{2}\int f(x)\, d_x L_t^x \\ &+ \sum_{0<s\leqslant t} \{F(X_s) - F(X_{s-}) - f(X_{s-})\Delta X_s\} \, . \end{aligned} \tag{3}$$

Applying this formula to $\hat{X}$ and using Remark (2), we immediately obtain:

$$\begin{aligned} &\int f(x)\, d_x L_t^x \\ &= \int_0^t f(X_{s-})\, dX_s + \int_{1-t}^1 f(\hat{X}_{s-})\, d\hat{X}_s + \sum_{0\leqslant s\leqslant t} \{f(X_s) - f(X_{s-})\}\Delta X_s \, . \end{aligned}$$

Note that the process $\big(X_t - \sum_{0\leqslant s\leqslant t} \Delta X_s, 0 \leqslant t \leqslant 1\big)$ is a continuous semimartingale $(M_t + V_t, 0 \leqslant t \leqslant 1)$ where M is a continuous local martingale and V is a continuous process with bounded variations. Note also that the process $\hat{M}$ is a continuous semimartingale with respect to the natural filtration of $\hat{X}$. Hence, we obtain:

$$\int f(x)\, d_x L_t^x = \int_0^t f(X_{s-})\, dM_s + \int_{1-t}^1 f(\hat{X}_{s-})\, d\hat{M}_s.$$

Choosing then $f(x) = 1_{(-\infty,a]}(x)$, we obtain the following lemma.

Lemma 2.1. *The local time process* $(L_t^x, x \in \mathbb{R}, 0 \leqslant t \leqslant 1)$ *of* X *satisfies*

$$L_t^x = \int_0^t 1_{(X_{s-}\leqslant x)}\, dM_s + \int_{1-t}^1 1_{(\hat{X}_{s-}\leqslant x)}\, d\hat{M}_s.$$

Let f_Δ be an elementary function. Thanks to Lemma 2.1, we have

$$\int_0^t\int_{\mathbb{R}} f_\Delta(x,s)\,dL_s^x = \sum_{(x_i,s_j)\in\Delta} f_{ij}\big(L_{s_{j+1}}^{x_{i+1}} - L_{s_j}^{x_{i+1}} - L_{s_{j+1}}^{x_i} + L_{s_j}^{x_i}\big)$$

$$= \sum_{(x_i,s_j)\in\Delta} f_{ij}\Big\{\int_{s_j}^{s_{j+1}} 1_{(x_i,x_{i+1}]}(X_{s-})\,dM_s + \int_{1-s_{j+1}}^{1-s_j} 1_{(x_i,x_{i+1}]}(\hat{X}_{s-})\,d\hat{M}_s\Big\}$$

$$= \int_0^t f_\Delta(X_{s-},s)\,dM_s + \int_{1-t}^1 f_\Delta(\hat{X}_{s-},1-s)\,d\hat{M}_s\ .$$

Let f be a bounded deterministic function. Then there exists a bounded sequence of elementary functions $(f_n, n\geqslant 0)$ converging simply to f. The corresponding sequences $\big(\int_0^t f_n(X_{s-},s)\,dM_s, n\geqslant 0\big)$ and $\big(\int_{1-t}^1 f_n(\hat{X}_{s-},1-s)\,d\hat{M}_s, n\geqslant 0\big)$ are converging in probability to, respectively, $\int_0^t f(X_{s-},s)\,dM_s$ and $\int_{1-t}^1 f(\hat{X}_{s-},1-s)\,d\hat{M}_s$. These limits do not depend on the choice of the sequence $(f_n, n\geqslant 0)$. Hence $\big(\int_0^t\int_{\mathbb{R}} f(x,s)\,dL_s^x, t\in[0,1]\big)$ is well defined and satisfies

$$\int_0^t\int_{\mathbb{R}} f(x,s)\,dL_s^x = \int_0^t f(X_{s-},s)\,dM_s + \int_{1-t}^1 f(\hat{X}_{s-},1-s)\,d\hat{M}_s;\ 0\leqslant t\leqslant 1.$$

The following theorem can then be proved similarly as in [4].

Theorem 2.2. *Let f be a locally bounded measurable function from $\mathbb{R}\times[0,1]$ to $\mathbb{R}$. Then $\int_0^t\int_a^b f(x,s)\,dL_s^x$ is well defined for any couple (a,b) and converges in probability as a and b tend, respectively, to $-\infty$ and $+\infty$. We define the integral $\int_0^t\int_{\mathbb{R}} f(x,s)\,dL_s^x$ as this limit. Moreover we have:*

$$\int_0^t\int_{\mathbb{R}} f(x,s)\,dL_s^x = \int_0^t f(X_{s-},s)\,dM_s + \int_{1-t}^1 f(\hat{X}_{s-},1-s)\,d\hat{M}_s;\ 0\leqslant t\leqslant 1.$$

The following lemma gives simple rules to compute integrals with respect to local times.

The covariation of two processes Y and Z on the time interval $[0,t]$ is defined as the following limit when it exists in probability

$$[Y,Z]_t = \lim_{n\to\infty}\sum_{i=1}^n (Y_{t_{i+1}} - Y_{t_i})(Z_{t_{i+1}} - Z_{t_i})\ ,$$

where the limit is taken over all sequences of the subdivisions $0=t_1<t_2<t_3<\cdots<t_n=t$ such that $\sup_{1\leqslant i\leqslant n}|t_{i+1}-t_i|$ tends to 0 when n tends to ∞.

Lemma 2.3. *(i) Let f be a continuous function on $\mathbb{R}\times[0,1]$; then*

$$\int_0^t\int_{\mathbb{R}} f(x,s)\,dL_s^x = -[f(X_.,.),M_.]_t\ .$$

(ii) Let f be a function on $\mathbb{R} \times [0,1]$ admitting a continuous derivative $\frac{\partial f}{\partial x}$; then

$$\int_0^1 \int_{\mathbb{R}} f(x,s)\, dL_s^x$$

exists and

$$\int_0^1 \int_{\mathbb{R}} f(x,s)\, dL_s^x = -\int_0^1 \frac{\partial f}{\partial x}(X_s,s)\, d[X]_s^c \,.$$

The proof of Lemma 2.3 (i) is similar to that of Lemma 2.6 (i) [4]. The proof of (ii) will be given in the proof of Lemma 3.1.

3 Local times on curves for reversible semimartingales

Similarly to what has been done in [4], we define for X a local time process on any Borel curve $(b(t), 0 \leqslant t \leqslant 1)$ as follows:

$$L_t^{b(.)} = \int_0^t \int_{\mathbb{R}} 1_{(-\infty, b(s))}(x)\, dL_s^x.$$

The legitimacy of this definition is not as obvious as in the particular case of a Lévy process. To show that for a continuous curve $(b(t), 0 \leqslant t \leqslant 1)$ the limit of $\frac{1}{2\epsilon}\int_0^t 1_{(|X_s - b(s)| < \epsilon)}\, d[X]_s^c$ always exists in probability when ϵ tends to 0 and coincides with the above expression, we first need to establish the following lemma.

Lemma 3.1. *Let b be a continuous function from $[0,1]$ to $\mathbb{R}$. Let f be a continuous function on $\mathbb{R} \times \mathbb{R}_+$, admitting a continuous derivative $\frac{\partial f}{\partial x}$. Then we have:*

$$\int_0^t \int_{\mathbb{R}} f(x,s) 1_{(x \leqslant b(s))}\, dL_s^x = \int_0^t f(b(s),s)\, d_s L_s^{b(.)} - \int_0^t \frac{\partial f}{\partial x}(X_s,s) 1_{(X_s \leqslant b(s))}\, d[X]_s^c.$$

Proof. Similarly to the proof of Lemma 3.1 in [4], we can first establish that for any $a < b$

$$\int_0^t \int_{\mathbb{R}} 1_{[a,b]}(x) f(x,s)\, dL_s^x = \int_0^t f(b,s)\, d_s L_s^b - \int_0^t f(a,s)\, d_s L_s^a - \int_0^t \frac{\partial f}{\partial x}(X_s,s) 1_{[a,b]}(X_s)\, d[X]_s^c.$$

We note then that for $a < \inf_{s \in [0,t]} X_s$, the right-hand term of this identity is equal to $\int_0^t f(b,s)\, d_s L_s^b - \int_0^t \frac{\partial f}{\partial x}(X_s,s) 1_{(X_s \leqslant b)}\, d[X]_s^c$, hence the left-hand term of the identity converges pointwise as a tends to $-\infty$, which leads to

$$\int_0^t \int_{\mathbb{R}} f(x,s) 1_{(x \leqslant b)}\, dL_s^x = \int_0^t f(b,s)\, d_s L_s^b - \int_0^t \frac{\partial f}{\partial x}(X_s,s) 1_{(X_s \leqslant b)}\, d[X]_s^c \,. \quad (4)$$

Similarly, letting b tend to $+\infty$ one obtains

$$\int_0^t \int_{\mathbb{R}} f(x,s)\, dL_s^x = -\int_0^t \frac{\partial f}{\partial x}(X_s,s)\, d[X]_s^c \,.$$

which establishes Lemma 2.3 (ii).

Let b_Δ be the curve defined by $b_\Delta(s) = \sum_j \big(\sup_{u\in[s_j,s_{j+1})} b(u)\big) 1_{[s_j,s_{j+1})}(s)$, where $\Delta = (s_j)$ is a finite subdivision of $[0,1]$. Using (4), we have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} f(x,s) 1_{(x\leqslant b_\Delta(s))}\, dL_s^x &= \sum_j \int_{s_j}^{s_{j+1}} \int_{\mathbb{R}} f(x,s) 1_{(x\leqslant b_\Delta(s_j))}\, dL_s^x \\ &= \sum_j \int_{s_j}^{s_{j+1}} f(b_\Delta(s_j),s)\, d_s L_s^{b_\Delta(s_j)} \\ &\quad - \int_{s_j}^{s_{j+1}} \frac{\partial f}{\partial x}(X_s,s) 1_{(X_s\leqslant b_\Delta(s_j))}\, d[X]_s^c.\end{aligned}$$

Note that Lemma 2.1 together with Theorem 2.2 give for any Borel function h

$$\int_0^t \int_{\mathbb{R}} h(s) 1_{(x\leqslant a)}\, dL_s^x = \int_0^t h(s)\, d_s L_s^a \,.$$

Hence

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} f(x,s) 1_{(x\leqslant b_\Delta(s))}\, dL_s^x &= \int_0^t \int_{\mathbb{R}} f(b_\Delta(s),s) 1_{(x\leqslant b_\Delta(s))}\, dL_s^x \\ &\quad - \int_0^t \frac{\partial f}{\partial x}(X_s,s) 1_{(X_s\leqslant b_\Delta(s))}\, d[X]_s^c.\end{aligned}$$

We note that both $f(x,s)1_{(x\leqslant b_\Delta(s))}$ and $f(b_\Delta(s),s)1_{(x\leqslant b_\Delta(s))}$ converge simply as $|\Delta|$ tends to 0 to, respectively, $f(x,s)1_{(x\leqslant b(s))}$ and $f(b(s),s)1_{(x\leqslant b(s))}$. Consequently letting $|\Delta|$ tend to 0 in the above identity provides

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} f(x,s) 1_{(x\leqslant b(s))}\, dL_s^x &= \int_0^t \int_{\mathbb{R}} f(b(s),s) 1_{(x\leqslant b(s))}\, dL_s^x \\ &\quad - \int_0^t \frac{\partial f}{\partial x}(X_s,s) 1_{(X_s\leqslant b(s))}\, d[X]_s^c \,. \end{aligned} \tag{5}$$

Thanks to the continuity of b, one can easily prove that

$$\int_0^t f(b(s),s)\, d_s L_s^{b(.)} = \int_0^t \int_{\mathbb{R}} f(b(s),s) 1_{(x\leqslant b(s))}\, dL_s^x,$$

indeed $\int_0^t f(b(s),s)\, d_s L_s^{b(.)}$ is a Stieltjes integral with respect to a continuous process. Since $(f(b(s),s), s\in[0,1])$ is continuous, $\sum_j f(b(s_j),s_j)1_{[s_j,s_{j+1})}(s)$ is bounded (uniformly in Δ) and converges pointwise to $f(b(s),s)$ as $|\Delta|$ tends to 0. Hence thanks to the definition of $L^{b(.)}$

$$\begin{aligned}\int_0^t f(b(s),s)\,d_s L_s^{b(.)} &= \lim_{|\Delta|\to 0} \sum_j f(b(s_j),s_j)\big(L_{s_{j+1}}^{b(.)} - L_{s_j}^{b(.)}\big)\\ &= \lim_{|\Delta|\to 0} \int_0^1 \int_{\mathbb{R}} \sum_j f(b(s_j),s_j) 1_{[s_j,s_{j+1})}(s) 1_{(-\infty,b(s))}(x)\,dL_s^x\\ &= \int_0^t \int_{\mathbb{R}} f(b(s),s) 1_{(x\leqslant b(s))}\,dL_s^x\end{aligned}$$

thanks to Theorem 2.2. This together with (5) prove Lemma 3.1. □

Lemma 3.2. *Let b be a continuous function from $[0,1]$ to $\mathbb{R}$. We then have*

$$\frac{1}{2\epsilon}\int_0^t 1_{(|X_s-b(s)|<\epsilon)}\,d[X]_s^c \underset{n\to\infty}{\overset{Probability}{\longrightarrow}} L_t^{b(.)}.$$

Proof. Let f_ϵ be the function defined on $\mathbb{R}\times[0,1]$ by

$$f_\epsilon(x,t) = \frac{1}{2\epsilon}\int_x^{+\infty} 1_{(|y-b(t)|<\epsilon)}\,dy.$$

This function is continuous and uniformly bounded by 1. It admits a continuous derivative with respect to x on $\{x \neq b(t)\}$, but does not satisfy the assumptions of Lemma 2.3. Keeping the notations of the proof of Lemma 2.3 (ii), we set

$$f_{n,\epsilon}(x,t) = \iint_{\mathbb{R}^2} f_\epsilon\Big(x-\frac{y}{n}, t-\frac{s}{n}\Big)\,g(y)\,h(s)\,dy\,ds.$$

The sequence $(f_{n,\epsilon}(x,t), n\geqslant 0)$ converges to $f_\epsilon(x,t)$, consequently thanks to Theorem 2.2

$$\int_0^t\int_{\mathbb{R}} f_{n,\epsilon}(x,s)\,dL_s^x \underset{n\to\infty}{\overset{\text{Probability}}{\longrightarrow}} \int_0^t\int_{\mathbb{R}} f_\epsilon(x,s)\,dL_s^x.$$

We have thanks to Lemma 2.3 (ii)

$$\int_0^t\int_{\mathbb{R}} f_{n,\epsilon}(x,s)\,dL_s^x = -\int_0^t \frac{\partial f_{n,\epsilon}}{\partial x}(X_s,s)\,d[X]_s^c.$$

But as a consequence of (5), if b is continuous and H is any continuous function on $\mathbb{R}\times[0,1]$, then

$$\int_0^t H(X_s,s) 1_{(X_s=b(s))}\,d[X]_s^c = 0. \tag{6}$$

The above identity is obtained by subtracting (5) for the curve b to (5) for the curve $b-\epsilon$ and by letting ϵ decrease to 0.

Hence: $\int_0^t \int_{\mathbb{R}} f_{n,\epsilon}(x,s)\, dL_s^x = -\int_0^t \frac{\partial f_{n,\epsilon}}{\partial x}(X_s,s) 1_{(X_s \neq b(s))}\, d[X]_s^c$.

This last integral converges as n tends to ∞ to $-\int_0^t \frac{\partial f_\epsilon}{\partial x}(X_s,s) 1_{(X_s \neq b(s))}\, d[X]_s^c$. Consequently:

$$\int_0^t \int_{\mathbb{R}} f_\epsilon(x,s)\, dL_s^x = -\int_0^t \frac{\partial f_\epsilon}{\partial x}(X_s,s) 1_{(X_s \neq b(s))}\, d[X]_s^c\ .$$

Now, since $f_\epsilon(x,s)$ converges pointwise to $1_{(b(s)>x)} + \frac{1}{2} 1_{(x=b(s))}$ as ϵ tends to 0, we have thanks to Theorem 2.2

$$\int_0^t \int_{\mathbb{R}} f_\epsilon(x,s)\, dL_s^x \overset{\text{Probability}}{\underset{n\to\infty}{\longrightarrow}} L_t^{b(.)} + \frac{1}{2}\int_0^t \int_{\mathbb{R}} 1_{(x=b(s))}\, dL_s^x. \tag{7}$$

On the one hand, we use (8) for the constant function $H(x,t) = 1$ to note that $\int_0^t 1_{(X_s=b(s))}\, d[X]_s^c = 0$. This leads to

$$\int_0^t \int_{\mathbb{R}} f_\epsilon(x,s)\, dL_s^x = \frac{1}{2\epsilon}\int_0^t 1_{(|X_s-b(s)|<\epsilon)}\, d[X]_s^c$$

On the other hand, applying Lemma 3.1 to the constant function $f(x,t) = 1$, we obtain: $\int_0^t \int_{\mathbb{R}} 1_{(x\leqslant b(s))}\, dL_s^x = L_t^{b(.)}$. Consequently, $\int_0^t \int_{\mathbb{R}} 1_{(x=b(s))}\, dL_s^x = 0$, and we have finally obtained thanks to (7)

$$\frac{1}{2\epsilon}\int_0^t 1_{(|X_s-b(s)|<\epsilon)}\, d[X]_s^c \overset{\text{Probability}}{\underset{n\to\infty}{\longrightarrow}} L_t^{b(.)}\ . \qquad \square$$

4 An Itô formula involving local time-space integrals

In the case of a semimartingale satisfying (H1) and (H2), Errami, Russo and Vallois [6] have established the following Itô formula for a function F in $\mathcal{C}^1(\mathbb{R}^2)$

$$\begin{aligned} F(X_t,t) &= F(X_0,0) + \int_0^t \frac{\partial F}{\partial t}(X_s,s)\, ds + \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)\, dX_s \\ &\quad + \sum_{0<s\leqslant t}\Bigg\{F(X_s,s) - F(X_{s-},s) \\ &\qquad - \frac{1}{2}\left(\frac{\partial F}{\partial x}(X_s,s) + \frac{\partial F}{\partial x}(X_{s-},s)\right)\Delta X_s\Bigg\} \\ &\quad + \frac{1}{2}\left[\frac{\partial F}{\partial x}(X_.,.), X_.\right]_t\ . \end{aligned}$$

We note that

$$\frac{1}{2}\left[\frac{\partial F}{\partial x}(X_{.},.), M_{.}\right]_t$$
$$= \frac{1}{2}\left[\frac{\partial F}{\partial x}(X_{.},.), X_{.}\right]_t - \frac{1}{2}\sum_{0<s\leqslant t}\left(\frac{\partial F}{\partial x}(X_s, s) - \frac{\partial F}{\partial x}(X_{s-}, s)\right)\Delta X_s.$$

Besides, thanks to Lemma 2.3 (i), we know that:

$$\left[\frac{\partial F}{\partial x}(X_{.},.), X_{.}\right]_t = -\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)\, dL_s^x \,.$$

Hence, we finally see that

$$\begin{aligned} F(X_t,t) = F(X_0,0) &+ \int_0^t \frac{\partial F}{\partial t}(X_s,s)\,ds + \int_0^t \frac{\partial F}{\partial x}(X_{s-},s)\,dX_s \\ &+ \sum_{0<s\leqslant t}\left\{F(X_s,s) - F(X_{s-},s) - \frac{\partial F}{\partial x}(X_{s-},s)\Delta X_s\right\} \\ &- \frac{1}{2}\int_0^t\int_{\mathbb{R}} \frac{\partial F}{\partial x}(x,s)\,dL_s^x \,. \end{aligned} \tag{8}$$

In [4] we could prove that in the particular case of Lévy processes (8) remains valid for functions F such that $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial x}$ are only required to be locally bounded. The problem is that in the general case X might live with positive probability on the discontinuous points of the partial derivatives. In the special case of a function F everywhere $\mathcal{C}^2$ except along a curve $\{x = b(t)\}$, one should see the solution of that problem established by Peskir [9].

References

1. Azéma J., Yor M. (1978): Temps locaux. *Astérisque 52/53, Société Mathématique de France*
2. Bouleau N., Yor M. (1981): Sur la variation quadratique des temps locaux de certaines semi-martingales. *C.R. Acad. Sc. Paris* 292 (491–494)
3. Eisenbaum N. (2000): Integration with respect to local time. *Potential Anal.* 13 (303–328)
4. Eisenbaum N. (2006): Local time-space stochastic calculus for Lévy processes. *Stochastic Process. Appl.* **116** (757–778)
5. Elworthy K.D., Truman A., Zhao H.Z. (2005): A generalized Itô formula and space-time Lebesgue–Stieltjes integrals of local times, *in this volume*
6. Errami M., Russo F., Vallois P. (2002): Itô formula for $C^{1,\lambda}$-functions of a càdlàg process. *Probab. Theory Relat. Fields* 122 (191–221)
7. Ghomrasni R., Peskir G. (2003): Local time-space calculus and extensions of Itô's formula. *Proc. High Dim. Probab.* III (Sandbjerg 2002) *Prog. Probab.* 55, Birkhäuser Basel, (177–192)

8. Meyer P.A. (1976): Un cours sur les intégrales stochastiques. *Séminaire de Probabilités X* – Lect. Notes 511 (246–354)
9. Peskir G. (2005): A change-of-variable formula with local time on curves. *J. Theor. Probab.* **18** (499–535)

Elements of Stochastic Calculus via Regularization

A la mémoire de Paul André Meyer

Francesco Russo[1] **and Pierre Vallois**[2]

[1] Institut Galilée, Mathématiques, Université Paris 13
99 avenue J.B. Clément, 93430 Villetaneuse, France
e-mail: russo@math.univ-paris13.fr
[2] Institut de Mathématiques Elie Cartan, Université Henri Poincaré
B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France
e-mail: vallois@iecn.u-nancy.fr

Summary. This paper first summarizes the foundations of stochastic calculus via regularization and constructs through this procedure Itô and Stratonovich integrals. In the second part, a survey and new results are presented in relation with finite quadratic variation processes, Dirichlet and weak Dirichlet processes.

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1 Introduction

Stochastic integration via regularization is a technique of integration developed in a series of papers by the authors starting from [46], continued in [47, 48, 49, 50, 45] and later carried out by other authors, among them [51, 12, 13, 55, 54, 56, 58, 17, 16, 18, 19, 24]. Among some recent applications to finance, we refer for instance to [32, 4].

This approach constitutes a counterpart of a discretization approach initiated by Föllmer [20] and continued by many authors, see for instance [2, 22, 15, 14, 11, 23].

The two theories run parallel and, at the axiomatic level, almost all the results we obtained via regularization can essentially be translated in the language of discretization.

The advantage of using regularization lies in the fact that this approach is natural and relatively simple, and easily connects to other approaches. We now list some typical features of stochastic calculus via regularization.

- Two fundamental notions are the quadratic variation of a process, see Definition 2 and the forward integral, see Definition 1. Calculus via regularization is first of all a calculus related to finite quadratic variation processes, see Section 4. Itô integrals with respect to continuous semimartingales can be defined through forward integrals, see Section 3; this makes classical stochastic calculus appear as a particular instance of calculus via regularization. Let the integrator be a classical Brownian motion W and the integrand a measurable adapted process H such that $\int_0^T H_t^2 dt < \infty$ a.s., where a.s. means almost surely. We will show in Section 3.5 that the forward integral $\int_0^\cdot H d^- W$ coincides with the Itô integral $\int_0^\cdot H dW$. On the other hand, the discretization approach constitutes a sort of Riemann–Stieltjes type integral and only allows integration of processes that are not too irregular, see Remark 14.
- Calculus via regularization constitutes a bridge between noncausal and causal calculus operating through substitution formula, see Section 3.6. A precise link between forward integration and the theory of enlargement of filtrations may be given, see [47]. Our integrals can be connected to the well-known Skorohod type integrals, see again [47].
- With the help of symmetric integrals a calculus with respect to processes with a variation higher than 2 may be developed. For instance fractional Brownian motion is the prototype of such processes.
- This stochastic calculus constitutes somehow a barrier separating the pure pathwise calculus in the sense of T. Lyons and coauthors, see e.g., [36, 35, 31, 28], and any stochastic calculus taking into account an underlying probability, see Section 6.

This paper will essentially focus on the first item.

The paper is organized as follows. First, in Section 2, we recall the basic definitions and properties of forward, backward, symmetric integrals and covariations. Justifying the related definitions and properties needs no particular effort. A significant example is the Young integral, see [57]. In Section 3 we redefine Itô integrals in the spirit of integrals via regularization and we prove some typical properties. We essentially define Itô integrals as forward integrals in a subclass and we then extend this definition through functional analysis methods. Section 4 is devoted to finite quadratic variation processes. In particular we establish C^1-stability properties and an Itô formula of C^2-type. Section 5 provides some survey material with new results related to the class of weak Dirichlet processes introduced by [12] with later developments discussed by [24, 7]. Considerations about Itô formula under C^1-conditions are discussed as well.

2 Stochastic integration via regularization

2.1 Definitions and fundamental properties

In this paper T will be a fixed positive real number. By convention, any real continuous function f defined either on $[0,T]$ or $\mathbb{R}_+$ will be prolongated (with the same name) to the real line, setting

$$f(t) = \begin{cases} f(0) \text{ if } t \leqslant 0 \\ f(T) \text{ if } t > T. \end{cases} \tag{1}$$

Let $(X_t)_{t\geqslant 0}$ be a continuous process and $(Y_t)_{t\geqslant 0}$ be a process with paths in $L^1_{loc}(\mathbb{R}_+)$, i.e., for any $a > 0$, $\int_0^a |Y_t|dt < \infty$ a.s.

Our generalized stochastic integrals and covariations will be defined through a regularization procedure. More precisely, let $I^-(\varepsilon, Y, dX)$ (resp. $I^+(\varepsilon, Y, dX)$, $I^0(\varepsilon, Y, dX)$ and $C(\varepsilon, Y, X)$) be the ε-forward integral (resp. ε-backward integral, ε-symmetric integral and ε-covariation):

$$I^-(\varepsilon, Y, dX)(t) = \int_0^t Y(s)\frac{X(s+\varepsilon) - X(s)}{\varepsilon}ds; \quad t \geqslant 0, \tag{2}$$

$$I^+(\varepsilon, Y, dX)(t) = \int_0^t Y(s)\frac{X(s) - X(s-\varepsilon)}{\varepsilon}ds; \quad t \geqslant 0, \tag{3}$$

$$I^0(\varepsilon, Y, dX)(t) = \int_0^t Y(s)\frac{X(s+\varepsilon) - X(s-\varepsilon)}{2\varepsilon}ds; \quad t \geqslant 0, \tag{4}$$

$$C(\varepsilon, X, Y)(t) = \int_0^t \frac{\big(X(s+\varepsilon) - X(s)\big)\big(Y(s+\varepsilon) - Y(s)\big)}{\varepsilon}ds; \quad t \geqslant 0. \tag{5}$$

Observe that these four processes are continuous.

Definition 1.

1) *A family of processes* $(H_t^{(\varepsilon)})_{t\in[0,T]}$ *is said to converge to* $(H_t)_{t\in[0,T]}$ *in the* **ucp sense**, *if* $\sup_{0\leqslant t\leqslant T} |H_t^{(\varepsilon)} - H_t|$ *goes to* 0 *in probability, as* $\varepsilon \to 0$.

2) *Provided the corresponding limits exist in the ucp sense, we define the following integrals and covariations by the following formula:*

   a) **Forward integral:** $\int_0^t Y d^- X = \lim_{\varepsilon\to 0^+} I^-(\varepsilon, Y, dX)(t)$.

   b) **Backward integral:** $\int_0^t Y d^+ X = \lim_{\varepsilon\to 0^+} I^+(\varepsilon, Y, dX)(t)$.

   c) **Symmetric integral:** $\int_0^t Y d^\circ X = \lim_{\varepsilon\to 0+} I^\circ(\varepsilon, Y, dX)(t)$.

   d) **Covariation:** $[X,Y]_t = \lim_{\varepsilon\to 0^+} C(\varepsilon, X, Y)(t)$. *When* $X = Y$ *we often put* $[X] = [X,X]$.

Remark 1. Let X, X', Y, Y' be four processes with X, X' continuous and Y, Y' having paths in $L^1_{loc}(\mathbb{R}_+)$. $\star$ will stand for one of the three symbols $-$, $+$ or $\circ$.

1. $(X, Y) \mapsto \int_0^{\cdot} Y d^\star X$ and $(X, Y) \mapsto [X, Y]$ are bilinear operations.
2. The covariation of continuous processes is a symmetric operation.
3. When it exists, $[X]$ is an increasing process.
4. If τ is a random time, $[X^\tau, X^\tau]_t = [X, X]_{t\wedge\tau}$ and
$$\int_0^t Y 1_{[0,\tau]} d^\star X = \int_0^t Y d^\star X^\tau = \int_0^t Y^\tau d^\star X^\tau = \int_0^{t\wedge\tau} Y d^\star X,$$
where X^τ is the process X stopped at time τ, defined by $X^\tau_t = X_{t\wedge\tau}$.
5. If ξ and η are two fixed r.v., $\displaystyle\int_0^{\cdot} (\xi Y_s) d^\star (\eta X_s) = \xi\eta \int_0^{\cdot} Y_s d^\star X_s$.
6. Integrals via regularization also have the following localization property. Suppose that $X_t = X'_t, Y_t = Y'_t, \forall t \in [0, T]$ on some subset Ω_0 of Ω. Then
$$1_{\Omega_0} \int_0^t Y_s d^\star X_s = 1_{\Omega_0} \int_0^t Y'_s d^\star X'_s, \quad t \in [0, T].$$
7. If Y is an elementary process of the type $Y_t = \sum_{i=1}^{N} A_i 1_{I_i}$, where A_i are random variables and (I_i) a family of real intervals with end-points $a_i < b_i$, then
$$\int_0^t Y_s d^\star X_s = \sum_{i=1}^{N} A_i (X_{b_i\wedge t} - X_{a_i\wedge t}).$$

Definition 2.

1) If $[X]$ exists, X is said to be a **finite quadratic variation process** *and $[X]$ is called the* **quadratic variation** *of X.*
2) If $[X] = 0$, X is called a **zero quadratic variation process**.
3) A vector $(X^1, \ldots, X^n)$ of continuous processes is said to have all its **mutual covariations** *if $[X^i, X^j]$ exists for all $1 \leqslant i, j \leqslant n$.*
We will also use the terminology **bracket** *instead of covariation.*

Remark 2.

1) If $(X^1, \ldots, X^n)$ has all its mutual covariations, then
$$[X^i + X^j, X^i + X^j] = [X^i, X^i] + 2[X^i, X^j] + [X^j, X^j]. \tag{6}$$
From the previous equality, it follows that $[X^i, X^j]$ is the difference of two increasing processes, having therefore bounded variation; consequently the bracket is a classical integrator in the Lebesgue–Stieltjes sense.
2) Relation (6) holds as soon as three brackets among the four exist. More generally, by convention, an identity of the type $I_1 + \cdots + I_n = 0$ has the following meaning: if $n - 1$ terms among the I_j exist, the remaining one also makes sense and the identity holds true.

3) We will see later, in Remark 23, that there exist processes X and Y such that $[X,Y]$ exists but does not have finite variation; in particular (X,Y) does not have all its mutual brackets.

The properties below follow elementarily from the definition of integrals via regularization.

Proposition 1. *Let $X = (X_t)_{t\geqslant 0}$ be a continuous process and $Y = (Y_t)_{t\geqslant 0}$ be a process with paths in $L^1_{loc}(\mathbb{R}_+)$. Then*

1) $[X,Y]_t = \int_0^t Y d^+X - \int_0^t Y d^-X.$

2) $\int_0^t Y d^\circ X = \frac{1}{2}\left(\int_0^t Y d^+X + \int_0^t Y d^-X\right).$

3) **Time reversal**. *Set $\hat{X}_t = X_{T-t}, t \in [0,T]$. Then*

   1. $\int_0^t Y d^\pm X = -\int_{T-t}^T \hat{Y} d^\mp \hat{X}, \quad 0 \leqslant t \leqslant T;$

   2. $\int_0^t Y d^\circ X = -\int_{T-t}^T \hat{Y} d^\circ \hat{X}, \quad 0 \leqslant t \leqslant T;$

   3. $[\hat{X},\hat{Y}]_t = [X,Y]_T - [X,Y]_{T-t}, \quad 0 \leqslant t \leqslant T.$

4) **Integration by parts**. *If Y is continuous,*

$$\begin{aligned} X_tY_t &= X_0Y_0 + \int_0^t X d^-Y + \int_0^t Y d^+X \\ &= X_0Y_0 + \int_0^t X d^-Y + \int_0^t Y d^-X + [X,Y]_t. \end{aligned}$$

5) **Kunita–Watanabe inequality**. *If X and Y are finite quadratic variation processes,*

$$\big|[X,Y]\big| \leqslant \big\{[X]\ [Y]\big\}^{1/2}.$$

6) If X is a finite quadratic variation process and Y is a zero quadratic variation process then (X,Y) has all its mutual brackets and $[X,Y] = 0$.

7) Let X be a bounded variation process and Y be a process with locally bounded paths, and at most countably many discontinuities. Then

   a) $\int_0^t Y d^+X = \int_0^t Y d^-X = \int_0^t Y dX$, *where* $\int_0^t Y dX$ *is a Lebesgue–Stieltjes integral.*

   b) $[X,Y] = 0$. *In particular a bounded variation and continuous process is a zero quadratic variation process.*

8) Let X be an absolutely continuous process and Y be a process with locally bounded paths. Then

$$\int_0^t Y d^+X = \int_0^t Y d^-X = \int_0^t Y X' ds.$$

Remark 3. If Y has uncountably many discontinuities, 7) may fail. Take for instance $Y = 1_{\operatorname{supp} dV}$, where V is an increasing continuous function such that $V'(t) = 0$ a.e. (almost everywhere) with respect to Lebesgue measure. Then $Y = 0$ Lebesgue a.e., and $Y = 1, dV$ a.e. Consequently

$$\int_0^t Y dV = V(t) - V(0), \quad I^-(\varepsilon, Y, dV)(t) = 0 \quad \int_0^t Y d^- V = 0.$$

Remark 4. Point 2) of Proposition 1 states that the symmetric integral is the average of the forward and backward integrals.

Proof of Proposition 1. Points 1), 2), 3), 4) follow immediately from the definition. For illustration, we only prove 3); operating a change of variable $u = T - s$, we obtain

$$\int_0^t Y_s \frac{X_s - X_{s-\varepsilon}}{\varepsilon} ds = - \int_{T-t}^T \hat{Y}_u \frac{\hat{X}_{u+\varepsilon} - \hat{X}_u}{\varepsilon} du, \quad 0 \leqslant t \leqslant T.$$

Since X is continuous, one can take the limit of both members and the result follows.

5) follows by Cauchy–Schwarz inequality which says that

$$\begin{aligned} &\frac{1}{\varepsilon} \left| \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) ds \right| \\ &\quad \leqslant \left\{ \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)^2 ds \; \frac{1}{\varepsilon} \int_0^t (Y_{s+\varepsilon} - Y_s)^2 ds \right\}^{\frac{1}{2}}. \end{aligned}$$

6) is a consequence of 5).
7) Using Fubini, one has

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^t Y_s (X_{s+\varepsilon} - X_s) ds &= \frac{1}{\varepsilon} \int_0^t ds\, Y_s \int_s^{s+\varepsilon} dX_u \\ &= \int_0^{t+\varepsilon} dX_u \, \frac{1}{\varepsilon} \int_{u-\varepsilon}^{u \wedge t} Y_s ds. \end{aligned}$$

Since the jumps of Y are at most countable, $\frac{1}{\varepsilon} \int_{u-\varepsilon}^u Y_s ds \to Y_u, d|X|$ a.e. where $|X|$ denotes the total variation of X. Since $t \to Y_t$ is locally bounded, Lebesgue's convergence theorem implies that $\int_0^t Y d^- X = \int_0^t Y dX$.

The fact that $\int_0^t Y d^+ X = \int_0^t Y dX$ follows similarly.
b) is a consequence of point 1).
8) can be reached using similar elementary integration properties. □

2.2 Young integral in a simplified framework

We will consider the integral defined by Young ([57]) in 1936, and implemented in the stochastic framework by Bertoin, see [3]. Here we will restrict ourselves to the case when integrand and integrator are Hölder continuous processes. As a result, that integral will be shown to coincide with the forward integral, but also with backward and symmetric ones.

Definition 3.

1. *Let* C^α *be the set of Hölder continuous functions defined on* $[0,T]$*, with index* $\alpha > 0$*. Recall that* $f : [0,T] \mapsto \mathbb{R}$ *belongs to* C^α *if*

$$N_\alpha(f) := \sup_{0 \leqslant s,t \leqslant T} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < \infty.$$

2. *If* $X, Y : [0,T] \mapsto \mathbb{R}$ *are two functions of class* C^1*, the Young integral of* Y *with respect to* X *on* $[a,b] \subset [0,T]$ *is defined as:*

$$\int_a^b Y d^{(y)} X := \int_a^b Y(t) X'(t) dt, \quad 0 \leqslant a \leqslant b \leqslant T.$$

To extend the Young integral to Hölder functions we need some estimate of $\displaystyle\int_0^T Y d^{(y)} X$ in terms of the Hölder norms of X and Y. More precisely, let X and Y be as in Definition 3 above; then in [15], it is proved:

$$\left| \int_a^T (Y - Y(a)) d^{(y)} X \right| \leqslant C_\rho T^{1+\rho} N_\alpha(X) N_\beta(Y), \quad 0 \leqslant a \leqslant T, \tag{7}$$

where $\alpha, \beta > 0$, $\alpha + \beta > 1$, $\rho \in]0, \alpha + \beta - 1[$, and C_ρ is a universal constant.

Proposition 2.

1. *The map* $(X,Y) \in C^1([0,T]) \times C^1([0,T]) \mapsto \displaystyle\int_0^\cdot Y d^{(y)} X$ *with values in* C^α*, extends to a continuous bilinear map from* $C^\alpha \times C^\beta$ *to* C^α*. The value of this extension at point* $(X,Y) \in C^\alpha \times C^\beta$ *will still be denoted by* $\displaystyle\int_0^\cdot Y d^{(y)} X$ *and called the* **Young integral** *of* Y *with respect to* X*.*
2. *Inequality (7) is still valid for any* $X \in C^\alpha$ *and* $Y \in C^\beta$*.*

Proof. 1. Let X, Y be of class $C^1([0,T])$ and

$$F(t) = \int_0^t Y d^{(y)} X = \int_0^t Y(s) X'(s) ds, \quad t \in [0,T].$$

For any $a, b \in [0,T],\ a < b$, we have

$$F(b) - F(a) = \int_a^b \big(Y(t) - Y(a)\big) d^{(y)} X + Y(a)\big(X(b) - X(a)\big).$$

Then (7) implies

$$|F(b) - F(a)| \leqslant C_\rho (b-a)^{1+\rho} N_\alpha(X) N_\beta(Y) + \sup_{0 \leqslant t \leqslant T} |Y(t)|\ N_\alpha(X)(b-a)^\alpha; \quad (8)$$

consequently $F \in C^\alpha$.

Then the map $(X, Y) \in C^1([0,T]) \times C^1([0,T]) \mapsto \int_0^\cdot Y d^{(y)} X$, which is bilinear, extends to a continuous bilinear map from $C^\alpha \times C^\beta$ to C^α.

2. is a consequence of point 1. □

Before discussing the relation between Young integrals and integrals via regularization, here is useful technical result.

Lemma 1. *Let* $0 < \gamma' < \gamma \leqslant 1, \varepsilon > 0$. *With* $Z \in C^\gamma$ *we associate*

$$Z_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \big(Z(u+\varepsilon) - Z(u)\big) du,\ t \in [0,T].$$

Then Z_ε *converges to* Z *in* $C^{\gamma'}$, *as* $\varepsilon \to 0$.

Proof. For any $0 \leqslant t \leqslant T$,

$$Z_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \big(Z(u+\varepsilon) - Z(u)\big) du = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} Z(u) du - \frac{1}{\varepsilon} \int_0^\varepsilon Z(u) du.$$

Setting $\Delta_\varepsilon(t) = Z_\varepsilon(t) - Z(t)$, we get

$$\begin{aligned} \Delta_\varepsilon(t) - \Delta_\varepsilon(s) &= \frac{1}{\varepsilon} \int_t^{t+\varepsilon} Z(u) du - Z(t) - \frac{1}{\varepsilon} \int_s^{s+\varepsilon} Z(u) du + Z(s) \\ &= \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \big(Z(u) - Z(t)\big) du - \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \big(Z(u) - Z(s)\big) du, \end{aligned}$$

where $0 \leqslant s \leqslant t \leqslant T$.

a) Suppose $0 \leqslant s < s + \varepsilon < t$. The above inequality implies

$$|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \leqslant \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \big|Z(u) - Z(t)\big| du + \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \big|Z(u) - Z(s)\big| du.$$

Since $Z \in C^\gamma$, then

$$\begin{aligned}|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| &\leqslant \frac{N_\gamma(Z)}{\varepsilon}\left(\int_t^{t+\varepsilon}(u-t)^\gamma du + \int_s^{s+\varepsilon}(u-s)^\gamma du\right)\\ &\leqslant \frac{2N_\gamma(Z)}{\gamma+1}\varepsilon^\gamma.\end{aligned}$$

But $\varepsilon < t - s$, consequently

$$|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \leqslant \frac{2N_\gamma(Z)}{\gamma+1}\varepsilon^{\gamma-\gamma'}|t-s|^{\gamma'}. \tag{9}$$

b) We now investigate the case $0 \leqslant s < t < s + \varepsilon$. The difference $\Delta_\varepsilon(t) - \Delta_\varepsilon(s)$ may be decomposed as follows:

$$\begin{aligned}\Delta_\varepsilon(t) - \Delta_\varepsilon(s) = \frac{1}{\varepsilon}\int_{s+\varepsilon}^{t+\varepsilon}\big(Z(u) - Z(s+\varepsilon)\big)du - \frac{1}{\varepsilon}\int_s^t\big(Z(u) - Z(s)\big)du\\ + \frac{t-s}{\varepsilon}\big(Z(s+\varepsilon) - Z(s)\big) + Z(s) - Z(t).\end{aligned}$$

Proceeding as in the previous step and using the inequality $0 < t - s < \varepsilon$, we obtain

$$\begin{aligned}|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| &\leqslant N_\gamma(Z)\left(\frac{2}{\gamma+1}\frac{(t-s)^{\gamma+1}}{\varepsilon} + \frac{t-s}{\varepsilon^{1-\gamma}} + (t-s)^\gamma\right)\\ &\leqslant 2N_\gamma(Z)\frac{\gamma+2}{\gamma+1}\varepsilon^{\gamma-\gamma'}|t-s|^{\gamma'}.\end{aligned}$$

At this point, the above inequality and (9) directly imply that $N_{\gamma'}(Z_\varepsilon - Z) \leqslant C\varepsilon^{\gamma-\gamma'}$ and the claim is finally established. □

In the sequel of this section X and Y will denote stochastic processes.

Remark 5. If X and Y have a.s. Hölder continuous paths, respectively, of order α and β with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta > 1$, then one can easily prove that $[X, Y] = 0$.

Proposition 3. *Let X, Y be two real processes indexed by $[0, T]$ whose paths are, respectively, a.s. in C^α and C^β, with $\alpha > 0, \beta > 0$ and $\alpha + \beta > 1$. Then the three integrals $\int_0^\cdot Y d^+X$, $\int_0^\cdot Y d^-X$ and $\int_0^\cdot Y d^\circ X$ exist and coincide with the Young integral $\int_0^\cdot Y d^{(y)}X$.*

Proof. We establish that the forward integral coincides with the Young integral. The equality concerning the two other integrals is a consequence of Proposition 1 1., 2. and Remark 5.

By additivity we can suppose, without loss of generality, that $Y(0) = 0$. Set

$$\Delta_\varepsilon(t) := \int_0^t Y d^{(y)} X - \int_0^t Y dX_\varepsilon,$$

where

$$X_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \big(X(u+\varepsilon) - X(u)\big) du, \ t \in [0,T].$$

Since $t \mapsto X_\varepsilon(t)$ is of class $C^1([0,T])$, then $\int_0^t Y dX_\varepsilon$ is equal to the Young integral $\int_0^t Y d^{(y)} X_\varepsilon$ and therefore

$$\Delta_\varepsilon(t) = \int_0^t Y d^{(y)} \big(X - X_\varepsilon\big).$$

Let α' be such that: $0 < \alpha' < \alpha$ and $\alpha' + \beta > 1$. Applying inequality (7) we obtain

$$\sup_{0 \leqslant t \leqslant T} |\Delta_\varepsilon(t)| \leqslant C_\rho T^{1+\rho} N_{\alpha'}(X - X_\varepsilon) N_\beta(Y), \quad \rho \in]0, \alpha' + \beta - 1[.$$

Lemma 1 with $Z = X$ and $\gamma = \alpha$ directly implies that $\Delta_\varepsilon(t)$ goes to 0, uniformly a.s. on $[0,T]$, as $\varepsilon \to 0$, concluding the proof of the proposition. □

3 Itô integrals and related topics

The section presents the construction of Itô integrals with respect to continuous local martingales; it is based on McKean's idea (see Section 2.1 of [37]), which fits the spirit of calculus via regularization.

3.1 Some reminders on martingale theory

In this section, we recall basic notions related to martingale theory, essentially without proofs, except when they help the reader. For detailed complements, see [30], Chap. 1, in particular for definition of adapted and progressively measurable processes.

Let $(\mathcal{F}_t)_{t \geqslant 0}$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions, see Definition 2.25, Chap. 1 in [30].

An adapted process (M_t) of integrable random variables, i.e., verifying $E(|M_t|) < \infty, \ \forall t \geqslant 0$ is:

- An $(\mathcal{F}_t)$-martingale if $E(M_t|\mathcal{F}_s) = M_s, \quad \forall t \geqslant s$
- A $(\mathcal{F}_t)$-submartingale if $E(M_t|\mathcal{F}_s) \geqslant M_s, \quad \forall t \geqslant s$

In this paper, all submartingales (and therefore all martingales) will be supposed to be continuous.

Remark 6. It follows from the definition that if $(M_t)_{t\geqslant 0}$ is a martingale, then $E(M_t) = E(M_0)$, $\forall t \geqslant 0$. If $(M_t)_{t\geqslant 0}$ is a supermartingale (respectively, submartingale) then $t \longrightarrow E(M_t)$ is decreasing (respectively, increasing).

Definition 4. *A process X is said to be* **square integrable** *if $E(X_t^2) < \infty$ for each $t \geqslant 0$.*

When we speak of a martingale without specifying the σ-fields, we refer to the *canonical* filtration generated by the process and satisfying the usual conditions.

Definition 5. *1. A (continuous) process $(X_t)_{t\geqslant 0}$, is called a $(\mathcal{F}_t)$-***local martingale** *(resp. $(\mathcal{F}_t)$-***local submartingale***) if there exists an increasing sequence (τ_n) of stopping times such that $X^{\tau_n} 1_{\tau_n > 0}$ is an $(\mathcal{F}_t)$-martingale (respectively, submartingale) and $\lim_{n\to\infty} \tau_n = +\infty$ a.s.*

Remark 7.

- An $(\mathcal{F}_t)$-martingale is an $(\mathcal{F}_t)$-local martingale. A bounded $(\mathcal{F}_t)$-local martingale is an $(\mathcal{F}_t)$-martingale.
- The set of $(\mathcal{F}_t)$-local martingales is a linear space.
- If M is an $(\mathcal{F}_t)$-local martingale and τ a stopping time, then M^τ is again an $(\mathcal{F}_t)$-local martingale.
- If M_0 is bounded, in the definition of a local martingale one can choose a localizing sequence (τ_n) such that each M^{τ_n} is bounded.
- A convex function of an $(\mathcal{F}_t)$-local submartingale is an $(\mathcal{F}_t)$-local submartingale.

Definition 6. *A process S is called a (continuous) $(\mathcal{F}_t)$-***semimartingale** *if it is the sum of an $(\mathcal{F}_t)$-local martingale and an $(\mathcal{F}_t)$-adapted continuous bounded variation process.*

A basic decomposition in stochastic analysis is the following.

Theorem 1 (Doob decomposition of a submartingale).
Let X be a $(\mathcal{F}_t)$-local submartingale. Then, there is an $(\mathcal{F}_t)$-local martingale M and an adapted, continuous, and finite variation process V (such that $V_0 = 0$) with $X = M + V$. The decomposition is unique.

Definition 7. *Let M be an $(\mathcal{F}_t)$-local martingale. We denote by $\langle M \rangle$ the bounded variation process featuring in the Doob decomposition of the local submartingale M^2. In particular $M^2 - \langle M \rangle$ is an $(\mathcal{F}_t)$-local martingale.*

In Corollary 2, we will prove that $\langle M \rangle$ coincides with $[M, M]$, so that the skew bracket $\langle M \rangle$ does not depend on the underlying filtration.

The following result will be needed in Section 3.2.

Lemma 2. *Let $\left(M^n_{t\in[0,T]}\right)$ be a sequence of $(\mathcal{F}_t)$ local martingales such that $M^n_0 = 0$ and $\langle M^n\rangle_T$ converges to 0 in probability as $n\to\infty$. Then $M^n \to 0$ ucp, when $n\to\infty$.*

Proof. It suffices to apply to $N = M^n$ the following inequality stated in [30], Problem 5.25 Chap. 1, which holds for any $(\mathcal{F}_t)$-local martingale (N_t) such that $N_0 = 0$:

$$P\left(\sup_{0\leqslant u\leqslant t}|N_u| \geqslant \lambda\right) \leqslant P\left(\langle N\rangle_t \geqslant \delta\right) + \frac{1}{\lambda^2}E\big[\delta\wedge\langle N\rangle_t\big], \tag{10}$$

for any $t\geqslant 0,\ \lambda,\delta > 0$. □

An immediate consequence of the previous lemma is the following.

Corollary 1. *Let M be an $(\mathcal{F}_t)$-local martingale vanishing at zero, with $\langle M\rangle = 0$. Then M is identically zero.*

3.2 The Itô integral

Let M be an $(\mathcal{F}_t)$-local martingale. We construct here the Itô integral with respect to M using stochastic calculus via regularization. We will proceed in two steps. First we define the Itô integral $\int_0^\cdot HdM$ for a smooth integrand process H as the forward integral $\int_0^\cdot Hd^-M$. Second, we extend $H\mapsto\int_0^\cdot HdM$ via functional analytical arguments. We remark that the classical theory of Itô integrals first defines the integral of simple step processes H, see Remark 9, for details.

Observe first that the forward integral of a continuous process H of bounded variation is well defined because Proposition 1 4), 7) imply that

$$\int_0^t Hd^-M = H_tM_t - H_0M_0 - \int_0^t Md^+H = H_tM_t - H_0M_0 - \int_0^t M_s dH_s. \tag{11}$$

Call $\mathcal{C}$ the vector algebra of adapted processes whose paths are of class C^0. This linear space, equipped with the metrizable topology which governs the ucp convergence, is an F-space. For the definition and properties of F-spaces, see [10], Chap. 2.1. Remark that the set $\mathcal{M}_{\mathrm{loc}}$ of continuous $(\mathcal{F}_t)$-local martingales is a closed linear subspace of $\mathcal{C}$, see for instance [24].

Denote by $\mathcal{C}^{BV}$ the $\mathcal{C}$ subspace of processes whose paths are a.s. continuous with bounded variation. The next observation is crucial.

Lemma 3. *If H is an adapted process in $\mathcal{C}^{BV}$ then $\left(\int_0^\cdot Hd^-M\right)$ is an $(\mathcal{F}_t)$-local martingale whose quadratic variation is given by*

$$\left\langle\int_0^\cdot Hd^-M\right\rangle_t = \int_0^\cdot H_s^2 d\langle M\rangle_s.$$

Proof. We only sketch the proof. We restrict ourselves to prove that if M is a local martingale then $Y = \int_0^{\cdot} H d^- M$ is a local martingale.

By localization, we can suppose that H, its total variation $\|H\|$ and M are bounded processes.

Let $0 \leqslant s < t$. Since $H_t = H_0 + \int_s^t dH_u$, (11) implies

$$Y_t = H_s M_t - H_0 M_0 - \int_0^s M_u dH_u + \int_s^t (M_t - M_u) dH_u. \tag{12}$$

Let (π_n) be a sequence of subdivisions of $[s,t]$, such that the mesh of (π_n) goes to zero when $n \to +\infty$. Since M is continuous, M and $\|H\|$ are bounded,

$$\Delta_n := \sum_{\pi_n} (M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i}),$$

goes to $\int_s^t (M_t - M_u) dH_u$ a.s. and in L^1. Consequently,

$$E\left(\int_s^t (M_t - M_u) dH_u\right) = \lim_{n\to\infty} E(\Delta_n | \mathcal{F}_s)$$

and

$$E(\Delta_n | \mathcal{F}_s) = \sum_{\pi_n} E\left((M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i}) | \mathcal{F}_s\right).$$

But one has

$$\begin{aligned} &E\left((M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i}) | \mathcal{F}_s\right) \\ &\quad = E\left(E((M_t - M_{u_{i+1}})(H_{u_{i+1}} - H_{u_i}) | \mathcal{F}_{u_{i+1}}) | \mathcal{F}_s\right) \qquad (13) \\ &\quad = E\left((H_{u_{i+1}} - H_{u_i}) E(M_t - M_{u_{i+1}} | \mathcal{F}_{u_{i+1}}) | \mathcal{F}_s\right) \\ &\quad = 0, \qquad (14) \end{aligned}$$

since H is adapted and M is a martingale.

Finally, taking the conditional expectation with respect to $\mathcal{F}_s$ in (12) yields

$$E\big[Y_t | \mathcal{F}_s\big] = H_s M_s - H_0 M_0 - \int_0^s M_u dH_u = Y_s.$$

Similar arguments show that $Y^2 - \int_0^{\cdot} H^2 d\langle M \rangle$ is a martingale. □

The previous lemma allows to extend the map $H \mapsto \int_0^t H d^- M$. Let $\mathcal{L}^2(d\langle M \rangle)$ denote the set of progressively measurable processes such that

$$\int_0^T H^2 d\langle M \rangle < \infty \text{ a.s.} \tag{15}$$

$\mathcal{L}^2(d\langle M \rangle)$ is an F-space with respect to the metrizable topology d_2 defined as follows: (H^n) converges to H when $n \to \infty$ if $\int_0^T (H_s^n - H_s)^2 d\langle M \rangle_s \to 0$ in probability, when $n \to \infty$.

Remark 8. $\mathcal{C}^{BV}$ is dense in $\mathcal{L}^2(d\langle M\rangle)$. Indeed, according to [30], Lemma 2.7 Section 3.2, simple processes are dense into $\mathcal{L}^2(d\langle M\rangle)$. On the other hand, a simple process of the form $H_t = \xi 1_{]a,b]}$, ξ being $\mathcal{F}_a$ measurable, can be expressed as a limit of $H^n_t = \xi\phi^n$ where ϕ^n are continuous functions with bounded variation.

Let $\Lambda : \mathcal{C}^{BV} \to \mathcal{M}_{\mathrm{loc}}$ be the map defined by $\Lambda H = \int_0^\cdot H d^- M$.

Lemma 4. *If $\mathcal{C}^{BV}$ (respectively, $\mathcal{M}_{\mathrm{loc}}$) is equipped with d_2 (respectively, the ucp topology) then Λ is continuous.*

Proof. Let H^k be a sequence of processes in $\mathcal{C}^{BV}$, converging to 0 for d_2 when $k \to \infty$. Set $N^k = \int_0^\cdot H^k d^- M$. Lemma 3 implies that $\langle N^k\rangle_T$ converges to 0 in probability. Finally Lemma 2 concludes the proof. □

We can now easily define the Itô integral. Since $\mathcal{C}^{BV}$ is dense in $\mathcal{L}^2(d\langle M\rangle)$ for d_2, Lemma 4 and standard functional analysis arguments imply that Λ uniquely and continuously extends to $\mathcal{L}^2(d\langle M\rangle)$.

Definition 8. *If H belongs to $\mathcal{L}^2(d\langle M\rangle)$, we put $\int_0^\cdot HdM := \Lambda H$ and we call this the* **Itô integral of H with respect to M**.

Proposition 4. *If H belongs to $\mathcal{L}^2(d\langle M\rangle)$, then $(\int_0^\cdot HdM)$ is an $(\mathcal{F}_t)$-local martingale with bracket*

$$\left\langle \int_0^\cdot HdM \right\rangle = \int_0^\cdot H^2 d\langle M\rangle. \tag{16}$$

Proof. Let $H \in \mathcal{L}^2(d\langle M\rangle)$. From Definition 8, $(\int_0^\cdot HdM)$ is an $(\mathcal{F}_t)$-local martingale. It remains to prove (16).

Since H belongs to $\mathcal{L}^2(d\langle M\rangle)$, then there exists a sequence (H_n) of elements in $\mathcal{C}^{BV}$, such that $H_n \to H$ in $\mathcal{L}^2(d\langle M\rangle)$.

Introduce $N_n = \displaystyle\int_0^\cdot H_n dM$ and $N'_n = N_n^2 - \langle N_n\rangle$. According to lemma 4, $\langle N_n\rangle = \displaystyle\int_0^\cdot H_n^2 d\langle M\rangle$; now $N_n \to N$, ucp, $n \to \infty$ and $\langle N_n\rangle$ goes to $\displaystyle\int_0^\cdot H^2 d\langle M\rangle$ in the ucp sense, as $n \to \infty$. Therefore N'_n converges with respect to the ucp topology, to the local martingale $N^2 - \displaystyle\int_0^\cdot H^2 d\langle M\rangle$. This actually proves (16). □

Remark 9.

1. Recall that whenever $H \in \mathcal{C}^{BV}$

$$\int_0^\cdot HdM = \int_0^\cdot Hd^- M.$$

This property will be generalized in Proposition 6 and 10.

2. We emphasize that Itô stochastic integration based on adapted simple step processes and the previous construction, finally lead to the same object. If H is of the type $Y1_{]a,b]}$ where Y is an $\mathcal{F}_a$ measurable random variable, it is easy to show that $\int_0^t HdM = Y(M_{t\wedge b} - M_{t\wedge a})$. Since the class of elementary processes obtained by linear combination of previous processes is dense in $\mathcal{L}^2(d\langle M\rangle)$ and the map Λ is continuous, then $\int_0^\cdot HdM$ equals the classical Itô integral.

In Proposition 5 below we state the chain rule property.

Proposition 5. *Let $(M_t, t \geqslant 0)$ be an $(\mathcal{F}_t)$-local martingale, $(H_t, t \geqslant 0)$ be in $\mathcal{L}^2(d\langle M\rangle)$, $N := \displaystyle\int_0^\cdot H_s dM_s$ and $(K_t, t \geqslant 0)$ be a $(\mathcal{F}_t)$-progressively measurable process such that $\displaystyle\int_0^T (H_s K_s)^2 d\langle M\rangle_s < \infty$ a.s. Then*

$$\int_0^t K_s dN_s = \int_0^t H_s K_s dM_s, \quad 0 \leqslant t \leqslant T. \tag{17}$$

Proof. Since the map $\Lambda : H \in \mathcal{L}^2(d\langle M\rangle) \mapsto \int_0^\cdot HdM$ is continuous, it suffices to prove (17) for H and K continuous and with bounded variation.

For simplicity we suppose $M_0 = H_0 = K_0 = 0$.

One has

$$\int_0^t KdN = \int_0^t (N_t - N_u) dK_u,$$

and

$$\begin{aligned} N_t - N_u &= \int_0^t (M_t - M_v) dH_v - \int_0^u (M_u - M_v) dH_v \\ &= (M_t - M_u) H_u + \int_u^t (M_t - M_v) dH_v, \end{aligned}$$

where $0 \leqslant u \leqslant t$.

Using Fubini's theorem one gets

$$\begin{aligned} \int_0^t KdN &= \int_0^t (M_t - M_u)(H_u dK_u + K_u dH_u) \\ &= \int_0^t (M_t - M_u) d(HK)_u = \int_0^t HKdM. \end{aligned}$$ □

3.3 Connections with calculus via regularizations

The next proposition will show that, under suitable conditions, the Itô integral is a forward integral.

Proposition 6. *Let X be an $(\mathcal{F}_t)$-local martingale and suppose that (H_t) is progressively measurable and locally bounded.*

1. *If H has a left limit at each point then* $\int_0^{\cdot} H_s d^- X_s = \int_0^{\cdot} H_{s-} dX_s$.
2. *If* $H_t = H_{t-}$, $d\langle X\rangle_t$ *a.e. (in particular if H is càdlàg), then* $\int_0^{\cdot} H_s d^- X_s = \int_0^{\cdot} H_s dX_s$.

Proof. Since $s \mapsto \int_{s-\varepsilon}^{s} H_u du$ is continuous with bounded variation,

$$\begin{aligned}\int_0^t \left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} H_u du\right) dX_s &= \int_0^t \left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} H_u du\right) d^- X_s \\ &= X_t\left(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_u du\right) - H_0 X_0 \\ &\quad - \frac{1}{\varepsilon}\int_0^t (H_s - H_{s-\varepsilon}) X_s ds.\end{aligned}$$

The second integral in the right-hand side can be modified as follows:

$$\begin{aligned}-\int_0^t (H_s - H_{s-\varepsilon}) X_s ds = \int_0^t H_s(X_{s+\varepsilon} - X_s) ds - \int_{t-\varepsilon}^{t} H_s X_{s+\varepsilon} ds \\ + H_0 \int_0^{\varepsilon} X_s ds.\end{aligned}$$

Consequently

$$\int_0^t \left(\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} H_u du\right) dX_s = \frac{1}{\varepsilon}\int_0^t H_s(X_{s+\varepsilon} - X_s) ds + R_\varepsilon(t), \tag{18}$$

where

$$\begin{aligned}R_\varepsilon(t) &= X_t\left(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_s ds\right) - \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_s X_{s+\varepsilon} ds + H_0\left(\frac{1}{\varepsilon}\int_0^{\varepsilon} X_s ds - X_0\right) \\ &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} H_s(X_t - X_{s+\varepsilon}) ds + H_0\left(\frac{1}{\varepsilon}\int_0^{\varepsilon} X_s ds - X_0\right) \qquad (19)\end{aligned}$$

converges to zero ucp.

Under assumption 1, Lebesgue's dominated convergence theorem implies that $\frac{1}{\varepsilon}\int_{\cdot-\varepsilon}^{\cdot} H_s ds$ converges to H_- according to $\mathcal{L}^2(d\langle M\rangle)$, so the left-hand side of equality (18) converges to the Itô integral $\int_0^{\cdot} H_{s-} dX_s$. This forces the right-hand side to converge to $\int_0^{\cdot} H_s d^- X_s$.

The proof of 2 is similar, remarking that $H_s = H_{s-}$, for $d\langle M\rangle_s$ a.e. □

When the integrator is a Brownian motion W, we will see in Theorem 2 below that the forward integral coincides with the Itô integral for any integrand in $\mathcal{L}^2(d\langle W\rangle)$. This is no longer true when the integrator is a general semimartingale. The following example provides a martingale (M_t) and a deterministic integrand h such that the Itô integral $\int_0^t hdM$ and the forward integral $\int_0^t hd^-M$ exist, but are different.

Example 1. Let $\psi : [0,\infty[\mapsto \mathbb{R}$ verify $\psi(0) = 0$, ψ is continuous, increasing, and $\psi'(t) = 0$ a.e. (with respect to the Lebesgue measure). Let (M_t) be the process: $M_t = W_{\psi(t)},\ t \geqslant 0$, and h be the indicator function of the support of the positive measure $d\psi$. Since $W_t^2 - t$ is a martingale, $\langle W\rangle_t = t$. Clearly (M_t) is a martingale and $\langle M\rangle_t = \psi(t),\ t \geqslant 0$. Observe that $h = 0$ a.e. with respect to Lebesgue measure. Then $\int_0^\cdot h(s)\frac{M(s+\varepsilon)-M(s)}{\varepsilon}ds = 0$ and so $\int_0^\cdot hd^-M = 0$.

On the other hand, $h = 1,\ d\psi$ a.e., implies $\int_0^t hdM = M_t,\ t \geqslant 0$.

Remark 10. A significant result of classical stochastic calculus is the Bichteler–Dellacherie theorem, see [43] Th. 22, Section III.7. In the regularization approach, an analogous property occurs: if the forward integral exists for a rich class of adapted integrands, then the integrator is forced to be a semimartingale. More precisely we recall the significant statement of [47], Proposition 1.2.

Let $(X_t, t \geqslant 0)$ be an $(\mathcal{F}_t)$-adapted and continuous process such that for any càdlàg, bounded and adapted process (H_t), the forward integral $\int_0^\cdot Hd^-X$ exists. Then (X_t) is an $(\mathcal{F}_t)$-semimartingale.

From Proposition 6 we deduce the relation between skew and square bracket.

Corollary 2. *Let M be an $(\mathcal{F}_t)$-local martingale. Then $\langle M\rangle = [M]$ and*

$$M_t^2 = M_0^2 + 2\int_0^t Md^-M + \langle M\rangle_t. \tag{20}$$

Proof. The proof of (20) is very simple and is based on the following identity:

$$(M_{s+\varepsilon} - M_s)^2 = M_{s+\varepsilon}^2 - M_s^2 - 2M_s(M_{s+\varepsilon} - M_s).$$

Integrating on $[0,t]$ leads to

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (M_{s+\varepsilon} - M_s)^2 ds &= \frac{1}{\varepsilon}\int_0^t M_{s+\varepsilon}^2 ds - \frac{1}{\varepsilon}\int_0^t M_s^2 ds - \frac{2}{\varepsilon}\int_0^t M_s(M_{s+\varepsilon} - M_s)ds \\ &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} M_s^2 ds - \frac{1}{\varepsilon}\int_0^\varepsilon M_s^2 ds - \frac{2}{\varepsilon}\int_0^t M_s(M_{s+\varepsilon} - M_s)ds.\end{aligned}$$

Therefore, taking the limit when $\varepsilon \to 0$, one obtains

$$[M]_t = M_t^2 - M_0^2 - 2\int_0^t M_s d^- M_s.$$

Since $t \mapsto M_t$ is continuous, the forward integral $\int_0^{\cdot} M d^- M$ coincides with the corresponding Itô integral. Consequently $M_t^2 - M_0^2 - [M]_t$ is a local martingale. This proves both $[M] = \langle M \rangle$ and (20). □

Corollary 3. *Let M, M' be two $(\mathcal{F}_t)$-local martingales. Then (M, M') has all its mutual covariations.*

Proof. Since M, M' and $M + M'$ are continuous local martingales, Corollary 2 directly implies that they have finite quadratic variation. The bilinearity property of the covariation directly implies that $[M, M']$ exists and equals

$$\frac{1}{2}\left([M + M'] - [M] - [M']\right). \qquad \square$$

Proposition 7. *Let M and M' be two $(\mathcal{F}_t)$-local martingales, H and H' be two progressively measurable processes such that*

$$\int_0^{\cdot} H^2 d\langle M \rangle < \infty, \quad \int_0^{\cdot} H^2 d\langle M' \rangle < \infty.$$

Then

$$\left[\int_0^{\cdot} H dM, \int_0^{\cdot} H' dM'\right]_t = \int_0^t H H' d[M, M']_t.$$

The next proposition provides a simple example of two processes (M_t) and (Y_t) such that $[M, Y]$ exists even though the vector (M, Y) has no mutual covariation.

Proposition 8. *Let (M_t) be an continuous $(\mathcal{F}_t)$-local martingale, (Y_t) a càdlàg and an $(\mathcal{F}_t)$-adapted process. If M and Y are independent then $[M, Y] = 0$.*

Proof. Let $\mathcal{Y}$ be the σ-field generated by (Y_t), and denote by $(\tilde{\mathcal{M}}_t)$ the smallest filtration satisfying the usual conditions and containing $(\mathcal{F}_t)$ and $\mathcal{Y}$, i.e., $\sigma(M_s, s \leqslant t) \vee \mathcal{Y} \subset \tilde{\mathcal{M}}_t, \forall t \geqslant 0$. It is not difficult to show that (M_t) is also an $(\tilde{\mathcal{M}}_t)$-martingale.

Thanks to Proposition 1 1., it is sufficient to prove that

$$\int_0^t Y d^- M = \int_0^t Y d^+ M. \tag{21}$$

Proposition 6 implies that the left-hand side coincides with the $(\mathcal{M}_t)$-Itô integral $\int_0^t Y dM$.

Without restricting generality we suppose $M_0 = 0$. We proceed as in the proof of Proposition 6. Since a.s. $s \mapsto \int_s^{s+\varepsilon} Y_u du$ is continuous with bounded variation,

$$\int_0^t \left(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} Y_u du\right) d^- M_s = M_t \left(\frac{1}{\varepsilon}\int_s^{s+\varepsilon} Y_u du\right) - \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s) M_s ds.$$

As the processes Y and M are independent, the forward integral in the left-hand side above is actually an Itô integral. Therefore, taking the limit when $\varepsilon \to 0$ and using Proposition 6, one gets

$$\int_0^t Y dM = \int_0^t Y d^- M = Y_t M_t - \int_0^t M d^- Y.$$

According to point 4) of Proposition 1, the right-hand side is equal to $\int_0^t Y d^+ M$; this proves (21). □

3.4 The semimartingale case

We begin this section with a technical lemma which implies that the decomposition of a semimartingale is unique.

Lemma 5. *Let $(M_t, t \geqslant 0)$ be a $(\mathcal{F}_t)$-local martingale with bounded variation. Then (M_t) is constant.*

Proof. Since M has bounded variation, then Proposition 1, 7) implies that $[M] = 0$. Consequently Corollaries 1 and 2 imply that $M_t = M_0,\ t \geqslant 0$. □

It is now easy to define stochastic integration with respect to continuous semimartingales.

Definition 9. *Let $(X_t, t \geqslant 0)$ be an $(\mathcal{F}_t)$-semimartingale with canonical decomposition $X = M + V$, where M (respectively, V) is a continuous $(\mathcal{F}_t)$-local martingale (respectively, bounded variation, continuous and $(\mathcal{F}_t)$-adapted process) vanishing at 0. Let $(H_t, t \geqslant 0)$ be an $(\mathcal{F}_t)$-progressively measurable process, satisfying*

$$\int_0^T H_s^2 d[M,M]_s < \infty, \quad \textit{and} \quad \int_0^T |H_s| d\|V\|_s < \infty, \tag{22}$$

where $\|V\|_t$ is the total variation of V over $[0,t]$.

We set

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dV_s, \quad 0 \leqslant t \leqslant T.$$

Remark 11.

1. In the previous definition, the integral with respect to M (respectively, V) is an Itô-type (respectively, Stieltjes-type) integral.
2. It is clear that $\int_0^{\cdot} H_s dX_s$ is again a continuous $(\mathcal{F}_t)$-semimartingale, with martingale part $\int_0^{\cdot} H_s dM_s$ and bounded variation component $\int_0^{\cdot} H_s dV_s$.

Once we have introduced stochastic integrals with respect to continuous semimartingales, it is easy to define Stratonovich integrals.

Definition 10. *Let $(X_t, t \geqslant 0)$ be an $(\mathcal{F}_t)$-semimartingale and $(Y_t, t \geqslant 0)$ an $(\mathcal{F}_t)$-progressively measurable process. The* **Stratonovich** *integral of Y with respect to X is defined as follows:*

$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s dX_s + \frac{1}{2}[Y, X]_t; \quad t \geqslant 0, \tag{23}$$

if $[Y, X]$ and $\int_0^{\cdot} Y_s dX_s$ exist.

Remark 12.

1. Recall that conditions of type (22) ensure existence of the stochastic integral with respect to X.
2. If (X_t) and (Y_t) are $(\mathcal{F}_t)$-semimartingales, then $\int_0^{\cdot} Y_s \circ dX_s$ exists and is called the **Fisk–Stratonovich** integral.
3. Suppose that (X_t) is an $(\mathcal{F}_t)$-semimartingale and (Y_t) is a left continuous and $(\mathcal{F}_t)$-adapted process such that $[Y, X]$ exists. We already have observed (see Proposition 6) that $\int_0^{\cdot} Y_s dX_s$ coincides with $\int_0^{\cdot} Y_s d^- X_s$. Proposition 1 1) and 2) imply that the Stratonovich integral $\int_0^{\cdot} Y_s \circ dX_s$ is equal to the symmetric integral $\int_0^{\cdot} Y_s d^\circ X_s$.

At this point we can easily identify the covariation of two semimartingales.

Proposition 9. *Let $S^i = M^i + V^i$ be two $(\mathcal{F}_t)$-semimartingales, $i = 1, 2$, where M^i are local martingales and V^i bounded variation processes. One has $[S^1, S^2] = [M^1, M^2]$.*

Proof. The result follows directly from Corollary 3, Proposition 1 7), and the bilinearity of the covariation. □

Corollary 4. *Let S^1, S^2 be two $(\mathcal{F}_t)$-semimartingales such that their martingale parts are independent. Then $[S^1, S^2] = 0$.*

Proof. It follows from Proposition 8. □

The statement of Proposition 6 can be adapted to semimartingale integrators as follows.

Proposition 10. *Let X be an $(\mathcal{F}_t)$-semimartingale and suppose that (H_t) is adapted, with left limits at each point. Then* $\int_0^\cdot H_s d^- X_s = \int_0^\cdot H_{s-} dX_s$. *If H is càdlàg then* $\int_0^\cdot H d^- X = \int_0^\cdot H dX$.

Remark 13.

1. Forward integrals generalize not only classical Itô integrals but also the integral obtained from the theory of enlargements of filtrations, see e.g., [29]. Let $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ be two filtrations fulfilling the usual conditions with $\mathcal{F}_t \subset \mathcal{G}_t$ for all t. Let X be a $(\mathcal{G}_t)$-semimartingale which is $(\mathcal{F}_t)$-adapted. By Stricker's theorem, X is also an $(\mathcal{F}_t)$-semimartingale. Let H be a càdlàg bounded $(\mathcal{F}_t)$-adapted process. According to Proposition 10, the $(\mathcal{F}_t)$-Itô integral $\int_0^\cdot H dX$ equals the $(\mathcal{G}_t)$-Itô integral and it coincides with the forward integral $\int_0^\cdot H d^- X$.
2. The result stated above is false when H has no left limits at each point. Using a tricky example in [42], it is possible to exhibit a filtration $(\mathcal{G}_t)$, a $(\mathcal{G}_t)$-semimartingale $(X_t)_{t\geqslant 0}$ with natural filtration $\mathcal{F}_t^X$, a bounded and $(\mathcal{F}_t^X)$-progressively measurable process H, such that $\int_0^\cdot H d^- X$ equals the $(\mathcal{F}_t^X)$-Itô integral but differs from the $(\mathcal{G}_t)$-Itô integral. More precisely one has:
   a) X is a 3-dimensional Bessel process with decomposition
   $$X_t = W_t + \int_0^t \frac{1}{X_s} ds, \tag{24}$$
   where W is an $(\mathcal{F}_t^X)$-Brownian motion,
   b) X is a $(\mathcal{G}_t)$-semimartingale with decomposition $M + V$ where M is the local martingale part,
   c) $H_t(\omega) = 1$ for $dt \otimes dP$-almost all $(t,\omega) \in [0,T] \times \Omega$,
   d) $\beta_t = \int_0^t H dX$ is a $(\mathcal{G}_t)$-Brownian motion.

   Property (d) implies that $I^-(\varepsilon, H, dX) = I^-(\varepsilon, 1, dX)$ so that $\int_0^t H d^- X = X_t$. The $(\mathcal{F}_t^X)$-Itô integral $\int_0^t H dX$ equals $\int_0^t H dW + \int_0^t \frac{H_s}{X_s} ds$; Theorem 2 below and Proposition 1 8) imply that this integral coincides with $\int_0^t H d^- X$. Since a Bessel process cannot be equal to a Brownian motion, the $(\mathcal{G}_t)$-Itô integral $\int_0^t H dX$ differs from the $(\mathcal{F}_t^X)$-Itô integral $\int_0^t H dX$. Indeed, the pathology comes from the integration with respect to the bounded variation process. In fact, according to ii), $[X]_t = [W]_t = t$; therefore M is a $(\mathcal{G}_t)$-Brownian motion. Theorem 2 below says that $\int_0^\cdot H d^- M = \int_0^\cdot H dM$; the additivity of forward integrals and Itô integrals imply that $\int_0^\cdot H d^- V \neq \int_0^\cdot H dV$. Consequently it can be deduced from Proposition 1 7) a) that the discontinuities of H are not a.s. countable. It can even be shown that the discontinuities of H are not negligible with respect to dV.

3.5 The Brownian case

In this section we will investigate the link between forward and Itô integration with respect to a Brownian motion. In this section (W_t) will denote a $(\mathcal{F}_t)$-Brownian motion.

The main result of this section is the following.

Theorem 2. *Let $(H_t, t \geqslant 0)$ be an $(\mathcal{F}_t)$-progressively measurable process satisfying $\int_0^T H_s^2 ds < \infty$ a.s. Then the Itô integral $\int_0^{\cdot} H_s dW_s$ coincides with the forward integral $\int_0^{\cdot} H_s d^- W_s$.*

Remark 14.

1. We would like to illustrate the advantage of using regularization instead of discretization ([20]) through the following example.
   Let g be the indicator function of $\mathbb{Q} \cap \mathbb{R}_+$.
   Let $\Pi = \{t_0 = 0, t_1, \cdots, t_N = T\}$ be a subdivision of $[0, T]$ and
   $$I(\Pi, g, dW)_t := \sum_i g(t_i)\big(W(t_{i+1} \wedge t) - W(t_i \wedge t)\big); \quad 0 \leqslant t \leqslant T.$$
   We remark that
   $$I(\Pi, g, dW)_t = \begin{cases} 0 & \text{if } \Pi \subset \mathbb{R} \setminus \mathbb{Q} \\ W_t & \text{if } \Pi \subset \mathbb{Q}. \end{cases}$$
   Therefore there is no canonical definition of $\int_0^t gdW$ through discretization. This is not surprising since g is not a.e. continuous and so is not Riemann integrable. On the contrary, integration via regularization seems drastically more adapted to define $\int_0^t gd^- W$, for any $g \in L^2([0, T])$, since this integral coincides with the classical Itô-Wiener integral.
2. In order to overcome this problem, McShane pointed out an alternative approximation scheme, see [38] Chaps. 2 and 3. McShane's stochastic integration makes use of the so-called *belated* partition; the integral is then even more general than Itô's one, and it includes in particular the function g above.

Proof (of Theorem 2). 1) First, suppose in addition that H is a continuous process. Replacing X by W in (18) one gets

$$\int_0^t \left(\frac{1}{\varepsilon} \int_{s-\varepsilon}^s H_u du \right) dW_s = \frac{1}{\varepsilon} \int_0^t H_s (W_{s+\varepsilon} - W_s) ds + R_\varepsilon(t), \tag{25}$$

where the remainder term $R_\varepsilon(t)$ is given by (19).

Recall the maximal inequality [52, Chap. I.1]: there exists a constant C such that for any $\phi \in L^2([0,T])$,

$$\int_0^T \left(\sup_{0<\eta<1} \left\{ \frac{1}{\eta} \int_{(v-\eta)_+}^v \phi_v dv \right\} \right)^2 du \leqslant C \int_0^T \phi_v^2 dv. \tag{26}$$

2) We claim that (25) may be extended to any progressively measurable process (H_t) satisfying $\int_0^{\cdot} H_s^2 ds < \infty$.

Set $H_t^n = n \int_{t-1/n}^t H_u du$ for $t \geqslant 0$. It is clear that as $n \to \infty$

- For a.e. t, H_t^n converges to H_t
- (H_t^n) converges to (H_t) in $\mathcal{L}^2(d\langle W \rangle)$ (i.e., $\int_0^{\cdot} (H_s^n - H_s)^2 ds$ goes to 0 in the ucp sense)

Since
$$\left\langle \int_0^{\cdot} \Big(\frac{1}{\varepsilon} \int_{s-\varepsilon}^s H_u du \Big) dW_s \right\rangle_t = \int_0^{\cdot} \left(\frac{1}{\varepsilon} \int_{s-\varepsilon}^s H_u du \right)^2 ds,$$

(26) and Lemma 2 imply that (25) and (19) are valid.

3) Letting $\varepsilon \to 0$ in (25) and using once more (26), Lemma 2 allows to conclude the proof of Theorem 2. □

3.6 Substitution formulae

We conclude Section 3 by observing that discretization makes it possible to integrate nonadapted integrands in a context which is covered neither by Skorohod integration theory nor by enlargement of filtrations. A class of examples is the following.

Let $(X(t,x), t \geqslant 0, x \in \mathbb{R}^d)$ and $(Y(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ be two families of continuous $(\mathcal{F}_t)$ semimartingales depending on a parameter x and $(H(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ an $(\mathcal{F}_t)$ progressively measurable processes depending on x. Let Z be a $\mathcal{F}_T$-measurable r.v., taking its values in $\mathbb{R}^d$.

Under some minimal conditions of Garsia–Rodemich–Rumsey type, see for instance [49, 50], one has

$$\int_0^t H(s,Z)\, d^- X(s,Z) = \int_0^t H(s,x)\, dX(s,x) \Big|_{x=Z},$$
$$[X(\cdot,Z), Y(\cdot,Z)] = [X(\cdot,x), Y(\cdot,x)] \Big|_{x=Z}.$$

The first result is useful to prove existence results for SDEs driven by semimartingales, with anticipating initial conditions.

It is significant to remark that these substitution formula give rise to anticipating calculus in a setting which is not covered by Malliavin noncausal calculus since our integrators may be general semimartingales, while Skorohod integrals apply essentially to Gaussian integrators or eventually to Poisson type processes. Note that the usual causal Itô calculus does not apply here since $(X(s, Z))_s$ is not a semimartingale (take for instance a r.v. Z which generates $\mathcal{F}_T$.)

4 Calculus for finite quadratic variation processes

4.1 Stability of the covariation

A basic tool of calculus via regularization is the stability of finite quadratic variation processes under C^1 transformations.

Proposition 11. *Let (X^1, X^2) be a vector of processes having all its mutual covariations and $f, g \in C^1(\mathbb{R})$. Then $[f(X^1), g(X^2)]$ exists and is given by*

$$[f(X^1), g(X^2)]_t = \int_0^t f'(X_s^1) g'(X_s^2) d[X^1, X^2]_s$$

Proof. By polarization and bilinearity, it suffices to consider the case when $X = X^1 = X^2$ and $f = g$. Using Taylor's formula, one can write

$$f(X_{s+\varepsilon}) - f(X_s) = f'(X_s)(X_{s+\varepsilon} - X_s) + R(s,\varepsilon)(X_{s+\varepsilon} - X_s), \quad s \geqslant 0,\ \varepsilon > 0,$$

where $R(s, \varepsilon)$ denotes a process which converges in the ucp sense to 0 when $\varepsilon \to 0$. Since f' is uniformly continuous on compacts,

$$(f(X_{s+\varepsilon}) - f(X_s))^2 = f'(X_s)^2 (X_{s+\varepsilon} - X_s)^2 + R(s,\varepsilon)(X_{s+\varepsilon} - X_s)^2.$$

Integrating from 0 to t yields

$$\frac{1}{\varepsilon} \int_0^t (f(X_{s+\varepsilon}) - f(X_s))^2 ds = I_1(t,\varepsilon) + I_2(t,\varepsilon)$$

where

$$I_1(t,\varepsilon) = \int_0^t f'(X_s)^2 \frac{(X_{s+\varepsilon} - X_s)^2}{\varepsilon} ds,$$
$$I_2(t,\varepsilon) = \frac{1}{\varepsilon} \int_0^t R(s,\varepsilon)(X_{s+\varepsilon} - X_s)^2 ds.$$

Clearly one has

$$\sup_{t\leqslant T}|I_2(t,\varepsilon)| \leqslant \sup_{s\leqslant T}|R(s,\varepsilon)|\frac{1}{\varepsilon}\int_0^T (X_{s+\varepsilon}-X_s)^2 ds.$$

Since $[X]$ exists, $I_2(\cdot,\varepsilon) \xrightarrow{\text{ucp}} 0$. The result will follow if we establish

$$\frac{1}{\varepsilon}\int_0^{\cdot} Y_s d\mu_\varepsilon(s) \xrightarrow{\text{ucp}} \int_0^{\cdot} Y_s d[X,X]_s \tag{27}$$

where $\mu_\varepsilon(t) = \int_0^t \frac{ds}{\varepsilon}(X_{s+\varepsilon}-X_s)^2$ and Y is a continuous process. It is not difficult to verify that a.s., $\mu_\varepsilon(dt)$ converges to $d[X,Y]$, when $\varepsilon \to 0$; this finally implies (27). □

4.2 Itô formulae for finite quadratic variation processes

Even though all Itô formulae that we will consider can be stated in the multi-dimensional case, see for instance [49], we will only deal here with dimension 1. Let $X = (X_t)_{t\geqslant 0}$ be a continuous process.

Proposition 12. *Suppose that $[X,X]$ exists and let $f \in C^2(\mathbb{R})$. Then*

$$\int_0^{\cdot} f'(X)d^-X \quad \text{and} \quad \int_0^{\cdot} f'(X)d^+X \quad \text{exist.} \tag{28}$$

Moreover

a) $f(X_t) = f(X_0) + \int_0^t f'(X)d^{\mp}X \pm \frac{1}{2}\int_0^t f''(X_s)d[X,X]_s$,

b) $f(X_t) = f(X_0) + \displaystyle\int_0^t f'(X)d^{\mp}X \pm \frac{1}{2}[f'(X),X]_t$,

c) $f(X_t) = f(X_0) + \displaystyle\int_0^t f'(X)d^{\circ}X$.

Proof. c) Follows from b) summing up + and −.
b) follows from a), since Proposition 11 implies that

$$[f'(X),X]_t = \int_0^t f''(X)d[X,X].$$

The proof of a) and (28) is similar to that of Proposition 11, but with a second-order Taylor expansion. □

The next lemma emphasizes that the existence of a quadratic variation is closely connected with the existence of some related forward and backward integrals.

Lemma 6. *Let X be a continuous process. Then $[X,X]$ exists $\Longleftrightarrow \displaystyle\int_0^{\cdot} Xd^-X$ exists $\Longleftrightarrow \displaystyle\int_0^{\cdot} Xd^+X$ exists.*

Proof. Start with the identity

$$(X_{s+\varepsilon} - X_s)^2 = X_{s+\varepsilon}^2 - X_s^2 - 2X_s(X_{s+\varepsilon} - X_s) \tag{29}$$

and observe that, when $\varepsilon \to 0$,

$$\frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon}^2 - X_s^2)ds \to X_t^2 - X_0^2.$$

Integrating (29) from 0 to t and dividing by ε easily gives the equivalence between the first two assertions.

The equivalence between the first and third ones is similar, replacing ε with $-\varepsilon$ in (29). □

Lemma 6 admits the following generalization.

Corollary 5. *Let X be a continuous process. The following properties are equivalent*

a) $[X, X]$ *exists;*

b) $\forall g \in C^1, \quad \int_0^{\cdot} g(X)d^-X$ *exists;*

c) $\forall g \in C^1, \quad \int_0^{\cdot} g(X)d^+X$ *exists.*

Proof. The Itô formula stated in Proposition 12 1) implies a) ⇒ b). b) ⇒ a) follows setting $g(x) = x$ and using Lemma 6.

b) ⇔ c) because of Proposition 1 1) which states that

$$\int_0^{\cdot} g(X)d^+X = \int_0^{\cdot} g(X)d^-X + [g(X), X],$$

and Proposition 11 saying that $[g(X), X]$ exists. □

When X is a semimartingale, the Itô formula seen above becomes the following.

Proposition 13. *Let $(S_t)_{t\geqslant 0}$ be a continuous $(\mathcal{F}_t)$-semimartingale and f a function in $C^2(\mathbb{R})$. One has the following.*

1. $$f(S_t) = f(S_0) + \int_0^t f'(S_u)dS_u + \frac{1}{2}\int_0^t f''(S_u)d[S,S]_u.$$

2. *Let (S_t^0) be another continuous $(\mathcal{F}_t)$-semimartingale. The following integration by parts holds:*

$$S_tS_t^0 = S_0S_0^0 + \int_0^t S_u dS_u^0 + \int_0^t S_u^0 dS_u + [S, S^0]_t.$$

Proof. We recall that Itô and forward integrals coincide, see Proposition 6; therefore point 1 is a consequence of Proposition 12.

Point 2 stems from the integration by parts formula in Proposition 1 4). □

4.3 Lévy area

In Corollary 5, we have seen that $\int_0^t g(X)d^-X$ exists when X is a one-dimensional finite quadratic variation process and $g \in C^1(\mathbb{R})$.

If $X = (X^1, X^2)$ is two-dimensional and has all its mutual covariations, consider $g \in C^1(\mathbb{R}^2; \mathbb{R}^2)$. We naturally define, if it exists,

$$\int_0^t g(X) \cdot d^-X = \lim_{\varepsilon \to 0^+} I^-(\varepsilon, g(X) \cdot dX)(t),$$

where

$$I^-(\varepsilon, g(X) \cdot dX)(t) = \int_0^t g(X)(s) \cdot \frac{X(s+\varepsilon) - X(s)}{\varepsilon} ds; \quad 0 \leqslant t \leqslant T, \quad (30)$$

and $\cdot$ denotes the scalar product in $\mathbb{R}^2$.

With a 2-dimensional Itô formula of the same type as in Proposition 12, it is possible to show that $\int_0^t g(X) \cdot d^-X$ exists if $g = \nabla u$, where u is a potential of class C^2. If g is a general $C^1(\mathbb{R}^2)$ function, one cannot expect in general that $\int_0^t g(X) \cdot d^-X$ exists.

Lyons' rough paths approach, see for instance [36, 35, 31, 28, 8] has considered in detail the problem of the existence of integrals of the type $\int_0^t g(X) \cdot dX$. In this theory, the concept of Lévy area plays a significant role. Translating this in the present context one would say that the essential assumption is that $X = (X^1, X^2)$ has a Lévy area type process. This section will only make some basic observations on that topic from the perspective of stochastic calculus via regularization.

Given two classical semimartingales S^1, S^2, the classical notion of Lévy area is defined by

$$L(S^1, S^2)_t = \int_0^t S^1 dS^2 - \int_0^t S^2 dS^1,$$

where both integrals are of Itô type.

Definition 11. *Given two continuous processes X and Y, we put*

$$L(X,Y)_t = \lim_{\varepsilon \to 0^+} \int_0^t \frac{X_s Y_{s+\varepsilon} - X_{s+\varepsilon} Y_s}{\varepsilon} ds.$$

where the limit is understood in the ucp sense. $L(X,Y)$ is called the **Lévy area** *of the processes X and Y.*

Remark 15. The following properties are easy to establish.

1. $L(X,X) \equiv 0$.
2. The Lévy area is an antisymmetric operation, i.e.,

$$L(X,Y) = -L(Y,X).$$

Using the approximation of symmetric integral we can easily prove the following.

Proposition 14. $\int_0^{\cdot} X d^\circ Y$ *exists if and only if* $L(X,Y)$ *exists. Moreover*

$$2\int_0^t X d^\circ Y = X_t Y_t - X_0 Y_0 + L(X,Y)_t.$$

Recalling the convention that an equality among three objects implies that at least two among the three are defined, we have the following.

Proposition 15.

1. $L(X,Y)_t = \int_0^t X d^\circ Y - \int_0^t Y d^\circ X.$
2. $L(X,Y)_t = \int_0^t X d^- Y - \int_0^t Y d^- X.$

Proof.

1. From Proposition 14 applied to X, Y and Y, X, and by antisymmetry of Lévy areas we have

$$2\int_0^t X d^\circ Y = X_t Y_t - X_0 Y_0 + L(X,Y)_t,$$
$$2\int_0^t Y d^\circ X = X_t Y_t - X_0 Y_0 - L(X,Y)_t.$$

   Taking the difference gives 1.
2. Follows from the definition of forward integrals. □

Remark 16. If $[X,Y]$ exists, point 2 of Proposition 15 is a consequence of point 1 and of Proposition 1 1, 2.

For a real-valued process $(X_t)_{t\geqslant 0}$, Lemma 6 says that

$$[X,X] \quad \text{exists} \Leftrightarrow \int_0^{\cdot} X d^- X \quad \text{exists}.$$

Given a vector of processes $\underline{X} = (X^1, X^2)$ we may ask wether the following statement is true:

(X^1, X^2) has all its mutual brackets if and only if

$$\int_0^{\cdot} X^i d^- X^j \quad \text{exists},$$

for $i, j = 1, 2$. In fact the answer is negative if the two-dimensional process X does not have a Lévy area.

Remark 17. Suppose that (X^1, X^2) has all its mutual covariations. Let $*$ stand for $\circ$, or $-$, or $+$. The following are equivalent.

1. The Lévy area $L(X^1, X^2)$ exists.
2. $\int_0^\cdot X^i d^* X^j$ exists for any $i, j = 1, 2$.

By Lemma 6, we first observe that $\int X^i d^\circ X^i$ exists since X^i is a finite quadratic variation process. In point 2, equivalence between the three cases $\circ, -$ and $+$ is obvious using Proposition 1 1 2. Equivalence between the existence of $\int_0^\cdot X^1 d^\circ X^2$ and $L(X^1, X^2)$ was already established in Proposition 14.

5 Weak Dirichlet processes

5.1 Generalities

Weak Dirichlet processes constitute a natural generalization of Dirichlet processes, which in turn naturally extend semimartingales. Dirichlet processes have been considered by many authors, see for instance [21, 2].

Let $(\mathcal{F}_t)_{t \geqslant 0}$ be a fixed filtration fulfilling the usual conditions. In the present section 5, (W_t) will denote a classical $(\mathcal{F}_t)$-Brownian motion. For simplicity, we shall stick to the framework of continuous processes.

Definition 12.

1. *An $(\mathcal{F}_t)$-**Dirichlet process** is the sum of an $(\mathcal{F}_t)$-local martingale M and a zero quadratic variation process A.*
2. *An $(\mathcal{F}_t)$-**weak Dirichlet process** is the sum of an $(\mathcal{F}_t)$-local martingale M and a process A such that $[A, N] = 0$ for every continuous $(\mathcal{F}_t)$-local martingale N.*

In both cases, we will suppose $A_0 = 0$ a.s.

Remark 18.

1. The process (A_t) in the latter decomposition is $(\mathcal{F}_t)$-adapted.
2. Any $(\mathcal{F}_t)$-semimartingale is an $(\mathcal{F}_t)$-Dirichlet process.

The statement of the following proposition is essentially contained in [13].

Proposition 16.

1. *Any $(\mathcal{F}_t)$-Dirichlet process is an $(\mathcal{F}_t)$-weak Dirichlet process.*
2. *The decomposition $M + A$ is unique.*

Proof. Point 1 follows from Proposition 1 6).

Concerning point 2, let X be a weak Dirichlet process with decompositions $X = M^1 + A^1 = M^2 + A^2$. Then $0 = M + A$ where $M = M^1 - M^2, A = A^1 - A^2$. We evaluate the covariation of both members against M to obtain

$$0 = [M] + [M, A^1] - [M, A^2] = [M].$$

Since $M_0 = A_0 = 0$ and M is a local martingale, Corollary 1 gives $M = 0$. □

The class of semimartingales with respect to a given filtration is known to be stable with respect to C^2 transformations, as Proposition 13 implies. Proposition 11 says that finite quadratic variation processes are stable under C^1 transformations.

It is possible to show that the class of weak Dirichlet processes with finite quadratic variation (as well as Dirichlet processes) is stable with respect to the same type of transformations. We start with a result which is a slight improvement (in the continuous case) of a result obtained by [7].

Proposition 17. *Let X be a finite quadratic variation process which is $(\mathcal{F}_t)$-weak Dirichlet, and $f \in C^1(\mathbb{R})$. Then $f(X)$ is also weak Dirichlet.*

Proof. Let $X = M + A$ be the corresponding decomposition. We express $f(X_t) = M^f + A^f$ where

$$M_t^f = f(X_0) + \int_0^t f'(X)dM, \quad A_t^f = f(X_t) - M_t^f.$$

Let N be a local martingale. We have to show that $[f(X) - M^f, N] = 0$.

By additivity of the covariation, and the definition of weak Dirichlet process, $[X, N] = [M, N]$ so that Proposition 11 implies $[f(X), N]_t = \int_0^t f'(X_s)d[M, N]_s$.

On the other hand, Proposition 7 gives

$$[M^f, N]_t = \int_0^t f'(X_s)d[M, N]_s,$$

and the result follows. □

Remark 19.

1. If X is an $(\mathcal{F}_t)$- Dirichlet process, it can be proved similarly that $f(X)$ is an $(\mathcal{F}_t)$- Dirichlet process; see [2] and [51] for details.
2. The class of Lyons–Zheng processes introduced in [51] constitutes a natural generalization of reversible semimartingales, see Definition 13. The authors proved that this class is also stable through C^1 transformations.
3. Suppose that $(\mathcal{F}_t)$ is the canonical filtration associated with a Brownian motion W. Then a continuous $(\mathcal{F}_t)$-adapted process D is weak Dirichlet if and only if D is the sum of an $(\mathcal{F}_t)$-local martingale and a process A such that $[A, W] = 0$. See [9], Corollary 3.10.

We also report a Girsanov type theorem established by [7] at least in a discretization framework.

Proposition 18. *Let $X = (X_t)_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-weak Dirichlet process, and Q a probability equivalent to P on $\mathcal{F}_T$. Then $X = (X_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-weak Dirichlet process with respect to Q.*

Proof. We set $D_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$; D is a positive local martingale.

Let L be the local martingale such that $D_t = \exp(L_t - \frac{1}{2}[L]_t)$. Let $X = M + A$ be the corresponding decomposition. It is well-known that $\tilde{M} = M - [M, L]$ is a local martingale under Q. So, X is a Q-weak Dirichlet process. □

As mentioned earlier, Dirichlet processes are stable with respect to C^1 transformations. In applications, in particular to control theory, one often needs to know the nature of a process $(u(t, D_t))$ where $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$ and D is a Dirichlet process. The following result was established in [24].

Proposition 19. *Let (S_t) be a continuous $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation; let $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$. Then $(u(t, S_t))$ is a $(\mathcal{F}_t)$-weak Dirichlet process.*

Remark 20. There is no reason for $(u(t, S_t))$ to have a finite quadratic variation since the dependence of u on the first argument t may be very rough. A fortiori $(u(t, S_t))$ will not be Dirichlet. Consider for instance u only depending on time, deterministic, with infinite quadratic variation.

Examples of Dirichlet processes (respectively, weak Dirichlet processes) arise directly from classical Brownian motion W.

Example 2. Let f be of class $C^0(\mathbb{R})$, $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$.

1. If f is C^1, then $X = f(W)$ is a $(\mathcal{F}_t)$-Dirichlet process.
2. $u(t, W_t)$ is an $(\mathcal{F}_t)$-weak Dirichlet process, but not Dirichlet in general.
3. $f(W)$ is not always a Dirichlet process, not even of finite quadratic variation as shown by Proposition 20.

The Example and Remark above easily show that the class of $(\mathcal{F}_t)$-Dirichlet processes strictly includes the class of $(\mathcal{F}_t)$-semimartingales.

More sophisticated examples of weak Dirichlet processes may be found in the class of the so-called *Volterra* type processes, see e.g., [12, 13].

Example 3. Let $(N_t)_{t \geqslant 0}$ be an $(\mathcal{F}_t)$-local martingale, $G : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ a continuous random field such that $G(t, \cdot)$ is $(\mathcal{F}_s)$-adapted for each t. Set

$$X_t = \int_0^t G(t, s) dN_s.$$

Then (X_t) is an $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $M + A$, where $M_t = \int_0^t G(s, s) dN_s$.

Suppose that $[G(\cdot, s_1), G(\cdot, s_2)]$ exists for any s_1, s_2. With some additional technical assumption, one can show that A is a finite quadratic variation process with

$$[A]_t = 2 \int_0^t \left(\int_0^{s_2} [G(\cdot, s_1), G(\cdot, s_2)] \circ dM_{s_1} \right) \circ dM_{s_2};$$

this iterated Stratonovich integral can be expressed as the sum $C_1(t) + C_2(t)$ where

$$C_1(t) = \int_0^t [G(\cdot, s), G(\cdot, s)] d[M]_s,$$

$$C_2(t) = 2 \int_0^t \left(\int_0^{s_2} [G(\cdot, s_1), G(\cdot, s_2)] dM_{s_1} \right) dM_{s_2}.$$

Example 4. Take for N a Brownian motion W and $G(t,s) = B_{(t-s)\vee 0}$ where B is a Brownian motion independent of W. Then $[A] = \int_0^t (t-s) ds = \frac{t^2}{2}$.

One significant motivation for considering Dirichlet (respectively, weak Dirichlet) processes comes from the study of generalized diffusion processes, typically solutions of stochastic differential equations with distributional drift.

Such processes were investigated using stochastic calculus via regularization by [18, 19]. We try to express here just a guiding idea. The following particular case of such equations is motivated by random media modelization:

$$dX_t = dW_t + b'(X_t)dt, \quad X_0 = x_0 \tag{31}$$

where b is a continuous function. Typically, b could be the realization of a continuous process, independent of W, stopped outside a finite interval.

We shall not recall the precise meaning of the solution of (31). In [18, 19] a rigorous sense is given to a solution (in the distribution laws) and existence and uniqueness are established for any initial conditions.

Here we shall just attempt to convince the reader that the solution is a Dirichlet process. For this we define the real function h of class C^1 by

$$h(x) = \int_0^x e^{-b(y)} dy.$$

We set $\sigma_0 = h' \circ h^{-1}$. We consider the unique solution in law of the equation

$$dY_t = \sigma_0(Y_t) dW_t, Y_0 = h(x_0)$$

which exists because of classical Stroock–Varadhan arguments ([53]); so Y is clearly a semimartingale, thus a Dirichlet process. The process $X = h^{-1}(Y)$ is a Dirichlet process since h^{-1} is of class C^1. If b were of class C^1, (31) would be an ordinary stochastic differential equation, and it could be shown that X is the unique solution of that equation. In the present case X will still be the solution of (31), considered as a generalized stochastic differential equation.

We now consider the case when the drift is time inhomogeneous as follows:

$$dX_t = dW_t + \partial_x b(t, X_t)dt, X_0 = x_0 \tag{32}$$

where $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function of class C^1 in time. Then it is possible to find a $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ such that the *solution* (X_t) of (32) can be expressed as $(k(t, Y_t))$ for some semimartingale Y; so X will be an $(\mathcal{F}_t)$-weak Dirichlet process. For this and more general situations, see [44].

5.2 Itô formula under weak smoothness assumptions

In this section, we formulate and prove an Itô formula of C^1 type. As for the C^2 type Itô formula, the next Theorem is stated in the one-dimensional framework only in spite of its validity in the multidimensional case.

Let $(S_t)_{t\geqslant 0}$ be a semimartingale and $f \in C^2$. We recall the classical Itô formula, as a particular case of Proposition 13:

$$f(S_t) = f(S_0) + \int_0^t f'(S_s)dS_s + \frac{1}{2}\int_0^t f''(S_s)d[S,S]_s.$$

Using Proposition 6 and Definition 10 (Stratonovich integrals), we obtain

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S_s)dS_s + \frac{1}{2}[f'(S),S]_t \\ &= f(S_0) + \int_0^t f'(S) \circ dS. \end{aligned} \tag{33}$$

We observe that in formulae (33), only the first derivative of f appears. Besides, we know that $f(S)$ is a Dirichlet process if $f \in C^1(\mathbb{R})$.
At this point we may ask if formulae (33) remain valid when f is in $C^1(\mathbb{R})$ only; a partial answer will be given in Theorem 3 below.

Definition 13. *Let (S_t) be a continuous semimartingale; set $\hat{S}_t = S_{T-t}$ for $t \in [0,T]$. S is called a* **reversible semimartingale** *if $(\hat{S}_t)_{t\in[0,T]}$ is again a semimartingale.*

Theorem 3. *([45]) Let S be a reversible semimartingale indexed by $[0,T]$ and $f \in C^1(\mathbb{R})$. Then one has*

$$f(S_t) = f(S_0) + \int_0^t f'(S)dS + R_t = f(S_0) + \int_0^t f'(S) \circ dS$$

where $R = \frac{1}{2}[f'(S),S]$.

Remark 21. After the pioneering work of [5], which expressed the remainder term (R_t) with the help of generalized integral with respect to local time, two papers appeared: [22] in the case of Brownian motion and [22] and [45] for multidimensional reversible semimartingales. Later, an incredible amount of contributions on that topic have been published. We cannot give the precise content of each paper; a non-exhaustive list is [1, 14, 15, 23, 24, 39, 40]. Among the C^1-type Itô formulae in the framework of generalized Stratonovich integral with respect to Lyons–Zheng processes, it is also important to quote [33, 34, 51].

Example 5.

i) Classical $(\mathcal{F}_t)$-Brownian motion W is a reversible semimartingale, see for instance [22, 41, 19]. More precisely $\hat{W}_t = W_T + \beta_t + \displaystyle\int_0^t \frac{\hat{W}_s}{T-s} ds$, where β is a $(\mathcal{G}_t)$-Brownian motion and $(\mathcal{G}_t)$ is the natural filtration associated with $\hat{W}_t$.

ii) Let (X_t) be the solution of the stochastic differential equation

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt,$$

with $\sigma, b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ Lipschitz with at most linear growth, $\sigma \geqslant c > 0$. Then (X_t) is a reversible semimartingale; see for instance [19]. Moreover if $f \in W^{1,2}_{\text{loc}}$, it is proved in [19] that $(f(X_t))$ is an $(\mathcal{F}_t)$-Dirichlet process.

Proof (of Theorem 3). We use in an essential way the Banach–Steinhaus theorem for F-spaces; see for instance [10] Chap. 2.1.

Define two maps $T^{\pm}_{\varepsilon}$ from the F-space $C^0(\mathbb{R})$ to the F-space $\mathcal{C}([0,T])$, which consists of all continuous processes indexed by $[0,T]$, by

$$T^-_{\varepsilon} g = \int_0^{\cdot} g(S_s) \frac{S_{s+\varepsilon} - S_s}{\varepsilon} ds,$$

$$T^+_{\varepsilon} g = \int_0^{\cdot} g(S_s) \frac{S_s - S_{s-\varepsilon}}{\varepsilon} ds.$$

These operators are linear and continuous. Moreover, for each $g \in C^0$ we have

$$\lim_{\varepsilon \to 0} T^-_{\varepsilon} g = \int_0^{\cdot} g(S)dS,$$

because of Proposition 6 which says that $\int_0^t g(S)dS$ is also an Itô integral.

Since $\hat{S}$ is a semimartingale, for the same reasons as above,

$$\int_{T-t}^{T} g(\hat{S})d^-\hat{S} \tag{34}$$

also exists and equals an Itô integral.

Using Proposition 1 3), it follows that $\displaystyle\int_0^{\cdot} g(S)d^+S$ also exists.

Therefore the Banach–Steinhaus theorem implies that

$$g \mapsto \int_0^{\cdot} g(S)d^-S, \quad g \mapsto \int_0^{\cdot} g(S)d^+S,$$

are continuous maps from $C^0(\mathbb{R})$ to $\mathcal{C}([0,T])$; by additivity, so are also

$$g \mapsto [g(S), S], \quad g \mapsto \int_0^{\cdot} g(S)d^{\circ}S.$$

Let $f \in C^1(\mathbb{R})$, $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers converging to the Dirac measure at zero. We set $f_\varepsilon = f \star \rho_\varepsilon$ where $\star$ denotes convolution. Since f_ε is of class C^2, by the "smooth" Itô formula stated at Proposition 13 and by Proposition 1 1) and 2), we have

$$f_\varepsilon(S_t) = f_\varepsilon(S_0) + \int_0^t f'_\varepsilon(S)dS + \frac{1}{2}[f'_\varepsilon(S), S],$$
$$f_\varepsilon(S_t) = f_\varepsilon(S_0) + \int_0^t f'_\varepsilon(S)d^\circ S.$$

Since f'_ε goes to f' in $C^0(\mathbb{R})$, we can take the limit term by term and

$$\begin{aligned} f(S_t) &= f(S_0) + \int_0^t f'(S)dS + \frac{1}{2}[f'(S), S], \\ f(S_t) &= f(S_0) + \int_0^t f'(S)d^\circ S. \end{aligned} \tag{35}$$

Remark 12 says that the latter symmetric integral is in fact a Stratonovich integral. □

Corollary 6. *If $(S_t)_{t\in[0,T]}$ is a reversible semimartingale and $g \in C^0(\mathbb{R})$, then $[g(S), S]$ exists and has zero quadratic variation.*

Proof. Let $g \in C^0(\mathbb{R})$ and let $S = M + V$ be the decomposition of S as a sum of a local martingale M and a finite variation process V, such that $V_0 = 0$. Let $f \in C^1(\mathbb{R})$ such that $f' = g$. We know that $f(S)$ is a Dirichlet process with local martingale part

$$M_t^f = f(S_0) + \int_0^t g(S)dM.$$

Let A^f be its zero quadratic variation component. Using Theorem 3, we have

$$A_t^f = \int_0^t g(S)dV + \frac{1}{2}[g(S), S].$$

$\int_0^\cdot g(S)dV$ has finite variation, therefore it has zero quadratic variation; since so does also A^f, the result follows immediately. □

Proposition 20. *Let $g \in C^0(\mathbb{R})$ such that $g(W)$ is a finite quadratic variation process. Then g has bounded variation on compacts.*

Proof. Suppose that $g(W)$ is of finite quadratic variation. We already know that W is a reversible semimartingale. By Corollary 6, $[W, g(W)]$ exists and it is a zero quadratic variation process. Since $[W]$ exists, we deduce that $(g(W), W)$ has all its mutual covariations. In particular $[g(W), W]$ has bounded variation because of Remark 2. Let f be such that $f' = g$; Theorem 3 implies that $f(W)$ is a semimartingale. A celebrated result of Çinlar, Jacod, Protter and Sharpe [6] asserts that $f(W)$ is a $(\mathcal{F}_t)$-semimartingale if and only if f is a difference of two convex functions; this finally allows to conclude that g has bounded variation on compacts. □

Remark 22. Given two processes X and Y, the covariations $[X]$ and $[X, Y]$ may exist even if Y is not of finite quadratic variation. In particular (X, Y) may not have all its mutual covariations. For instance, if X has bounded variation, and Y is any continuous process, then $[X, Y] = 0$, see Proposition 1 7 b). A less trivial example is provided by $X = W$, $Y = g(W)$ where g is continuous but not of bounded variation, see Proposition 20.

Remark 23 ([22]). When S is a Brownian motion, Theorem 3 and Corollary 6 are in fact, respectively, valid for $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R})$.

6 Final remarks

We conclude this paper with some considerations about calculus related to processes having no quadratic variation. On this, the reader can consult [13, 27, 26]. In [13] one defines a notion of n-covariation $[X^1, \ldots, X^n]$ of n processes $X^1, \ldots, X^n$ and the n-variation of a process X.

We recall some basic significant results related to those papers.

1. For a process X having a 3-variation, it is possible to write an Itô formula of the type
$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X_s - \frac{1}{12} \int_0^t f^{(3)}(X_s) d[X, X, X]_s.$$
Moreover one-dimensional stochastic differential equations driven by a strong 3-variation were considered in [13].
2. Let $B = B^H$ be a fractional Brownian motion with Hurst index $H > \frac{1}{6}$ and f a function of class C^6. It is shown in [27, 26] that
$$f(B_t) = f(B_0) + \int_0^t f'(B) d^\circ B.$$
3. Using more sophisticated integrals via regularization, other types of Itô formulae can be written for any H in $]0, 1[$; see [26].
4. In [25], it is shown that stochastic calculus via regularization is *almost pathwise*. Suppose for instance that X is a semimartingale or a fractional Brownian motion, with Hurst index $H > \frac{1}{2}$; then its quadratic variation $[X]$ is a limit of $C(\varepsilon, X, X)$ not only ucp as in (5), but also *uniformly a.s.* Similarly, if X is semimartingale and Y is a suitable integrand, the Itô integral $\int_0^\cdot Y dX$ is approximated by $I^-(\varepsilon, Y, dX)$ not only ucp as in (2), but also uniformly a.s.

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References

1. X. Bardina and M. Jolis. An extension of Itô's formula for elliptic diffusion processes. *Stochastic Process. Appl.*, 69(1):83–109, 1997
2. J. Bertoin. Les processus de Dirichlet en tant qu'espace de Banach. *Stochastics*, 18(2):155–168, 1986
3. J. Bertoin. Sur une intégrale pour les processus à α-variation bornée. *Ann. Probab.*, 17(4):1521–1535, 1989
4. F. Biagini and B. Øksendal. Minimal variance hedging for insider trading. *Preprint Oslo 2004*
5. N. Bouleau and M. Yor. Sur la variation quadratique des temps locaux de certaines semimartingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(9):491–494, 1981
6. E. Çinlar, J. Jacod, P. Protter and M.J. Sharpe. Semimartingales and Markov processes. *Z. Wahrsch. Verw. Gebiete*, 54(2):161–219, 1980
7. F. Coquet, A. Jakubowki, J. Mémin and L. Slomiński. Natural decomposition of processes and weak Dirichlet processes. *To appear: Séminaire de Probabilités*
8. Laure Coutin and Zhongmin Qian. Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields*, 122(1):108–140, 2002
9. R. Coviello and F. Russo. Non-Semimartingales processes: stochastic differential equations and weak Dirichlet processes. *Preprint LAGA 2005-11. Accepted for publication: Annals of Probability*
10. N. Dunford and J.T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. Wiley, New York, 1988 General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original. Wiley, New York
11. N. Eisenbaum. Integration with respect to local time. *Potential Anal.*, 13(4):303–328, 2000
12. M. Errami and F. Russo. Covariation de convolution de martingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(5):601–606, 1998
13. M. Errami and F. Russo. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2):259–299, 2003
14. M. Errami, F. Russo and P. Vallois. Itô's formula for $C^{1,\lambda}$-functions of a càdlàg process and related calculus. *Probab. Theory Related Fields*, 122(2):191–221, 2002
15. D. Feyel and A. de La Pradelle. On fractional Brownian processes. *Potential Anal.*, 10(3):273–288, 1999
16. F. Flandoli and F. Russo. Generalized calculus and SDEs with nonregular drift. *Stoch. Stoch. Rep.*, 72(1–2):11–54, 2002
17. F. Flandoli and F. Russo. Generalized integration and stochastic ODEs. *Ann. Probab.*, 30(1):270–292, 2002
18. F. Flandoli, F. Russo and J. Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003

19. F. Flandoli, F. Russo and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004
20. H. Föllmer. Calcul d'Itô sans probabilités. In *Séminaire de Probabilités, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 143–150. Springer, Berlin Heidelberg New York, 1981
21. H. Föllmer. Dirichlet processes. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 476–478. Springer, Berlin Heidelberg New York, 1981
22. H. Föllmer, P. Protter and A.N. Shiryaev. Quadratic covariation and an extension of Itô's formula. *Bernoulli*, 1(1–2):149–169, 1995
23. R. Ghomrasni and G. Peskir. Local time-space calculus and extensions of Itô's formula. In *High dimensional probability, III (Sandjberg, 2002)*, volume 55 of *Progr. Probab.*, pages 177–192. Birkhäuser, Basel, 2003
24. F. Gozzi and F. Russo. Weak Dirichlet processes with a stochastic control perspective. *Preprint LAGA-Paris 13, 2005-13*
25. M. Gradinaru and I. Nourdin. Approximation at first and second order of m-order integrals of the fractional Brownian motion and of certain semimartingales. *Electron. J. Probab.*, 8:no. 18, 26 pp. (electronic), 2003
26. M. Gradinaru, I. Nourdin, F. Russo and P. Vallois. m-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.*, 41(4):781–806, 2005
27. M. Gradinaru, F. Russo and P. Vallois. Generalized covariations, local time and Stratonovich Itô's formula for fractional Brownian motion with Hurst index $H \geqslant \frac{1}{4}$. *Ann. Probab.*, 31(4):1772–1820, 2003
28. M. Gubinelli. Controlling rough paths. *J. Funct. Analysis*, 216(1):86–140, 2004
29. Th. Jeulin and M. Yor, editors. *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*. Springer, Berlin Heidelberg New York, 1985 Papers from the seminar on stochastic calculus held at the Université de Paris VI, Paris, 1982/1983
30. I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer, Berlin Heidelberg New York, second edition, 1991
31. A. Lejay. An introduction to rough paths. In *Séminaire de Probabilités XXXVII*, volume 1832 of *Lecture Notes in Math.*, pages 1–59. Springer, Berlin Heidelberg New York, 2003
32. J.A. León, R. Navarro and D. Nualart. An anticipating calculus approach to the utility maximization of an insider. *Math. Finance*, 13(1):171–185, 2003 Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001)
33. T.J. Lyons and T.S. Zhang. Decomposition of Dirichlet processes and its application. *Ann. Probab.*, 22(1):494–524, 1994
34. T.J. Lyons and W.A. Zheng. A crossing estimate for the canonical process on a Dirichlet space and a tightness result. *Astérisque*. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987)
35. T. Lyons and Z. Qian. *System control and rough paths*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2002
36. T.J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998
37. H.P. McKean, Jr. *Stochastic integrals*. Probability and Mathematical Statistics, No. 5. Academic Press, New York, 1969

38. E.J. McShane. *Stochastic calculus and stochastic models*. Academic Press, New York, 1974 Probability and Mathematical Statistics, Vol. 25.
39. S. Moret and D. Nualart. Quadratic covariation and Itô's formula for smooth nondegenerate martingales. *J. Theoret. Probab.*, 13(1):193–224, 2000
40. S. Moret and D. Nualart. Generalization of Itô's formula for smooth nondegenerate martingales. *Stochastic Process. Appl.*, 91(1):115–149, 2001
41. É. Pardoux. Grossissement d'une filtration et retournement du temps d'une diffusion. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 48–55. Springer, Berlin Heidelberg New York, 1986
42. E. Perkins. Stochastic integrals and progressive measurability—an example. In *Seminaire de Probabilités, XVII*, volume 986 of *Lecture Notes in Math.*, pages 67–71. Springer, Berlin Heidelberg New York, 1983
43. P. Protter. *Stochastic integration and differential equations, a new approach*, volume 21 of *Applications of Mathematics (New York)*. Springer, Berlin Heidelberg New York, 1990
44. F. Russo and G. Trutnau. Time inhomogeneous one-dimensional stochastic differential equations with distributional drift. *In preparation*.
45. F. Russo and P. Vallois. Itô formula for C^1-functions of semimartingales. *Probab. Theory Related Fields*, 104(1):27–41, 1996
46. F. Russo and P. Vallois. Intégrales progressive, rétrograde et symétrique de processus nonadaptés. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(8):615–618, 1991
47. F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3):403–421, 1993
48. F. Russo and P. Vallois. Noncausal stochastic integration for làd làg processes. In *Stochastic analysis and related topics (Oslo, 1992)*, volume 8 of *Stochastics Monogr.*, pages 227–263. Gordon and Breach, Montreux, 1993
49. F. Russo and P. Vallois. The generalized covariation process and Itô formula. *Stochastic Process. Appl.*, 59(1):81–104, 1995
50. F. Russo and P. Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1–2):1–40, 2000
51. F. Russo, P. Vallois and J. Wolf. A generalized class of Lyons–Zheng processes. *Bernoulli*, 7(2):363–379, 2001
52. E.M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, NJ, 1970
53. D.W. Stroock and S.R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin Heidelberg New York, 1979
54. J. Wolf. An Itô formula for local Dirichlet processes. *Stochastics Stochastics Rep.*, 62(1–2):103–115, 1997
55. J. Wolf. Transformations of semimartingales and local Dirichlet processes. *Stochastics Stochastics Rep.*, 62(1–2):65–101, 1997
56. J. Wolf. A representation theorem for continuous additive functionals of zero quadratic variation. *Probab. Math. Statist.*, 18(2, Acta Univ. Wratislav. No. 2111):385–397, 1998
57. L.C. Young. An inequality of Hölder type, connected with Stieltjes integration. *Acta Math.*, 67:251–282, 1936
58. M. Zähle. Integration with respect to fractal functions and stochastic calculus. II. *Math. Nachr.*, 225:145–183, 2001

On the Smooth-Fit Property for One-Dimensional Optimal Switching Problem

Huyên Pham

Laboratorie de Probabilités et Modèles Aléatoires, CNRS
UMR 7599, Université Paris 7
e-mail: pham@math.jussieu.fr

Summary. This paper studies the problem of optimal switching for a one-dimensional diffusion, which may be regarded as a sequential optimal stopping problem with changes of regimes. The resulting dynamic programming principle leads to a system of variational inequalities, and the state space is divided into continuation regions and switching regions. By a viscosity solutions approach, we prove the smooth-fit C^1 property of the value functions.

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Key words: Optimal switching, System of variational inequalities, Viscosity solutions, Smooth-fit principle

1 Introduction

In this paper, we consider the optimal switching problem for a one-dimensional stochastic process X. The diffusion process X may take a finite number of regimes that are switched at time decisions. The evolution of the controlled system is governed by

$$dX_t = b(X_t, I_t)dt + \sigma(X_t, I_t)dW_t,$$

with the indicator process of the regimes:

$$I_t = \sum_n \kappa_n 1_{\tau_n \leqslant t < \tau_{n+1}}.$$

Here W is a standard Brownian motion on a filtered probability space

$$(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}, P),$$

b, σ are given maps, $(\tau_n)_n$ is a sequence of increasing stopping times representing the switching regimes time decisions, and κ_n is $\mathcal{F}_{\tau_n}$-measurable and valued in a finite set, representing the new chosen value of the regime at time τ_n and until τ_{n+1}.

Our problem consists in maximizing over the switching controls (τ_n, κ_n) the gain functional

$$E\left[\int_0^\infty e^{-\rho t} f(X_t, I_t)dt - \sum_n e^{-\rho \tau_n} g_{\kappa_{n-1},\kappa_n}\right]$$

where f is some running profit function depending on the current state and the regime, and g_{ij} is the cost for switching from regime i to j. We then denote by $v_i(x)$ the value function for this control problem when starting initially from state x and regime i.

Optimal switching problems for stochastic systems were studied by several authors, see [1, 4, 7]. These control problems lead via the dynamic programming principle to a system of second-order variational inequalities for the value functions v_i. Since the v_i are not smooth C^2 in general, a first mathematical point is to give a rigorous meaning to these variational PDE, either in Sobolev spaces as in [4], or by means of viscosity notion as in [7]. We also see that for each fixed regime i, the state space is divided into a switching region where it is optimal to change from regime i to some regime j, and the continuation region where it is optimal to stay in the current regime i. Optimal switching problem may be viewed as sequential optimal stopping problems with regimes shifts. It is well known that optimal stopping problem leads to a free-boundary problem related to a variational inequality that divides the state space into the stopping region and the continuation region. Moreover, there is the so-called smooth-fit principle for optimal stopping problems that states the smoothness C^1 regularity of the value function through the boundary of the stopping region, once the reward function is smooth C^1 or is convex, see, e.g. [6]. Smooth-fit principle for optimal stopping problems may be proved by different arguments and we mention recent ones in [2] or [5] based on local time and extended Itô's formula. Our main concern is to study such smooth-fit principle in the context of optimal switching problem, which has not yet been considered in the literature to the best of our knowledge.

Here, we use viscosity solutions arguments to prove the smooth-fit C^1 property of the value functions through the boundaries of the switching regions. The main difficulty with regard to optimal stopping problems, comes from the fact that the switching region for the value function v_i depend also on the other value functions v_j for which one does not know a priori C^1 regularity (this is what we want to prove!) or convexity property. For this reason, it is an open question to see how extended Itô's formula and local time may be used to derive such smooth-fit property for optimal switching problems. Our proof arguments are relatively simple and do not require any specific knowledge on viscosity solutions theory.

The plan of this paper is organized as follows. In Section 2, we formulate our optimal switching problem and make some assumptions. Section 3 is devoted to the dynamic programming PDE characterization of the value functions by viscosity solutions, through a system of variational inequalities. In Section 4, we prove the smooth-fit property of the value functions.

2 Problem formulation and assumptions

We start with the mathematical framework for our optimal switching problem. The stochastic system X is valued in the state space $\mathcal{X} \subset \mathbb{R}$ assumed to be an interval with endpoints $-\infty \leqslant \ell < r \leqslant \infty$. We call $\mathbb{I}_d = \{1, \ldots, d\}$ the finite set of regimes. The dynamics of the controlled stochastic system is modelled as follows. We are given maps $b, \sigma : \mathcal{X} \times \mathbb{I}_d \to \mathbb{R}$ satisfying a Lipschitz condition in x:

$$(\mathbf{H1}) \qquad |b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \leqslant C|x-y|, \quad \forall x, y \in \mathcal{X},\ i \in \mathbb{I}_d,$$

for some positive constant C, and we require

$$(\mathbf{H2}) \qquad \sigma(x,i) > 0, \quad \forall x \in \operatorname{int}(\mathcal{X}) \ = \ (\ell, r),\ i \in \mathbb{I}_d.$$

We set $b_i(.) = b(., i)$, $\sigma_i(.) = \sigma(., i)$, $i \in \mathbb{I}_d$, and we assume that for any $x \in \mathcal{X}$, $i \in \mathbb{I}_d$, there exists a unique strong solution valued in $\mathcal{X}$ to the s.d.e.

$$dX_t = b_i(X_t)dt + \sigma_i(X_t)dW_t, \quad X_0 = x. \tag{2.1}$$

where W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}, P)$ satisfying the usual conditions.

A switching control α consists of a double sequence $\tau_1, \ldots, \tau_n, \ldots, \kappa_1, \ldots, \kappa_n, \ldots$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where τ_n are stopping times, $\tau_n < \tau_{n+1}$ and $\tau_n \to \infty$ a.s., and κ_n is $\mathcal{F}_{\tau_n}$-measurable valued in $\mathbb{I}_d$. We denote by $\mathcal{A}$ the set of all such switching controls. Now, for any initial condition $(x, i) \in \mathcal{X} \times \mathbb{I}_d$, and any control $\alpha = (\tau_n, \kappa_n)_{n \geqslant 1} \in \mathcal{A}$, there exists a unique strong solution valued in $\mathcal{X} \times \mathbb{I}_d$ to the controlled stochastic system:

$$X_0 = x, \quad I_{0^-} \ = \ i, \tag{2.2}$$

$$dX_t = b_{\kappa_n}(X_t)dt + \sigma_{\kappa_n}(X_t)dW_t, \quad I_t \ = \ \kappa_n, \quad \tau_n \leqslant t < \tau_{n+1}, \quad n \geqslant 0. \tag{2.3}$$

Here, we set $\tau_0 = 0$ and $\kappa_0 = i$. We denote by $(X^{x,i}, I^i)$ this solution (as usual, we omit the dependence in α for notational simplicity). We notice that $X^{x,i}$ is a continuous process and I^i is a cadlag process, possibly with a jump at time 0 if $\tau_1 = 0$ and so $I_0 = \kappa_1$.

We are given a running profit function $f : \mathcal{X} \times \mathbb{I}_d \to \mathbb{R}$, and we assume a Lipschitz condition:

$$(\mathbf{H3}) \qquad |f(x,i) - f(y,i)| \leqslant C|x-y|, \quad \forall x, y \in \mathcal{X},\ i \in \mathbb{I}_d,$$

for some positive constant C. We also set $f_i(.) = f(., i)$, $i \in \mathbb{I}_d$. The cost for switching from regime i to j is constant equal to g_{ij}. We assume that:

(H4) $$0 < g_{ik} \leqslant g_{ij} + g_{jk}, \quad \forall i \neq j \neq k \neq i \in \mathbb{I}_d.$$

This last condition means that the switching cost is positive and it is no more expensive to switch directly in one step from regime i to k than in two steps via an intermediate regime j.

The expected total profit of running the system when initial state is (x, i) and using the switching control $\alpha = (\tau_n, \kappa_n)_{n \geqslant 1} \in \mathcal{A}$ is

$$J(x, i, \alpha) = E\left[\int_0^\infty e^{-\rho t} f\left(X_t^{x,i}, I_t^i\right) dt - \sum_{n=1}^\infty e^{-\rho \tau_n} g_{\kappa_{n-1}, \kappa_n}\right],$$

where $\kappa_0 = i$. Here $\rho > 0$ is a positive discount factor, and we use the convention that $e^{-\rho \tau_n(\omega)} = 0$ when $\tau_n(\omega) = \infty$. We shall see below in Lemma 3.1 that the expectation defining $J(x, i, \alpha)$ is well defined for ρ large enough (independent of x, i, α). The objective is to maximize this expected total profit over all strategies α. Accordingly, we define the function

$$v(x, i) = \sup_{\alpha \in \mathcal{A}} J(x, i, \alpha), \quad x \in \mathcal{X}, \ i \in \mathbb{I}_d. \tag{2.4}$$

and we denote $v_i(.) := v(., i)$ for $i \in \mathbb{I}_d$. The goal of this paper is to study the smoothness property of the value functions v_i. Our main result is the following:

Theorem 2.1 *Assume that* **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *hold. Then, for all* $i \in \mathbb{I}_d$, *the value function* v_i *is continuously differentiable on* $\mathrm{int}(\mathcal{X}) = (\ell, r)$.

3 Dynamic programming, viscosity solutions and system of variational inequalities

We first show the Lipschitz continuity of the value functions v_i.

Lemma 3.1 *Under* **(H1)** *and* **(H3)**, *there exists some positive constant* $C > 0$ *such that for all* $\rho \geqslant C$, *we have:*

$$|v_i(x) - v_i(y)| \leqslant C|x - y|, \quad \forall x, y \in \mathcal{X}, \ i \in \mathbb{I}_d. \tag{3.1}$$

Proof. In the sequel, for notational simplicity, the C denote a generic constant in different places, depending on the constants appearing in the Lipschitz conditions in **(H1)** and **(H3)**. For any $\alpha \in \mathcal{A}$, the solution to (2.2)–(2.3) is written as:

$$X_t^{x,i} = x + \int_0^t b\left(X_s^{x,i}, I_s^i\right) ds + \int_0^t \sigma\left(X_s^{x,i}, I_s^i\right) dW_s$$

$$I_t^i = \sum_{n=0}^\infty \kappa_n 1_{\tau_n \leqslant t < \tau_{n+1}}, \quad (\tau_0 = 0, \ \kappa_0 = i).$$

By standard estimate for s.d.e. applying Itô's formula to $\left|X_t^{x,i}\right|^2$ and using Gronwall's lemma, we then obtain from the linear growth condition on b and σ in **(H1)** the following inequality for any $\alpha \in \mathcal{A}$:

$$E\left|X_t^{x,i}\right|^2 \leqslant Ce^{Ct}(1+|x|^2), \quad t \geqslant 0.$$

Hence, by the linear growth condition on f in **(H3)**, this proves that for any $\alpha \in \mathcal{A}$:

$$\begin{aligned} E\left[\int_0^\infty e^{-\rho t}\left|f\left(X_t^{x,i}, I_t^i\right)\right|dt\right] &\leqslant CE\left[\int_0^\infty e^{-\rho t}\left(1+\left|X_t^{x,i}\right|\right)dt\right] \\ &\leqslant C\int_0^\infty e^{-\rho t}e^{Ct}(1+|x|)dt \\ &\leqslant C(1+|x|), \end{aligned}$$

for ρ larger than C. Recalling that the g_{ij} are non-negative, this last inequality proves in particular that for all $(x,i,\alpha) \in \mathcal{X} \times \mathbb{I}_d \times \mathcal{A}$, $J(x,i,\alpha)$ is well defined, valued in $[-\infty, \infty)$.

Moreover, by standard estimate for s.d.e. applying Itô's formula to $\left|X_t^{x,i} - X_t^{y,i}\right|^2$ and using Gronwall's lemma, we then obtain from the Lipschitz condition **(H1)** the following inequality uniformly in $\alpha \in \mathcal{A}$:

$$E\left|X_t^{x,i} - X_t^{y,i}\right|^2 \leqslant e^{Ct}|x-y|^2, \quad \forall x, y \in \mathcal{X},\ t \geqslant 0.$$

From the Lipschitz condition **(H3)**, we deduce

$$\begin{aligned} |v_i(x) - v_i(y)| &\leqslant \sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-\rho t}\left|f\left(X_t^{x,i}, I_t^i\right) - f\left(X_t^{y,i}, I_t^i\right)\right|dt\right] \\ &\leqslant C\sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-\rho t}\left|X_t^{x,i} - X_t^{y,i}\right|dt\right] \\ &\leqslant C\int_0^\infty e^{-\rho t}e^{Ct}|x-y|dt \leqslant C|x-y|, \end{aligned}$$

for ρ larger than C. This proves (3.1). □

In the rest of this paper, we shall now assume that ρ is large enough so that from the previous Lemma, the expected gain functional $J(x,i,\alpha)$ is well defined for all x, i, α, and also the value functions v_i are continuous.

The dynamic programming principle is a well-known property in stochastic optimal control. In our optimal switching control problem, it is formulated as follows:

Dynamic programming principle: For any $(x,i) \in \mathcal{X} \times \mathbb{I}_d$, we have

$$\begin{aligned} v(x,i) = \sup_{(\tau_n,\kappa_n)_n\in\mathcal{A}} E\Bigg[&\int_0^\theta e^{-\rho t}f\left(X_t^{x,i}, I_t^i\right)dt + e^{-\rho\theta}v(X_\theta^{x,i}, I_\theta^i) \\ &- \sum_{\tau_n\leqslant\theta} e^{-\rho\tau_n}g_{\kappa_{n-1},\kappa_n}\Bigg], \end{aligned} \tag{3.2}$$

where θ is any stopping time, possibly depending on $\alpha \in \mathcal{A}$ in (3.2). This principle was formally stated in [1] and proved rigorously for the finite horizon case in [7]. The arguments for the infinite horizon case may be adapted in a straightforward way.

The dynamic programming principle combined with the notion of viscosity solutions are known to be a general and powerful tool for characterizing the value function of a stochastic control problem via a PDE representation, see [3]. We recall the definition of viscosity solutions for a PDE in the form

$$F(x, v, D_x v, D^2_{xx} v) = 0, \quad x \in \mathcal{O}, \tag{3.3}$$

where $\mathcal{O}$ is an open subset in $\mathbb{R}^n$ and F is a continuous function and nonincreasing in its last argument (with respect to the order of symmetric matrices).

Definition 3.1 *Let v be a continuous function on $\mathcal{O}$. We say that v is a viscosity solution to* (3.3) *on $\mathcal{O}$ if it is*

(i) a viscosity supersolution to (3.3) *on $\mathcal{O}$: for any $x_0 \in \mathcal{O}$ and any C^2 function φ in a neighborhood of x_0 s.t. x_0 is a local minimum of $v - \varphi$ and $(v - \varphi)(x_0) = 0$, we have:*

$$F(x_0, \varphi(x_0), D_x\varphi(x_0), D^2_{xx}\varphi(x_0)) \geqslant 0,$$

and

(ii) a viscosity subsolution to (3.3) *on $\mathcal{O}$: for any $x_0 \in \mathcal{O}$ and any C^2 function φ in a neighborhood of x_0 s.t. x_0 is a local maximum of $v - \varphi$ and $(v - \varphi)(x_0) = 0$, we have:*

$$F(x_0, \varphi(x_0), D_x\varphi(x_0), D^2_{xx}\varphi(x_0)) \leqslant 0.$$

We shall denote by $\mathcal{L}_i$ the second-order operator on the interior (ℓ, r) of $\mathcal{X}$ associated to the diffusion X solution to (2.1):

$$\mathcal{L}_i \varphi = \frac{1}{2}\sigma_i^2 \varphi'' + b_i \varphi', \quad i \in \mathbb{I}_d.$$

Theorem 3.1 *Assume that* **(H1)** *and* **(H3)** *hold. Then, for each $i \in \mathbb{I}_d$, the value function v_i is a continuous viscosity solution on (ℓ, r) to the variational inequality:*

$$\min\Big\{\rho v_i - \mathcal{L}_i v_i - f_i \;,\; v_i - \max_{j \neq i}(v_j - g_{ij})\Big\} = 0, \quad x \in (\ell, r). \tag{3.4}$$

This means that for all $i \in \mathbb{I}_d$, we have both supersolution and subsolution properties:

(1) Viscosity supersolution property: for any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a local minimum of $v_i - \varphi$, $v_i(\bar{x}) = \varphi(\bar{x})$, we have

$$\min\Big\{\rho\varphi(\bar{x}) - \mathcal{L}_i\varphi(\bar{x}) - f_i(\bar{x}) \;,\; v_i(\bar{x}) - \max_{j \neq i}(v_j - g_{ij})(\bar{x})\Big\} \geqslant 0, \tag{3.5}$$

(2) Viscosity subsolution property: for any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a local maximum of $v_i - \varphi$, $v_i(\bar{x}) = \varphi(\bar{x})$, we have

$$\min\Big\{\rho\varphi(\bar{x}) - \mathcal{L}_i\varphi(\bar{x}) - f_i(\bar{x})\ ,\ v_i(\bar{x}) - \max_{j \neq i}(v_j - g_{ij})(\bar{x})\Big\} \leqslant 0. \tag{3.6}$$

Proof. The arguments of this proof are standard, based on the dynamic programming principle and Itô's formula. We defer the proof to the appendix. □

For any regime $i \in \mathbb{I}_d$, we introduce the switching region:

$$\mathcal{S}_i = \Big\{x \in (\ell, r)\ : v_i(x) = \max_{j \neq i}(v_j - g_{ij})(x)\Big\}.$$

$\mathcal{S}_i$ is a closed subset of (ℓ, r) and corresponds to the region where it is optimal to change of regime. The complement set $\mathcal{C}_i$ of $\mathcal{S}_i$ in (ℓ, r) is the so-called continuation region:

$$\mathcal{C}_i = \Big\{x \in (\ell, r)\ : v_i(x) > \max_{j \neq i}(v_j - g_{ij})(x)\Big\},$$

where one remains in regime i.

Remark 3.1 Let us consider the following optimal stopping problem:

$$v(x) = \sup_{\tau \text{ stopping times}} E\left[\int_0^\tau e^{-\rho\tau} f\left(X_t^x\right) dt + e^{-\rho\tau} h\left(X_\tau^x\right)\right]. \tag{3.7}$$

It is well known that the dynamic programming principle for (3.7) leads to a variational inequality for v in the form:

$$\min\left\{\rho v - \mathcal{L}v - f\ ,\ v - h\right\} = 0,$$

where $\mathcal{L}$ is the infinitesimal generator of the diffusion X. Moreover, the state space domain of X is divided into the stopping region

$$\mathcal{S} = \left\{x\ : v(x) = h(x)\right\},$$

and its complement set, the continuation region:

$$\mathcal{C} = \left\{x\ : v(x) > h(x)\right\}.$$

The smooth-fit principle for optimal stopping problems states that the value function v is smooth C^1 through the boundary of the stopping region, the so-called free boundary, as soon as h is C^1 or convex.

Our aim is to state similar results for optimal switching problems. The main difficulty comes from the fact that we have a system of variational inequalities, so that the switching region for v_i depend also on the other value functions v_j which are a priori not convex or known to be C^1.

4 The smooth-fit property

We first show, like for optimal stopping problems, that the value functions are smooth C^2 in their continuation regions. We provide here a quick proof based on viscosity solutions arguments.

Lemma 4.1 *Assume that* **(H1)***,* **(H2)** *and* **(H3)** *hold. Then, for all $i \in \mathbb{I}_d$, the value function v_i is smooth C^2 on $\mathcal{C}_i$ and satisfies in a classical sense:*

$$\rho v_i(x) - \mathcal{L}_i v_i(x) - f_i(x) = 0, \quad x \in \mathcal{C}_i. \tag{4.1}$$

Proof. We first check that v_i is a viscosity solution to (4.1). Let $\bar{x} \in \mathcal{C}_i$ and φ be a C^2 function on $\mathcal{C}_i$ s.t. $\bar{x}$ is a local maximum of $v_i - \varphi$, $v_i(\bar{x}) = \varphi(\bar{x})$. Then, by definition of $\mathcal{C}_i$, we have $v_i(\bar{x}) > \max_{j \neq i}(v_j - g_{ij})(\bar{x})$, and so from the subsolution viscosity property (3.6) of v_i, we have:

$$\rho\varphi(\bar{x}) - \mathcal{L}_i\varphi(\bar{x}) - f_i(\bar{x}) \leqslant 0.$$

The supersolution inequality for (4.1) is immediate from (3.5).

Now, for an arbitrary bounded interval $(x_1, x_2) \subset \mathcal{C}_i$, consider the Dirichlet boundary linear problem:

$$\rho w(x) - \mathcal{L}_i w(x) - f_i(x) = 0, \quad \text{on } (x_1, x_2) \tag{4.2}$$

$$w(x_1) \ = \ v_i(x_1), \quad w(x_2) \ = \ v_i(x_2). \tag{4.3}$$

Under the nondegeneracy condition **(H2)**, classical results provide the existence and uniqueness of a smooth C^2 function w solution on (x_1, x_2) to (4.2)–(4.3). In particular, this smooth function w is a viscosity solution of (4.1) on (x_1, x_2). From standard uniqueness results on viscosity solutions (here for a linear PDE in a bounded domain), we deduce that $v_i = w$ on (x_1, x_2). From the arbitrariness of $(x_1, x_2) \subset \mathcal{C}_i$, this proves that v_i is smooth C^2 on $\mathcal{C}_i$, and so satisfies (4.1) in a classical sense. □

We now state an elementary partition property on the switching regions.

Lemma 4.2 *Assume that* **(H1)***,* **(H3)** *and* **(H4)** *hold. Then, for all $i \in \mathbb{I}_d$, we have $\mathcal{S}_i = \cup_{j \neq i}\mathcal{S}_{ij}$ where*

$$\mathcal{S}_{ij} = \{x \in \mathcal{C}_j \ : v_i(x) = (v_j - g_{ij})(x)\}.$$

Proof. Denote $\tilde{\mathcal{S}}_i = \cup_{j \neq i}\mathcal{S}_{ij}$. Since we always have $v_i \geqslant \max_{j \neq i}(v_j - g_{ij})$, the inclusion $\tilde{\mathcal{S}}_i \subset \mathcal{S}_i$ is clear.

Conversely, let $x \in \mathcal{S}_i$. Then there exists $j \neq i$ s.t. $v_i(x) = v_j(x) - g_{ij}$. We have two cases:

⋆ if x lies in $\mathcal{C}_j$, then $x \in \mathcal{S}_{ij}$ and so $x \in \tilde{\mathcal{S}}_i$.

⋆ if x does not lie in $\mathcal{C}_j$, then x would lie in $\mathcal{S}_j$, which means that one could find some $k \neq j$ s.t. $v_j(x) = v_k(x) - g_{jk}$, and so $v_i(x) = v_k(x) - g_{ij} - g_{jk}$. From condition **(H4)** and since we always have $v_i \geqslant v_k - g_{ik}$, this would imply $v_i(x) = v_k(x) - g_{ik}$. Since cost g_{ik} is positive, we also get that $k \neq i$. Again, we have two cases: if x lies in $\mathcal{C}_k$, then x lies in $\mathcal{S}_{ik}$ and so in $\tilde{\mathcal{S}}_i$. Otherwise, we repeat the above argument and since the number of states is finite, we should necessarily find some $l \neq i$ s.t. $v_i(x) = v_l(x) - g_{il}$ and $x \in \mathcal{C}_l$. This shows finally that $x \in \tilde{\mathcal{S}}_i$. □

Remark 4.1 $\mathcal{S}_{ij}$ represents the region where it is optimal to switch from regime i to regime j and stay here for a moment, i.e. without changing instantaneously from regime j to another regime.

We can finally prove the smooth-fit property of the value functions v_i through the boundaries of the switching regions.

Theorem 4.1 *Assume that* **(H1)**, **(H2)**, **(H3)** *and* **(H4)** *hold. Then, for all* $i \in \mathbb{I}_d$, *the value function* v_i *is continuously differentiable on* (ℓ, r). *Moreover, at* $x \in \mathcal{S}_{ij}$, *we have* $v_i'(x) = v_j'(x)$.

Proof. We already know from Lemma 4.1 that v_i is smooth C^2 on the open set $\mathcal{C}_i$ for all $i \in \mathbb{I}_d$. We have to prove the C^1 property of v_i at any point of the closed set $\mathcal{S}_i$. We denote for all $j \in \mathbb{I}_d$, $j \neq i$, $h_j = v_j - g_{ij}$ and we notice that h_j is smooth C^1 (actually even C^2) on $\mathcal{C}_j$.

1. We first check that v_i admits a left and a right derivative $v'_{i,-}(x_0)$ and $v'_{i,+}(x_0)$ at any point x_0 in $\mathcal{S}_i = \cup_{j \neq i} \mathcal{S}_{ij}$. We distinguish the two following cases:

• *Case (a)* x_0 lies in the interior $\mathrm{Int}(\mathcal{S}_i)$ of $\mathcal{S}_i$. Then, we have two subcases:

⋆ $x_0 \in \mathrm{Int}(\mathcal{S}_{ij})$ for some $j \neq i$, i.e. there exists some $\delta > 0$ s.t. $[x_0 - \delta, x_0 + \delta] \subset \mathcal{S}_{ij}$. By definition of $\mathcal{S}_{ij}$, we then have $v_i = h_j$ on $[x_0 - \delta, x_0 + \delta] \subset \mathcal{C}_j$, and so v_i is differentiable at x_0 with $v_i'(x_0) = h_j'(x_0)$.

⋆ There exists $j \neq k \neq i$ in $\mathbb{I}_d$ and $\delta > 0$ s.t. $[x_0 - \delta, x_0] \subset \mathcal{S}_{ij}$ and $[x_0, x_0 + \delta] \subset \mathcal{S}_{ik}$. We then have $v_i = h_j$ on $[x_0 - \delta, x_0] \subset \mathcal{C}_j$ and $v_i = h_k$ on $[x_0, x_0 + \delta] \subset \mathcal{C}_k$. Thus, v_i admits a left and a right derivative at x_0 with $v'_{i,-}(x_0) = h_j'(x_0)$ and $v'_{i,+}(x_0) = h_k'(x_0)$.

• *Case (b)* x_0 lies in the boundary $\partial \mathcal{S}_i = \mathcal{S}_i \setminus \mathrm{Int}(\mathcal{S}_i)$ of $\mathcal{S}_i$. We assume that x_0 lies in the left-boundary of $\mathcal{S}_i$, i.e. there exists $\delta > 0$ s.t. $[x_0 - \delta, x_0) \subset \mathcal{C}_i$ (the other case where x_0 lies in the right-boundary is dealt with similarly). Recalling that on $\mathcal{C}_i$, v_i is solution to: $\rho v_i - \mathcal{L} v_i - f_i = 0$, we deduce that on $[x_0 - \delta, x_0)$, v_i is equal to w_i the unique smooth C^2 solution to the o.d.e.: $\rho w_i - \mathcal{L} w_i - f_i = 0$ with the boundaries conditions: $w_i(x_0 - \delta) = v_i(x_0 - \delta)$, $w_i(x_0) = v_i(x_0)$. Therefore, v_i admits a left derivative at x_0 with $v'_{i,-}(x_0) = w_i'(x_0)$. In order to prove that v_i admits a right derivative, we distinguish the two subcases:

⋆ There exists $j \neq i$ in $\mathbb{I}_d$ and $\delta' > 0$ s.t. $[x_0, x_0 + \delta'] \subset \mathcal{S}_{ij}$. Then, on $[x_0, x_0 + \delta']$, v_i is equal to h_j. Hence v_i admits a right derivative at x_0 with $v'_{i,+}(x_0) = h_j'(x_0)$.

⋆ Otherwise, for all $j \neq i$, we can find a sequence (x_n^j) s.t. $x_n^j \geqslant x_0$, $x_n^j \notin \mathcal{S}_{ij}$ and $x_n^j \to x_0$. By a diagonalization procedure, we construct then a sequence (x_n) s.t. $x^n \geqslant x_0$, $x_n \notin \mathcal{S}_{ij}$ for all $j \neq i$, i.e. $x_n \in \mathcal{C}_i$, and $x_n \to x_0$. Since $\mathcal{C}_i$ is open, there exists then $\delta'' > 0$ s.t. $[x_0, x_0 + \delta''] \subset \mathcal{C}_i$. We deduce that on $[x_0, x_0 + \delta'']$, v_i is equal to $\hat{w}_i$ the unique smooth C^2 solution to the o.d.e. $\rho \hat{w}_i - \mathcal{L}\hat{w}_i - f_i = 0$ with the boundaries conditions $\hat{w}_i(x_0) = v_i(x_0)$, $\hat{w}_i(x_0 + \delta'') = v_i(x_0 + \delta'')$. In particular, v_i admits a right derivative at x_0 with $v'_{i,+}(x_0) = \hat{w}'_i(x_0)$.

2. Consider now some point in $\mathcal{S}_i$ eventually on its boundary. We recall again that from Lemma 4.2, there exists some $j \neq i$ s.t. $x_0 \in \mathcal{S}_{ij} : v_i(x_0) = h_j(x_0)$, and h_j is smooth C^1 on x_0 in $\mathcal{C}_j$. Since $v_j \geqslant h_j$, we deduce that

$$\frac{v_i(x) - v_i(x_0)}{x - x_0} \leqslant \frac{h_j(x) - h_j(x_0)}{x - x_0}, \quad \forall\, x < x_0$$
$$\frac{v_i(x) - v_i(x_0)}{x - x_0} \geqslant \frac{h_j(x) - h_j(x_0)}{x - x_0}, \quad \forall\, x > x_0,$$

and so:

$$v'_{i,-}(x_0) \leqslant h'_j(x_0) \leqslant v'_{i,+}(x_0).$$

We argue by contradiction and suppose that v_i is not differentiable at x_0. Then, in view of the above inequality, one can find some $p \in (v'_{i,-}(x_0), v'_{i,+}(x_0))$. Consider, for $\varepsilon > 0$, the smooth C^2 function:

$$\varphi_\varepsilon(x) = v_i(x_0) + p(x - x_0) + \frac{1}{2\varepsilon}(x - x_0)^2.$$

Then, we see that v_i dominates locally in a neighborhood of x_0 the function φ_ε, i.e x_0 is a local minimum of $v_i - \varphi_\varepsilon$. From the supersolution viscosity property of v_i to the PDE (3.4), this yields:

$$\rho\varphi_\varepsilon(x_0) - \mathcal{L}_i\varphi_\varepsilon(x_0) - f_i(x_0) \geqslant 0,$$

which is written as:

$$\rho v_i(x_0) - b_i(x_0)p - f_i(x_0) - \frac{1}{2\varepsilon}\sigma_i^2(x_0) \geqslant 0.$$

Sending ε to zero provides the required contradiction under **(H2)**. We have then proved that for $x_0 \in \mathcal{S}_{ij}$, $v'_i(x_0) = h'_j(x_0) = v'_j(x_0)$. □

Appendix: Proof of Theorem 3.1

A.1 Viscosity supersolution property

Fix $i \in \mathbb{I}_d$. Consider any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a minimum of $v_i - \varphi$ in a neighbourhood $B_\varepsilon(\bar{x}) = (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ of $\bar{x}$, $\varepsilon > 0$, and $v_i(\bar{x}) = \varphi(\bar{x})$.

By taking the immediate switching control $\tau_1 = 0$, $\kappa_1 = j \neq i$, $\tau_n = \infty$, $n \geqslant 2$, and $\theta = 0$ in the relation (3.2), we obtain

$$v_i(\bar{x}) \geqslant v_j(\bar{x}) - g_{ij}, \quad \forall j \neq i. \tag{A.1}$$

On the other hand, by taking the no-switching control $\tau_n = \infty$, $n \geqslant 1$, i.e. $I_t^i = i$, $t \geqslant 0$, $X^{\bar{x},i}$ stays in regime i with diffusion coefficients b_i and σ_i, and $\theta = \tau_\varepsilon \wedge h$, with $h > 0$ and $\tau_\varepsilon = \inf\{t \geqslant 0 \ : X_t^{\bar{x},i} \notin B_\varepsilon(\bar{x})\}$, we get from (3.2):

$$\begin{aligned}\varphi(\bar{x}) \ = \ v_i(\bar{x}) &\geqslant E\left[\int_0^\theta e^{-\rho t} f_i\big(X_t^{\bar{x},i}\big)dt + e^{-\rho\theta} v_i\big(X_\theta^{\bar{x},i}\big)\right] \\ &\geqslant E\left[\int_0^\theta e^{-\rho t} f_i\big(X_t^{\bar{x},i}\big)dt + e^{-\rho\theta} \varphi\big(X_\theta^{\bar{x},i}\big)\right]\end{aligned}$$

By applying Itô's formula to $e^{-\rho t}\varphi\big(X_t^{\bar{x},i}\big)$ between 0 and $\theta = \tau_\varepsilon \wedge h$ and plugging into the last inequality, we obtain:

$$\frac{1}{h}E\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t}\left(\rho\varphi - \mathcal{L}_i\varphi - f_i\right)\left(X_t^{\bar{x},i}\right)\right] \geqslant 0.$$

From the dominated convergence theorem, this yields by sending h to zero:

$$(\rho\varphi - \mathcal{L}_i\varphi - f_i)(\bar{x}) \geqslant 0.$$

By combining with (A.1), we obtain the required supersolution inequality (3.5).

A.2 Viscosity subsolution property

Fix $i \in \mathbb{I}_d$, and consider any $\bar{x} \in (\ell, r)$ and $\varphi \in C^2(\ell, r)$ s.t. $\bar{x}$ is a maximum of $v_i - \varphi$ in a neighbourhood $B_\varepsilon(\bar{x}) = (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ of $\bar{x}$, $\varepsilon > 0$, and $v_i(\bar{x}) = \varphi(\bar{x})$. We argue by contradiction by assuming on the contrary that (3.6) does not hold so that by continuity of v_i, v_j, $j \neq i$, φ and its derivatives, there exists some $0 < \delta \leqslant \varepsilon$ s.t.

$$(\rho\varphi - \mathcal{L}_i\varphi - f_i)(x) \geqslant \delta, \quad \forall x \in B_\delta(\bar{x}) = (x - \delta, x + \delta) \tag{A.2}$$

$$v_i(x) \ - \ \max_{j \neq i}(v_j - g_{ij})(x) \geqslant \delta, \quad \forall x \in B_\delta(\bar{x}). \tag{A.3}$$

For any $\alpha = (\tau_n, \kappa_n)_{n \geqslant 1} \in \mathcal{A}$, consider the exit time $\tau_\delta = \inf\{t \geqslant 0 :$ $X_t^{\bar{x},i} \notin B_\delta(\bar{x})\}$. By applying Itô's formula to $e^{-\rho t}\varphi\big(X_t^{\bar{x},i}\big)$ between 0 and $\theta = \tau_1 \wedge \tau_\delta$, we have by noting that before θ, $X^{x,i}$ stays in regime i and in the ball $B_\delta(\bar{x}) \subset B_\varepsilon(\bar{x})$:

$$\begin{aligned} v_i(\bar{x}) \;=\; \varphi(\bar{x}) &= E\left[\int_0^\theta e^{-\rho t}\big(\rho\varphi - \mathcal{L}_i\varphi\big)\big(X_t^{\bar{x},i}\big)dt + e^{-\rho\theta}\varphi\big(X_\theta^{\bar{x},i}\big)\right] \\ &\geqslant E\left[\int_0^\theta e^{-\rho t}\big(\rho\varphi - \mathcal{L}_i\varphi\big)\big(X_t^{\bar{x},i}\big)dt + e^{-\rho\theta}v_i\big(X_\theta^{\bar{x},i}\big)\right]. \end{aligned} \tag{A.4}$$

Now, since $\theta = \tau_\delta \wedge \tau_1$, we have

$$\begin{aligned} e^{-\rho\theta}v\big(X_\theta^{\bar{x},i}, I_\theta^i\big) - \sum_{\tau_n \leqslant \theta} g_{\kappa_{n-1},\kappa_n} &= e^{-\rho\tau_1}\left(v\left(X_{\tau_1}^{\bar{x},i}, \kappa_1\right) - g_{i\kappa_1}\right)1_{\tau_1\leqslant\tau_\delta} \\ &\qquad + e^{-\rho\tau_\delta}v_i\left(X_{\tau_\delta}^{\bar{x},i}\right)1_{\tau_\delta<\tau_1} \\ &\leqslant e^{-\rho\tau_1}\left(v_i\left(X_{\tau_1}^{\bar{x},i}\right) - \delta\right)1_{\tau_1\leqslant\tau_\delta} \\ &\qquad + e^{-\rho\tau_\delta}v_i\left(X_{\tau_\delta}^{\bar{x},i}\right)1_{\tau_\delta<\tau_1} \\ &= e^{-\rho\theta}v_i\big(X_\theta^{\bar{x},i}\big) - \delta e^{-\rho\tau_1}1_{\tau_1\leqslant\tau_\delta}, \end{aligned}$$

where the inequality follows from (A.3). By plugging into (A.4) and using (A.2), we get:

$$\begin{aligned} v_i(\bar{x}) \geqslant E&\left[\int_0^\theta e^{-\rho t}f_i\big(X_t^{\bar{x},i}\big)dt + e^{-\rho\theta}v\big(X_\theta^{\bar{x},i}, I_\theta^i\big) - \sum_{\tau_n\leqslant\theta} g_{\kappa_{n-1},\kappa_n}\right] \\ &+ \delta\, E\left[\int_0^\theta e^{-\rho t}dt + e^{-\rho\tau_1}1_{\tau_1\leqslant\tau_\delta}\right]. \end{aligned} \tag{A.5}$$

We now claim that there exists some positive constant $c_0 > 0$ s.t.:

$$E\left[\int_0^\theta e^{-\rho t}dt + e^{-\rho\tau_1}1_{\tau_1\leqslant\tau_\delta}\right] \geqslant c_0, \quad \forall\alpha \in \mathcal{A}.$$

For this, we construct a smooth function w s.t.

$$\max\{\rho w(x) - \mathcal{L}_i w(x) - 1\;,\; w(x) - 1\} \leqslant 0,\ \forall x \in B_\delta(\bar{x}) \tag{A.6}$$
$$w(x) = 0,\ \forall x \in \partial B_\delta(\bar{x}) = \{x\ : |x - \bar{x}| = \delta\} \tag{A.7}$$
$$w(\bar{x}) > 0. \tag{A.8}$$

For instance, we can take the function $w(x) = c_0\left(1 - \frac{|x-\bar{x}|^2}{\delta^2}\right)$, with

$$0 < c_0 \leqslant \min\left\{\left(\rho + \frac{2}{\delta}\sup_{x\in B_\delta(\bar{x})}|b_i(x)| + \frac{1}{\delta^2}\sup_{x\in B_\delta(\bar{x})}|\sigma_i(x)|^2\right)^{-1}, 1\right\}.$$

Then, by applying Itô's formula to $e^{-\rho t}w\big(X_t^{\bar{x},i}\big)$ between 0 and $\theta = \tau_\delta \wedge \tau_1$, we have:

$$0 \;<\; c_0 \;=\; w(\bar{x}) = E\left[\int_0^{\theta} e^{-\rho t}\big(\rho w - \mathcal{L}_i w\big)\big(X_t^{\bar{x},i}\big)dt + e^{-\rho\theta} w\big(X_\theta^{\bar{x},i}\big)\right]$$
$$\leqslant E\left[\int_0^{\theta} e^{-\rho t}dt + e^{-\rho\tau_1} 1_{\tau_1 \leqslant \tau_\delta}\right],$$

from (A.6), (A.7) and (A.8). By plugging this last inequality (uniform in α) into (A.5), we then obtain:

$$v_i(\bar{x}) \geqslant \sup_{\alpha\in\mathcal{A}} E\left[\int_0^{\theta} e^{-\rho t} f_i\big(X_t^{\bar{x},i}\big)dt + e^{-\rho\theta} v\big(X_\theta^{\bar{x},i}, I_\theta^i\big) - \sum_{\tau_n\leqslant\theta} g_{\kappa_{n-1},\kappa_n}\right]$$
$$+ \delta c_0,$$

which is in contradiction with the dynamic programming principle (3.2).

References

1. Bensoussan A. et J.L. Lions (1982): Contrôle impulsionnel et inéquations variationnelles, Dunod
2. Jacka S. (1993): "Local times, optimal stopping and semimartingales", *Ann. Probab.*, 21, 329–339
3. Fleming W. and M. Soner (1993): Controlled Markov processes and viscosity solutions, Springer, Berlin Heidelberg New York
4. Lenhart S. and S. Belbas (1983): "A system of nonlinear partial differential equations arising in the optimal control of stochastic systems with switching costs", *SIAM J. Appl. Math.*, 43, 465–475
5. Peskir G. (2002): "A change-of-variable formula with local time on curves", *J. Theoret. Probab.*, 18, 499–535
6. Shiryaev A. (1978): Optimal stopping rules, Springer, Berlin Heidelberg New York
7. Tang S. and J. Yong (1993): "Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach", *Stoch. and Stoch. Reports*, 45, 145–176

Part III

Other Contributions

A Strong Form of Stable Convergence

Irene Crimaldi[1], Giorgio Letta[2], and Luca Pratelli[3]

[1] Department of Mathematics, University of Bologna
Piazza di Porta San Donato 5, 40126 Bologna, Italy
e-mail: crimaldi@dm.unibo.it

[2] Department of Mathematics, University of Pisa
Largo Bruno, Pontecorvo 5, 56127 Pisa, Italy
e-mail: letta@dm.unipi.it

[3] Accademia Navale
Viale Italia 72, 57100 Livorno, Italy
e-mail: pratel@mail.dm.unipi.it

Summary. We introduce and study a strengthening of the notion of stable convergence.

Key words: Conditional distribution, Conditional expectation, Convergence in L^1, Stable convergence, Central limit theorems.

1 Introduction

The notion of stable convergence for a family (X_t) of random variables was introduced by Rényi [12] and, successively, studied by many other authors (e.g. Aldous and Eagleson [1], Feigin [5], Hall and Heyde [6], Jacod and Mémin [8], Letta [10]). It is a strengthening of the notion of convergence in distribution and, in its most general formulation, it is defined with respect to a given sub-σ-field $\mathcal{H}$ (see Definitions 1 and 2). In this paper, we shall start by introducing a notion of convergence which extends that of stable convergence in a "natural" way: more precisely, we shall replace the single sub-σ-field $\mathcal{H}$ by a family $\mathcal{G} = (\mathcal{G}_t)_t$ of sub-σ-fields (see Definition 3). Then, we shall study a strengthening of this "extended form" of stable convergence, for which we propose the name *$\mathcal{G}$-stable convergence in the strong sense* (see Definition 4). We shall give different characterizations of this last type of convergence (see Theorems 3 and 4), which will then be applied in order to obtain some convergence results (see Theorem 5, Examples 4, 5 and 6).

2 Preliminaries

In the sequel, we shall suppose given a probability space $(\Omega, \mathcal{A}, P)$ and we shall denote by $\mathcal{N}$ the sub-σ-field of $\mathcal{A}$ generated by the P-negligible events. Moreover, we shall suppose given a Polish space E (endowed with its Borel σ-field $\mathcal{B}(E)$), a directed set T and a family $(X_t)_{t\in T}$ of random variables on $(\Omega, \mathcal{A}, P)$ with values in E. All the limits concerning a family with T as the set of indexes are taken with respect to the "section filter" of T. We shall call a *distribution* on E a probability measure on $\mathcal{B}(E)$ and we shall consider the space of distributions on E endowed with the topology of weak convergence, i.e. the topology generated by the functionals of the form $\mu \mapsto \langle \mu, f\rangle = \int f\,\mathrm{d}\mu$ with f in $\mathcal{C}_b(E)$. According to the terminology used in [2], we shall call a *determining class* a set $\mathcal{K}$ of bounded Borel functions on E such that, for each pair μ, ν of distributions on E, the equality $\langle \mu, f\rangle = \langle \nu, f\rangle$ for all f in $\mathcal{K}$ implies $\mu = \nu$. Moreover, we shall call a *convergence determining class* a subset $\mathcal{K}$ of $\mathcal{C}_b(E)$ such that, in the space of distributions on E, the topology of weak convergence coincides with the topology generated by the functionals of the form $\mu \mapsto \langle \mu, f\rangle$ with f in $\mathcal{K}$. (Such a class is a particular determining class.) We shall call a *kernel* a family

$$N = \big(N(\omega, \cdot)\big)_{\omega\in\Omega}$$

of distributions on E such that, for each bounded Borel function f on E, the function Nf defined on Ω by

$$Nf(\omega) = \int N(\omega, \mathrm{d}x) f(x)$$

is measurable with respect to $\mathcal{A}$. Given a sub-σ-field $\mathcal{H}$ of $\mathcal{A}$, a kernel N will be said to be *measurable with respect to* $\mathcal{H}$, or, more briefly, $\mathcal{H}$*-measurable* if, for each bounded Borel function f on E, the random variable Nf is measurable with respect to $\mathcal{H}$. We shall denote by $\sigma(N)$ the smallest σ-field with respect to which N is measurable. Given a kernel N and a probability measure Q on $(\Omega, \mathcal{A})$, we shall denote by QN the distribution on E which is defined by

$$\langle QN, f\rangle = \langle Q, Nf\rangle,$$

for each bounded Borel function f on E. Given a kernel N, a sub-σ-field $\mathcal{H}$ of $\mathcal{A}$, and a class $\mathcal{K}$ of bounded Borel functions on E such that $\mathcal{K}\cup\{1\}$ is total in $\mathcal{L}^1(PN)$, then, in order that N is measurable with respect to $\mathcal{H}\vee\mathcal{N}$, it suffices that such a measurability condition is satisfied by each random variable of the form Nf with f in $\mathcal{K}$. (This criterion applies, for instance, when $\mathcal{K}$ is $\mathcal{C}_b(E)$ or, more generally, any determining class.) We shall consider the space of kernels endowed with the *weak topology*, i.e. the topology generated by the functionals of the form $N \mapsto \langle P, I_H Nf\rangle$, with H in $\mathcal{A}$ and f in $\mathcal{C}_b(E)$ (e.g. [10], p. 196). It is not a Hausdorff space in general. However, its quotient with respect to P-equivalence is a Hausdorff space. (In the sequel a kernel will be used interchangeably with its P-equivalence class.) If Y is a random variable with values in E and $\mathcal{H}$ is a sub-σ-field of $\mathcal{A}$, a version of the *conditional distribution* of Y given $\mathcal{H}$ is a kernel N such that, for each bounded Borel function f on E, the random variable Nf is a version of the conditional expectation $\mathrm{E}[f(Y)|\mathcal{H}]$. (Such a kernel is obviously $\mathcal{H}$-measurable.)

Finally, let us recall the following definition of $\mathcal{H}$-stable convergence.

Definition 1. *Let $\mathcal{H}$ be a sub-σ-field of $\mathcal{A}$. We say that $(X_t)_{t\in T}$ converges* $\mathcal{H}$-stably *if the following* Aldous-Rényi condition *holds: for each event H in $\mathcal{H}$ with $P(H) > 0$, the family $(X_t)_{t\in T}$ converges in distribution, under P_H, to a limit distribution (which generally depends on H).*

The following two results are substantially known (e.g. [10]).

Theorem 1. *Let $\mathcal{H}$ be a sub-σ-field of $\mathcal{A}$. Further, for each t in T, denote by N_t a version of the conditional distribution of X_t given $\mathcal{H}$. Then the following conditions are equivalent:*

(a) The family $(X_t)_{t\in T}$ converges $\mathcal{H}$-stably.
(b) For each probability measure Q of the form $Q = Z \cdot P$, where Z is $\mathcal{H}$-measurable, the family $(X_t)_{t\in T}$ converges in distribution under Q.
(c) The family $(N_t)_{t\in T}$ converges with respect to the weak topology in the space of kernels.

Proposition 1. *Under the same assumptions as in the previous theorem, let N be a kernel. Then the following conditions are equivalent:*

(a) For each function f in $\mathcal{C}_b(E)$, the conditional expectation $\mathrm{E}[f(X_t)|\mathcal{H}]$ *converges in $\sigma(L^1, L^\infty)$ to the random variable Nf.*
(b) The kernel N is measurable with respect to $\mathcal{H} \vee \mathcal{N}$ and, for each event H in $\mathcal{H}$ with $P(H) > 0$, the family $(X_t)_{t\in T}$ converges in distribution, under P_H, to the probability measure $P_H N$.
(c) The kernel N is measurable with respect to $\mathcal{H} \vee \mathcal{N}$ and, for each probability measure Q of the form $Q = Z \cdot P$, where Z is $\mathcal{H}$-measurable, the family $(X_t)_{t\in T}$ converges in distribution, under Q, to the probability measure QN.
(d) The kernel N is the limit of $(N_t)_{t\in T}$ with respect to the weak topology in the space of kernels.

Definition 2. *If the equivalent conditions (a), (b), (c), (d) of the above proposition hold, we say that $(X_t)_{t\in T}$ converges $\mathcal{H}$-stably to the kernel N.*

Remark 1. By means of condition (b) of Proposition 1, it is easy to see that, if $(X_t)_{t\in T}$ converges $\mathcal{H}$-stably to N, then it also converges $\mathcal{J}$-stably to N, for each σ-field $\mathcal{J}$ such that $\sigma(N) \subset \mathcal{J} \vee \mathcal{N} \subset \mathcal{H} \vee \mathcal{N}$.

3 Stable convergence with respect to a conditioning system

The following definition extends the notion of stable convergence, from the case in which a single sub-σ-field of $\mathcal{A}$ is fixed, to the case in which a *family* of sub-σ-fields of $\mathcal{A}$ is given.

Definition 3. *We shall call a* conditioning system *a family $\mathcal{G} = (\mathcal{G}_t)_{t\in T}$ of σ-fields of $\mathcal{A}$. (A* constant *conditioning system, i.e. a conditioning system $\mathcal{G}$ such that all the σ-fields $\mathcal{G}_t$ are equal to a σ-field $\mathcal{H}$, will be used interchangeably, in the notation and terminology, with the σ-field $\mathcal{H}$ itself.) Given a conditioning system $\mathcal{G}$ and a kernel N, we shall say that $(X_t)_{t\in T}$ converges* stably *with respect to the conditioning system $\mathcal{G}$ (or, more briefly, $\mathcal{G}$-*stably*) to N, if, for each function f in $\mathcal{C}_b(E)$, the conditional expectation* $\mathrm{E}[f(X_t)|\mathcal{G}_t]$ *converges to the random variable Nf in $\sigma(L^1, L^\infty)$.* (Let us observe that, in particular, this condition implies the convergence in distribution of $(X_t)_{t\in T}$ to the distribution PN.)

In the sequel, we shall employ a fixed conditioning system $\mathcal{G} = (\mathcal{G}_t)_{t\in T}$ and we shall set

$$\mathcal{G}_* = \liminf_t \mathcal{G}_t = \bigvee_t \bigcap_{u\geqslant t} \mathcal{G}_u, \qquad \mathcal{G}^* = \limsup_t \mathcal{G}_t = \bigcap_t \bigvee_{u\geqslant t} \mathcal{G}_u. \tag{1}$$

We shall say that a family $(Z_t)_{t\in T}$ of random variables on $(\Omega, \mathcal{A})$ is $\mathcal{G}$-*adapted* if, for each t, the random variable Z_t is $\mathcal{G}_t$-measurable. Further, we shall say that a family $(H_t)_{t\in T}$ of events in $\mathcal{A}$ is $\mathcal{G}$-*adapted* if the family $(I_{H_t})_{t\in T}$ of the corresponding indicator functions is $\mathcal{G}$-adapted. Hence, it is possible to extend Theorem 1 and Proposition 1 in the following way:

Theorem 2. *For each t in T, let us denote by N_t a version of the conditional distribution of X_t given $\mathcal{G}_t$. Then the following conditions are equivalent:*

(a) The family $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably to a suitable kernel.

(b) For each probability measure Q of the form $Q = Z \cdot P$, where Z is $\mathcal{G}^*$-measurable, if we set

$$Z_t = \mathrm{E}[Z|\mathcal{G}_t], \qquad Q_t = Z_t \cdot P, \tag{2}$$

then the distribution of X_t under Q_t converges weakly to a suitable limit distribution.

(c) The family $(N_t)_{t\in T}$ converges with respect to the weak topology in the space of kernels.

Proof. A family $(N_t)_{t\in T}$ of kernels converges weakly to the kernel N if, and only if, for each f in $\mathcal{C}_b(E)$, the family $(N_t f)_{t\in T}$ of random variables converges to Nf in $\sigma(L^1, L^\infty)$ (see [10], p. 197). Thus, the equivalence between (a) and (c) is obvious.

Let us prove implication (a)$\Rightarrow$(b). Assume that $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably to a kernel, say N. Given a probability measure of the form $Q = Z \cdot P$, where Z is $\mathcal{G}^*$-measurable, it suffices to prove that, with the notation (2), the distribution of X_t under Q_t converges weakly to QN. To this end, fix f in $\mathcal{C}_b(E)$. Then we have $N_t f = \mathrm{E}[f(X_t)|\mathcal{G}_t]$. Moreover, the family $(N_t f)_{t\in T}$ is uniformly bounded and it converges to Nf in $\sigma(L^1, L^\infty)$. Therefore, we have

$$\langle Q_t, f(X_t)\rangle = \langle P, Z_t f(X_t)\rangle = \langle P, Z_t N_t f\rangle = \langle P, ZN_t f\rangle \to \langle P, ZNf\rangle = \langle QN, f\rangle.$$

Finally, let us prove implication (b)⇒(c). Assuming condition (b), it is enough to verify that, if we fix a probability measure of the form $Q = Y\cdot P$, then the family of distributions $(QN_t)_{t\in T}$ converges weakly to a suitable distribution (see [10], (3.2), p. 198). To this end, let us set

$$Y_t = \mathrm{E}[Y|\textstyle\bigvee_{u\geqslant t}\mathcal{G}_u], \qquad Z = \mathrm{E}[Y|\mathcal{G}^*], \qquad Z_t = \mathrm{E}[Z|\mathcal{G}_t], \qquad Q_t = Z_t\cdot P.$$

Then, by a well-known convergence result for inverse martingales, the family $(Y_t)_{t\in T}$ converges in L^1 to Z. Moreover, by assumption (b), the distribution of X_t under Q_t converges weakly to a distribution, say μ. Thus, if we fix f in $\mathcal{C}_b(E)$, we have

$$\langle P, ZN_t f\rangle = \langle P, Z_t N_t f\rangle = \langle P, Z_t f(X_t)\rangle \to \langle \mu, f\rangle. \tag{3}$$

Therefore, we can write

$$\langle QN_t, f\rangle = \langle P, YN_t f\rangle = \langle P, Y_t N_t f\rangle \to \langle \mu, f\rangle, \tag{4}$$

where the convergence of the last term follows from (3) by the inequality

$$|\langle P, Y_t N_t f\rangle - \langle P, ZN_t f\rangle| \leqslant \|f\|_\infty \|Y_t - Z\|_{L^1(P)}.$$

Since, by (4), the family $(QN_t)_{t\in T}$ converges weakly to μ, we are done. □

Proposition 2. *Under the same assumptions as in the previous theorem, let N be a kernel. Then the following conditions are equivalent.*

(a) The family $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably to N.
(b) The kernel N is measurable with respect to $\mathcal{G}^*\vee\mathcal{N}$ and, for each probability measure Q of the form $Q = Z\cdot P$, where Z is $\mathcal{G}^*$-measurable, we have, using the notation (2), that the distribution of X_t under Q_t converges weakly to the distribution QN.
(c) The kernel N is the limit of $(N_t)_{t\in T}$ with respect to the weak topology in the space of kernels.

Proof. As we have already observed in the proof of Theorem 2, for the equivalence between (a) and (c), we refer to [10] (p. 197). Moreover, it is easy to verify that condition (a) (or, equivalently, condition (c)) implies the measurability of N with respect to $\mathcal{G}^*\vee\mathcal{N}$. Thus, the proof of implication (a)⇒(b) is exactly the same as the one we have just done for implication (a)⇒(b) of Theorem 2. Finally, in order to prove implication (b)⇒(c), it is sufficient to repeat the proof of implication (b)⇒(c) of Theorem 2, taking into account the fact that, since N is $(\mathcal{G}^*\vee\mathcal{N})$-measurable, for each probability measure of the form $Y\cdot P$, we have $(Y\cdot P)N = (Z\cdot P)N$, where Z denotes $\mathrm{E}[Y|\mathcal{G}^*]$. □

Remark 2. Under the assumptions of the previous proposition, let $\mathcal{D}$ be a class of $\mathcal{G}^*$-measurable P-probability densities such that the linear space spanned by $\mathcal{D}$ is dense in $L^1(P_{|\mathcal{G}^*})$ (where $P_{|\mathcal{G}^*}$ denotes the restriction of P to $\mathcal{G}^*$). Then condition (b) is equivalent to the condition which is obtained by replacing in (b) the whole class of probability measures of the form $Q = Z \cdot P$, with any $\mathcal{G}^*$-measurable P-probability density Z, by the class of probability measures of the form $Q = Z \cdot P$ with Z in $\mathcal{D}$.

Corollary 1. *Let us suppose that the conditioning system $\mathcal{G} = (\mathcal{G}_t)_{t\in T}$ is monotone (increasing or decreasing). Then the two σ-fields $\mathcal{G}_*$ and $\mathcal{G}^*$ are equal and, if we set $\mathcal{H} = \mathcal{G}_* = \mathcal{G}^*$, in order that $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably to the kernel N, it is necessary and sufficient that it converges $\mathcal{H}$-stably to N (in the sense of Definition 2).*

Proof. By Proposition 2, in order that $(X_t)_t$ converges $\mathcal{G}$-stably to the kernel N, it is necessary and sufficient that N is $(\mathcal{H} \vee \mathcal{N})$-measurable and that, for each f in $\mathcal{C}_b(E)$ and each $\mathcal{H}$-measurable P-probability density Z, we have, using the notation (2),

$$\langle P, Z_t f(X_t)\rangle \to \langle P, ZNf\rangle.$$

Since the monotonicity of $\mathcal{G}$ implies the convergence in L^1 of Z_t to Z, we are allowed to replace Z_t by Z in the above relation. Thus the relation is reduced to condition (c) of Proposition 1; that is, to the $\mathcal{H}$-stable convergence of $(X_t)_t$ to N. □

4 Stable convergence in the strong sense

The following definition introduces a strengthening of the notion of stable convergence (with respect to a conditioning system) studied in Section 3.

Definition 4. *Given a conditioning system $\mathcal{G}$ and a kernel N, we shall say that $(X_t)_{t\in T}$ converges* stably in the strong sense *with respect to the conditioning system $\mathcal{G}$ (or, more briefly, $\mathcal{G}$-*stably in the strong sense*) to N if, for each function f in $\mathcal{C}_b(E)$, the conditional expectation* $\mathrm{E}[f(X_t)|\mathcal{G}_t]$ *converges to the random variable Nf in L^1 (i.e. in probability).*

The condition imposed in the above definition entails, in particular, that, for each f in $\mathcal{C}_b(E)$, the random variable Nf is the limit in L^1 of a $\mathcal{G}$-adapted family of bounded random variables. This fact suggests introducing some useful terminology.

Definition 5. *A bounded random variable which is the limit in L^1 of a $\mathcal{G}$-adapted family of bounded random variables will be called* $\mathcal{G}$-regular. *An event will be called* $\mathcal{G}$-regular *if its indicator function is $\mathcal{G}$-regular.*

Proposition 3. *Let Z be a bounded random variable and let us set $U_t =$* $\mathrm{E}[Z|\mathcal{G}_t]$. *Then the following conditions are equivalent:*

(a) Z is $\mathcal{G}$-regular.
(b) Z is the limit in L^1 of $(U_t)_{t\in T}$.
(c) We have $\mathrm{E}[Z^2] \leqslant \liminf_t \mathrm{E}[U_t^2]$.

Proof. Implication (b)⇒(a) is obvious. In order to prove the opposite implication, it is enough to observe that, if $(V_t)_{t\in T}$ is a $\mathcal{G}$-adapted family of bounded random variables which converges in L^1 to Z, then the difference $U_t - Z$ converges in L^1 to zero because it can be put in the form

$$U_t - Z = \mathrm{E}[Z - V_t|\mathcal{G}_t] + V_t - Z.$$

Condition (b) is equivalent to the convergence in L^2 of U_t to Z. Therefore it is equivalent to condition (c) because of the following equalities:

$$\mathrm{E}\big[\,|Z - U_t|^2\,\big] = \mathrm{E}[Z^2] + \mathrm{E}[U_t^2] - 2\mathrm{E}[ZU_t] = \mathrm{E}[Z^2] - \mathrm{E}[U_t^2]. \qquad \square$$

By Proposition 3, one immediately verifies that the class of bounded $\mathcal{G}$-regular random variables is a Dedekind σ-complete Riesz space. Therefore, the $\mathcal{G}$-regular events form a σ-field, which we shall briefly call the *$\mathcal{G}$-regular σ-field*, and the bounded random variables which are measurable with respect to this σ-field are exactly the bounded $\mathcal{G}$-regular random variables. Obviously, the $\mathcal{G}$-regular σ-field is contained in the σ-field $\mathcal{G}^* \vee \mathcal{N}$.

Definition 6. *A* random variable *(with values in any measurable space) will be said to be* $\mathcal{G}$-regular *if it is measurable with respect to the $\mathcal{G}$-regular σ-field. Similarly, a* kernel N *will be said to be* $\mathcal{G}$-regular *if it is measurable with respect to the $\mathcal{G}$-regular σ-field.*

Using this terminology, we can state the three following simple results (the first two of which without proof).

Proposition 4. *Let $\mathcal{K}$ be a determining class. Then, in order that a kernel N is $\mathcal{G}$-regular, it is necessary and sufficient that each random variable of the form Nf, with f in $\mathcal{K}$, is $\mathcal{G}$-regular.*

Proposition 5. *In order that a family $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably in the strong sense to a kernel N, it is necessary that N is $\mathcal{G}$-regular.*

Proposition 6. *The σ-field $\mathcal{G}_* \vee \mathcal{N}$ is contained in the $\mathcal{G}$-regular σ-field.*

Proof. It is sufficient to observe that, by means of the classical convergence theorem for closed martingales, a bounded $(\mathcal{G}_* \vee \mathcal{N})$-measurable random variable Z is the limit in L^1 of the $\mathcal{G}$-adapted family $(W_t)_{t\in T}$ defined as:

$$W_t = \mathrm{E}[Z|\textstyle\bigcap_{u\geqslant t} \mathcal{G}_u]. \qquad \square$$

As we shall see in Section 6 (example 3), the above inclusion may be strict.

Using the notion of $\mathcal{G}$-regularity, we can now prove the following characterization of the $\mathcal{G}$-stable convergence in the strong sense.

Theorem 3. *Let N be a kernel and $\mathcal{K}$ a convergence determining class. Then the following conditions are equivalent:*

(a) The family $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably in the strong sense to N.
(b) The family $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably to N and moreover, for each f in $\mathcal{K}$, setting

$$V_t = \mathrm{E}[f(X_t)|\mathcal{G}_t], \tag{5}$$

we have $\mathrm{E}[V_t^2] \to \mathrm{E}[(Nf)^2]$.
(c) For each f in $\mathcal{K}$, the conditional expectation $\mathrm{E}[f(X_t)|\mathcal{G}_t]$ converges in L^1 (i.e. in probability) to Nf.
(d) The kernel N is $\mathcal{G}$-regular and moreover, for each f in $\mathcal{K}$ and each $\mathcal{G}$-adapted family $(Z_t)_{t\in T}$ of real random variables in $L^\infty(P)$, with $\limsup_t \|Z_t\|_{L^\infty} < +\infty$, we have

$$\int \bigl[f(X_t) - Nf\bigr] Z_t \,\mathrm{d}P \to 0. \tag{6}$$

(e) The kernel N is $\mathcal{G}$-regular and, for each f in $\mathcal{K}$ and each $\mathcal{G}$-adapted family $(H_t)_{t\in T}$ of events, with $\inf_t P(H_t) > 0$, we have

$$\int \bigl[f(X_t) - Nf\bigr] \,\mathrm{d}P_{H_t} \to 0. \tag{7}$$

Proof. Implication (a)⇒(b) is trivial. Implication (b)⇒(c) follows from the equality

$$\mathrm{E}\bigl[\,|V_t - Nf|^2\,\bigr] = \mathrm{E}[V_t^2] + \mathrm{E}[\,(Nf)^2\,] - 2\mathrm{E}[V_t(Nf)]. \tag{8}$$

In order to prove implication (c)⇒(d), let us assume condition (c). Then, by Proposition 4, the kernel N is $\mathcal{G}$-regular. Moreover, if $(Z_t)_{t\in T}$ is a family of real random variables as in condition (d), and f is an element in $\mathcal{K}$, then we have, using (5),

$$\Bigl|\int \bigl[f(X_t) - Nf\bigr] Z_t \,\mathrm{d}P\Bigr| = \Bigl|\int \bigl[V_t - Nf\bigr] Z_t \,\mathrm{d}P\Bigr| \leqslant \|Z_t\|_{L^\infty} \int |V_t - Nf| \,\mathrm{d}P \to 0.$$

Condition (d) is so verified.

Implication (d)⇒(e) is trivial: it suffices to take $Z_t = P(H_t)^{-1} I_{H_t}$.

Finally, let us prove implication (e)⇒(a). Given condition (e), we first prove a preliminary result: *for each $\mathcal{G}$-adapted family $(H_t)_{t\in T}$ of events, with $\inf_t P(H_t) > 0$, the convergence (7) holds for each f in $\mathcal{C}_b(E)$*. To this end, having a fixed $\mathcal{G}$-adapted family $(H_t)_{t\in T}$ of events, with $\inf_t P(H_t) > 0$, let us set

$$Z_t = P(H_t)^{-1} I_{H_t}, \qquad Q_t = Z_t \cdot P, \qquad \nu_t = Q_t N$$

and denote by μ_t the distribution of X_t under Q_t. Then, by assumption (e), we have

$$\langle \mu_t - \nu_t, f\rangle \to 0 \qquad \text{for each } f \text{ in } \mathcal{K}. \tag{9}$$

On the other hand, since we have $\inf_t P(H_t) > 0$, the probability densities Z_t are uniformly bounded and so they form, in the space $L^1(P)$, a relatively compact set with respect to the topology $\sigma(L^1, L^\infty)$. Therefore, if $\mathcal{U}$ is an ultrafilter on T, which is finer than the section filter, then, with respect to $\mathcal{U}$, the family $(Z_t)_{t\in T}$ converges in $\sigma(L^1, L^\infty)$ to a probability density Z. Thus, setting $Q = Z \cdot P$, the family $(\nu_t)_{t\in T}$ converges weakly (with respect to $\mathcal{U}$) to the probability measure QN. The same holds for $(\mu_t)_{t\in T}$ (because of (9) and the fact that $\mathcal{K}$ is a convergence determining class). It follows that, for each f in $\mathcal{C}_b(E)$, the bounded family of real numbers $(\langle \mu_t - \nu_t, f\rangle)_{t\in T}$ converges to zero with respect to the ultrafilter $\mathcal{U}$, and so (since $\mathcal{U}$ is arbitrary) also with respect to the section filter. Thus, the convergence (7) holds for each f in $\mathcal{C}_b(E)$. After proving this preliminary result, let us now prove that condition (a) is satisfied. To this end, having a fixed function f in $\mathcal{C}_b(E)$, let us adopt the notation (5) and, in addition, let us set $U_t = \mathrm{E}[Nf|\mathcal{G}_t]$. By Proposition 3, we can affirm that U_t converges in L^1 to Nf. Therefore, it is sufficient to prove that the difference $D_t = V_t - U_t$ converges in L^1 to zero. Actually, it is sufficient to prove that the above convergence holds for $D_t^+ = D_t \vee 0$ (because, passing to the function $-f$, we may obtain the analogous relation for $D_t^- = (-D_t)\vee 0$). Thus, let us prove that we have $\mathrm{E}[D_t^+] \to 0$ with respect to any ultrafilter $\mathcal{U}$ which is finer than the section filter of T. To this end, let us denote by λ the limit of the family $\big(P\{D_t > 0\}\big)_{t\in T}$ with respect to $\mathcal{U}$. If λ is zero, then, by the obvious inequality $\mathrm{E}[D_t^+] \leqslant 2\|f\|P\{D_t > 0\}$, the assertion is immediate. Hence, we are allowed to assume $\lambda > 0$, and so we can choose an element A in $\mathcal{U}$ such that $P\{D_t > 0\} > \lambda/2$ for each t in A. If we set

$$H_t = \begin{cases} \{D_t > 0\} & \text{if } t \in A, \\ \Omega & \text{if } t \in T \setminus A, \end{cases}$$

then we have $\inf_t P(H_t) > 0$ and, for each t in A,

$$\begin{aligned} \mathrm{E}[D_t^+] &= \textstyle\int_{H_t} (V_t - U_t)\,\mathrm{d}P = \int_{H_t} \big[f(X_t) - Nf\big]\,\mathrm{d}P \\ &= P(H_t) \textstyle\int \big[f(X_t) - Nf\big]\,\mathrm{d}P_{H_t}. \end{aligned}$$

In order to finish, it is enough to observe that, by the preliminary result proved above, the last term of this relation converges to zero. □

Corollary 2. *Let N be a kernel and $\mathcal{G}$, $\mathcal{G}'$ two conditioning systems such that $\mathcal{G}_t \subset \mathcal{G}'_t$ for each t in T. Let us suppose that N is $\mathcal{G}$-regular and that $(X_t)_{t\in T}$ converges $\mathcal{G}'$-stably in the strong sense to N. Then $(X_t)_{t\in T}$ also converges to N $\mathcal{G}$-stably in the strong sense.*

Proof. It is an immediate consequence of Theorem 3. □

Corollary 3. *Let us suppose that $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably in the strong sense to the kernel N. Let $(Y_t)_{t\in T}$ be a family of random variables, with values in a Polish space F. Assume that $(Y_t)_{t\in T}$ converges in probability to a*

$\mathcal{G}$-regular random variable Y. Then the family $([X_t, Y_t])_{t\in T}$ converges $\mathcal{G}$-stably in the strong sense to the product kernel $N \otimes \epsilon_Y$, where ϵ_Y denotes the Dirac kernel associated with Y, i.e. the family $\left(\epsilon_{Y(\omega)}\right)_{\omega\in\Omega}$ of Dirac distributions on F.

Proof. A convergence determining class for the distributions on $E \times F$ is given by the functions of the form $f \otimes g$, with f in $\mathcal{C}_b(E)$ and g in $\mathcal{C}_b(F)$. By Theorem 3, it suffices to prove that, for each function $f \otimes g$ of this type, we have

$$\mathrm{E}[f(X_t)g(Y_t)\,|\,\mathcal{G}_t] \xrightarrow{L^1} g(Y)Nf.$$

On the other hand, since $g(Y)$ is $\mathcal{G}$-regular, there exists a $\mathcal{G}$-adapted family $(U_t)_{t\in T}$ of bounded random variables which converges in L^1 to $g(Y)$. Then we have that $U_t - g(Y_t)$ converges in L^1 to zero and so we may replace $g(Y_t)$ by U_t in the above relation. Thus, what we have to prove is the relation

$$U_t\,\mathrm{E}[f(X_t)\,|\,\mathcal{G}_t] \xrightarrow{L^1} g(Y)Nf.$$

This is true by the assumptions. □

Corollary 4. *Let N be a kernel. Then the following conditions are equivalent:*

(a) The family $(X_t)_{t\in T}$ converges $\mathcal{A}$-stably in the strong sense to N.
(b) There exists a random variable X on $(\Omega, \mathcal{A}, P)$, with values in E, such that N is P-equivalent to the Dirac kernel ϵ_X and $(X_t)_{t\in T}$ converges in probability to X.

Proof. Implication (b)⇒(a) is trivial. Let us assume condition (a). Then $(X_t)_t$ converges in distribution to PN and, setting $\mathcal{G}_t = \mathcal{A}$ for each t in T, condition (b) of Theorem 3 holds. Therefore, we have $\mathrm{E}[Nf^2] = \mathrm{E}[(Nf)^2]$ for each f in $\mathcal{C}_b(E)$. It follows that N is P-equivalent to a Dirac kernel ϵ_X. Applying Corollary 3, we obtain the $\mathcal{A}$-stable convergence in the strong sense of $([X_t, X])_t$ to the kernel $\epsilon_X \otimes \epsilon_X$, and so the convergence in distribution of $([X_t, X])_t$ to $[X, X]$; that is, the convergence in probability of X_t to X. □

5 Relationship with the classical notion of stable convergence

Strong stable convergence can also be expressed as a strengthening of stable convergence in the classical sense (i.e. stable convergence with respect to a suitable fixed sub-σ-field of $\mathcal{A}$). This is immediate when the conditioning system $\mathcal{G}$ is monotone (increasing or decreasing). Indeed, if this is the case and we set $\mathcal{H} = \mathcal{G}_* = \mathcal{G}^*$, then, by Corollary 1, we may change condition (b) of Theorem 3 by replacing $\mathcal{G}$-stable convergence by $\mathcal{H}$-stable convergence. In the general case, the following theorem holds.

Theorem 4. *Let N be a kernel and $\mathcal{K}$ a convergence determining class. Then the following conditions are equivalent:*

(a) The family $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably in the strong sense to N.
(b) The family $(X_t)_{t\in T}$ converges $\sigma(N)$-stably to N and N is $\mathcal{G}$-regular. Moreover, for each f in $\mathcal{K}$, we have, using the notation (5), $\mathrm{E}[V_t^2] \to \mathrm{E}[(Nf)^2]$.

Proof. (a)⇒(b): Given condition (a), it is sufficient to prove the $\sigma(N)$-stable convergence of $(X_t)_t$ to N (the last part of condition (b) being trivially satisfied). To this end, let us fix f in $\mathcal{C}_b(E)$, use the notation (5) and set

$$W_t = \mathrm{E}[f(X_t)|\sigma(N)]. \tag{10}$$

Next, let us denote by $\mathcal{P}$ the set of finite products of functions of the form Ng with g in $\mathcal{C}_b(E)$. Since $\sigma(N)$ is generated by $\mathcal{P}$ and $\mathcal{P}$ is closed with respect to products, in order to prove that W_t converges to Nf in $\sigma(L^1, L^\infty)$, by a monotone class argument, it suffices to prove, for each K in $\mathcal{P}$, the convergence $\mathrm{E}[KW_t] \to \mathrm{E}[K\,Nf]$; that is,

$$\mathrm{E}[Kf(X_t)] \to \mathrm{E}[K\,Nf]. \tag{11}$$

On the other hand, the random variable K is bounded and $\mathcal{G}$-regular. In fact it is the product of a finite number of bounded $\mathcal{G}$-regular random variables. Thus, it is possible to find a uniformly bounded $\mathcal{G}$-adapted family $(K_t)_{t\in T}$ of random variables which converges in L^1 to K. Therefore (11) is equivalent to each of the two following limits:

$$\mathrm{E}[K_t f(X_t)] \to \mathrm{E}[K\,Nf], \qquad \mathrm{E}[K_t V_t] \to \mathrm{E}[K\,Nf].$$

The second of these two limits obviously holds: indeed, by assumption (a), $K_t V_t$ converges in probability (and so in L^1) to $K\,Nf$.

(b)⇒(a): Given condition (b), it is sufficient to prove that condition (c) of Theorem 3 holds. To this end, let us fix f in $\mathcal{K}$ and adopt the notation of (5) and (10). Because of the equality (8) and assumption (b), in order to prove the convergence in probability of V_t to Nf, it is enough to prove the convergence

$$\mathrm{E}[V_t\,Nf] \to \mathrm{E}[(Nf)^2]. \tag{12}$$

On the other hand, the random variable Nf is obviously $\sigma(N)$-measurable and so, using the fact that, by assumption (b), W_t converges to Nf in $\sigma(L^1, L^\infty)$, we may write

$$\mathrm{E}[f(X_t)\,Nf] = \mathrm{E}[W_t\,Nf] \to \mathrm{E}[(Nf)^2]. \tag{13}$$

Thus, we see that, in order to prove (12), it suffices to verify that we have

$$\mathrm{E}\big[\,(V_t - f(X_t))\,Nf\,\big] \to 0. \tag{14}$$

Now, by the assumption of $\mathcal{G}$-regularity of N, the random variable Nf is the limit in L^1 of a uniformly bounded $\mathcal{G}$-adapted family $(U_t)_{t\in T}$ of random variables. Since the difference $V_t - f(X_t)$, by (5), is orthogonal to U_t, the relation (14) is equivalent to

$$\mathrm{E}\big[\,(V_t - f(X_t))(Nf - U_t)\,\big] \to 0.$$

This last convergence obviously holds: indeed the random variables $V_t - f(X_t)$ are uniformly bounded and the difference $Nf - U_t$ converges in L^1 to zero. □

As we shall see in the next Section 6 (example 2), the $\mathcal{G}$-regularity of N is essential in condition (b) of the theorem we have just proved.

The following corollary concerns a very special case: the case in which the kernel N is measurable with respect to $\mathcal{G}_* \vee \mathcal{N}$. Remember that, by Proposition 6, this strong assumption of measurability implies automatically the $\mathcal{G}$-regularity of N.

Corollary 5. *Under the same assumptions as in the previous theorem, let us suppose that N is measurable with respect to $\mathcal{G}_* \vee \mathcal{N}$. Then, for each sub-$\sigma$-field $\mathcal{J}$ such that $\sigma(N) \subset \mathcal{J} \vee \mathcal{N} \subset \mathcal{G}_* \vee \mathcal{N}$, the following conditions are equivalent:*

(a) The family $(X_t)_{t\in T}$ converges $\mathcal{G}$-stably in the strong sense to N.
(b) The family $(X_t)_{t\in T}$ converges $\mathcal{J}$-stably to N and, for each f in $\mathcal{K}$, we have, using the notation (5), $\mathrm{E}[V_t^2] \to \mathrm{E}[(Nf)^2]$.

Proof. Implication (b)⇒(a) is trivial: indeed, if condition (b) holds, then, by Remark 1, condition (b) of Theorem 4 holds as well.

Let us prove implication (a)⇒(b). Given condition (a), it is sufficient (by Remark 1) to prove that $(X_t)_t$ converges $\mathcal{G}_*$-stably to N. To this end, let us fix f in $\mathcal{C}_b(E)$, use the notation (5), and set $W_t = \mathrm{E}[f(X_t)|\mathcal{G}_*]$. Further, let us set

$$\mathcal{I} = \textstyle\bigcup_s \bigcap_{t\geqslant s} \mathcal{G}_t.$$

Since Nf and W_t are $(\mathcal{G}_* \vee \mathcal{N})$-measurable and since $\mathcal{I}$ is an algebra which generates the σ-field $\mathcal{G}_*$, in order to prove that W_t converges to Nf in $\sigma(L^1, L^\infty)$, by a monotone class argument, it is enough to check that, for each H in $\mathcal{I}$, we have

$$\textstyle\int_H W_t \,\mathrm{d}P \to \int_H Nf \,\mathrm{d}P.$$

This fact obviously holds: indeed, for a fixed H in $\mathcal{I}$, we have $H \in \mathcal{G}_t$ for t sufficiently large and so, for t sufficiently large, we have

$$\textstyle\int_H W_t \,\mathrm{d}P = \int_H f(X_t) \,\mathrm{d}P = \int_H V_t \,\mathrm{d}P,$$

where the last term, by assumption (a), converges to $\int_H Nf \,\mathrm{d}P$. □

6 Some counter-examples

We note that, for a $\mathcal{G}$-adapted family $(X_t)_{t\in T}$ of random variables, $\mathcal{G}^*$-stable convergence implies $\mathcal{A}$-stable convergence. (It is a consequence of a known result regarding stable convergence in the classical sense.) The following example shows that we do not have a similar result for stable convergence in the strong sense: indeed, even if $(X_t)_{t\in T}$ is a $\mathcal{G}$-adapted family, $\mathcal{G}^*$-stable convergence in the strong sense does not imply $\mathcal{A}$-stable convergence in the strong sense.

Example 1. Let us take $T = \mathbb{N}^*$, $E = \mathbb{R}$. Given a sequence $(Y_j)_{j\geqslant 1}$ of independent and identically distributed real random variables, with mean zero and variance 1, let us set $X_n = (Y_1 + \cdots + Y_n)/\sqrt{n}$ and $\mathcal{G}_n = \sigma(X_k : k \geqslant n)$ for each $n \geqslant 1$. Then the σ-field $\mathcal{G}^*$ is contained in the symmetric σ-field of $(Y_j)_{j\geqslant 1}$, which is degenerate (because of the Hewitt-Savage theorem and the Kolmogorov 0–1 law). Therefore, by the central limit theorem, we have the strong $\mathcal{G}^*$-stable convergence of $(X_n)_n$ to the constant kernel N with $N(\omega, \cdot) = \mathcal{N}(0, 1)$ for each ω. Hence, $(X_n)_n$ also converges $\mathcal{A}$-stably to N, but, by Corollary 4, it cannot converge $\mathcal{A}$-stably in the strong sense to any kernel.

With the next example we show that we cannot replace condition (b) of Theorem 4 by the weaker condition obtained by omitting the part concerning the $\mathcal{G}$-regularity of N. More precisely, this weaker condition is not even sufficient in order to assure the simple $\mathcal{G}$-stable convergence of $(X_t)_{t\in T}$ to N.

Example 2. Let us take $T = \mathbb{N}^*$, $E = \mathbb{R}$. Then, given two independent and identically distributed real random variables R, S, which are not degenerate, let us set

$$X_n = R + S \text{ for each } n, \quad \mathcal{G}_n = \sigma(R) \text{ if } n \text{ is odd}, \quad \mathcal{G}_n = \sigma(S) \text{ if } n \text{ is even}.$$

Further, let us denote by M a version of the conditional distribution of $R+S$ given $\sigma(R)$ and by N a version of the conditional distribution of $R+S$ given $\sigma(S)$. Then, for each f in $\mathcal{C}_b(E)$ and each $n \geqslant 1$, a version of the conditional expectation $\mathrm{E}[f(X_n)|\sigma(N)]$ is Nf. Thus $(X_n)_{n\geqslant 1}$ obviously converges $\sigma(N)$-stably to N. Moreover, for each f in $\mathcal{C}_b(E)$ and each n, a version of the conditional expectation $\mathrm{E}[f(X_n)|\mathcal{G}_n]$ is the random variable V_n defined by:

$$V_n = Mf \quad \text{if } n \text{ is odd}, \qquad V_n = Nf \quad \text{if } n \text{ is even}. \tag{15}$$

By the exchangeability of the pair (R, S), the two random variables Mf and Nf are identically distributed and so we have $\mathrm{E}[V_n^2] = \mathrm{E}[(Nf)^2]$, for each n. Therefore, the relation $\mathrm{E}[V_n^2] \to \mathrm{E}[(Nf)^2]$ is obviously satisfied. Nevertheless, the sequence $(X_n)_n$ cannot converge $\mathcal{G}$-stably to N. Indeed, if that happened, then, for each f in $\mathcal{C}_b(E)$, the sequence $(V_n)_{n\geqslant 1}$ would converge to

Nf in $\sigma(L^1, L^\infty)$ and so, since Mf and Nf are independent and identically distributed, we would have

$$\mathrm{E}[(Nf)^2] = \lim_k \mathrm{E}[V_{2k+1}\, Nf] = \mathrm{E}[(Mf)(Nf)] = \big(\mathrm{E}[Nf]\big)^2.$$

This would imply that, for each f in $\mathcal{C}_b(E)$, the random variable Nf is degenerate and thus that the two random variables $R+S$, S are independent, in contrast to the fact that S is not degenerate.

Finally, the following example shows that the inclusion stated in Proposition 6 may be strict.

Example 3. Let us take $T = \mathbb{N}^*$, $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$, $P = \lambda$ (where λ denotes the Lebesgue measure on $\mathcal{B}([0,1])$). Moreover, let $(\mathcal{P}_n)_{n\geqslant 1}$ be a sequence of finite partitions of Ω such that, for each n, $\mathcal{P}_n$ is formed by intervals whose maximum length is smaller than $1/n$ and let us take $\mathcal{G}_n = \sigma(\mathcal{P}_n)$. Then the $\mathcal{G}$-regular σ-field coincides with $\mathcal{A}$. Indeed, for each x in $[0,1]$, if we denote by A_n the union of the elements of $\mathcal{P}_n$ which are contained in $[0,x]$, we have $0 \leqslant x - \lambda(A_n) \leqslant 1/n$ and so $I_{[0,x]}$ is the limit in L^1 of the $\mathcal{G}$-adapted family $(I_{A_n})_{n\geqslant 1}$. However, it is always possible to choose the partitions $\mathcal{P}_n$ in such a way that we have $\mathcal{G}_m \cap \mathcal{G}_n = \{\emptyset,\, \Omega\}$ for $m \neq n$ and so, if this is the case, it follows that $\mathcal{G}_* = \{\emptyset,\, \Omega\}$ and, consequently, $\mathcal{G}_* \vee \mathcal{N} = \mathcal{N}$.

7 Application to a triangular array of martingales

In this section we shall take $T = \mathbb{N}^*$ and $E = \mathbb{R}^d$ with $d \in \mathbb{N}^*$. Further, for each real matrix a, we shall denote by $|a|$ the sum of the absolute values of its entries and by a' the transpose of a. With this notation, if x is a real vector (which we consider as a column-matrix), we have $|xx'| = |x|^2$. If U is a random variable, on a probability space $(\Omega, \mathcal{A}, P)$, with values in the space of positive semidefinite $d \times d$-matrices, the family of Gaussian distributions

$$\big(\mathcal{N}\big(0, U(\omega)\big)\big)_{\omega\in\Omega}$$

is a kernel which we shall call the *Gaussian kernel* associated with U and denote it by $\mathcal{N}(0,U)$. We can now state the following theorem.

Theorem 5. *Let $(l_n)_{n\geqslant 1}$ be a sequence of strictly positive integers. On a probability space $(\Omega, \mathcal{A}, P)$, for each $n \geqslant 1$, let $(\mathcal{F}_{n,j})_{0\leqslant j\leqslant l_n}$ be a filtration and $(M_{n,j})_{n\geqslant 1, 0\leqslant j\leqslant l_n}$ be a triangular array of random variables on $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{R}^d$ such that, for each n, the family $(M_{n,j})_{0\leqslant j\leqslant l_n}$ is a d-dimensional martingale with respect to $(\mathcal{F}_{n,j})_{0\leqslant j\leqslant l_n}$ and $M_{n,0} = 0$. For each pair (n,j), with $n \geqslant 1$, $1 \leqslant j \leqslant l_n$, let us set $X_{n,j} = M_{n,j} - M_{n,j-1}$ and*

$$S_n = \textstyle\sum_{j=1}^{l_n} X_{n,j} = M_{n,l_n}, \quad U_n = \sum_{j=1}^{l_n} X_{n,j} X'_{n,j}, \quad X_n^* = \sup_{1\leqslant j\leqslant l_n} |X_{n,j}|.$$

Let us suppose that the sequence $(U_n)_{n\geqslant 1}$ *converges in probability to a random variable* U *with values in the space of positive semidefinite* $d\times d$*-matrices. Further, let* $(k_n)_{n\geqslant 1}$ *be a sequence of strictly positive integers, with* $k_n X_n^* \xrightarrow{L^1} 0$ *and let us denote by* $\mathcal{G}$ *the conditioning system* $(\mathcal{F}_{n,k_n\wedge l_n})_{n\geqslant 1}$.

Then, in order that the sequence $(S_n)_{n\geqslant 1}$ *converges* $\mathcal{G}$*-stably in the strong sense to the gaussian kernel* $\mathcal{N}(0,U)$*, it is (necessary and) sufficient that* U *is* $\mathcal{G}$*-regular (in the sense of Definition 6).*

For the proof of this theorem, we need the following lemma (which is inspired by a well-known technique introduced by McLeish [11]).

Lemma 1. *Given a finite family* $(X_j)_j$ *of real random variables on a probability space* $(\Omega,\mathcal{A},P)$*, let us set*

$$S=\textstyle\sum_j X_j,\qquad U=\sum_j X_j^2,\qquad X^*=\sup_j |X_j|.$$

Further, given two real numbers b *and* t*, with* $b>0$*, and a random variable* V *with values in* $[0,b]$*, let us set*

$$L=\textstyle\prod_j(1+itX_j),\;\; D=\exp(itS)-L\exp(-\frac{1}{2}t^2V),\;\; B=\{|t|X^*\leqslant 1,\, U\leqslant b\}.$$

Then, on the set B*, we have*

$$|D|\leqslant \kappa(b,t)\big(|U-V|+2b|t|X^*\big),\quad \textit{with}\quad \kappa(b,t)=\tfrac{1}{2}t^2\exp(\tfrac{7}{2}bt^2).$$

Proof. Since, for each real number x, we have $|1+ix|^2=1+x^2\leqslant\exp(x^2)$, it follows that $|L|^2\leqslant\exp(t^2U)$; that is,

$$|L|\leqslant\exp(\tfrac{1}{2}t^2U). \tag{16}$$

Let us denote the principal value of the complex logarithm by Log and set

$$\operatorname{Log}(1+ix)=ix+\tfrac{1}{2}x^2+r(x),$$

for each real number x. Thus, we have

$$|r(x)|\leqslant|x|^3\quad\text{for } |x|\leqslant 1,\qquad 1+ix=\exp[ix+\tfrac{1}{2}x^2+r(x)]\quad\text{for } x\in\mathbb{R}. \tag{17}$$

By the first of these two relations, we obtain that, on the set $\{|t|X^*\leqslant 1\}$, we have

$$\textstyle\sum_j|r(tX_j)|\leqslant\sum_j|tX_j|^3\leqslant|t|X^*t^2U\leqslant t^2U. \tag{18}$$

From the second of the relations (17), we get $L=\exp\big[itS+\frac{1}{2}t^2U+\sum_j r(tX_j)\big]$; that is,

$$\exp(itS)=L\exp\big[-\tfrac{1}{2}t^2U-\textstyle\sum_j r(tX_j)\big],$$

and so, by the definition of D, we have

$$D = L\Delta \quad \text{where} \quad \Delta = \exp\bigl[-\tfrac{1}{2}t^2U - \textstyle\sum_j r(tX_j)\bigr] - \exp(-\tfrac{1}{2}t^2V).$$

Since we have $|\exp(y) - \exp(z)| \leqslant |y - z| \exp\bigl[\frac{3}{2}(|y| + |z|)\bigr]$ for each pair y, z of complex numbers, we find

$$|\Delta| \leqslant \bigl|\tfrac{1}{2}t^2(U - V) + \textstyle\sum_j r(tX_j)\bigr| \exp\Bigl[\tfrac{3}{2}\bigl(|\tfrac{1}{2}t^2U + \textstyle\sum_j r(tX_j)| + \tfrac{1}{2}t^2V\bigr)\Bigr]. \quad (19)$$

On the other hand, since V takes values in $[0, b]$ and, on the set B, inequalities (18) hold, on B we have

$$\begin{aligned} |\Delta| &\leqslant \exp(3bt^2)\bigl|\tfrac{1}{2}t^2(U - V) + \textstyle\sum_j r(tX_j)\bigr| \\ &\leqslant \tfrac{1}{2}t^2 \exp(3bt^2)\bigl(|U - V| + 2b|t|X^*\bigr), \end{aligned}$$

and so, by (16),

$$|D| = |L\Delta| \leqslant \tfrac{1}{2}t^2 \exp(\tfrac{7}{2}bt^2)\bigl(|U - V| + 2b|t|X^*\bigr).$$

The lemma is thus proved. □

Proof (of Theorem 5). Without loss of generality, we may suppose $k_n \leqslant l_n$, for each n, and so, we may write $\mathcal{G}_n = \mathcal{F}_{n,k_n}$. If we assume that U is $\mathcal{G}$-regular, by Theorem 3(c), in order to prove that $(S_n)_n$ converges $\mathcal{G}$-stably in the strong sense to the Gaussian kernel $\mathcal{N}(0, U)$, it suffices to verify that, for each x in $\mathbb{R}^d$, we have

$$\mathrm{E}\bigl[\exp(ix'S_n)|\mathcal{G}_n\bigr] \xrightarrow{L^1} \exp(-\tfrac{1}{2}x'Ux).$$

This fact allows us to limit ourselves to proving the theorem in the particular case in which $d = 1$. Therefore, let us assume this is the case and let us prove that, for each t in $\mathbb{R}$, we have

$$\mathrm{E}\bigl[\exp(itS_n)|\mathcal{G}_n\bigr] \xrightarrow{L^1} \exp(-\tfrac{1}{2}t^2U),$$

or, equivalently, $\mathrm{E}\bigl[\exp(itS_n)|\mathcal{G}_n\bigr] - \exp(-\frac{1}{2}t^2U_n) \xrightarrow{L^1} 0$.

Let us set $S_{n,h} = \sum_{j=1}^h X_{n,j} = M_{n,h}$ and $U_{n,h} = \sum_{j=1}^h X_{n,j}^2$ for $0 \leqslant h \leqslant l_n$. Then, fixing $\epsilon > 0$ and choosing a real positive number a such that $P\{U \geqslant a\} < \epsilon$, let us define the stopping time J_n (with respect to the filtration $(\mathcal{F}_{n,h})_{0 \leqslant h \leqslant l_n}$) in the following way:

$$J_n(\omega) = l_n \wedge \inf\{h \in \mathbb{N} : \ h \leqslant l_n,\, U_{n,h}(\omega) \geqslant a\}.$$

Thus, it suffices to prove the relation

$$\mathrm{E}\bigl[\exp(itS_{n,J_n})|\mathcal{G}_n\bigr] - \exp\bigl[-\tfrac{1}{2}t^2(U_n \wedge a)\bigr] \xrightarrow{L^1} 0. \quad (20)$$

Indeed, the random variables S_{n,J_n} and $U_n \wedge a$ coincide with S_n and U_n, respectively, on the set $\{U_n < a\}$ and, for n sufficiently large, we have $P\{U_n \geqslant a\} < \epsilon$.

In order to prove (20), let us define the complex-valued martingale $(L_{n,h})_{0\leqslant h\leqslant l_n}$ (with respect to the filtration $(\mathcal{F}_{n,h})_{0\leqslant h\leqslant l_n}$) in the following way:

$$L_{n,h} = \prod_{j=1}^{h}(1 + itX_{n,j}I_{\{j\leqslant J_n\}}) \qquad \text{for} \quad 0 \leqslant h \leqslant l_n.$$

We observe that

$$k_n X_n^* \xrightarrow{L^1} 0$$

implies

$$X_n^* \xrightarrow{L^1} 0$$

and that we have

$$\begin{aligned} 1 \leqslant |L_{n,k_n}| &\leqslant \exp\left[\tfrac{1}{2}t^2(k_n X_n^*)^2\right] \xrightarrow{P} 1, \\ |L_{n,J_n}| &\leqslant \exp(\tfrac{1}{2}t^2 a)(1 + |t|X_n^*). \end{aligned} \tag{21}$$

Moreover, since the martingale $(L_{n,h})_{0\leqslant h\leqslant l_n}$ is stopped at J_n (and so closed by L_{n,J_n}), we have

$$\mathrm{E}[L_{n,J_n} \,|\, \mathcal{G}_n] = \mathrm{E}[L_{n,J_n} \,|\, \mathcal{F}_{n,k_n}] = L_{n,k_n} \xrightarrow{L^1} 1, \tag{22}$$

where the convergence in L^1 of the last term to the constant 1 follows from the relations (21), from the inequality $|L_{n,k_n}| \leqslant |L_{n,J_n}|$ and from the relation

$$\begin{aligned} |\mathrm{Arg}(L_{n,k_n})| &\leqslant \textstyle\sum_{j=1}^{k_n\wedge J_n} |\mathrm{Arg}(1 + itX_{n,j})| = \sum_{j=1}^{k_n\wedge J_n} |\arctan(tX_{n,j})| \\ &\leqslant |t| k_n X_n^* \xrightarrow{L^1} 0. \end{aligned}$$

Now, let us fix a real number b with $b > a$ and set $V_n = \mathrm{E}[U \wedge b|\mathcal{G}_n]$. Since the random variable $U \wedge b$ is $\mathcal{G}$-regular, the sequence $(V_n)_n$ converges in L^1 to $U \wedge b$ and so we have

$$V_n - (U_n \wedge b) \xrightarrow{L^1} 0. \tag{23}$$

Finally, let us set

$$B_n = \{\, |t|X_n^* \leqslant 1,\, X_n^* \leqslant \sqrt{b-a}\,\}, \quad D_n = \exp(itS_{n,J_n}) - L_{n,J_n}\exp(-\tfrac{1}{2}t^2 V_n).$$

We have $P(B_n) \to 1$, and so, by the second of the inequalities (21), we get

$$I_{B_n^c}\,|D_n| \leqslant 2I_{B_n^c}\,|L_{n,J_n}| \xrightarrow{L^1} 0. \tag{24}$$

Further, since we have $B_n \subset \{|t|X_n^* \leqslant 1,\, U_{n,J_n} \leqslant b\}$, applying Lemma 1 to the finite family $(X_{n,j} I_{\{j \leqslant J_n\}})_{1 \leqslant j \leqslant l_n}$, we find (with the same notation as in Lemma 1)

$$I_{B_n}\,|D_n| \leqslant \kappa(b,t)\big(\,|(U_{n,J_n} \wedge b) - V_n| + 2b|t|X_n^*\,\big).$$

From this inequality, taking into account (23) and (24), we obtain

$$\begin{aligned}
\limsup_n \|D_n\|_{L^1} &= \limsup_n \|I_{B_n} D_n\|_{L^1} \\
&\leqslant \kappa(b,t) \limsup_n \|(U_{n,J_n} \wedge b) - (U_n \wedge b)\|_{L^1} \\
&= \kappa(b,t) \limsup_n \textstyle\int_{\{J_n < l_n\}} |(U_{n,J_n} \wedge b) - (U_n \wedge b)|\,\mathrm{d}P \\
&\leqslant \kappa(b,t)(b-a).
\end{aligned}$$

By (22) and (23), we finally get

$$\begin{aligned}
&\limsup_n \|\,\mathrm{E}\,[\exp(itS_{n,J_n})|\mathcal{G}_n] - \exp[-\tfrac{1}{2}t^2(U_n \wedge b)]\,\|_{L^1} \\
&\quad = \limsup_n \|\,\mathrm{E}\,[\exp(itS_{n,J_n})|\mathcal{G}_n] - L_{n,k_n}\,\exp(-\tfrac{1}{2}t^2 V_n)\,\|_{L^1} \\
&\quad = \limsup_n \|\,\mathrm{E}[D_n|\mathcal{G}_n]\,\|_{L^1} \leqslant \kappa(b,t)(b-a).
\end{aligned}$$

In order to obtain the desired relation (20), it remains to let b go to a. The theorem is thus proved. □

Corollary 6. *Under the same assumptions as in Theorem 5, let us suppose that U is $\mathcal{G}$-regular. Then $([S_n, U_n])_{n\geqslant 1}$ converges $\mathcal{G}$-stably in the strong sense to the product kernel $\mathcal{N}(0,U) \otimes \epsilon_U$, where ϵ_U denotes the Dirac kernel associated with U.*

If we assume $P\{\det U > 0\} > 0$ and if we denote by V_n the random matrix which coincides with $U_n^{-1/2}$ on $\{\det U_n > 0\}$ and which is null elsewhere, then, under the probability measure $P(\,\cdot\,|\{\det U > 0\})$, the sequence $(V_n S_n)_{n\geqslant 1}$ converges $\mathcal{G}$-stably in the strong sense to the constant gaussian kernel $\mathcal{N}(0, I_d)$, where I_d denotes the d-dimensional identity matrix.

Proof. By Corollary 3, the first assertion is a consequence of the previous theorem. The second assertion follows immediately. □

Remark 3. In the proof of Theorem 5, we observed that, the condition $k_n X_n^* \xrightarrow{L^1} 0$ implies the condition $X_n^* \xrightarrow{L^1} 0$. Conversely, if we assume this last condition, then it is always possible to find a sequence $(k_n)_{n\geqslant 1}$ of strictly positive integers such that the first condition holds and, in addition, such that $k_n \uparrow +\infty$: more precisely, if we exclude the trivial case in which the random variables X_n^* are eventually negligible, then it suffices to take $k_n = \lceil 1/\sqrt{a_n}\,\rceil$, where $a_n = \sup_{m\geqslant n} \mathrm{E}[X_m^*]$.

The previous remark allows us to obtain as a corollary of Theorem 5 the following result (the first part of which is already known (see [3]).

Corollary 7. *Let us change the assumptions of Theorem 5 by replacing the condition* $k_n X_n^* \xrightarrow{L^1} 0$ *by the condition* $X_n^* \xrightarrow{L^1} 0$. *Moreover, let us set*

$$\mathcal{H}_j = \liminf_n \mathcal{F}_{n,j\wedge l_n} \quad \text{for } j \geqslant 0, \qquad \mathcal{H} = \bigvee_{j\geqslant 0} \mathcal{H}_j. \tag{25}$$

(a) If U is measurable with respect to the σ-field $\mathcal{H} \vee \mathcal{N}$, then, for each σ-field $\mathcal{J}$ with $\sigma(U) \subset \mathcal{J} \vee \mathcal{N} \subset \mathcal{H} \vee \mathcal{N}$, the sequence $(S_n)_{n\geqslant 1}$ converges $\mathcal{J}$-stably to the gaussian kernel $\mathcal{N}(0, U)$.

(b) If $j \geqslant 0$ is such that U is regular with respect to the conditioning system $(\mathcal{F}_{n,j\wedge l_n})_{n\geqslant 1}$, then, for each $h \geqslant j$, the sequence $(S_n)_{n\geqslant 1}$ converges $(\mathcal{F}_{n,h\wedge l_n})_{n\geqslant 1}$-stably in the strong sense to $\mathcal{N}(0, U)$.

Proof. (a) By Remark 3, there exists a sequence $(k_n)_{n\geqslant 1}$ of strictly positive integers with $k_n \uparrow +\infty$ and $k_n X_n^* \xrightarrow{L^1} 0$. Hence, if we denote by $\mathcal{G}$ the conditioning system $(\mathcal{F}_{n,k_n\wedge l_n})_{n\geqslant 1}$, we can easily see that the σ-field $\mathcal{H}$ is contained in $\mathcal{G}_*$ (indeed, each $\mathcal{H}_j$ is contained in $\mathcal{G}_*$). By Proposition 6, it follows that U is $\mathcal{G}$-regular. Therefore, by Theorem 5, we have that $(S_n)_n$ converges $\mathcal{G}$-stably in the strong sense to $\mathcal{N}(0, U)$. Then, by Corollary 5, we can conclude the proof of statement (a).

(b) We observe that U is regular with respect to each conditioning system of the form $(\mathcal{F}_{n,h\wedge l_n})_{n\geqslant 1}$, with $h \geqslant j$. Therefore, the conclusion easily follows by applying Theorem 5 with $k_n = h \vee 1$ for each n, and then, if $h = j = 0$, Corollary 2 with $\mathcal{G}_n = \mathcal{F}_{n,h\wedge l_n}$, $\mathcal{G}'_n = \mathcal{F}_{n,k_n\wedge l_n}$. □

Remark 4. Without changing the proof, it is possible to extend the statement of Theorem 5 by replacing the sequence $(k_n)_{n\geqslant 1}$ by a sequence $(K_n)_{n\geqslant 1}$ of random variables with values in $\mathbb{N}^*$ such that, for each n, the random variable $K_n \wedge l_n$ is a stopping time with respect to the filtration $(\mathcal{F}_{n,j})_{0\leqslant j\leqslant l_n}$ and such that $X_n^* + \sum_{j=1}^{K_n\wedge l_n} |X_{n,j}| \xrightarrow{L^1} 0$. Furthermore, we note that, with some small changes, the statement of Theorem 5 (and so of Corollary 7) remains true if we let $(l_n)_{n\geqslant 1}$ be a sequence of elements of $\mathbb{N}^* \cup \{\infty\}$ and assume $\mathcal{F}_{n,\infty} = \bigvee_{j\in\mathbb{N}} \mathcal{F}_{n,j}$.

8 Further applications

In this section we shall give further examples which will show how the previous theory can be applied in order to obtain results of stable convergence in the strong sense. In the first example we shall obtain a strengthening of the result in [3] (Th. 2.2), which is already an improvement of some results by Heyde [7] and Küchler and Sørensen [9] in the setting of likelihood theory for stochastic processes.

Example 4. On a probability space $(\Omega, \mathcal{A}, P)$, let $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be a filtration which satisfies the usual conditions and let us set $\mathcal{F}_\infty = \bigvee_{t\in\mathbb{R}_+} \mathcal{F}_t$. Further, let $M = (M_t)_{t \in \mathbb{R}_+}$ be a d-dimensional real martingale such that $M_0 = 0$ and such that its trajectories are right-continuous and with limits from the left. Let us denote by Q the process $[M, M]$ (with values in the space of positive semidefinite $d \times d$-matrices). In addition, let $(a_t)_{t \in T}$ be a family of $d \times d$-matrices such that (using the same notation as in the previous section) the following conditions hold:

$$|a_t| \longrightarrow 0, \qquad \sup_{0 \leqslant s \leqslant t} |a_t \Delta_s M| \xrightarrow{L^1} 0, \qquad a_t Q_t a_t' \xrightarrow{P} U$$

(where U is a $\mathcal{F}_\infty$-measurable random variable with values in the space of positive semidefinite $d \times d$-matrices). It is known that, under these assumptions, the family $(a_t M_t)_{t \in \mathbb{R}_+}$ converges $\mathcal{A}$-stably to the gaussian kernel $\mathcal{N}(0, U)$ (see [3], Th. 2.2). Moreover, it is known (see [3], Lemma 3.2) that, under the same assumptions, it is possible to construct, for each positive real number t, a stopping time τ_t in such a way that the following conditions hold:

$$\tau_t \leqslant t, \qquad \tau_t \to +\infty, \qquad |a_t M_{\tau_t}| \xrightarrow{L^1} 0, \qquad |a_t Q_{\tau_t} a_t'| \xrightarrow{P} 0. \tag{26}$$

Applying the theory of the previous sections, we can verify that, if $(\tau_t)_{t \in \mathbb{R}_+}$ is a family of stopping times with the above properties, then $(a_t M_t)_{t \in \mathbb{R}_+}$ converges $(\mathcal{F}_{\tau_t})_{t \in \mathbb{R}_+}$-stably in the strong sense to the Gaussian kernel $\mathcal{N}(0, U)$.

In order to prove this assertion, it suffices to prove that, for any fixed increasing sequence $(t_n)_{n \geqslant 1}$ of real positive numbers with $t_n \uparrow +\infty$, if we denote the stopping time τ_{t_n} by τ_n, the sequence $(a_{t_n} M_{t_n})_{n \geqslant 1}$ converges $(\mathcal{F}_{\tau_n})_{n \geqslant 1}$-stably in the strong sense to the gaussian kernel $\mathcal{N}(0, U)$. We note that, by the third relation of (26), it is enough to prove that $(a_{t_n}(M_{t_n} - M_{\tau_n}))_{n \geqslant 1}$ converges $(\mathcal{F}_{\tau_n})_{n \geqslant 1}$-stably in strong sense to the kernel $\mathcal{N}(0, U)$. To this end, let us construct (in the same way as in Th. 2.2 of [3]) a triangular array $(T_{n,j})_{n \geqslant 1, j \geqslant 0}$ of stopping times and a sequence $(l_n)_{n \geqslant 1}$ of strictly positive integers such that, for each n, the sequence $(T_{n,j})_{j \geqslant 0}$ is an increasing sequence of stopping times with $T_{n,0} = \tau_n$, $\tau_n \leqslant T_{n,j} \leqslant t_n$ and such that, setting

$$X_{n,j} = a_{t_n}\big(M_{T_{n,j}} - M_{T_{n,j-1}}\big) \quad \text{for} \quad 1 \leqslant j \leqslant l_n,$$

the following properties hold:

$$P\Big\{ \big| a_{t_n}(M_{t_n} - M_{\tau_n}) - \textstyle\sum_{j=1}^{l_n} X_{n,j} \big| > n^{-1} \Big\} < n^{-1}, \tag{27}$$

$$P\Big\{ \big| a_{t_n}(Q_{t_n} - Q_{\tau_n}) a_{t_n}' - \textstyle\sum_{j=1}^{l_n} X_{n,j} X_{n,j}' \big| > n^{-1} \Big\} < n^{-1},$$

$$\sup_{1 \leqslant j \leqslant l_n} |X_{n,j}| \leqslant n^{-1} + \sup_{0 \leqslant s \leqslant t_n} |a_{t_n} \Delta_s M|.$$

Setting $\mathcal{F}_{n,j} = \mathcal{F}_{T_{n,j}}$ for each $n \geqslant 1,\ j \geqslant 0$, it follows that the triangular array

$$\big(\textstyle\sum_{j=1}^{h} X_{n,j} \big)_{n \geqslant 1,\, 0 \leqslant h \leqslant l_n}$$

satisfies the assumptions of Corollary 7(b) with $j = 0$. Indeed, since $\tau_n \to +\infty$, using notation (25), we have that the σ-field $\mathcal{H}_0$ coincides with $\mathcal{F}_\infty$ (see [3], Lemma 3.4) and so U is $(\mathcal{F}_{n,0})_{n\geqslant 1}$-regular. Thus, we can affirm that the sequence

$$\big(\textstyle\sum_{j=1}^{l_n} X_{n,j}\big)_{n\geqslant 1} \tag{28}$$

converges stably in the strong sense with respect to $(\mathcal{F}_{n,0})_{n\geqslant 1}$ (i.e. with respect to $(\mathcal{F}_{\tau_n})_{n\geqslant 1}$) to the kernel $\mathcal{N}(0, U)$. Finally, it suffices to observe that, by (27), the sequence

$$\big(a_{t_n}(M_{t_n} - M_{S_n})\big)_{n\geqslant 1}$$

differs from the sequence (28) by a sequence which converges in probability to zero. Let us note that we have used the condition $\tau_t \to +\infty$ only in order to prove the $(\mathcal{F}_{\tau_n})_{n\geqslant 1}$-regularity of U. Therefore, the convergence result for $(a_t M_t)_{t\in\mathbb{R}_+}$ is still true if we assume this last condition instead of $\tau_t \to +\infty$.

Example 5. Let us consider a triangular array $(X_{n,j})_{n\geqslant 1,\,1\leqslant j\leqslant l_n}$ of real random variables, and let us set

$$S_n = \textstyle\sum_{j=1}^{l_n} X_{n,j}, \qquad U_n = \sum_{j=1}^{l_n} X_{n,j}^2, \qquad X_n^* = \sup_{1\leqslant j\leqslant l_n} |X_{n,j}|.$$

Further, let $(\mathcal{C}_n)_n$ be a sequence of sub-σ-fields of $\mathcal{A}$ such that, for each fixed n, the random vector $[X_{n,j}]_{1\leqslant j\leqslant l_n}$ is $\mathcal{C}_n$-conditionally jointly symmetric; that is, for each fixed n and each $H \in \mathcal{C}_n$ with $P(H) \neq 0$, all the random vectors of the form $[\epsilon_j X_{n,j}]_{1\leqslant j\leqslant l_n}$, with $\epsilon_j \in \{-1, 1\}$, are identically distributed under P_H. For $n \geqslant 1$ and $0 \leqslant j \leqslant l_n$, let us denote by $\mathcal{S}_{n,j}$ the σ-field generated by the random variables of the form $g(X_{n,1}, \ldots, X_{n,l_n})$, where g is a Borel real function on $\mathbb{R}^{l_n}$ such that $g(\epsilon_1 x_1, \ldots, \epsilon_{l_n} x_{l_n}) = g(x_1, \ldots, x_{l_n})$ for each family $(\epsilon_i)_{1\leqslant i\leqslant l_n}$ of coefficients in $\{-1, 1\}$ with $\epsilon_i = 1$ for $1 \leqslant i \leqslant j$. Moreover, let us set

$$V_n = I_{\{U_n>0\}} U_n^{-1/2}, \qquad Y_{n,j} = V_n X_{n,j}, \qquad Y_n^* = \sup_{1\leqslant j\leqslant l_n} |Y_{n,j}| = V_n X_n^*.$$

Since, for each n, the random variable V_n is $\mathcal{S}_{n,0}$-measurable, it is easy to see that, for each fixed n, the family $\big(\sum_{j=1}^{h} Y_{n,j}\big)_{0\leqslant h\leqslant l_n}$ is a martingale with respect to the filtration $(\mathcal{C}_n \vee \mathcal{S}_{n,j})_{0\leqslant j\leqslant l_n}$. Further, we have $\sum_{j=1}^{l_n} Y_{n,j} = V_n S_n$, $\sum_{j=1}^{l_n} Y_{n,j}^2 = I_{\{U_n>0\}}$ and that $(Y_n^*)_{n\geqslant 1}$ is uniformly bounded. Therefore, if $P\{U_n > 0\} \to 1$ and $(Y_n^*)_n$ converges in probability (and so in L^1) to zero, then, by Corollary 7(b), we obtain that $(V_n S_n)_{n\geqslant 1}$ converges $(\mathcal{C}_n \vee \mathcal{S}_{n,h})_n$-stably in the strong sense to the constant kernel $\mathcal{N}(0, 1)$, for each $h \geqslant 0$. More generally, by Theorem 5, we have that, if $P\{U_n > 0\} \to 1$ and there exists an increasing sequence $(k_n)_{n\geqslant 1}$ of strictly positive integers such that $k_n Y_n^* = k_n V_n X_n^* \xrightarrow{L^1} 0$, then $(V_n S_n)_{n\geqslant 1}$ converges $(\mathcal{C}_n \vee \mathcal{S}_{n,k_n\wedge l_n})_n$-stably in the strong sense to $\mathcal{N}(0, 1)$. These results improve Th. 4.2 and Cor. 4.3 in [4].

Example 6. On a probability space $(\Omega, \mathcal{A}, P)$, let $(Y_n)_{n\geqslant 1}$ be an exchangeable sequence of integrable real random variables and let us denote by $\mathcal{T}$ its tail σ-field and by $\mathcal{G} = (\mathcal{G}_n)_{n\geqslant 0}$ its natural filtration. Moreover, let us set

$$W = \mathrm{E}[Y_1|\mathcal{T}], \qquad W_n = \mathrm{E}[Y_{n+1}|\mathcal{G}_n] = \mathrm{E}[W|\mathcal{G}_n].$$

We assume that the following conditions hold:

(a) $n \sum_{k\geqslant n} (W_k - W_{k+1})^2 \xrightarrow{P} U,$

where U is a positive random variable (which is obviously measurable with respect to $\mathcal{G}_\infty \vee \mathcal{N} = \bigvee_{n\geqslant 0} \mathcal{G}_n \vee \mathcal{N}$).

(b) $\sqrt{n} \sup_{k\geqslant n} |W_k - W_{k+1}| \xrightarrow{L^1} 0.$

If we set $S_n = \sqrt{n}(W_n - W)$, then the sequence $(S_n)_{n\geqslant 1}$ converges $\mathcal{G}$-stably in the strong sense to $\mathcal{N}(0, U)$. In order to prove this assertion, it suffices to apply Corollary 7(b) (together with Remark 4), after observing that the random variables S_n can be put in the form $S_n = \sqrt{n} \sum_{k\geqslant n}(W_k - W_{k+1})$ and that we have $\mathcal{G}_* = \mathcal{G}^* = \mathcal{G}_\infty$ (because $\mathcal{G}$ is increasing).

Let us now suppose, in particular, that, the sequence $(Y_n)_{n\geqslant 1}$ is the one associated with the "Pólya urn". Then conditions (a), (b) are satisfied with $U = W - W^2$. Furthermore, if we set $M_n = (Y_1 + \cdots + Y_n)/n$, we have $\sqrt{n}(W_n - M_n) \xrightarrow{L^1} 0$. Thus, if we set $S'_n = \sqrt{n}(M_n - W)$, then the sequence $(S'_n)_{n\geqslant 1}$ also converges $\mathcal{G}$-stably in the strong sense to $\mathcal{N}(0, W - W^2)$. Let us note that this sequence also converges to the previous kernel $\mathcal{A}$-stably (but not in the strong sense because of Corollary 4).

References

1. Aldous, D.J., Eagleson, G.K. (1978): On mixing and stability of limit theorems. Ann. Prob., 6, 325–331
2. Billingsley, P. (1968): Convergence of Probability Measures. Wiley, New York
3. Crimaldi, I., Pratelli, L. (2005): Convergence Results for Multivariate Martingales. Stochastic Processes Appl., 115/4, 571–577
4. Crimaldi, I. (2002): Convergence Results for a Normalized Triangular Array of Symmetric Random Variables. Expo. Math., 20(4), 375–384
5. Fegin, P.D. (1985): Stable Convergence of Semimartingales. Stochastic Processes Appl., 19, 125–134
6. Hall, P., Heyde, C.C. (1980): Martingale limit theory and its application. Academic Press, New York
7. Heyde, C.C. (1997): Quasi-Likelihood and its Application. Springer Series in Statistics, Springer, Berlin Heidelberg New York
8. Jacod, J., Memin, J. (1981): Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité. Sem. de Prob. XV, Lecture Notes in Math. 850, Springer, Berlin Heidelberg New York, 529–546

9. Küchler, U., Sørensen, M. (1997): Exponential Families of Stochastic Processes. Springer Series in Statistics, Springer, Berlin Heidelberg New York
10. Letta, G. (1998): Convergence stable et applications. Atti Sem. Mat. Fis. Univ. Modena, supplemento al vol. XLVI, 191–211
11. McLeish, D.L. (1974): Dependent central limit theorems and invariance principles. Ann. Prob. 2, 620–628
12. Rényi, A. (1963): On Stable Sequences of Events. Sankhya Ser. A. 25, 293–302

Product of Harmonic Maps is Harmonic: A Stochastic Approach

Pedro J. Catuogno[1] and Paulo R.C. Ruffino[2]

Departamento de Matemática, Universidade Estadual de Campinas
13.081-970 Campinas, SP, Brazil

Summary. Let $\phi_j : M_j \to G$, $j = 1, 2, \ldots, n$, be harmonic mappings from Riemannian manifolds M_j to a Lie group G. Then the product $\phi_1 \phi_2 \cdots \phi_n$ is a harmonic mapping between $M_1 \times M_2 \times \cdots \times M_n$ and G. The proof is a combination of properties of Brownian motion in manifolds and Itô formulae for stochastic exponential and logarithm of product of semimartingales in Lie groups.

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1 Introduction

Let (M, g) and (N, h) be two compact Riemannian manifolds and consider $\phi : M \to N$ a C^∞-differentiable map. The *energy functional* (or *action integral*) of ϕ is defined as the integral of the *density energy function* $e(\phi)(x)$

$$E(\phi) = \int_M e(\phi)(x) dv_g,$$

where $e(\phi)(x) = 1/2\mathrm{tr}_g(\phi^* h)(x)$ and v_g is the Riemannian volume on M. A physical interpretation of the energy functional E could be the accumulated elastic energy on M, when this space is stretched on N.

We say that ϕ is a *harmonic mapping* if ϕ is a critical point of the functional E. The Euler–Lagrange formula in this case can be written in terms of the connections ∇^M and ∇^N on M and N, respectively: namely ϕ is harmonic

[1] E-mail: pedrojc@ime.unicamp.br. Research partially supported by FAPESP grant n° 01/13158-4 and 02/10246-2.
[2] E-mail: ruffino@ime.unicamp.br. Research partially supported by CNPq grant n° 301112/2003-7 and FAPESP 02/10246-2.

if and only if the tension field $\tau(\phi)(x) = \mathrm{tr}(\widetilde{\nabla}\phi_* - \phi_*\nabla^M)$ vanishes everywhere in M, where $\widetilde{\nabla}$ is the induced connection on the induced bundle $\phi^{-1}TN$ (see, e.g., Urakawa [11]). The definition of harmonic mappings extends naturally to noncompact manifolds just considering the critical property of the energy functional locally on M. In particular, we recall that geodesics are harmonic mappings from the real line to a Riemannian manifold.

Variational problems have been, historically, a non trivial area of interest both for mathematicians and physicists. In the last couple of decades many important contributions on harmonic mappings were done, see, e.g., in one of the seminal papers, Calabi construction of harmonic mappings from two spheres into symmetric spaces [1]. For a classical text we refer to Eells and Lemaire [3]. For an approach of harmonic mappings into Lie groups see, e.g., Uhlenbeck [10]. This article is a contribution in the topic which comes as an application of stochastic tools in geometry.

Here we consider harmonic mappings with image in a Lie group G with a bi-invariant Riemannian metric. In this case the associated (Levi-Civita) left invariant connection on G, denoted by ∇^L, satisfies $\nabla^L_X Y = \frac{1}{2}[X,Y]$ for all X, Y left invariant vector fields in the Lie algebra $\mathcal{G}$, see, e.g., Cheeger and Ebin [2].

The aim of this paper is to present a direct stochastic proof that a product of harmonic mappings is a harmonic mapping. More precisely: given $\phi_j : M_j \to G$, $j = 1, 2, \ldots, n$, harmonic mappings between Riemannian manifolds M_j and a Lie group G with bi-invariant Riemannian metric, then the product $\phi_1\phi_2\cdots\phi_n$ is a harmonic mapping between $M_1 \times M_2 \times \cdots \times M_n$ and G. In the best of our knowledge, this is a new result in harmonic mappings theory.

Our proof is a combination of properties of Brownian motion in manifolds and Itô formulae for stochastic exponential and logarithm of products of semimartingales in Lie groups. Although we assume that the group has a bi-invariant metric, one can easily verify that our argument also holds for any Lie group considering the left connection $\nabla^L_X Y = \frac{1}{2}[X,Y]$ if $X, Y \in \mathcal{G}$.

We recall that a product of harmonic mappings appears also in other contexts: we mention here harmonic functions on Lie groups with respect to a Radon probability measure μ on G, see, e.g., Furstenberg [5]. A function $f : G \to \mathbb{R}$ is called *μ-harmonic* if

$$f(g) = \int_G f(gh)\ d\mu(h),$$

for every g in G. If $f_1 : (G_1, \mu_1) \to \mathbb{R}$ and $f_2 : (G_2, \mu_2) \to \mathbb{R}$ are harmonic functions, then Fubini theorem implies that $f_1 f_2 : G_1 \times G_2 \to \mathbb{R}$ is a $\mu_1 \times \mu_2$-harmonic function.

In Section 2 we recall some basic facts and formulae on stochastic calculus in Lie groups. We refer mainly to Hakim-Dowek and Lépingle [6], nevertheless, it may happen that someone finds the proofs presented here simpler than in [6]. We shall use these formulae in Section 3, where we prove the main results.

By convention, all martingales, semimartingales, and local martingales are assumed to be continuous.

2 Preliminary results

Let M be a Riemannian manifold and consider $\theta_{X_t} \in T^*_{X_t}M$ an adapted stochastic one-form along X_t, an M-valued semimartingale. The integral of the form θ along X was proposed by Ikeda and Manabe [7] (see also, among others, Emery [4] or Meyer [9]). This integral is geometrically intrinsic, and it has a natural description in local charts: let $(U, x^1, \dots, x^n)$ be a local system of coordinates in M, then θ can be written as $\theta_x = \theta^1(x)\,dx^1 + \cdots + \theta^n(x)\,dx^n$, where $\theta^i(x)$, $i = 1, 2, \dots, n$, are (C^∞, say) functions in M. The Stratonovich integral of θ along X_t is given by:

$$\int \theta \circ dX_t = \sum_{i=1}^{n} \int \theta^i(X_t) \circ dX_t^i.$$

Let G be a Lie group with the corresponding Lie algebra $\mathcal{G}$. We denote by ω the (left) Maurer–Cartan form in G, i.e., if $v \in T_gG$, then $\omega_g(v) = L_{g^{-1}*}(v)$. It corresponds to the unique $\mathcal{G}$-valued left invariant one-form in G.

The logarithm of a semimartingale X_t on G (with $X_0 = e$) is the integral of the Maurer–Cartan form along X_t, namely, it is the following semimartingale in the Lie algebra:

$$(\log X)_t = \int_0^t \omega \circ dX_s.$$

Conversely, consider a semimartingale M_t in the Lie algebra $\mathcal{G}$. We recall that the (left) stochastic exponential $\epsilon(M)$ of M_t is the stochastic process X_t which is solution of the Stratonovich left invariant equation on G:

$$\begin{cases} dX_t = L_{X_t*} \circ dM_t, \\ X_0 = e. \end{cases}$$

An interesting geometric characterization of the exponential $\epsilon(M)$ is the fact that it corresponds to the stochastic development of $M_t \in T_eG$ to the group G with respect to the left invariant connection ∇^L. One easily checks that the logarithm is the inverse of the stochastic exponential ϵ.

Martingales in G (with respect to ∇^L-connection) and local martingales in the Lie algebra $\mathcal{G}$ are related by the following characterization, see Hakim-Dowek and Lépingle [6]:

Theorem 1. *A process X_t on G is a ∇^L-martingale if and only if $X_t = X_0 \cdot \epsilon(M)$ for some local martingale M in $\mathcal{G}$.*

The pull-back of Maurer–Cartan forms by homomorphisms of Lie groups is easily described by:

Lemma 1. *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then the pull-back $\varphi^*\omega_H$ satisfies, for $v \in T_gG$:*

$$(\varphi^*\omega_H)v = \varphi_*(\omega_G(v)).$$

In particular, if X is a semimartingale in G, then $\varphi_*(\log X) = \log(\varphi(X))$.

Proof. Once $\varphi(L_{g^{-1}}(h)) = L_{\varphi(g)^{-1}}(\varphi(h))$, the chain rule implies that

$$L_{\varphi(g)^{-1}*}(\varphi_*(v)) = \varphi_*(L_{g^{-1}*}(v)).$$

For the last formula, by definition: $\log\varphi(X) = \int \varphi^*\omega_H \circ dX$. The result follows directly by the first part of the Lemma and the very definition of $\varphi_* \log X$. □

Denote by $I_g : G \to G$ the adjoint in the group G given by $h \mapsto ghg^{-1}$. The map I_g is an automorphism of G and its derivative corresponds to the isomorphism of the Lie algebra called adjoint in $\mathcal{G}$ denoted by $Ad(g) = I_{g*} : \mathcal{G} \to \mathcal{G}$. We have that $R_g^*\,\omega = Ad(g^{-1})\omega$ (see, e.g., Kobayashi and Nomizu [8]). The pull-back of the canonical form by multiplication and inverse is given by:

Proposition 1. *Let $m : G \times G \to G$ be the multiplication and $i : G \to G$ be the inverse in the group. Then the pull-backs satisfy:*

a) $m^*\omega = Ad^{-1}(\pi_2)(\pi_1^*\omega) + \pi_2^*\omega$.
b) $i^*\omega = -Ad\ \omega$.

Proof. Let $w = (u, v) \in T_{(g,h)}G \times G \simeq T_gG \times T_hG$. Then

$$\begin{aligned} m^*\omega(w) &= \omega(m_*w) = \omega(R_{h*}u + L_{g*}v) \\ &= L_{(gh)^{-1}*}(R_{h*}u + L_{g*}v) \\ &= L_{h^{-1}*}R_{h*}L_{g^{-1}*}u + L_{h^{-1}*}L_{g^{-1}*}L_{g*}v \\ &= Ad(h^{-1})\omega(u) + \omega(v). \end{aligned}$$

For the inverse function, consider the diagonal map $\Delta : G \to G \times G$ given by $\Delta(g) = (g, g)$. We have that $m \circ (Id \times i) \circ \Delta = e$, then the pull-back $(m \circ (Id \times i) \circ \Delta)^*\omega = 0$ which implies, using the formula of item (a), that

$$Ad\omega + i^*\omega = 0.$$ □

Lemma 2. *Given semimartingales X and Y in G, we have the following Itô formulas:*

a) $\log(XY) = \int Ad(Y^{-1}) \circ d(\log X) + \log Y$.
b) $\log(X^{-1}) = -\int Ad(X) \circ d(\log X)$.

Proof. The first formula follows from the calculation:

$$\begin{aligned}\log(XY) = \int \omega \circ dm(X,Y) &= \int m^*\omega \circ d(X,Y) \\ &= \int \left(Ad^{-1}(\pi_2)\pi_1^*\omega + \pi_2^*\omega\right) \circ d(X,Y) \\ &= \int Ad(Y^{-1}) \circ d\left(\int \omega \circ dX\right) + \int \omega \circ dY \\ &= \int Ad(Y^{-1}) \circ d\log X + \log Y.\end{aligned}$$

For the second formula, apply the identity (a) with $Y = X^{-1}$. □

We have now a direct way to prove the stochastic Campbell–Hausdorff formula (cf. Hakim-Dowek and Lépingle [6]).

Theorem 2. *We have that:*

a) $\epsilon(M+N) = \epsilon\left(\int Ad(\epsilon(N)) \circ dM\right)\epsilon(N)$.
b) $\epsilon(M)^{-1} = \epsilon\left(-\int Ad(\epsilon(M)) \circ dM\right)$.

3 Harmonic mappings

Consider M and N two Riemannian manifolds and let $f : M \to N$ be a C^∞-differentiable map. The key point in stochastic geometry which matters in the question addressed in this article is the following result, due originally to Bismut:

Theorem 3. *A mapping $f : M \to N$ is a harmonic mapping if and only if for all Brownian motion B_t in M, $f(B_t)$ is a ∇^N-martingale in N.*

See, e.g., Emery [4].

Theorem 4 (Main result). *Let $\phi_j : M_j \to G$, $j = 1, 2, \ldots, n$, be harmonic mappings from Riemannian manifolds M_j to a Lie group G (with respect to the connection ∇^L). Then the product $\phi_1\phi_2\cdots\phi_n$ is a harmonic mapping between $M_1 \times M_2 \times \cdots \times M_n$ and G.*

Proof. It is enough to take $n = 2$. Consider $f_1 : M_1 \to G$ and $f_2 : M_2 \to G$ two harmonic mappings. Let B^1 and B^2 be independent Brownian motions in M_1 and M_2, respectively. Then (B^1, B^2) is a Brownian motion in the product space $M_1 \times M_2$. We have to prove that the product $f_1(B^1)f_2(B^2)$ is a martingale in the group G. By Theorem 1 this product is a martingale if and only if its logarithm is a local martingale.

By the Itô formula (a) of Lemma 2 we have that:

$$\log(f_1(B^1)f_2(B^2)) = \int Ad(f_2(B^2)^{-1}) \circ d(\log f_1(B^1)) + \log f_2(B^2). \tag{1}$$

By hypothesis, the integrator $\log f_1(B^1)$ and the last term $\log f_2(B^2)$ are local martingales. Moreover, the Stratonovich integral reduces to an Itô integral, since the correction term vanishes by independence of the Brownian motions. Hence $\log(f_1(B^1)f_2(B^2))$ is a local martingale in the Lie algebra $\mathcal{G}$, hence the product $f_1 \cdot f_2$ is harmonic. □

If the group G is Abelian, the proof above is straightforward since the adjoint is the identity.

Example 1 (Product of geodesics is harmonic). Let G be a Lie group with a bi-invariant metric. Consider $X_1, \ldots, X_n$ elements of the Lie algebra $\mathcal{G}$. Then the map $f : (\mathbb{R}^n, <,>) \to G$ defined by

$$f(t_1, \ldots, t_n) = \exp(t_1 X_1) \cdot \ldots \cdot \exp(t_n X_n)$$

is harmonic. Again, with $n = 2$, given (B_t^1, B_t^2) a Brownian motion on the plane $\mathbb{R}^2$, then:

$$\log\left(\exp(B_t^1 X_1)\exp(B_t^2 X_2)\right) = \int_0^t Ad\left((\exp(B_s^1 X_1))^{-1}\right) \circ d(B_s^2 X_2) + B_t^2 X_2,$$

where a direct calculation shows that the correction term of the Stratonovich integral is $[X_2, X_1]\, d[B_t^1, B_t^2] = 0$. Note that this example also holds for general geodesics (not only starting at the identity).

A corollary of the proof of the theorem shows a partial converse of the theorem.

Corollary 1. *Let $f_1 : M \to G$ and $f_2 : N \to G$ be two C^∞-differentiable map. If the product $f_1 \cdot f_2$ is harmonic and one of the two mappings, f_1 or f_2, is harmonic, then the other map is also harmonic.*

Proof. The proof follows from (1), where the Stratonovich integral reduces to an Itô integral. The left-hand side is a local martingale by hypothesis and Theorem 1. If f_1 is harmonic then the integrator is a local martingale, hence $\log f_2(B^2)$ is also a local martingale and f_2 is harmonic.

On the other hand, if f_2 is harmonic, then the (Itô) integral is a local martingale. By Doob-Meyer decomposition, and the fact that the adjoint Ad is an isomorphism it follows that $\log(f_1(B^1))$ is a local martingale, hence f_1 is harmonic. □

The factorization result of the corollary cannot be improved: a harmonic product may not be product of harmonic components. Consider, for example, the harmonic function in the Abelian group $(\mathbb{R}^2, +)$ given by $f(x, y) = x^2 - y^2$.

Example 2 (Invariance by geodesic translations). Let $f : M \to G$ be a C^∞-differentiable map, and let $X_1, \ldots, X_n$ be elements of the Lie algebra $\mathcal{G}$. The map f is harmonic if and only if $\exp(t_1X_1)\cdot\ldots\cdot\exp(t_nX_n)\cdot f : \mathbb{R}^n \times M \to G$ is harmonic. Yet, f is harmonic if and only if $f \cdot \exp(t_1X_1) \cdot \ldots \cdot \exp(t_nX_n) : M \times \mathbb{R}^n \to G$ is harmonic.

Finally, we remark that if the group G is endowed with a bi-invariant metric then the inverse $i : g \mapsto g^{-1}$ is an isometry, hence a harmonic morphism. Therefore, a map $f : M \to G$ is harmonic if and only if f^{-1} is also harmonic.

References

1. Calabi, E. Minimal immersions of surfaces in Euclidean spheres. *J. Diff. Geom.*, **1**, 111–125, 1967
2. Cheeger, J. and Ebin, D. *Comparison theorems in Riemannian geometry.* North-Holland, Amsterdam, 1975
3. Eells, J. and Lemaire, L. *Selected topics in harmonic maps.* CBMS. Regional Conference Series in Math. Vol. **50**. American Mathematical Society, Providence, RI, 1983
4. Emery, M. *Stochastic calculus in manifolds.* Springer, Berlin Heidelberg New York, 1990
5. Furstenberg, H. A Poisson formula for semi-simple Lie groups. *Ann. Math.*, **77**, 335–386, 1963
6. Hakim-Dowek, M. and Lépingle, D. L'exponentielle stochastique des groupes de Lie. Séminaire de Probabilités **XX**, *Lecture Notes in Mathematics*, **1059**, 352–374. Springer, Berlin Heidelberg New York, 1986
7. Ikeda, N. and Manabe, S. *Integral of differential forms along the path of diffusion processes.* Publ. R.I.M.S., **15**, 827–852, 1979
8. Kobayashi, S. and Nomizu, K. *Foundations of differential geometry.* Vol. **1**. Interscience, New York, 1969
9. Meyer, P.A. Géométrie Stochastique sans Larmes. Séminaire de Probabilités **XVI**, *Lecture Notes in Mathematics*, **850**, 44–102. Springer, Berlin Heidelberg New York, 1981
10. Uhlenbeck, K. Harmonic maps into Lie groups (Classical solution of the chiral model). *J. Diff. Geom.*, **30**, 1–50, 1989
11. Urakawa, H. *Calculus of variations and harmonic maps.* Translations on Mathematical Monographs AMS, Vol. **132**, 1993

More Hypercontractive Bounds for Deformed Orthogonal Polynomial Ensembles

Michel Ledoux

Institute de Mathématiques, Université Paul Sabatier
31062 Toulouse, France
e-mail: ledoux@math.ups-tlse.fr

Summary. We illustrate one further use of hypercontractivity to non-asymptotic small deviation inequalities on the largest eigenvalue of non-null Wishart matrices and deformed orthogonal polynomial ensembles.

The study of the asymptotic properties of the largest eigenvalues of random matrices gave recently rise to a number of important developments. To illustrate one typical example in the context of sample covariance matrices, let $G^N = G$ be an $N \times N$ random matrix whose entries are independent standard complex Gaussian random variables, and denote by $X^N = X = GG^*$ the so-called Wishart matrix (with covariance the identity matrix). Both the global and local asymptotic regimes of the (real non-negative) eigenvalues $\lambda_1^N, \ldots, \lambda_N^N$ of $(1/4N)X^N$ as $N \to \infty$ have been carefully examined. In particular, the empirical measure on the eigenvalues is known to converge to the square of the semicircle law (the Marchenko-Pastur distribution) with support $[0,1]$, and the largest eigenvalue $\lambda_{\max}^N$ to converge almost surely to the right-hand side of the support (cf. e.g. [1] and the references therein, covering non-Gaussian entries). Fluctuations of $\lambda_{\max}^N$ have been described recently in [9, 10] where it is shown in particular that

$$\lim_{N\to\infty} \mathbb{P}\big(4^{-1/3}N^{2/3}[\lambda_{\max}^N - 1] \leqslant t\big) = F(t), \quad t \in \mathbb{R},$$

where F is the Tracy-Widom distribution. The result holds more generally for rectangular matrices G (provided the ratio between rows and columns is suitably controlled). Such a remarkable result at the unusual $N^{2/3}$ fluctuation rate was proven first for the Gaussian Unitary Ensemble by Forrester [7] and Tracy and Widom [15], and is part of the theory of orthogonal polynomial ensembles (cf. [5, 9, 11]).

In [12], it was observed that hypercontractive methods may be used to produce simple non-asymptotic bounds on the largest eigenvalue at the Tracy-Widom rate in the form of the (upper) small deviation inequality

$$\mathbb{P}\big\{\lambda^N_{\max} \geqslant 1+\varepsilon\big\} \leqslant C\,\varepsilon^{-1/2}\,\mathrm{e}^{-cN\varepsilon^{3/2}} \tag{1}$$

for numerical constants C, $c > 0$ and all $0 < \varepsilon \leqslant 1$ and $N \geqslant 1$.

Motivated by questions by Johnstone on the statistical analysis of large eigenvalues, Wishart matrices with arbitrary covariance matrices have been recently investigated by Baik, Ben Arous and Péché [4] who detected a striking phase transition in the asymptotic regime of the largest eigenvalues. As one simple instance, let $\Sigma = \mathrm{diag}\,(\alpha, 1, \ldots, 1)$, $\alpha > 0$, be a diagonal covariance matrix, and let now $X^N = X = G\Sigma G^*$. Then, it was shown in [4] that whenever $\alpha < 2$, the behavior of the largest eigenvalue $\lambda^N_{\max}$ of $(1/4N)X^N$ is similar to the null case $\Sigma = \mathrm{Id}$, whereas when $\alpha > 2$, $\lambda^N_{\max}$ jumps outside the support of the limiting eigenvalue distribution (still the square of the semicircle law) and fluctuates normally around its limiting value $\alpha^2/4(\alpha-1)$. At the critical value $\alpha = 2$, the fluctuation regime $N^{2/3}$ is still in force, but with a different limiting distribution (of the Tracy-Widom type). More general results for rectangular matrices G and a finite number of values different from 1 in Σ are actually described in [4].

The point of this note is to observe that hypercontractive tools may again be used in this context to obtain simple non-asymptotic upper bounds on the probability that the largest eigenvalue exceeds its limiting value. No delicate asymptotics of contour integrals or orthogonal polynomials are required (as it is the case in the proofs of the asymptotic results). The method moreover gives a hint on the critical value of the phase transition. For simplicity, we will restrict ourselves in this note to the preceding simple model $X = G\Sigma G^*$ with $\Sigma = \mathrm{diag}\,(\alpha, 1, \ldots, 1)$.

The starting point of the investigation is the joint eigenvalue distribution on $\mathbb{R}^N$ of the Wishart matrix $X = G\Sigma G^*$ given by the determinantal representation [4, 8]

$$Z^{-1}\Delta_N(x)\det\big(\phi_j(x_i)\big)\prod_{i=1}^N \mu(\mathrm{d}x_i), \tag{2}$$

where $\Delta_N(x)$ is the Vandermonde determinant, $\phi_j(x) = x^{j-1}$, $j \leqslant N-1$, $\phi_N(x) = \mathrm{e}^{\beta x}$, $\beta = 1-(1/\alpha) < 1$, and μ is the exponential law with parameter 1. As in the classical case [13], the analysis of this model is made possible by using the underlying orthogonal polynomials for μ, in this case the Laguerre polynomials P_k, $k \in \mathbb{N}$. In particular, the mean empirical measure $m^N = \mathbb{E}[(1/N)\sum_{i=1}^N \delta_{\lambda_i^N}]$ on the eigenvalues $\lambda_1^N, \ldots, \lambda_N^N$ of $(1/4N)X^N$ may then be described as

$$\int f\,\mathrm{d}m^N = \int f\left(\frac{x}{4N}\right)p_N(x)\,\mu(\mathrm{d}x)$$

where

$$p_N(x) = \left(1 - \frac{1}{N}\right)\frac{1}{N-1}\sum_{k=0}^{N-2} P_k^2(x)$$
$$+\frac{1}{N}\,\langle P_{N-1}\phi_N\rangle^{-1} P_{N-1}(x)\bigg(\phi_N(x) - \sum_{k=0}^{N-2}\langle P_k\phi_N\rangle P_k(x)\bigg).$$

Here $\langle\,\cdot\,\rangle$ denotes integration with respect to μ and the P_k's are assumed to be normalized in $\mathrm{L}^2(\mu)$. (In the null case $\Sigma = \mathrm{Id}$, replace ϕ_N by P_{N-1}.) Note that by the generating series of the Laguerre polynomials, for every k,

$$\langle P_k\phi_N\rangle = \frac{1}{1-\beta}\left(\frac{\beta}{\beta-1}\right)^k. \tag{3}$$

Now, for every $t \geqslant 0$,

$$\mathbb{P}\big\{\lambda_{\max}^N \geqslant t\big\} \leqslant N m^N\big([\,t,\infty)\big) = N\int_{4Nt}^{\infty} p_N(x)\,\mu(\mathrm{d}x).$$

Using hypercontractivity of the Laguerre operator, it was shown in [12] that, for every $N \geqslant 1$ and every $0 < \varepsilon \leqslant 1$,

$$\int_{4N(1+\varepsilon)}^{\infty}\sum_{k=0}^{N-1} P_k^2(x)\,\mu(\mathrm{d}x) \leqslant C\,\varepsilon^{-1/2}\,\mathrm{e}^{-cN\varepsilon^{3/2}} \tag{4}$$

where $C, c > 0$ are numerical, leading thus to (1). The possible new information on the deformed ensemble (2) has thus to be searched in the second piece of the density p_N. Two cases have to be considered.

If $\beta < 1/2$ ($\alpha < 2$), then $\phi_N = \mathrm{e}^{\beta x} \in \mathrm{L}^2(\mu)$ so that, by (3),

$$\langle P_{N-1}\phi_N\rangle^{-2}\,\bigg\|\phi_N - \sum_{k=0}^{N-2}\langle P_k\phi_N\rangle P_k\bigg\|_2^2$$

is uniformly bounded in N. Together with (4) and the Cauchy-Schwarz inequality, it is then an easy task to conclude that, as in the null case,

$$\mathbb{P}\big\{\lambda_{\max}^N \geqslant 1+\varepsilon\big\} \leqslant C\,\varepsilon^{-1/2}\,\mathrm{e}^{-cN\varepsilon^{3/2}}$$

for $N \geqslant 1$, $0 < \varepsilon \leqslant 1$ and constants $C, c > 0$ depending on α only.

If $\beta > 1/2$ ($\alpha > 2$), by (3) again

$$\langle P_{N-1}\phi_N\rangle^{-2}\,\bigg\|\sum_{k=0}^{N-2}\langle P_k\phi_N\rangle P_k\bigg\|_2^2$$

is uniformly bounded in N, so that the corresponding contribution of the density p_N is handled as in the preceding case by (4). With slightly more efforts, this is also the case when $\beta = 1/2$ ($\alpha = 2$). Therefore, the main contribution of p_N is concentrated in the term

$$\langle P_{N-1}\phi_N\rangle^{-1}\int_{4Nt}^{\infty} P_{N-1}\phi_N\,\mu(\mathrm{d}x).$$

By (3) and Hölder's inequality, for every $p > 1$ and $(1/p) + (1/q) = 1$ such that $\beta q < 1$, the (absolute value of the) latter is bounded above by

$$(1-\beta)(1-\beta q)^{-1/q}\left(\frac{1-\beta}{\beta}\right)^{N-1}\|P_{N-1}\|_p\,\mathrm{e}^{-4Nt(1-\beta q)/q}.$$

Here is the place where hypercontractivity comes again into play. As in [12], since P_{N-1} is an eigenfunction of the Laguerre operator with eigenvalue $N-1$, hypercontractivity of the Laguerre operator shows that for every $p > 2$,

$$\|P_{N-1}\|_p \leqslant (p-1)^{N-1}.$$

Therefore,

$$\begin{aligned}&\left(\frac{1-\beta}{\beta}\right)^{N-1}\|P_{N-1}\|_p\,\mathrm{e}^{-4Nt(1-\beta q)/q}\\ &\qquad\leqslant \exp\big((N-1)\log\big[(p-1)(1-\beta)/\beta\big] - 4Nt(1-\beta q)/q\big).\end{aligned}$$

Set $(p-1)(1-\beta)/\beta = 1+\delta$, $\delta > 0$, and optimize in $\delta \to 0$. Since

$$\frac{1-\beta q}{q} = \frac{\beta(1-\beta)\delta}{1+\beta\delta}\,,$$

it first appears that the critical value t_0 of t is given by

$$t_0 = \frac{1}{4\beta(1-\beta)} = \frac{\alpha^2}{4(\alpha-1)}\,.$$

This value is the phase transition identified in [4]. It then appears that when $\beta > 1/2$, i.e. $\alpha > 2$, the asymptotics is of the order of δ^2. However, when $\beta = 1/2$, i.e. $\alpha = 2$, the Taylor expansion has to be taken at the third order. It follows more precisely that, for every $N \geqslant 1$, $0 < \varepsilon \leqslant 1$, and some constants $C, c > 0$ only depending on α,

$$\langle P_{N-1}\phi_N\rangle^{-1}\int_{4N(t_0+\varepsilon)}^{\infty} P_{N-1}\phi_N\,\mu(\mathrm{d}x) \leqslant C\,\varepsilon^{-\beta}\,\mathrm{e}^{-cN\varepsilon^2}$$

when $\alpha > 2$, whereas

$$\langle P_{N-1}\phi_N\rangle^{-1}\int_{4N(t_0+\varepsilon)}^{\infty} P_{N-1}\phi_N\,\mu(\mathrm{d}x) \leqslant C\,\varepsilon^{-1/4}\,\mathrm{e}^{-cN\varepsilon^{3/2}}$$

when $\alpha = 2$. We thus conclude to the following small deviation bound on the largest eigenvalue $\lambda_{\max}^N$ for fixed N. (The polynomial factors are technical and probably not necessary, see [12].)

Proposition. *Let G be a standard complex Gaussian $N \times N$ random matrix, and let $\Sigma = \mathrm{diag}\,(\alpha, 1, \ldots, 1)$, $\alpha > 0$. Denote by $\lambda_{\max}^N$ the largest eigenvalue of $(1/4N)G\Sigma G^*$. Set $t_0(\alpha) = 1$ if $\alpha \leqslant 2$ and $t_0(\alpha) = \alpha^2/4(\alpha - 1)$ if $\alpha \geqslant 2$. Set also $\gamma = 1/\max(\alpha, 2)$. Then, for every $N \geqslant 1$, $0 < \varepsilon \leqslant 1$, and constants $C, c > 0$ depending on α only,*

$$\mathbb{P}\big\{\lambda_{\max}^N \geqslant t_0 + \varepsilon\big\} \leqslant C\,\varepsilon^{\gamma-1}\,\mathrm{e}^{-cN\varepsilon^{\kappa}}$$

where $\kappa = 2$ if $\alpha > 2$ and $\kappa = 3/2$ if $\alpha \leqslant 2$.

This dichotomy is in accordance with the results of [4].

The preceding could be extended to more general non-null sample covariance matrices with a bounded control on the ratio on rows and columns and a finite number of non-units on the diagonal Σ as studied in [4]. As explained also in [4], the preceding estimates may be interpreted as bounds on a last passage percolation model for exponential random variables. The hypercontractive method may be developed similarly for more general deformed orthogonal polynomial ensembles of the type (2) as obtained by the introduction of an (additive or multiplicative) external source in a unitary invariant ensemble (see [2]). At the fluctuation level, the Hermite ensemble (corresponding to the choice of a Gaussian distribution μ in (2)) is studied in [14] where similar phase transitions are observed. Hypercontractivity of the Hermite operator may be used as above to produce non-asymptotic bounds for these deformed Gaussian Unitary Ensembles. Applications to the Jacobi ensembles are probably also possible. Discrete ensembles might require moment methods to reach similar conclusions. Jumps of the largest eigenvalues outside the spectrum for non-Gaussian sample covariance matrices is analyzed in the recent contribution [3] by means of the Stieltjes transform, and for deformed Wigner matrices in [6]. It would be of interest to quantify these asymptotics results.

References

1. Z. Bai. Methodologies in spectral analysis of large-dimensional random matrices, a review. Statist. Sinica 9, pp. 611–677 (1999)
2. J. Baik, E.M. Rains. Limiting distributions for a polynuclear growth model with external sources. J. Statist. Phys. 100, pp. 523–541 (2000)
3. J. Baik, J.W. Silverstein. Eigenvalues of large sample covariance matrices of spiked population models. J. Multivariate Anal. 97, pp. 1382–1408 (2006)
4. J. Baik, G. Ben Arous, S. Péché. Phase transition of the largest eigenvalue for non-null complex sample covariance matrices. Ann. Probab. 33, pp. 1643–1697 (2005)

5. P.A. Deift. Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. CIMS Lecture Notes 3. Courant Institute of Mathematical Sciences (1999)
6. D. Feral. On the extremal eigenvalues of large deformed Wigner matrices (2005)
7. P.J. Forrester. The spectrum edge of random matrix ensembles. Nuclear Phys. B 402, pp. 709–728 (1993)
8. A. James. Distributions of matrix variates and latent roots derived from normal samples. Ann. Math. Statist. 35, pp. 475–501 (1964)
9. K. Johansson. Shape fluctuations and random matrices. Comm. Math. Phys. 209, pp. 437–476 (2000)
10. I.M. Johnstone. On the distribution of the largest principal component. Ann. Statist. 29, pp. 295–327 (2001)
11. W. König. Orthogonal polynomial ensembles in probability theory. Probability Surveys 2, pp. 385–447 (2005)
12. M. Ledoux. A remark on hypercontractivity and tail inequalities for the largest eigenvalues of random matrices. Séminaire de Probabilités XXXVII. Lecture Notes in Math. 1832, pp. 360–369 (2003). Springer, Berlin Heidelberg New York
13. M.L. Mehta. Random matrices. (1991). Academic Press, New York
14. S. Péché. The largest eigenvalues of Hermitian random matrices with non centered entries (2004)
15. C. Tracy, H. Widom. Level-spacing distribution and Airy kernel. Comm. Math. Phys. 159, pp. 151–174 (1994)

No Multiple Collisions for Mutually Repelling Brownian Particles

Emmanuel Cépa[1] and Dominique Lépingle[2]

MAPMO, Université d'Orléans
B.P.6759, 45067 Orléans Cedex 2, France
[1] *e-mail: Emmanuel.Cepa@univ-orleans.fr*
[2] *e-mail: dlepingl@univ-orleans.fr*

Summary. Although Brownian particles with small mutual electrostatic repulsion may collide, multiple collisions at positive time are always forbidden.

1 Introduction

A three-dimensional Brownian motion $B_t = (B_t^1, B_t^2, B_t^3)$ does not hit the axis $\{x_1 = x_2 = x_3\}$ except possibly at time 0. An easy proof is obtained by applying Ito's formula to $R_t = [(B_t^1 - B_t^2)^2 + (B_t^1 - B_t^3)^2 + (B_t^2 - B_t^3)^2]$ and remarking that up to the multiplicative constant 3 the process R is the square of a two-dimensional Bessel process for which $\{0\}$ is a polar state. This remark will be our guiding line in the sequel.

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ and for $N \geqslant 3$ the following system of stochastic differential equations

$$dX_t^i \;=\; dB_t^i \;+\; \lambda \sum_{1\leqslant j\neq i\leqslant N} \frac{dt}{X_t^i - X_t^j}\,,\; i = 1, 2, \ldots, N$$

with boundary conditions

$$X_t^1 \leqslant X_t^2 \leqslant \cdots \leqslant X_t^N\,, \quad 0 \leqslant t < \infty\,,$$

and a random, $\mathcal{F}_0$-measurable, initial value satisfying

$$X_0^1 \leqslant X_0^2 \leqslant \cdots \leqslant X_0^N\,.$$

Here $B_t = (B_t^1, B_t^2, \ldots, B_t^N)$ denotes a standard N-dimensional $(\mathcal{F}_t)$-Brownian motion and λ is a positive constant. This system has been extensively studied in [5], [7], [2], [1], [3], [6]. For comments on the relationship between this system and the spectral analysis of Brownian matrices, and also conditioning of Brownian particles, we refer to the introduction and the bibliography in [3].

When $\lambda \geqslant \frac{1}{2}$, establishing strong existence and uniqueness is not difficult, because particles never collide, as proved in [7]. The general case with arbitrary coupling strength is investigated in [2] and it is proved in [3] that collisions occur a.s. if and only if $0 < \lambda < \frac{1}{2}$. As for multiple collisions (three or more particles at the same location), it has been stated without proof in [9] and [4] that they are impossible. The proof we give below, with a Bessel process unexpectedly coming in, is just an exercise on Ito's formula.

2 A remarkable identity in law

We consider for any $t \geqslant 0$

$$S_t = \sum_{j=1}^{N}\sum_{k=1}^{N}(X_t^j - X_t^k)^2 \,.$$

Theorem 1. *For any $\lambda > 0$, the process S divided by the constant $2N$ is the square of a Bessel process with dimension $(N-1)(\lambda N+1)$.*

Proof. It is purely computational. Ito's formula provides for any $j \neq k$

$$\begin{aligned}(X_t^j - X_t^k)^2 = (X_0^j - X_0^k)^2 &+ 2\int_0^t (X_s^j - X_s^k)\, d(B_s^j - B_s^k) \\ &+ 2\lambda \sum_{1\leqslant l\neq j\leqslant N} \int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l}\, ds \\ &+ 2\lambda \sum_{1\leqslant m\neq k\leqslant N} \int_0^t \frac{X_s^k - X_s^j}{X_s^k - X_s^m}\, ds + 2\,t\,.\end{aligned}$$

Adding the $N(N-1)$ equalities we get

$$\begin{aligned}S_t = S_0 &+ 2\sum_{j=1}^{N}\sum_{k=1}^{N}\int_0^t (X_s^j - X_s^k)\, d(B_s^j - B_s^k) \\ &+ 4\lambda \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant l\neq j\leqslant N}\int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l}\, ds \;+\; 2N(N-1)t\,.\end{aligned}$$

But

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant l\neq j\leqslant N}\int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l}\, ds \\ &= \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant j\neq l\leqslant N}\left[\int_0^t \frac{X_s^j - X_s^l}{X_s^j - X_s^l}\, ds + \int_0^t \frac{X_s^l - X_s^k}{X_s^j - X_s^l}\, ds\right] \\ &= N^2(N-1)t - \sum_{l=1}^{N}\sum_{k=1}^{N}\sum_{1\leqslant l\neq j\leqslant N}\int_0^t \frac{X_s^l - X_s^k}{X_s^l - X_s^j}\, ds \\ &= \frac{1}{2}N^2(N-1)t\,.\end{aligned}$$

For the martingale term, we compute

$$\begin{aligned}
&\sum_{j=1}^{N}\left(\sum_{k=1}^{N}(X_s^j - X_s^k)\right)^2 \\
&\quad = \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}(X_s^j - X_s^k)(X_s^j - X_s^l) \\
&\quad = \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}(X_s^j - X_s^k)^2 \;+\; \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}(X_s^j - X_s^k)(X_s^k - X_s^l) \\
&\quad = \frac{N}{2} S_s \,.
\end{aligned}$$

Let B' be a linear Brownian motion independent of B. The process C defined by:

$$C_t \;=\; \int_0^t \mathbf{I}_{\{S_s>0\}} \, \frac{\displaystyle\sum_{j=1}^{N}\sum_{k=1}^{N}(X_s^j - X_s^k)\, dB_s^j}{\sqrt{\frac{N}{2} S_s}} \;+\; \int_0^t \mathbf{I}_{\{S_s=0\}} dB'_s$$

is a linear Brownian motion and we have

$$S_t \;=\; S_0 \;+\; 2\int_0^t \sqrt{2NS_s}\, dC_s \;+\; 2N(N-1)(\lambda N+1)t\,,$$

which completes the proof. □

3 Multiple collisions are not allowed

Since multiple collisions do not occur for Brownian particles without interaction, we can guess they do not either in case of mutual repulsion. Here is the proof.

Theorem 2. *For any $\lambda > 0$, multiple collisions cannot occur after time* 0.

Proof. i) For $3 \leqslant r \leqslant N$ and $1 \leqslant q \leqslant N-r+1$, let

$$I = \{q, q+1, \ldots, q+r-1\}$$

$$S_t^I = \sum_{j\in I}\sum_{k\in I}(X_t^j - X_t^k)^2$$

$$\tau^I = \inf\{t > 0 : S_t^I = 0\}\ .$$

ii) We first consider the initial condition X_0. From [2], Lemma 3.5, we know that for any $1 \leqslant i < j \leqslant N$ and any $t < \infty$, we have a.s.

$$\int_0^t \frac{du}{X_u^j - X_u^i} \; < \; \infty \, .$$

Therefore for any $u > 0$ there exists $0 < v < u$ such that $X_v^1 < X_v^2 < \cdots < X_v^N$ a.s. In order to prove $\mathbb{P}(\tau^I = \infty) = 1$, we may thus assume $X_0^1 < X_0^2 < \cdots < X_0^N$ a.s., which implies for any I that $S_0^I > 0$ and so $\tau^I > 0$ a.s.

iii) We know ([8], XI, section 1) that $\{0\}$ is polar for the Bessel process $\sqrt{S_t}/\sqrt{2N}$, which means that $\tau^I = \infty$ a.s. for $I = \{1, 2, \dots, N\}$. We will prove the same result for any I by backward induction on $r = \mathrm{card}(I)$. Assume there are no s-multiple collisions for any $s > r$. Then

$$\begin{aligned} S_t^I = S_0^I \; &+ \; 4 \sum_{j \in I} \sum_{k \in I} \int_0^t (X_s^j - X_s^k) \, dB_s^j \\ &+ 4\lambda \sum_{j \in I} \sum_{k \in I} \sum_{l \notin I} \int_0^t \frac{X_s^j - X_s^k}{X_s^j - X_s^l} \, ds \; + \; 2r(r-1)(\lambda r + 1)t \, . \end{aligned}$$

We set for $n \in \mathbb{N}^*$, $\tau_n^I \; = \; \inf\{t > 0 : S_t^I \leqslant 1/n\}$. For any $t \geqslant 0$,

$$\begin{aligned} \log S_{t\wedge\tau_n^I}^I = \log S_0^I \; &+ \; 4 \sum_{j \in I} \sum_{k \in I} \int_0^{t\wedge\tau_n^I} \frac{X_s^j - X_s^k}{S_s^I} \, dB_s^j \\ &+ 2\lambda \sum_{j \in I} \sum_{k \in I} \sum_{l \notin I} \int_0^{t\wedge\tau_n^I} \frac{(X_s^j - X_s^k)}{S_s^I} \left[\frac{1}{X_s^j - X_s^l} - \frac{1}{X_s^k - X_s^l} \right] ds \\ &+ 2r[(r-1)(\lambda r + 1) - 2] \int_0^{t\wedge\tau_n^I} \frac{ds}{S_s^I} > -\infty \, . \end{aligned}$$

From the induction hypothesis we deduce that for $j, k \in I$ and $l \notin I$, a.s. on $\{\tau^I < \infty\}$, $(X_{\tau^I}^j - X_{\tau^I}^l)(X_{\tau^I}^k - X_{\tau^I}^l) \; > \; 0$ and so

$$\begin{aligned} &\int_0^{t\wedge\tau^I} \frac{(X_s^j - X_s^k)}{S_s^I} \left[\frac{1}{X_s^j - X_s^l} - \frac{1}{X_s^k - X_s^l} \right] ds \\ &\quad = \; - \int_0^{t\wedge\tau^I} \frac{(X_s^j - X_s^k)^2}{S_s^I} \frac{ds}{(X_s^j - X_s^l)(X_s^k - X_s^l)} > -\infty \, . \end{aligned}$$

The martingale $(M_n, \mathcal{F}_{t\wedge\tau_n^I})_{n \geqslant 1}$ defined by

$$M_n \; = \; 4 \sum_{j \in I} \sum_{k \in I} \int_0^{t\wedge\tau_n^I} \frac{X_s^j - X_s^k}{S_s^I} \, dB_s^j$$

has associated increasing process $A_n = 8r \int_0^{t\wedge\tau_n^I} \frac{ds}{S_s^I}$. It follows that $M_n + \frac{1}{4}[(r-1)(\lambda r+1)-2]A_n$ either tends to a finite limit or to $+\infty$ as n tends to $+\infty$. Then for any $t \geqslant 0$, $\log S_{t\wedge\tau^I}^I > -\infty$ and so $\mathbb{P}(\tau^I = \infty) = 1$, which completes the proof. □

4 Brownian particles on the circle

We now turn to the popular model of interacting Brownian particles on the circle ([9], [3]). Consider the system of stochastic differential equations

$$dX_t^i \;=\; dB_t^i \;+\; \frac{\lambda}{2} \sum_{1\leqslant j\neq i\leqslant N} \cot\left(\frac{X_t^i - X_t^j}{2}\right) dt\,,\; i = 1,2,\ldots,N$$

with the boundary conditions

$$X_t^1 \leqslant X_t^2 \leqslant \cdots \leqslant X_t^N \leqslant X_t^1 \;+\; 2\pi\,, \qquad 0 \leqslant t < \infty\,.$$

As expected we can prove there are no multiple collisions for the particles $Z_t^j \;=\; e^{i\,X_t^j}$ that live on the unit circle. The proof is more involved and will be deduced by approximation from the previous one.

Theorem 3. *Multiple collisions for the particles on the circle do not occur after time* 0 *for any* $\lambda > 0$.

Sketch of the proof. For the sake of simplicity, we only deal with the N-collisions. Let

$$R_t = \sum_{j=1}^{N}\sum_{k=1}^{N} \sin^2\left(\frac{X_t^j - X_t^k}{2}\right)$$

$$\sigma_n = \inf\{t > 0 \;:\; R_t \leqslant \frac{1}{n}\}\;.$$

We apply Ito's formula to log R_t and get

$$\log\, R_{t\wedge\sigma_n} \;=\; \log\, R_0 \;+ \sum_{j=1}^{N} \int_0^{t\wedge\sigma_n} H_s^j\, dB_s^j \;+\; \int_0^{t\wedge\sigma_n} L_s\, ds$$

for some continuous processes H^j and L. We divide each integral into an integral over $\{R_s \geqslant \frac{1}{2}\}$ and an integral over $\{R_s < \frac{1}{2}\}$. The first type integrals do not pose any problem. When $R_s < \frac{1}{2}$, we replace X_s^j with

$$Y_s^j \;=\; X_s^j \qquad \text{or } Y_s^j \;=\; X_s^j \;-\; 2\pi$$

in such a way that for any j, k we have $|Y_s^j - Y_s^k| < \pi/3$. The processes H^j and L have the same expressions in terms of X or Y. With this change of variables we may approximate $\sin x$ by x, $\cos x$ by 1 and replace the trigonometric functions by approximations of the linear ones which we have met in the previous sections. We obtain that

$$\log R_{t\wedge\sigma_n} = \log R_0 + M_n + \frac{1}{4}[(N-1)(\lambda N+1)-2]A_n + \int_0^{t\wedge\sigma_n} D_s\, ds$$

where M_n is a martingale with associated increasing process A_n and D is a.s. a locally integrable process. Details are left to the reader as well as the case of an arbitrary subset I like those in Section 3.

References

1. Bonami A., Bouchut F., Cépa E., Lépingle D. *A non linear SDE involving Hilbert transform.* J. Funct. Anal. 165, 390–406, 1999
2. Cépa E., Lépingle D. *Diffusing particles with electrostatic repulsion.* Prob. Theory and Relat. Fields, 107, 429–449, 1997
3. Cépa E., Lépingle D. *Brownian particles with electrostatic repulsion on the circle: Dyson's model for unitary random matrices revisited.* Esaim Prob. Stat. 5, 203–224, 2001
4. Cépa E., Lépingle D. *Interacting Brownian particles with strong repulsion.* Monte Carlo Methods and Applications, 7, 93–98, 2001
5. Chan T. *The Wigner semi-circle law and eigenvalues of matrix-valued diffusions.* Prob. Theory Relat. Fields, 93, 249–272, 1992
6. Fontbona J. *Nonlinear martingale problems involving singular integrals.* J. Funct. Anal. 200, 198–236, 2003
7. Rogers L.C.G., Shi Z. *Interacting Brownian particles and the Wigner law.* Prob. Theory Relat. Fields, 95, 555–570, 1993
8. Revuz D., Yor M. *Continuous martingales and Brownian motion.* Springer, Berlin Heidelberg New York, vol. 293, 1991
9. Spohn H. *Dyson's model of interacting Brownian motions at arbitrary coupling strength.* Markov Proc. Relat. Fields, 4, 649–661, 1998

On the Joint Law of the L^1 and L^2 Norms of a 3-Dimensional Bessel Bridge

Larbi Alili[1] **and Pierre Patie**[2]*

[1] Department of Statistics, University of Warwick
Coventry CV4 7AL, United Kingdom
e-mail: l.alili@warwick.ac.uk
[2] RiskLab, Department of Mathematics, ETH-Zürich
8092 Zürich, Switzerland
e-mail: patie@math.ethz.ch

Summary. We give an analytical expression for the joint Laplace transform of the L^1 and L^2 norms of a 3-dimensional Bessel bridge. We derive the results by using merely probabilistic arguments. More precisely we show that the law of this functional is closely connected with the one of the first passage time of an Ornstein–Uhlenbeck process. The motivation for studying this problem are multiple; as an instance, they include the computation of the density of the first passage time of Brownian motion over some moving boundaries such as the square root and the quadratic ones.

Key words: Bessel bridges, Ornstein–Uhlenbeck process, Williams' time-reversal theorem, Feynman–Kac formula, Cylinder parabolic function, Boundary crossing

1 Introduction

Let $(r_s, s \leqslant t)$ be a 3-dimensional Bessel bridge over the interval $[0,t]$ between x and y, where x, y are some positive real numbers and t is a fixed time horizon. Introduce the couple of random variables

$$\left(N_t^{(1)}(r), N_t^{(2)}(r)\right) = \left(\int_0^t r_s ds, \int_0^t r_s^2 \, ds\right). \tag{1}$$

In this paper, we aim to compute explicitly its joint Laplace transform. Let $(W_t, t \geqslant 0)$ be a standard real-valued Brownian motion started at $x \in \mathbb{R}$ and set $H_a^{(\lambda)} = \inf\left\{s \geqslant 0;\ W_s = a\sqrt{1+2\lambda s}\right\}$, where $\lambda > 0$ and $a \in \mathbb{R}$. Doob's transform allows to relate $H_a^{(\lambda)}$ to the hitting time of the same

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level a by an Ornstein–Uhlenbeck process with parameter λ. That is, with $\sigma_a = \inf\{s \geqslant 0;\, U_s = a\}$ and

$$U_t = e^{-\lambda t}\left(x + \int_0^t e^{\lambda s}\, dB_s\right), \quad t \geqslant 0, \tag{2}$$

where B is another real-valued Brownian motion defined on the same probability space, we have $H_a = \frac{1}{2\lambda}\log(1 + 2\lambda\sigma_a)$ almost surely. We shall see that the determination of the distribution of σ_a, or equivalently that of H_a, amounts to the study of the joint distribution of the L^1 and L^2 norms of a 3-dimensional Bessel bridge. While we are interested in the joint law, we mention that there is a substantial literature devoted to the study of the law of the L^1 norm of the Brownian excursion, that is, when $x = y = 0$, see e.g. [18], [9], [21] and [12]. The L^2 norm of the Bessel bridge, which is closely related to the Lévy stochastic area formula, has also been intensively studied by many authors including for instance [22], [6] and the references therein.

Then, we establish a relation between the first passage times of Brownian motion over a large class of (smooth) curves and over the linear or quadratic ones. As a by-product, we establish some connections between certain stochastic objects and some special functions. We will show that this device applies to continuous time stochastic processes.

The paper is organized as follows. The next section recalls known facts concerning Bessel and Ornstein–Uhlenbeck processes. In particular, we give a probabilistic construction of the cylinder parabolic function which characterizes the Laplace transform of the first hitting time of fixed level by an Ornstein–Uhlenbeck process. In Section 3, we derive the sought joint law in terms of transforms via stochastic techniques for the case $y = 0$. For $y > 0$, we resort to the Feynman–Kac formula. Then, we establish some relation between stopping times for general stochastic processes which we apply to Brownian motion. This link yields asymptotic results for the parabolic cylinder functions. We end up this paper by making some connections between the studied law and the one of some other functionals.

2 Preliminaries and reminders

Let $(B_t, t \geqslant 0)$ be a 1-dimensional Brownian motion starting from 0. The 3-dimensional Bessel process, denoted by R, is defined as the unique strong solution to

$$dR_t = dB_t + \frac{1}{R_t}dt, \quad R_0 = x \geqslant 0.$$

This is a linear diffusion with speed measure $m(dy) = 2y^2 dy$. Its semi-group is absolutely continuous with respect to m with density

$$q_t(x, y) = \frac{1}{2\sqrt{2\pi t}}\frac{1}{yx}\left(e^{-\frac{1}{2t}(x-y)^2} - e^{-\frac{1}{2t}(x+y)^2}\right), \quad x, y, t > 0$$

and taking the limit as x tends to zero we obtain

$$q_t(0,x) = \frac{1}{\sqrt{2\pi t^3}}\, e^{-\frac{x^2}{2t}}, \quad x,t > 0.$$

We shall denote by $\mathbb{Q}_x$ the law of R when it is started at x and we simply write $\mathbb{Q}$ for $x = 0$. Next, for y and $t \geqslant 0$, the conditional measure $\mathbb{Q}^t_{x,y} = \mathbb{Q}_x[\,.\mid R_t = y]$, viewed as a probability measure on $\mathcal{C}\left([0,t],[0,\infty)\right)$, stands for the law of the 3-dimensional Bessel bridge starting at x and ending at y at time t. Since R is transient, we have $\mathbb{Q}^t_{x,y} = \mathbb{Q}_x[\,.\mid L_y = t]$ where $L_y = \sup\{s \geqslant 0;\ R_s = y\}$. Williams' time reversal relation states that, for $R_0 = 0$, $B_0 = x > 0$, the processes $(R_{L_x - s}, s \leqslant L_x)$ and $(B_s, s \leqslant T_0)$ are equivalent, where $T_0 = \inf\{s \geqslant 0;\ B_s = 0\}$.

We continue by recalling some facts on Ornstein–Uhlenbeck processes (OU processes for short). For an OU process with parameter $\lambda \in \mathbb{R}$, the realization given by (2) is also the unique strong solution to

$$dU_t = dB_t - \lambda U_t dt, \quad U_0 = x \in \mathbb{R}. \tag{3}$$

Denote by $\mathbb{P}^{(\lambda)}_x$ the law of U when $U_0 = x \in \mathbb{R}$ and write simply $\mathbb{P}^{(\lambda)}$ for $\mathbb{P}^{(\lambda)}_0$. By Girsanov's theorem, $\mathbb{P}^{(\lambda)}_x$ is absolutely continuous with respect to the Wiener measure $\mathbb{P}_x$ via

$$d\mathbb{P}^{(\lambda)}_{x|\mathcal{F}_t} = e^{-\frac{\lambda}{2}\left(B_t^2 - x^2 - t\right) - \frac{\lambda^2}{2}\int_0^t B_u^2\, du}\, d\mathbb{P}_{x|\mathcal{F}_t}, \quad t > 0, \tag{4}$$

where $(\mathcal{F}_t)_{t \geqslant 0}$ is the natural filtration of B. Obviously we can write

$$\int_0^t e^{\lambda s}\, dB_s = W_{\tau(t)} \text{ and } \quad W_t = \int_0^{A(t)} e^{\lambda s}\, dB_s, \quad t \geqslant 0,$$

where $\tau(t) = \frac{1}{2\lambda}(e^{2\lambda t} - 1)$, $A(t) = \frac{1}{2\lambda}\log(1 + 2\lambda t)$, and W is a Brownian motion thanks to the Lévy's characterization theorem, see [23]. Hence, Doob's representation

$$U_t = e^{-\lambda t}\left(x + W_{\tau(t)}\right), \quad t \geqslant 0, \tag{5}$$

holds. The relation between the stopping times σ_a and $H^{(\lambda)}_a$, discovered by Breiman [3] and recalled in the introduction, is a straightforward consequence of this fact. The process U is a linear diffusion. Moreover, when $\lambda > 0$, it is positively recurrent and its semi-group has a unique invariant measure which is the law of a centered Gaussian random variable with variance $1/2\lambda$. Next, for a fixed $a \in \mathbb{R}$, introduce the random variable $\sigma_a = \inf\{s \geqslant 0;\ U_s = a\}$. It is a stopping time whose law is absolutely continuous with respect to the Lebesgue measure with a probability density function $p^{(\lambda)}_{x\to a}$, i.e. $\mathbb{P}^{(\lambda)}_x(\sigma_a \in dt) = p^{(\lambda)}_{x\to a}(t)\, dt$. For Brownian motion, recovered when λ tends to 0, we recall that

$$p_{x\to a}(t) = \frac{|a - x|}{\sqrt{2\pi t^3}} e^{-\frac{(a-x)^2}{2t}}.$$

We are now ready to derive the expression of the Laplace transform of σ_a. This is a well-known result which can be found in Breiman [3]. However, we give a proof which relies on probabilistic arguments.

Proposition 1. *For any $x, a \in \mathbb{R}$ and $\beta \geqslant 0$, we have*

$$\mathbb{E}_x\left[e^{-\beta\sigma_a}\right] = \frac{e^{\lambda x^2/2} D_{-\beta/\lambda}(\varepsilon x\sqrt{2\lambda})}{e^{\lambda a^2/2} D_{-\beta/\lambda}(\varepsilon a\sqrt{2\lambda})}, \tag{6}$$

where $\varepsilon = sgn(x-a)$ and D_ν stands for the parabolic cylinder function which admits the following integral representation

$$D_\nu(z) = \frac{2^{\frac{\nu+1}{2}} e^{-z^2/4}}{\Gamma\left(\frac{1-\nu}{2}\right)} \int_0^\infty \left(t^2+z^2\right)^{\nu/2} t^{-\nu} e^{-t^2/2}\, dt, \tag{7}$$

where $Re(\nu) < 1, |arg(z)| < \frac{\pi}{2}$.

Proof. Doob's transformation implies the identity $H_{x\to a} = \tau(\sigma_a)$ almost surely, where $H_{x\to a} = \inf\left\{s \geqslant 0;\, W_s + x = a\sqrt{1+2\lambda s}\right\}$. Specializing on $a=0$ we deduce that $p^{(\lambda)}_{x\to 0}(t) = \tau'(t) p_{x\to 0}(\tau(t))$. Hence, one has

$$p^{(\lambda)}_{x\to 0}(t) = \frac{|x|}{\sqrt{2\pi}} \exp\left(-\frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)} + \frac{\lambda t}{2}\right)\left(\frac{\lambda}{\sinh(\lambda t)}\right)^{3/2}. \tag{8}$$

It follows that

$$\begin{aligned}\mathbb{E}_x\left[e^{-\beta\sigma_0}\right] &= \int_0^\infty e^{-\beta t}\tau'(t) p_{x\to 0}(\tau(t))\, dt \\ &= \frac{|x|}{\sqrt{2\pi}} \int_0^\infty (1+2\lambda t)^{-\beta/2\lambda} t^{-3/2} e^{-x^2/2t}\, dt \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty \left(t^2+\lambda x^2\right)^{-\beta/2\lambda} t^{\beta/\lambda} e^{-t^2}\, dt.\end{aligned}$$

The strong Markov property yields the following identity

$$\sigma_{x\to 0} \overset{(d)}{=} \sigma_{x\to a} + \hat{\sigma}_{a\to 0}, \quad x \leqslant a \leqslant 0,$$

where $\hat{\sigma}_{a\to 0}$ is an independent copy of $\sigma_{a\to 0}$. It follows that

$$\mathbb{E}_x\left[e^{-\beta\sigma_a}\right] = \frac{\int_0^\infty \left(t^2+\lambda x^2\right)^{-\beta/2\lambda} t^{\beta/\lambda} e^{-t^2}\, dt}{\int_0^\infty \left(t^2+\lambda a^2\right)^{-\beta/2\lambda} t^{\beta/\lambda} e^{-t^2}\, dt}.$$

By using the integral representation of the cylinder parabolic function (7), we get

$$\mathbb{E}_x\left[e^{-\beta\sigma_a}\right] = \frac{e^{\lambda x^2/2} D_{-\beta/\lambda}(-x\sqrt{2\lambda})}{e^{\lambda a^2/2} D_{-\beta/\lambda}(-a\sqrt{2\lambda})}, \quad x \leqslant a \leqslant 0.$$

Next, we observe that the symmetry of B in (2) allows to recover the case $x \geqslant a \geqslant 0$. The proof is then completed since we have computed the two functions, the increasing and decreasing one, which characterized the Laplace transform of σ_a, see Itô and McKean [11]. □

3 On the law of $\left(N_t^{(1)}(r), N_t^{(2)}(r)\right)$

For any $\beta > 0$, we introduce the resolvent kernel, or the Green's function, G_β given, for $\alpha \geqslant 0$ and λ real, by

$$G_\beta(x,y)dy = \int_0^\infty e^{-\beta t}\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}N_t^{(2)}(R)-\alpha N_t^{(1)}(R)}, R_t \in dy\right] dt,\ x,y \geqslant 0.$$

As we shall see below, we have $G_\beta(x,y) = w_\beta^{-1}m(y)\phi_\beta(x\wedge y)\psi_\beta(x\vee y)$ where ϕ_β (respectively, ψ_β) is the only solution (up to multiplicative positive constants) which is decreasing, positive and bounded at $+\infty$ (respectively, increasing, positive and bounded at 0) of the Sturm–Liouville equation

$$2^{-1}\varphi''(x) + x^{-1}\varphi'(x) - \left(2^{-1}\lambda^2x^2 + \alpha x + \beta\right)\varphi(x) = 0, \quad x > 0. \tag{9}$$

Note that in the case when $\lambda = 0$ (respectively $\alpha = 0$), the corresponding Green function is already known, see e.g. [2, Formula 5.1.8.5], (respectively Formula 5.1.9.5). For a fixed $t \geqslant 0$, let us introduce the notation

$$\Pi_{x\to y}^{\lambda,\alpha}(t) = \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}N_t^{(2)}(R)-\alpha N_t^{(1)}(R)} \,\middle|\, R_t = y\right], \quad x,y,\alpha \geqslant 0, \lambda \in \mathbb{R}.$$

We denote simply $\Pi_x^{\lambda,\alpha}(t)$ (respectively $\Pi^{\lambda,\alpha}(t)$) for $\Pi_{x\to 0}^{\lambda,\alpha}(t)$ (respectively, $\Pi_{0\to 0}^{\lambda,\alpha}(t)$).

Remark 1. We point out that, thanks to the scaling property of Bessel processes, we have the identity $\Pi_{x\to y}^{\lambda,\,\alpha}(t) = \Pi_{\frac{x}{\sqrt{t}}\to\frac{y}{\sqrt{t}}}^{\lambda t^2,\,\alpha t^{3/2}}(1)$.

3.1 Stochastic approach for the case $y = 0$

We shall now show how to exploit the results of the former section in order to compute $\Pi_x^{\lambda,\alpha}(t)$.

Proposition 2. *For $x, \lambda, \beta > 0$ and $\alpha \geqslant 0$, we have*

$$\int_0^\infty e^{-\beta t}q_t(x,0)\Pi_x^{\lambda,\alpha}(t)\,dt = \frac{1}{x}\,\frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda^2}\right)\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}.$$

Consequently, we have

$$\int_0^\infty \left(e^{-\beta t}-1\right) \varPi^{\lambda,\alpha}(t)\, \frac{dt}{\sqrt{2\pi t^3}}$$
$$= \sqrt{2\lambda}\left(\frac{D^{(x)}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} - \frac{D^{(x)}_{\frac{\alpha^2}{2\lambda^3}-\frac{1}{2}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{D_{\frac{\alpha^2}{2\lambda^3}-\frac{1}{2}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}\right),$$

where $D_\nu^{(x)}(y) = \frac{\partial D_\nu(x)}{\partial x}|_{x=y}$.

Proof. Fix $a = \alpha/\lambda^2$, observe that

$$\varPi_x^{\lambda,a\lambda^2}(t) = e^{a^2\lambda^2 t/2}\, \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u+a)^2\,du} \mid R_t = 0\right], \tag{10}$$

and recall that $L_x = \sup\{s \geqslant 0;\ R_s = x\}$ and $T_a = \inf\{s \geqslant 0;\ B_s = a\}$. Following a line of reasoning similar to [7], we get

$$\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u+a)^2\,du} \mid R_t = 0\right] = \mathbb{E}_{x+a}\left[e^{-\frac{\lambda^2}{2}\int_0^t B_u^2\,du} \mid T_a = t\right], \tag{11}$$

where we used the properties of Bessel bridges recalled in Section 2. Now, thanks to the absolute-continuity relation (4), we can write

$$p^{(\lambda)}_{x+a\to a}(t) = e^{\frac{\lambda}{2}(x^2+2ax+t)}\mathbb{E}_{x+a}\left[e^{-\frac{\lambda^2}{2}\int_0^t B_u^2\,du} \mid T_a = t\right] p_{x\to 0}(t)\ . \tag{12}$$

A combination of (10), (11) and (12) leads to

$$e^{\left(\frac{1}{2}a^2\lambda^2-\frac{\lambda}{2}\right)t} p^{(\lambda)}_{x+a\to a}(t) = e^{\frac{\lambda}{2}x^2+a\lambda x} p_{x\to 0}(t)\, \varPi_x^{\lambda,\alpha}(t).$$

By taking the Laplace transform with respect to the variable t on both sides and making use of (6) we get the first assertion. To prove the second one, it is enough to let x tend to 0 in the following formula

$$\int_0^\infty \left(e^{-\beta t}-1\right) e^{-x^2/2t}\varPi_x^{\lambda,\alpha}(t)\, \frac{dt}{\sqrt{2\pi t^3}}$$
$$= \frac{1}{x}\left(\frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda^2}\right)\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} - \frac{D_{-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda^2}\right)\right)}{D_{-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}\right).$$

□

Below, we give a straightforward reformulation of the previous result, which is based on the Laplace transform inversion formula. To this end, we recall the expression of the density of σ_a as a series expansion which can be found for instance in [1] and [17]. That is, for real x and a, we have

$$p^{(\lambda)}_{x\to a}(t) = -\lambda e^{\lambda(x^2-a^2)/2} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,\varepsilon\sqrt{2\lambda}a}}\left(\varepsilon\sqrt{2\lambda}x\right)}{D^{(\nu)}_{\nu_{n,\varepsilon\sqrt{2\lambda}a}}\left(\varepsilon\sqrt{2\lambda}a\right)}\, e^{-\lambda\nu_{n,\varepsilon\sqrt{2\lambda}a}t}, \qquad (13)$$

where we set $\varepsilon = \operatorname{sgn}(x-a)$, $D^{(\nu)}_{\nu_{n,b}}(b) = \frac{\partial D_\nu(b)}{\partial \nu}|_{\nu=\nu_{n,b}}$ and the sequence $(\nu_{j,b})_{j\geqslant 0}$ stands for the ordered positive zeros of the function $\nu \to D_\nu(b)$.

Corollary 1. *For $\lambda, x, t > 0$ and $\alpha \geqslant 0$, we have*

$$\Pi^{\lambda,\alpha}_x(t) = -\frac{\lambda}{x}\sqrt{2\pi t^3}\, e^{\left(\frac{\alpha^2}{\lambda^2}-\lambda\right)t/2 + x^2/2t} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,c}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda^2}\right)\right)}{D^{(\nu)}_{\nu_{n,c}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}\, e^{-t\lambda\nu_{n,c}}, \qquad (14)$$

where we set $c = \sqrt{2}\alpha\lambda^{-3/2}$.

The proof is omitted and left to the reader.

3.2 Extension to $y > 0$ using Feynman–Kac formula

Our aim here is to provide an extension of the previous result to all positive real y by using the Feynman–Kac formula.

Proposition 3. *For $y, x, \beta, \lambda > 0$ and $\alpha \geqslant 0$, we have*

$$\begin{aligned}
&\int_0^\infty e^{-\beta t} q_t(x,y)\, \Pi^{\lambda,\alpha}_{x\to y}(t)\, dt \\
&= \frac{\Gamma\left(\frac{\beta}{\lambda}+\frac{1}{2}-\frac{\alpha^2}{2\lambda^3}\right) y}{\sqrt{\lambda\pi}\, D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\alpha\lambda^{-2/3}\right) x} S_{-\frac{\beta}{2\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x\wedge y+\frac{\alpha}{\lambda^2}\right), \sqrt{2}\alpha\lambda^{2/3}\right) \\
&\quad \times D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x\vee y+\frac{\alpha}{\lambda^2}\right)\right),
\end{aligned}$$

where $S_\alpha(x,y) = D_\alpha(-x)D_\alpha(y) - D_\alpha(x)D_\alpha(-y)$.

Proof. We shall prove our statement by a method similar to that used by Shepp [26]. Set $F^y_\epsilon(x) = \frac{1}{2\epsilon}\mathbb{I}_{\{|x-y|<\epsilon\}}$ and $a(x) = \left(\frac{\lambda^2}{2}x^2 + \alpha x + \beta\right)$. First, note that

$$\lim_{\epsilon\to 0} \mathbb{E}_x\left[\int_0^\infty e^{-\beta t} e^{-\int_0^t a(R_s)\,ds} F^y_\epsilon(R_t)\,dt\right] = \int_0^\infty e^{-\beta t} q_t(x,y)\, \Pi^{\lambda,\alpha}_{x\to y}(t)\,dt.$$

Then, the Feynman–Kac formula states that

$$u_\epsilon(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\beta t} e^{-\int_0^t a(R_s)\,ds} F^y_\epsilon(R_t)\,dt\right]$$

is the bounded solution of

$$\frac{1}{2}u_\epsilon''(x) + \frac{1}{x}u_\epsilon'(x) - a(x)u_\epsilon(x) = F_\epsilon^y(x), \quad x > 0. \tag{15}$$

In order to solve this equation, we first consider the following homogeneous one

$$\frac{1}{2}u''(x) + \frac{1}{x}u'(x) - a(x)u(x) = 0, \quad x > 0.$$

Setting $u(x) = x^{-1}v(x)$, we get that v satisfies the Weber equation

$$\frac{1}{2}v''(x) = \left(\frac{\lambda^2}{2}\bar{x}^2 - \frac{\alpha^2}{2\lambda^2} + \beta\right)v(x), \quad x > 0, \tag{16}$$

where $\bar{x} = x + \frac{\alpha}{\lambda^2}$. A fundamental solution of (16) is expressed in terms of the parabolic cylinder function $D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(\sqrt{2\lambda}\bar{x}\big)$, see e.g. [8]. Thus, the solution of (16) which is positive and decreasing is given by

$$\varphi(x) = x^{-1}D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(\sqrt{2\lambda}\bar{x}\big), \quad x > 0.$$

The solution of (16) which is positive and increasing has the form

$$\psi(x) = x^{-1}\left(c_1 D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(-\sqrt{2\lambda}\bar{x}\big) + c_2 D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(\sqrt{2\lambda}\bar{x}\big)\right),$$

where c_1 and c_2 are constants. With the choice $c_1 = D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\big)$ and $c_2 = -D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(-\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\big)$, we check that $\psi(x)$ is bounded at 0. The two solutions are linearly independent and their Wronskian, normalized by the derivative of the scale function $s'(x) = x^{-2}$, is given by

$$w_\beta = D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\big)w_\beta^D$$

where $w_\alpha^D = \frac{2\sqrt{\lambda\pi}}{\Gamma\left(\frac{\beta}{\lambda}+\frac{1}{2}-\frac{\alpha^2}{2\lambda^3}\right)}$ is the Wronskian of the cylinder parabolic functions. Next, we recall the Green formula for the solution of the nonhomogeneous ode (15), that is with second member given by F_ϵ^y

$$u_\epsilon(x) = \frac{1}{w_\beta}\left(\varphi(x)\int_0^x \psi(r)F_\epsilon^y(r)\,m(dr) + \psi(x)\int_x^\infty \varphi(r)F_\epsilon^y(r)\,m(dr)\right),$$

where $m(dr) = 2r^2dr$ is the speed measure of the 3-dimensional Bessel process. The proof is then completed by passing to the limit as ϵ tends to 0. $\square$

Remark 2. Observing that $\lim_{x\to 0} x^{-1}S_\alpha(x,y) = w_\alpha^D$, we recover the result of Proposition 2.

Remark 3. In the same vein as Corollary 1, it is possible to derive an expression of the joint Laplace transform $\Pi_{x\to y}^{\lambda,\alpha}(t)$ as a series expansion.

4 Connection between the law of first passage times

Set $\lambda > 0$ and introduce the function $f_\delta(\lambda t) = \delta g(\lambda t) - \mu\lambda t - y$ where g is a twice continuously differentiable function on a neighbourhood of 0, and α, μ and y are some real numbers. Let Z be a continuous time stochastic process. Introduce the stopping times

$$\begin{aligned} T^\delta_{y,\mu} &= \inf\{s \geqslant 0;\ Z_s = f_\delta(\lambda s)\} \\ L^\alpha &= \inf\{s \geqslant 0;\ Z_s = \alpha s\} \\ S^\alpha &= \inf\left\{s \geqslant 0;\ Z_s = -\frac{\alpha}{2}s^2\right\}. \end{aligned}$$

We shall describe a device which allows to connect the law of the first passage times $T^\delta_{y,\mu}$, simply denoted by T^δ_y for $\mu = 0$, over the linear boundary and over the quadratic one. As an application, we shall apply this technique to the Brownian case and derive some limit results of the cylinder parabolic functions. This limit result can also be used as a test for checking the validity of the hitting time densities.

Proposition 4. *Let* $\delta^{(1)}_\lambda = \alpha/\lambda$*. Assume* $g'(0) \neq 0$*, then*

$$\lim_{\lambda\to 0} T^{\delta^{(1)}_\lambda}_{\delta^{(1)}_\lambda g(0)} = L^{\alpha g'(0)} \quad a.s.. \tag{17}$$

Next, let $\delta^{(2)}_\lambda = \alpha/\lambda^2$*. Assume* $g''(0) \neq 0$*, then*

$$\lim_{\lambda\to 0} T^{\delta^{(2)}_\lambda}_{\delta^{(2)}_\lambda g(0),\delta^{(2)}_\lambda g'(0)} = S^{-\alpha g''(0)} \quad a.s.. \tag{18}$$

Proof. The assertions follows from the following expansion

$$f_{\delta_\lambda}(\lambda t) = \delta_\lambda g(0) - y + \lambda(\delta_\lambda g'(0) - \mu)t + \frac{\lambda^2}{2}\delta_\lambda g''(0)t^2 + o(\lambda^2). \qquad \square$$

4.1 Brownian motion and the square root boundary

We apply the previous technique to the first passage time of Brownian motion over the curve $f_\delta(\lambda t) = \delta\sqrt{1+2\lambda t} - \mu\lambda t - y$ in order to evaluate some well-known limits of the ratio of parabolic cylinder functions.

Linear case

In this case, we set $\mu = 0$ and $\delta = y = \alpha/\lambda$ and state the following result.

Corollary 2. *Let* $\beta > 0$*,* $x, \alpha \in \mathbb{R}$ *then we have*

$$\lim_{\lambda\to 0} \frac{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2\lambda}\left(x + \frac{\alpha}{\lambda}\right)\right)}{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2}\alpha\lambda^{-1/2}\right)} = e^{-|x|\sqrt{\alpha^2+2\beta}}.$$

As a consequence, we also have

$$\lim_{\lambda\to 0} \lambda e^{\lambda\left(x^2-2\frac{\alpha}{\lambda}x\right)/2} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,\frac{\alpha}{\lambda}\sqrt{2\lambda}}}(\sqrt{2\lambda}x)}{D^{(\nu)}_{\nu_{n,\frac{\alpha}{\lambda}\sqrt{2\lambda}}}(\sqrt{2}\alpha\lambda^{-1/2})}\, e^{-\lambda\nu_{n,\varepsilon a\sqrt{2\lambda}}t} = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}(x-\alpha t)^2}.$$

Proof. First, by combining Doob's transformation with Proposition 1, we recover the result of Breiman [3] about the Mellin transform of T_δ^δ

$$\mathbb{E}_x\left[\left(1+2\lambda T_\delta^\delta\right)^{-\beta/2\lambda}\right] = e^{\alpha x}\frac{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda}\right)\right)}{D_{-\frac{\beta}{2\lambda}}\left(\sqrt{2}\alpha\lambda^{-1/2}\right)}.$$

Next, recall that the Laplace transform of L^α is specified by, see e.g. [13, p.197],

$$\mathbb{E}_x\left[e^{-\beta L^\alpha}\right] = e^{\alpha x-|x|\sqrt{\alpha^2+2\beta}}.$$

The statement follows readily from Proposition 4. □

Quadratic case

In what follows, we investigate the second-order expansion. We start by computing the law of S^α, the first passage time of Brownian motion over the second-order boundary. In the case $x\alpha > 0$, its law has been computed by Groeneboom [9] and Salminen [24] in terms of the Airy function, see e.g. [15]. For the sake of completeness we recall their approach.

Lemma 1. *For β and $\alpha, x > 0$, hold the relations*

$$\mathbb{E}_x\left[e^{-\beta S^\alpha}G(S^\alpha)\right] = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}$$

where $G(t) = e^{\frac{1}{6}\alpha^2 t^3}$ and

$$\mathbb{P}_x(S^\alpha \in dt) = (2\alpha^2)^{1/3} e^{-\frac{1}{6}\alpha^2 t^3} \sum_{k=0}^{\infty} \frac{Ai\left(z_k-(2\alpha)^{1/3}\right)}{Ai'(z_k)} e^{2^{-1/3}\alpha^{2/3}z_k t}\, dt,$$

where $(z_k)_{k\geqslant 0}$ is the decreasing sequence of negative zeros of the Airy function.

Proof. Let us denote by $\mathbb{P}^\alpha$ the law of the process $\left(B_t+\frac{\alpha}{2}t^2,\ t\geqslant 0\right)$. We have the following absolute continuity relation

$$\begin{aligned} d\mathbb{P}^\alpha_{x|\mathcal{F}_t} &= e^{\alpha\int_0^t s\,dB_s-\frac{\alpha^2}{6}t^3}\, d\mathbb{P}_{x|\mathcal{F}_t} \\ &= e^{\alpha t B_t-\alpha\int_0^t B_s\,ds-\frac{\alpha^2}{6}t^3}\, d\mathbb{P}_{x|\mathcal{F}_t}, \quad t>0, \end{aligned}$$

where the last line follows from Itô's formula. An application of Doob's optional stopping theorem yields

$$\mathbb{E}_x\left[e^{-\beta S^\alpha} G(S^\alpha)\right] = \mathbb{E}_x\left[e^{-\beta T_0 - \alpha\int_0^{T_0} B_s ds}\right].$$

As in the previous section, the expectation on the right-hand side can be computed via the Feynman–Kac formula. It is the solution to the boundary value problem

$$\frac{1}{2}u''(x) - (\alpha x + \beta)u(x) = 0, \quad x > 0,$$
$$u(0) = 1, \quad \lim_{x\to\infty} u(x) = 0,$$

which is given in terms of the Airy function, see e.g. [12]. The expression of the density is a consequence of the Laplace transform inversion formula and the residue theorem, see [9] or [24] for more details. □

Remark 4. By analogy to the results of Section 3, we have

$$\lim_{\lambda\to 0}\int_0^\infty e^{-\beta t} q_t(x,0) \Pi_x^{\lambda,\alpha}(t)\, dt = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)},$$

$$\Pi_x^{0,\alpha}(t) = \sqrt{2\pi t^3}\, e^{\frac{x^2}{2t}}\left(2\alpha^2\right)^{1/3}\sum_{k=0}^\infty \frac{Ai\left(z_k - (2\alpha)^{1/3}\right)}{Ai'(z_k)} e^{\left(\frac{\alpha^2}{2}\right)^{1/3} z_k t}\, dt$$

and finally

$$\int_0^\infty \left(e^{-\beta t} - 1\right) \Pi^{0,\alpha}(t) \frac{dt}{\sqrt{2\pi t^3}} = (2\alpha)^{1/3}\left(\frac{Ai'\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)} - \frac{Ai'(0)}{Ai(0)}\right).$$

Remark 5. We mention that the other case, i.e. $\alpha x < 0$, has been studied by Martin-Löf [19].

Next, we define the process $(U_t^{(\mu)},\ t \geqslant 0)$ as the solution to the stochastic differential equation

$$dU_t^{(\mu)} = \left(-\lambda U_t^{(\mu)} + \mu e^{\lambda t}\right) dt + dB_t, \qquad U_0^{(\mu)} = x \in \mathbb{R}.$$

Note that $U^{(\mu)}$ can also be expressed as follows

$$U_t^{(\mu)} = e^{-\lambda t}\left(x - \frac{\mu}{2\lambda} + \frac{\mu}{2\lambda}e^{2\lambda t} + \int_0^t e^{\lambda s} dB_s\right), \qquad t \geqslant 0.$$

For real numbers x and a, we introduce the stopping time $\sigma_a^{(\mu)} = \inf\left\{s \geqslant 0;\ U_s^{(\mu)} = a\right\}$ and denote by $p_{x\to a}^{(\lambda,\mu)}(t)$ its density. Let us also introduce the function $G_\lambda(t) = e^{\frac{\mu^2}{2}\tau_t - \mu e^{\lambda t} a}$, $t \geqslant 0$. The law of $\sigma_a^{(\mu)}$ is characterized in the following.

Proposition 5. *For $\beta > 0$, we have*

$$\mathbb{E}_x \left[e^{-\beta \sigma_a^{(\mu)}} G_\lambda(\sigma_a^{(\mu)}) \right] = \frac{e^{\lambda x^2/2 - \mu x} D_{-\frac{\beta}{\lambda}}\left(\varepsilon x \sqrt{2\lambda}\right)}{e^{\lambda a^2/2} D_{-\frac{\beta}{\lambda}}\left(\varepsilon a \sqrt{2\lambda}\right)}. \tag{19}$$

where we set $\varepsilon = sgn(x-a)$. In particular,

$$p_{x\to 0}^{(\lambda,\mu)}(t) = \frac{|\ x\ |}{\sqrt{2\pi}} e^{-\mu e^{\lambda t}(\frac{\mu}{2}\sinh(\lambda t) - a) - \mu x - \frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)} + \frac{\lambda t}{2}} \left(\frac{\lambda}{\sinh(\lambda t)} \right)^{3/2}. \tag{20}$$

Proof. The first assertion follows from the following absolutely continuity relation

$$d\mathbb{P}_{x|\mathcal{F}_t}^{(\lambda,\mu)} = e^{\mu e^{\lambda t} X_t - \mu x - \frac{\mu^2}{2}\tau_t} d\mathbb{P}_{x|\mathcal{F}_t}^{(\lambda)}, \quad t > 0 \tag{21}$$

and the application of Doob's optional stopping theorem. We point out that the exponential martingale is the one associated with the Gaussian martingale $\left(B_{\tau(t)}, t \geqslant 0\right)$. The expression of the density in the case $a = 0$ is obtained from the Laplace inversion formula of the parabolic cylinder function, see Formula (8). □

Remark 6. An expression of the density $p_{x\to a}^{(\lambda,\mu)}(t)$ is given in Daniels [4] as a contour integral. The author used a technique suggested by Shepp [25].

Let us now introduce the stopping times $H_{x\to a}^{(\lambda,\mu)} = \inf\left\{s \geqslant 0;\ x + B_s + \mu s = a\sqrt{1+2\lambda s}\right\}$ and $S_x^\alpha = \inf\left\{s \geqslant 0;\ B_s + x = -\frac{\alpha}{2}s^2\right\}$. We denote by $p_{x\to a}^{(\lambda,\mu)}$ (respectively, q_x^α) the density of $H_{x\to a}^{(\lambda,\mu)}$ (respectively, S_x^α). We proceed by giving some relationships between these different hitting times.

$$H_{x\to a}^{(\lambda,\mu)} = \tau\left(\sigma_{x\to a}^{(\lambda,\mu)}\right) \quad a.s., \tag{22}$$

$$\lim_{\lambda\to 0} H_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}^{\left(\lambda,\frac{\alpha}{\lambda}\right)} = S_x^\alpha \quad a.s.. \tag{23}$$

We are now ready to state the following limit result which can be found for instance in [5].

Corollary 3. *For β, α and $x > 0$, we have*

$$\lim_{\lambda\to 0} \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda^2}\right)\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}$$

Proof. Substituting β by $\beta - \frac{\alpha^2}{2\lambda^2}$, x by $x + \frac{\alpha}{\lambda^2}$ and setting $a = \frac{\alpha}{\lambda^2}$ and $\mu = \frac{\alpha}{\lambda}$ in (19), we get

$$\frac{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda^2}\right)\right)}{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}$$

$$= e^{-\frac{\lambda}{2}x^2+\frac{\alpha^2}{\lambda^3}}\int_0^\infty e^{-\left(\beta-\frac{\alpha^2}{2\lambda^2}\right)t+\frac{\alpha^2}{2\lambda^2}\tau_t-\frac{\alpha^2}{\lambda^3}e^{\lambda t}}p^{\left(\lambda,\frac{\alpha}{\lambda}\right)}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}(t)\,dt$$

Note that $\tau\left(H^{(\lambda,\alpha/\lambda)}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}\right)\to S^\alpha_x$ a.s., as $\lambda\to 0$. Thus, we have

$$\lim_{\lambda\to 0} e^{-\frac{\lambda}{2}x^2+\frac{\alpha^2}{\lambda^3}}\int_0^\infty e^{-\left(\beta-\frac{\alpha^2}{2\lambda^2}\right)t+\frac{\alpha^2}{2\lambda^2}\tau_t-\frac{\alpha^2}{\lambda^3}e^{\lambda t}}p^{\left(\lambda,\frac{\alpha}{\lambda}\right)}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}(t)\,dt$$

$$= \int_0^\infty e^{-\beta t+\frac{1}{6}\alpha^2t^3}q^\alpha_x(t)\,dt = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}.$$

where the last expression follows from Lemma 1. □

Remark 7. We mention that Lachal [14] establishes the following identity

$$\mathbb{E}_x\left[e^{-\beta\sigma_0-\alpha\int_0^{\sigma_0}U_s\,ds}\right] = e^{\frac{\lambda}{2}x^2}\frac{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\left(x+\frac{\alpha}{\lambda^2}\right)\right)}{D_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}$$

which gives the following relation

$$\int_0^\infty e^{-\beta t-x^2/2t}t^{-3/2}\Pi^{\lambda,\alpha,(1)}_x(t)\,dt = e^{-\frac{\lambda}{2}x^2}\mathbb{E}_x\left[e^{-\left(\beta+\frac{\lambda}{2}\right)\sigma_0-\alpha\int_0^{\sigma_0}U_s\,ds}\right].$$

We also indicate that the author computed the limit as $\lambda\to 0$ to recover the result of Lefebvre [16] stating that

$$\mathbb{E}_x\left[e^{-\beta T_0-\alpha\int_0^{T_0}B_s\,ds}\right] = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}.$$

In order to compute the expression of the limit of the Laplace transform, he used an asymptotic result of the cylinder parabolic function which has been derived by the steepest descent method in [5].

4.2 Another limit

From Proposition 2, we readily derive

$$\lim_{\alpha\to 0}\int_0^\infty e^{-\beta t}xe^{-x^2/2t}\Pi^{\lambda,\alpha}_x(t)\,\frac{dt}{\sqrt{2\pi t^3}} = \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}}\left(\sqrt{2\lambda}x\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}}(0)}.$$

We recall and show the following well-known results regarding the Laplace transform of the L^2 norm of Bessel bridges. In conjunction with (8), we extract the relation

$$\Pi_x^{\lambda}(t) = \left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{3}{2}} e^{-\frac{x^2}{2t}(\lambda t\ \coth(\lambda t)-1)}. \tag{24}$$

Since in this case the zeros of the function $\nu \mapsto D_\nu(0) = 2^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-\nu}{2})}$ correspond to the odd poles of the Γ function, we also have

$$\Pi_x^{\lambda}(t) = -\frac{\lambda}{x}\sqrt{2\pi t^3} e^{\frac{x^2}{2t}} \sum_{n=1}^{\infty} \frac{D_{2n+1}\big(x\sqrt{2\lambda}\big)}{D_{2n+1}^{(\nu)}(0)}\, e^{-2(n+1)\lambda t}.$$

From the expression (24), it is easy to derive the generalized Lévy stochastic area formula, see e.g. [22]. Indeed for any $\delta > 0$, denoting by $\Pi_x^{\lambda,(\delta)}$ the Laplace transform of the L^2 norm of a δ-dimensional Bessel process, thanks to the additivity property of the squared Bessel processes, we have

$$\Pi_x^{\lambda,(\delta)}(t) = \left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{\delta}{2}} e^{-\frac{x^2}{2t}(\lambda t\ \coth(\lambda t)-1)}. \tag{25}$$

In [6] the inverse of the Laplace transform $\Pi_x^{\lambda,(\delta)}(t)$ is given in terms of the parabolic cylinder functions.

5 Comments and some applications

Our aim here is first to examine the law of the studied functional when the fixed time T is replaced by some interesting stopping times. To a stopping time S we associate the following notation

$$\Sigma_x^{(\delta)}(S) = \mathbb{E}_x^{(\delta)}\left[e^{-\beta S - \frac{\lambda^2}{2}\int_0^S R_u^2\, du - \alpha \int_0^S R_u\, du}\right],$$

where $\beta, \lambda > 0$, $\alpha \geqslant 0$ and $\mathbb{E}_x^{(\delta)}$ denotes the expectation operator derived from $\mathbb{Q}_x^{\delta}$, the law of the δ-dimensional Bessel process starting from $x \geqslant 0$.

Next, with $H_y = \inf\{s \geqslant 0;\ R_s = y\}$ and $S = H_y$, we state the following result.

Proposition 6. *Let $x \geqslant y > 0$.*

$$\Sigma_x^{(3)}(H_y) = \frac{y}{x}\, \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\big)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\big(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\big)}. \tag{26}$$

Proof. First, recall the following absolute continuity relation

$$d\mathbb{P}_{x|\mathcal{F}_t} = (R_t/x)^{-1}\, d\mathbb{Q}^{(3)}_{x|\mathcal{F}_t}, \quad \text{on } \{H_0 > t\};$$

then observe that $H_y < H_0$ a.s. since $x \geqslant y$. Next, denote by $\sigma_x^{(\mu)}$ the first passage time to a fixed level $x \in \mathbb{R}$ of the OU process when the Brownian motion in the SDE (3) is replaced by a Brownian motion with drift $\mu \in \mathbb{R}$. The determination of its density, denoted by ${}^{(\mu)}p^{(\lambda)}_{x\to a}(t)$, can be reduced to the case $\mu = 0$ as follows

$${}^{(\mu)}p^{(\lambda)}_{x\to a}(t) = p^{(\lambda)}_{x-\frac{\mu}{\lambda}\to a-\frac{\mu}{\lambda}}(t), \quad t > 0.$$

Thus, we have

$$\begin{aligned}
\varSigma_x^{(3)}(H_y) &= \mathbb{E}_x^{(3)}\left[e^{-\beta H_y - \frac{\lambda^2}{2}\int_0^{H_y} R_s^2\, ds - \alpha\int_0^{H_y} R_s\, ds}\right]\\
&= \frac{y}{x}\, \mathbb{E}_x\left[e^{-\beta T_y - \frac{\lambda^2}{2}\int_0^{T_y} B_s^2\, ds - \alpha\int_0^{T_y} B_s\, ds}\right]\\
&= \frac{y}{x}\, e^{\frac{\lambda}{2}(y^2-x^2)}\mathbb{E}_x\left[e^{-\left(\beta+\frac{\lambda}{2}\right)\sigma_y - \alpha\int_0^{\sigma_y} U_s\, ds}\right]\\
&= \frac{y}{x}\, e^{\frac{\lambda}{2}(y^2-x^2)+\frac{\alpha}{\lambda}(y-x)}\, \mathbb{E}_x\left[e^{-\left(\beta+\frac{\lambda}{2}-\frac{\alpha^2}{2\lambda^2}\right)\sigma_y^{(\frac{\alpha}{\lambda})}}\right]\\
&= \frac{y}{x}\, \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}.
\end{aligned}$$

□

Corollary 4. *For any $x \geqslant y > 0$, we have*

$$\varSigma_x^{(1)}(H_y) = \frac{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}. \tag{27}$$

Proof. The result follows from the absolute continuity relation

$$d\mathbb{Q}^{(1)}_{x|\mathcal{F}_t} = (R_t/x)^{-1}\, d\mathbb{Q}^{(3)}_{x|\mathcal{F}_t}, \quad \text{on } \{H_0 > t\},$$

where $\mathbb{Q}^1$ stands for the law of reflected Brownian motion and H_0 is the first time when the canonical process hits 0. □

Next, let $(\tau_t, t \geqslant 0)$ be defined as the right continuous inverse process of the local time $(l_t, t \geqslant 0)$ at 0 of reflected Brownian motion. It is a stable subordinator, its Laplace exponent is given by

$$\mathbb{Q}^{(1)}\left[e^{-\beta\tau_t}\right] = e^{-t\sqrt{2\beta}}.$$

Denote, respectively, by n and $(e_u, 0 \leqslant u \leqslant V)$ the Itô measure associated with R^1 and the generic excursion process under n. Recall that with the choice of the normalization of the local time via the occupation formula with respect to the speed measure, we have $n(V \in dt) = \frac{dt}{\sqrt{2\pi t^3}}$, see e.g. [10].

Proposition 7. *Let* $\alpha, \beta \geqslant 0$ *and* $\lambda > 0$.

$$-\log\left(\Sigma^{(1)}(\tau_1)\right) = \sqrt{2}\lambda \frac{D^{(x)}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{D_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}. \tag{28}$$

Proof. The exponential formula of excursion theory, see e.g. [23], and the fact that conditionally on $V = t$ the process $(e_u, u \leqslant V)$ is a 3-dimensional Bessel bridge over $[0, t]$ between 0 and 0, give

$$\begin{aligned} -\log\left(\Sigma^{(1)}(\tau_1)\right) &= \int n(de)\left(1 - e^{-\beta V - \frac{\lambda^2}{2}\int_0^V e_u^2\,du - \alpha\int_0^V e_u\,du}\right) \\ &= \int_0^\infty \left(1 - e^{-\beta t}\Pi^{\lambda,\alpha}(t)\right)\frac{dt}{\sqrt{2\pi t^3}}. \end{aligned}$$

Next, set $K(\beta) = \int_0^\infty \left(1 - e^{-\beta t}\Pi^{\lambda,\alpha}(t)\right)\frac{dt}{\sqrt{2\pi t^3}}$. Thus, we have

$$K(\beta) - K(0) = \int_0^\infty \left(1 - e^{-\beta t}\right)\Pi^{\lambda,\alpha}(t)\frac{dt}{\sqrt{2\pi t^3}}.$$

The statement follows from Proposition 2. □

Finally, we shall extend the above computations to the radial part of a δ-dimensional Ornstein–Uhlenbeck process, denoted by X, with parameter $\theta \in \mathbb{R}^+$. The law of this process, when started at $x > 0$, is denoted by $\mathbb{P}_x^{(\theta),\delta}$. Girsanov's theorem gives

$$d\mathbb{P}^{(\theta),\delta}_{x|\mathcal{F}_t} = e^{-\frac{\theta}{2}\left(R_t^2 - x^2 - \delta t\right) - \frac{\theta^2}{2}\int_0^t R_u^2\,du}\, d\mathbb{Q}^\delta_{x|\mathcal{F}_t}, \quad t > 0. \tag{29}$$

We also shall need the densities of its semi-group which are given, see [2], by

$$\begin{aligned} p_t^{(3)}(0, x) &= \frac{\theta^{3/2} e^{\frac{3}{2}\theta t}}{\sqrt{2\pi}(\sinh(\theta t))^{3/2}} e^{-\frac{\theta x^2 e^{-\theta t}}{2\sinh(\theta t)}} \\ p_t^{(1)}(0, x) &= \frac{\theta^{1/2} e^{\frac{1}{2}\theta t}}{\sqrt{2\pi}(\sinh(\theta t))^{1/2}} e^{-\frac{\theta x^2 e^{-\theta t}}{2\sinh(\theta t)}}, \quad x > 0. \end{aligned}$$

With obvious notations, for a fixed $t \geqslant 0$, we set

$$\Lambda^{\lambda,\alpha,(\delta)}_{x\to y}(t) = \mathbb{E}_x^\delta\left[e^{-\frac{\lambda^2}{2}\int_0^t X_u^2\,du - \alpha\int_0^t X_u\,du} \mid X_t = y\right], \quad \lambda, x \text{ and } \alpha \geqslant 0.$$

Proposition 8. *Set* $\kappa = \lambda^2 + \theta^2$, $\omega_1 = \beta + \frac{\theta}{2}$ *and* $\omega_3 = \beta + \frac{3\theta}{2}$. *For* x *and* $\beta > 0$, *we have*

$$\int_0^\infty e^{-\beta t} p_t^{(1)}(0,x) \Lambda_x^{\lambda,\alpha,(1)}(t)\, dt = e^{-\frac{\theta}{2}x^2} \frac{D_{-\frac{\omega_1}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2\kappa}(x+\frac{\alpha}{\kappa^2})\right)}{D_{-\frac{\beta}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2}\alpha\kappa^{-3/2}\right)}$$

and

$$\int_0^\infty e^{-\beta t} p_t^{(3)}(0,x) \Lambda_x^{\lambda,\alpha,(3)}(t)\, dt = e^{-\frac{\theta}{2}x^2} x \frac{D_{-\frac{\omega_3}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2\kappa}(x+\frac{\alpha}{\kappa^2})\right)}{D_{-\frac{\beta}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2}\alpha\kappa^{-3/2}\right)}.$$

Proof. From the absolute continuity relation (29), we have

$$\begin{aligned}&\mathbb{E}_x^\delta\left[e^{-\frac{\lambda^2}{2}\int_0^t X_s^2\, ds - \alpha\int_0^t X_s\, ds}\right] \\ &\quad = \mathbb{E}_x^{(\delta)}\left[e^{-\frac{\theta}{2}(R_t^2 - x^2 - \delta t)} e^{-\left(\frac{\lambda^2+\theta^2}{2}\right)\int_0^t R_s^2\, ds - \alpha\int_0^t R_s\, ds}\right].\end{aligned}$$

The results follow by the same reasoning as in the proof of Proposition 2. □

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References

1. Alili, L., Patie, P. and Pedersen, J.L. Representations of first hitting time density of an Ornstein–Uhlenbeck process. *Stochastic Models*, 21(4), 2005
2. Borodin, A.N. and Salminen, P. *Handbook of Brownian Motion – Facts and Formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, 2nd edition, 2002
3. Breiman, L. First exit times from a square root boundary. *In Fifth Berkeley Symposium*, 2(2):9–16, 1967
4. Daniels, H.E. The minimum of a stationary Markov process superimposed on a U-shaped trend. *Journal of Applied Probability*, 6:399–408, 1969
5. Doering, D.A., Hagan, P.S. and Levermore, C.D. Mean exit times for particles driven by weakly colored noise. *SIAM Journal of Applied Mathematics*, 49(5):1480–1513, 1989
6. Ghomrasni, R. On distributions associated with the generalized Lévy stochastic area formula. *Studia Scientiarum Mathematicarum Hungarica. A Quarterly of the Hungarian Academy of Sciences*, 41(1):83–100, 2004
7. Göing-Jaeschke, A. and Yor, M. A clarification note about hitting times densities for Ornstein–Uhlenbeck processes. *Finance and Stochastics*, 7(3):413–415, 2003

8. Gradshteyn, I.S. and Ryshik, I.M. *Table of Integrals, Series and Products*. Academic Press, San Diego, 6th edition, 2000
9. Groeneboom, P. Brownian motion with a parabolic drift and Airy functions. *Probability Theory and Related Fields*, 81(1):79–109, 1989
10. Hu, Y., Shi, Z. and Yor, M. *Some Applications of Lévy Area Formula to Pseudo-Brownian and pseudo-Bessel bridges*, pp. 181–209. Exponentials Functionals and Principal Values Related to Brownian Motion. 1997
11. Itô, K. and McKean, H.P. *Diffusion Processes and their Sample Paths*. Springer, Berlin Heidelberg New York, 1996
12. Jeanblanc, M., Pitman, J. and Yor, M. Feynman-Kac's formula and decomposition of Brownian paths. *Matemática Aplicada e Computacional. Sociedade Brasiliera de Matemática Applicada e Computacional*, 16(1):27–52, 1997
13. Karatzas, I. and Shreve, S.E. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics, 113. Springer, Berlin Heidelberg New York, 2nd edition, 1991
14. Lachal, A. Quelques martingales associées à l'intégrale du processus d'Ornstein-Uhlenbeck. Application à l'étude des premiers instants d'atteinte. *Stochastic and Stochactic Report*, 3–4:285–302, 1993
15. Lebedev, N.N. *Special functions and their applications*. Dover, New York, 1972
16. Lefebvre, M. First-passage densities of a two-dimensional process. *SIAM Journal on Applied Mathematics*, 49(5):1514–1523, 1989
17. Linetsky, V. Computing hitting time densities for OU and CIR processes: Applications to mean-reverting models. *Journal of Computational Finance*, 7:1–22, 2004
18. Louchard, G. The Brownian excursion area. *Computational Mathematical Application*, 10:413–417, 1984. Erratum:A12 (1986) 375
19. Martin-Löf, A. The final size of a nearly critical epidemic, and the first passage time of a Wiener process to a parabolic barrier. *Journal of Applied Probability*, 35:671–682, 1998
20. Patie, P. *On some First Passage Time Problems Motivated by Financial Applications*. PhD thesis, ETH Zurich, 2004
21. Perman, M. and Wellner, J.A. On the distribution of the Brownian areas. *The Annals of Applied Probability*, 6(4):1091–1111, 1996
22. Pitman, J. and Yor, M. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, 59:425–457, 1982
23. Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*, volume 293. Springer, Berlin Heidelberg New York, 3rd edition, 1999
24. Salminen, P. On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. *Advances in Applied Probability*, 20(2): 411–426, 1988
25. Shepp, L.A. A first passage problem for the Wiener process. *Annals of Mathematics and Statistics*, 38:1912–1914, 1967
26. Shepp, L.A. On the integral of the absolute value of the pinned Wiener process. *The Annals of Probability*, 10(1):234–239, 1982

Tanaka Formula for Symmetric Lévy Processes

Paavo Salminen[1] and Marc Yor[2]

[1] Mathematical Department, Åbo Akademi University
20500, Åbo, Finland
e-mail: phsalmin@abo.fi

[2] Laboratoire de Probabilités et Modèles Aléatoires
Université Pierre et Marie Curie
4, Place Jussieu, Case 188, 75252 Paris Cedex 05, France

Summary. Starting from the potential theoretic definition of the local times of a Markov process – when these exist – we obtain a Tanaka formula for the local times of symmetric Lévy processes. The most interesting case is that of the symmetric α-stable Lévy process (for $\alpha \in (1, 2]$) which is studied in detail. In particular, we determine which powers of such a process are semimartingales. These results complete, in a sense, the works by K. Yamada [19] and Fitzsimmons and Getoor [8].

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1 Introduction and main results

It is well known that there are different constructions and definitions of local times corresponding to different classes of stochastic processes. For a large panorama of such definitions, see Geman and Horowitz [12].

The most common definition of the local times $L = \{L_t^x : x \in \mathbf{R}, t \geqslant 0\}$ of a given process $\{X_t : t \geqslant 0\}$ is as the Radon–Nikodym derivative of the occupation measure of X with respect to the Lebesgue measure in $\mathbf{R}$; precisely L satisfies

$$\int_0^t f(X_s)\, ds = \int_{-\infty}^{\infty} f(x) L_t^x\, dx \tag{1}$$

for every Borel function $f : \mathbf{R} \mapsto \mathbf{R}_+$.

There is also the well-known stochastic calculus approach developed by Meyer [16] in which one works with a general semimartingale $\{X_t : t \geqslant 0\}$, and defines $\Lambda = \{\lambda_t^x : x \in \mathbf{R},\, t \geqslant 0\}$ with respect to the Lebesgue measure from the formula

$$\int_0^t f(X_s)\, d\langle X^c \rangle_s = \int_{-\infty}^{\infty} f(x) \lambda_t^x\, dx. \tag{2}$$

Of course, in the particular case when $d\langle X^c\rangle_s = ds$, i.e., X^c is a Brownian motion, then the definitions of L and λ coincide. In other cases, e.g., if $X^c \equiv 0$, they will differ.

In this paper we focus on the potential theoretic approach applicable in the Markovian case in which the local times are defined as additive functionals whose p-potentials are equal to p-resolvent kernels of X. Local times can hereby be interpreted as the increasing processes in the Doob-Meyer decompositions of certain submartingales. Considering the p-resolvent kernels and passing to the limit, in an adequate manner, as $p \to 0$, we obtain a formula (3), which clearly extends Tanaka's original formula for the local times of Brownian motion to those of the symmetric α-stable processes, $\alpha \in (1,2]$, already obtained by T. Yamada [20] and further developed in K. Yamada [19]. Our approach may be simpler and may help to make these results better known to probabilists working with Lévy processes.

The formula (3) below and its counterparts about decompositions of powers of symmetric α-stable Lévy processes show at the same time similarities and differences with the well-known formulae for Brownian motion (see, in particular, Chapter 10 in [23] concerning the principal values of Brownian local times). We hope that the Tanaka representation of the local times in (3) may be useful to gain some better understanding for the Ray-Knight theorems of the local times of X as presented in Eisenbaum et al. [6], since in the Brownian case, Tanaka's formula has been such a powerful tool for this purpose, see, e.g., Jeulin [15].

We now state the main formulae and results for the symmetric α-stable Lévy process $X = \{X_t\}$. To be precise, we take X to satisfy

$$\mathbf{E}\left(\exp(\mathrm{i}\lambda X_t)\right) = \exp(-t|\lambda|^\alpha), \quad \lambda \in \mathbf{R},$$

in particular, for $\alpha = 2$, X equals $\sqrt{2}$ times a standard BM. General criteria can be applied to verify that X possesses a jointly continuous family of local times $\{L_t^x\}$ satisfying (1). The constants c_i appearing below and later in the paper will be computed precisely in Section 5; clearly, they depend on the index α and/or the exponent γ.

1) *For all* $t \geqslant 0$ *and* $x \in \mathbf{R}$

$$|X_t - x|^{\alpha-1} = |x|^{\alpha-1} + N_t^x + c_1\, L_t^x, \tag{3}$$

where N^x *is a martingale such that for* $0 \leqslant \gamma < \alpha/(\alpha-1)$, *especially for* $\gamma = 2$,

$$\mathbf{E}\left(\sup_{s\leqslant t} |N_s^x|^\gamma\right) < \infty. \tag{4}$$

Moreover, the continuous increasing process associated with N^x *is*

$$\langle N^x\rangle_t := c_2 \int_0^t \frac{ds}{|X_s - x|^{2-\alpha}}. \tag{5}$$

2) *For $\alpha - 1 < \gamma < \alpha$ the submartingale $\{|X_t - x|^\gamma\}$ has the decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{6}$$

where $N^{(\gamma)}$ is a martingale and $A^{(\gamma)}$ is the increasing process given by

$$A_t^{(\gamma)} := c_3 \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}}. \tag{7}$$

3) *For $0 < \gamma < \alpha - 1$ the process $\{|X_t - x|^\gamma\}$ is not a semimartingale but for $(\alpha-1)/2 < \gamma < \alpha-1$ it is a Dirichlet process with the canonical decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{8}$$

where $N^{(\gamma)}$ is a martingale and $A^{(\gamma)}$, which has zero quadratic variation, is given by the principal value integral

$$A_t^{(\gamma)} := c_4 \text{ p.v.} \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}} := c_4 \int \frac{dz}{|z|^{\alpha-\gamma}} \left(L_t^{x+z} - L_t^{x-z}\right). \tag{9}$$

The paper is organized so that in Section 2 some preliminaries about symmetric Lévy processes including their generators and some variants of the Itô formula are presented. In Section 3 we derive the Tanaka formula for general symmetric Lévy processes admitting local times. The above stated results for symmetric stable Lévy processes are proved and extended in Section 4. In Section 5 we compute explicitly the constants c_i featured above and also further ones appearing especially in Section 4. This is done by exhibiting some close relations between these constants and the known expressions of the moments $\mathbf{E}(|X_1|^\gamma)$ where X_1 denotes a standard symmetric α-stable variable. In Section 6, we consider, instead of $|X_t - x|^\gamma$, the process $\{(X_t - x)^{\gamma,*}\}$, where

$$a^{\gamma,*} := \text{sgn}(a)\,|a|^\gamma,$$

is the symmetric power of order γ, and we determine the parameter values for which these processes are semimartingales or Dirichlet processes, thus completing results (1), (2), and (3) above.

2 Preliminaries on symmetric Lévy processes

Throughout this paper, we consider a real-valued symmetric Lévy process $X = \{X_t\}$ and, if nothing else is stated, we assume $X_0 = 0$. The Lévy exponent Ψ of X is a nonnegative symmetric function such that

$$\mathbf{E}\left(\exp(\mathrm{i}\xi X_t)\right) = \mathbf{E}\left(\cos(\xi X_t)\right) = \exp\left(-t\Psi(\xi)\right). \tag{10}$$

The Lévy measure ν of X satisfies, as is well known, the integrability condition

$$\int_{-\infty}^{\infty} (1 \wedge z^2)\, \nu(dz) < \infty.$$

By symmetry, $\nu(A) = \nu(-A)$ for any $A \in \mathcal{B}$, the Borel σ-field on $\mathbf{R}$; hence,

$$\begin{aligned}\Psi(\xi) &= \frac{1}{2}\,\sigma^2\,\xi^2 - \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}\,\xi\,z} - 1 - \mathrm{i}\,\xi\,z\,\mathbf{1}_{\{|z|\leqslant 1\}}\right)\,\nu(dz) \\ &= \frac{1}{2}\,\sigma^2\,\xi^2 + 2\int_0^{\infty} (1 - \cos(\xi z))\,\nu(dz). \end{aligned} \tag{11}$$

Recall also (see, e.g., Ikeda and Watanabe [14, p. 65]) that X admits the Brownian–Poisson representation

$$X_t = \sigma\,B_t + \int_{(0,t]}\int_{\{|z|\geqslant 1\}} z\,\Pi(ds,dz) + \int_{(0,t]}\int_{\{|z|<1\}} z\,(\Pi - \pi)(ds,dz), \tag{12}$$

where the Brownian motion B and the Poisson random measure Π with the intensity

$$\pi(ds,dz) := \mathbf{E}(\Pi(ds,dz)) = ds\,\nu(dz)$$

are independent. The infinitesimal generator of X is given by

$$\begin{aligned}\mathcal{G}f(x) &:= \mathcal{G}^B f(x) + \mathcal{G}^{\Pi} f(x) \\ &:= \frac{1}{2}\,\sigma^2\,f''(x) + \int_{\mathbf{R}} \left(f(x+y) - f(x) - f'(x)\,y\,\mathbf{1}_{\{|y|<1\}}\right)\,\nu(dy), \end{aligned} \tag{13}$$

where $\mathcal{G}$ acts on regular functions f in particular those in the Schwartz space $S(\mathbf{R})$ of rapidly decreasing functions.

Given a smooth function f, the predictable form of the Itô formula (see Ikeda and Watanabe [14] and K. Yamada [19]) writes

$$\begin{aligned}& f(X_t) - f(X_0) - \int_0^t \mathcal{G}f(X_s)\,ds \\ &\quad = \sigma \int_0^t f'(X_s)\,dB_s + \int_0^t\int_{\mathbf{R}} \left(f(X_{s-} + z) - f(X_{s-})\right)\,(\Pi - \pi)(ds,dz). \end{aligned} \tag{14}$$

The formula (14) connects with the Itô formula for semimartingales, as developed by Meyer [16], and displayed as

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s-})\,dX_s + \frac{\sigma^2}{2}\int_0^t f''(X_s)\,ds \\ &\quad + \sum_{0<s\leqslant t} \left(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right). \end{aligned} \tag{15}$$

The sum of jumps $\sum_{0<s\leqslant t}(\dots)$ may be compensated by $\int_0^t \mathcal{G}^{\Pi} f(X_s)\,ds$, and, hence, we have recovered the integrated form of (13):

$$\int_0^t \mathcal{G} f(X_s)\,ds = \int_0^t \mathcal{G}^B f(X_s)\,ds + \int_0^t \mathcal{G}^{\Pi} f(X_s)\,ds.$$

We record also a more general compensator formula employed later in the paper. For this, let $\Phi : \mathbf{R}\times\mathbf{R} \mapsto \mathbf{R}_+$ be a Borel measurable function. Then

$$\mathbf{E}\left(\sum_{0<s\leqslant t} \Phi(X_{s-}, X_s)\mathbf{1}_{\{\Delta X_s \neq 0\}}\right) = \mathbf{E}\left(\int_0^t \int_{\mathbf{R}\setminus\{0\}} \pi(ds, dz)\Phi(X_s, X_s + z)\right). \quad (16)$$

3 Local times for symmetric Lévy processes

From now on, we assume that

$$\int_{-\infty}^{\infty} \frac{1}{1+\Psi(\xi)}\,d\xi < \infty. \quad (17)$$

From standard Fourier arguments (see Bertoin [1] and, e.g., Borodin and Ibragimov [2, p. 67]) one can show the existence of a jointly measurable family of local times $\{L_t^x : x \in \mathbf{R}, t \geqslant 0\}$ satisfying for every Borel-measurable function $f : \mathbf{R} \mapsto \mathbf{R}_+$ the occupation time formula

$$\int_0^t ds\, f(X_s) = \int_{-\infty}^{\infty} f(x) L_t^x\,dx.$$

For the condition (expressed in terms of the function v in (22)) under which $(t,x) \mapsto L_t^x$ is continuous, see Bertoin [1, p. 148]. In particular, the condition holds for symmetric α-stable Lévy processes; in fact it was shown by Boylan [3], see also Getoor and Kesten [13], that

$$|L_t^{x+y} - L_t^x| \leqslant K_t\,|y|^{\theta} \quad (18)$$

for any $\theta < (\alpha-1)/2$ and some random constant K_t.

Our approach toward a Tanaka formula for these local times is based on the potential theoretic construction which we now develop. It is well known, see Bertoin [1, p. 67], that for any $p > 0$

$$u^{(p)}(x) = \frac{1}{\pi}\int_0^{\infty} \frac{\cos(\xi x)}{p+\Psi(\xi)}\,d\xi \quad (19)$$

is a continuous version of the density of the resolvent

$$U^{(p)}(0,dx) = \mathbf{E}_0\left(\int_0^\infty \mathrm{e}^{-p\,t}\mathbf{1}_{\{X_t\in dx\}}\,dt\right).$$

Moreover, for every x the local time $\{L_t^x\}$ can be chosen as a continuous additive functional such that

$$u^{(p)}(y-x) = \mathbf{E}_y\left(\int_0^\infty \mathrm{e}^{-p\,t} d_t L_t^x\right). \tag{20}$$

From (20) we deduce the Doob-Meyer decomposition given in the next:

Proposition 1. *For every fixed x*

$$u^{(p)}(X_t - x) = u^{(p)}(X_0 - x) + M_t^{(p,x)} + p\int_0^t u^{(p)}(X_s - x)ds - L_t^x, \tag{21}$$

where $M^{(p,x)}$ is a martingale with respect to the natural filtration $\{\mathcal{F}_t\}$ of X. Moreover, for every fixed t, both the martingale $\{M_s^{(p,x)} : s \leqslant t\}$ and the random variable L_t^x belong to BMO; in particular, L_t^x has some exponential moments.

Proof. Straightforward computations using the Markov property show that for $y = X_0$

$$\mathbf{E}_y\left(\int_0^\infty \mathrm{e}^{-p\,t} d_t L_t^x \mid \mathcal{F}_s\right) = \int_0^s \mathrm{e}^{-p\,t} d_t L_t^x + \mathrm{e}^{-p\,s} u^{(p)}(X_s - x),$$

which together with an integration by parts yields (21). We leave the proofs of the remaining assertions to the reader. □

A variant of the Tanaka formula shall now be obtained by letting $p \to 0$ in (21). The result is stated in Proposition 2 but first we need an important ingredient.

Lemma 1. *For every $x \in \mathbf{R}$*

$$\lim_{p\to 0}\left(u^{(p)}(0) - u^{(p)}(x)\right) = \frac{1}{\pi}\int_0^\infty \frac{1-\cos(\xi x)}{\Psi(\xi)}\,d\xi =: v(x). \tag{22}$$

Proof. The statement follows from (19) by dominated convergence because (cf. (11))

$$\int_1^\infty \frac{1}{\Psi(\xi)}\,d\xi < \infty \quad \text{and} \quad \int_0^1 \frac{\xi^2}{\Psi(\xi)}\,d\xi < \infty.$$

Notice also that v is continuous. □

The formula (23) below generalizes in a sense the Tanaka formula for Brownian motion to symmetric Lévy processes. In the next section we study the particular case of symmetric stable processes.

Proposition 2. *Let v be the function introduced in (22) and $M^{(p,x)}$ the martingale defined in Proposition 1. Then*

$$v(X_t - x) = v(x) + \tilde{N}_t^x + L_t^x, \tag{23}$$

where $\tilde{N}_t^x := -\lim_{p\to 0} M_t^{(p,x)}$ defines a martingale.

Remark 1. Standard results about martingale additive functionals of X yield the following representations

$$\begin{aligned}\tilde{N}_t^x &= \sigma \int_0^t v'(X_s - x)\, dB_s \\ &\quad + \int_{(0,t]} \int_{\mathbf{R}} (v(X_{s-} - x + z) - v(X_{s-} - x))\ (\Pi - \pi)(ds, dz),\end{aligned}$$

and

$$\begin{aligned}\langle \tilde{N}^x \rangle_t &= \sigma^2 \int_0^t (v'(X_s - x))^2\, ds \\ &\quad + \int_0^t \int_{\mathbf{R}} (v(X_s - x + z) - v(X_s - x))^2\ \pi(ds,\, dz),\end{aligned}$$

where v' is a weak derivative of v.

Proof. Consider the identity (21). Let therein $p \to 0$ and use Lemma 1 to obtain

$$v(X_t - x) = v(x) - \lim_{p\to 0} \left(M_t^{(p,x)} + p \int_0^t u^{(p)}(X_s - x) ds \right) + L_t^x. \tag{24}$$

From (19) $u^{(p)}(y) \leqslant u^{(p)}(0)$, and, consequently,

$$0 \leqslant p \int_0^t u^{(p)}(X_s - x) ds \leqslant p\, u^{(p)}(0)\, t. \tag{25}$$

Next we show that

$$\lim_{p\to 0} p\, u^{(p)}(0) = 0. \tag{26}$$

Indeed, using (19) again,

$$\begin{aligned} p\, u^{(p)}(0) &= \frac{1}{\pi} \int_0^\infty \frac{p\, d\xi}{p + \Psi(\xi)} \\ &\leqslant \frac{1}{\pi} \int_0^1 \frac{p\, d\xi}{p + \Psi(\xi)} + \frac{p}{\pi} \int_1^\infty \frac{d\xi}{\Psi(\xi)}, \end{aligned} \tag{27}$$

and (26) results by dominated convergence. Hence, (24) yields (23) with $\tilde{N}^x$ as claimed. It remains to prove that $\tilde{N}^x$ is a martingale. For this it is enough to show that

$$\mathbf{E}\left(|\tilde{N}_t^x - M_t^{(p,x)}| \right) \to 0 \quad \text{as } p \to 0. \tag{28}$$

To prove (28) consider

$$|\tilde{N}_t^x - M_t^{(p,x)}| \leqslant p \int_0^t u^{(p)}(X_s - x) ds + |v(x) - (u^{(p)}(0) - u^{(p)}(x))|$$
$$+ |v(X_t - x) - (u^{(p)}(0) - u^{(p)}(X_t - x))|.$$

From (25) and (26), the integral term goes to 0 as $p \to 0$. Next, by Fubini's theorem and (10)

$$\begin{aligned}
&\mathbf{E}\left(\left|v(X_t - x) - (u^{(p)}(0) - u^{(p)}(X_t - x))\right|\right) \\
&= \frac{1}{\pi} p \int_0^\infty \frac{1 - \mathbf{E}(\cos(\xi(X_t - x)))}{\Psi(\xi)(p + \Psi(\xi))} d\xi \\
&= \frac{1}{\pi} p \int_0^\infty \frac{1 - \cos(\xi\, x) \exp(-t\Psi(\xi))}{\Psi(\xi)(p + \Psi(\xi))} d\xi \\
&\leqslant \frac{1}{\pi} p \left(\int_0^\infty \frac{1 - \cos(\xi\, x)}{\Psi(\xi)(p + \Psi(\xi))} d\xi + \int_0^\infty \frac{t\,\Psi(\xi)}{\Psi(\xi)(p + \Psi(\xi))} d\xi \right).
\end{aligned}$$

Applying the dominated convergence theorem for the first term above and (27) for the second one give

$$\lim_{p \to 0} \mathbf{E}\left(\left|v(X_t - x) - (u^{(p)}(0) - u^{(p)}(X_t - x))\right|\right) = 0,$$

completing the proof. □

Example 1. For standard Brownian motion B we have

$$u^{(p)}(x) = \frac{1}{\sqrt{2p}} \, \mathrm{e}^{-\sqrt{2p}|x|}.$$

Consequently,

$$v(x) := \lim_{p \to 0} \left(u^{(p)}(0) - u^{(p)}(x) \right) = |x|$$

and the formula (23) takes the familiar form

$$|B_t - x| = |x| + N_t^x + L_t^x,$$

where

$$\begin{aligned}
N_t^x &= \lim_{p \to 0} \int_0^t \mathrm{e}^{-\sqrt{2\,p}|B_s - x|} \operatorname{sgn}(B_s - x)\, dB_s \\
&= \int_0^t \operatorname{sgn}(B_s - x)\, dB_s.
\end{aligned}$$

4 Symmetric α-stable Lévy processes

Let $X = \{X_t\}$, $X_0 = 0$, denote the symmetric α-stable process with the Lévy exponent

$$\Psi(\xi) = |\xi|^\alpha, \quad \alpha \in (1,2).$$

We remark that the condition (17) is satisfied, and also that the local time of X has a jointly continuous version, as is discussed in Section 3. For clarity, we have excluded the Brownian motion from our study. However, the corresponding results for Brownian motion may be recovered by letting $\alpha \to 2$. Recall also that $\mathbf{E}(|X_t|^\gamma) < \infty$ for $\gamma < \alpha$, and that the Lévy measure is

$$\nu(dz) = c_5(\alpha)\,|z|^{-\alpha-1}\,dz, \quad \alpha \in (1,2). \tag{29}$$

Since $\int_1^\infty z\,\nu(dz) < \infty$ and ν is symmetric the infinitesimal generator of X (cf. (13)) is given by

$$\mathcal{G}f(x) = \frac{1}{2}\,\sigma^2\,f''(x) + \int_{\mathbf{R}} (f(x+y) - f(x) - f'(x)\,y)\;\nu(dy).$$

The function v introduced in Lemma 1 takes in this particular case the form

$$v(x) = c_6(\alpha)\,|x|^{\alpha-1}. \tag{30}$$

The results announced in the Introduction are now presented again and proven in a more complete form through the following three propositions. The first one treats the claim (1) in the Introduction.

Proposition 3. a) *For fixed x*

$$c_6(\alpha)\,\left(|X_t - x|^{\alpha-1} - |x|^{\alpha-1}\right) = \tilde{N}_t^x + L_t^x, \tag{31}$$

where $\{\tilde{N}_t^x\}$ is a square integrable martingale. In fact, for all $0 \leqslant \gamma < \alpha/(\alpha-1)$, especially for $\gamma = 2$,

$$\mathbf{E}\left(\sup_{s\leqslant t} |\tilde{N}_s^x|^\gamma\right) < \infty. \tag{32}$$

Moreover, the continuous increasing process associated with $\tilde{N}^x$ is

$$\langle \tilde{N}^x \rangle_t := c_7(\alpha) \int_0^t \frac{ds}{|X_s - x|^{2-\alpha}}. \tag{33}$$

b) *For every t and x the variable L_t^x belongs to BMO; in fact, for all $s \leqslant t$*

$$\mathbf{E}\,(L_t^x - L_s^x \,|\, \mathcal{F}_s) \leqslant K_{\alpha,t} \tag{34}$$

for some constant $K_{\alpha,t}$ which does not depend on s.

Proof. The fact that $\tilde{N}^x$ is a martingale is clear from Proposition 2. Because L_t^x has some exponential moments (cf. Proposition 1), it is seen easily from (31) that for $\gamma > 0$

$$\mathbf{E}(|\tilde{N}_t^x|^\gamma) < \infty$$

if

$$\mathbf{E}\left(|X_t - x|^{\gamma(\alpha-1)}\right) < \infty,$$

which is true for $\gamma(\alpha - 1) < \alpha$. Consequently, an extension of the Doob-Kolmogorov inequality, gives (32). The martingale $\tilde{N}^x$ has no continuous martingale part. Hence, letting

$$[\tilde{N}^x]_t := \sum_{s\leqslant t}(\Delta \tilde{N}_s^x)^2 := (c_6(\alpha))^2 \sum_{s\leqslant t}\left(|X_s - x|^{\alpha-1} - |X_{s-} - x|^{\alpha-1}\right)^2.$$

It holds that $\{ (\tilde{N}_t^x)^2 - [\tilde{N}^x]_t \}$ is a martingale. Consequently, $\langle \tilde{N}^x\rangle$ can be obtained as the dual predictable projection of $[\tilde{N}^x]$, and from the Lévy system of X, e.g., (16), we get

$$\langle \tilde{N}^x\rangle_t = (c_6(\alpha))^2\, c_5(\alpha) \int_0^t ds \int_{\mathbf{R}} \frac{dy}{|y|^{\alpha+1}} \left(|X_{s-} - x + y|^{\alpha-1} - |X_{s-} - x|^{\alpha-1}\right)^2.$$

Putting $z = X_{s-} - x$ and introducing $y = zu$ the latter integral takes the form

$$\int_{\mathbf{R}} \frac{dy}{|y|^{\alpha+1}} \left(|z+y|^{\alpha-1} - |z|^{\alpha-1}\right)^2 = \frac{1}{|z|^{2-\alpha}} \int_{\mathbf{R}} \frac{du}{|u|^{\alpha+1}} \left(|1+u|^{\alpha-1} - 1\right)^2.$$

Consequently, $\langle \tilde{N}^x\rangle$ is as claimed. To prove the second part of the proposition, notice that by the martingale property

$$\begin{aligned}\mathbf{E}\left(L_t^x - L_s^x \,|\, \mathcal{F}_s\right) &= c_6(\alpha)\ \mathbf{E}\left(|X_t - x|^{\alpha-1} - |X_s - x|^{\alpha-1} \,|\, \mathcal{F}_s\right)\\ &\leqslant c_6(\alpha)\ \mathbf{E}\left(|X_t - X_s|^{\alpha-1} \,|\, \mathcal{F}_s\right)\\ &\leqslant c_6(\alpha)\ \mathbf{E}\left(|X_{t-s}|^{\alpha-1}\right)\\ &\leqslant K'_\alpha\, t^{(\alpha-1)/\alpha},\end{aligned}$$

where also the scaling property and the inequality

$$|x^p - y^p| \leqslant |x-y|^p, \quad 0 < p \leqslant 1,$$

are used. □

The following corollary plays the same role for X as the classical Itô-Tanaka formula plays for Brownian motion. In fact, a large part of this paper discusses for which functions the identity (35), or some variant of it is valid.

Corollary 1. *Let f be a bounded Borel function with compact support and define*

$$F(y) := \int dx\, f(x)\, |y-x|^{\alpha-1}.$$

Then

$$F(X_t) = F(0) + \int dx\, f(x)\, N_t^x + c_1(\alpha) \int_0^t ds\, f(X_s) \tag{35}$$

expresses the canonical semimartingale decomposition of $\{F(X_t)\}$ with $\{\int dx\, f(x)\, N_t^x\}$ a martingale.

Proof. It suffices to integrate both sides of (31) (or rather (3)) with respect to the measure $f(x)\,dx$. □

Remark 2. **a)** In K. Yamada [19] the representation (31) of the local time (or Tanaka's formula for symmetric α-stable processes) is derived using the so-called "mollifier" approach as in Ikeda and Watanabe [14] in the Brownian motion case. In this case the martingale is given by

$$\tilde{N}_t^x = c_6(\alpha) \int_{(0,t]} \int_{\mathbf{R}} \left(|X_{s-} - x + z|^{\alpha-1} - |X_{s-} - x|^{\alpha-1} \right) \, (\Pi - \pi)(ds, dz),$$

where Π and π are the Poisson random measure and the corresponding intensity measure, respectively, associated with X.
b) The inequality (34) holds for all symmetric Lévy processes having local times. Indeed, it is proved in Bertoin [1, p. 147, Corollary 14] that the function v defined in (22), Lemma 1, induces a metric on $\mathbf{R}$, and, in particular, the triangle inequality holds. Consequently,

$$\mathbf{E}\left(L_t^x - L_s^x \,|\, \mathcal{F}_s\right) \leqslant \mathbf{E}(v(X_t - X_s)) = \mathbf{E}(v(X_{t-s})) \leqslant \mathbf{E}(v(X_t)) < \infty$$

because

$$\begin{aligned} \mathbf{E}(v(X_t)) &= \frac{1}{\pi} \int_0^\infty \frac{1 - \exp(-t\Psi(\xi))}{\Psi(\xi)}\, d\xi \\ &\leqslant \frac{1}{\pi} \left(t + \int_1^\infty \frac{d\xi}{\Psi(\xi)} \right) < \infty. \end{aligned}$$

c) We leave it to the reader to establish a version of Corollary 1 for general symmetric Lévy processes.

Proposition 4. *For a given x and $\alpha - 1 < \gamma < \alpha$ the submartingale $\{|X_t - x|^\gamma \,:\, t \geqslant 0\}$ has the decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{36}$$

where $N^{(\gamma)}$ *is a martingale and* $A^{(\gamma)}$ *is the increasing process given by*

$$A_t^{(\gamma)} = c_3(\alpha,\gamma) \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}}. \tag{37}$$

Moreover, when $\alpha - 1 \leqslant \gamma \leqslant \alpha/2$ *the increasing process* $\langle N^{(\gamma)} \rangle$ *is of the form*

$$\langle N^{(\gamma)} \rangle_t = c_8(\alpha,\gamma) \int_0^t \frac{ds}{|X_s - x|^{\alpha-2\gamma}}. \tag{38}$$

Proof. Formula (36) is obtained by integrating both sides of (31) (or (3) taken at level z with respect to the measure $dz/|z-x|^{\alpha-\gamma}$. The form of the left-hand side is obtained from the scaling argument. Because $A^{(\gamma)}$ is continuous the computation for finding $\langle N^{(\gamma)} \rangle$ is very similar to the computation of $\langle \tilde{N}^x \rangle$ in the proof of Proposition 3. We have

$$\langle N^{(\gamma)} \rangle_t = \int_0^t ds \int_{\mathbf{R}} \nu(dy) \left(|X_{s-} - x + y|^\gamma - |X_{s-} - x|^\gamma \right)^2, \tag{39}$$

which easily yields (38). □

For the next proposition, we recall the notion of Dirichlet process, i.e., a process which can be decomposed uniquely as the sum of a local martingale and a continuous process with zero quadratic variation (see, e.g., Föllmer [10] and Fukushima [11]).

Proposition 5. **a)**] *For* $0 < \gamma < \alpha - 1$ *the process* $|X - x|^\gamma$ *is not a semimartingale.*
b) *For* $(\alpha-1)/2 < \gamma < \alpha - 1$ *the process* $|X - x|^\gamma$ *is a Dirichlet process with the canonical decomposition*

$$|X_t - x|^\gamma = |x|^\gamma + N_t^{(\gamma)} + A_t^{(\gamma)}, \tag{40}$$

where $N^{(\gamma)}$ *is a martingale and* $A^{(\gamma)}$ *is given by the principal value integral*

$$\begin{aligned} A_t^{(\gamma)} &= c_4(\alpha,\gamma) \text{ p.v.} \int_0^t \frac{ds}{|X_s - x|^{\alpha-\gamma}} \\ &= c_4(\alpha,\gamma) \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left(L_t^{x+z} - L_t^{x-z} \right). \end{aligned} \tag{41}$$

Moreover, the increasing process $\langle N^{(\gamma)} \rangle$ *is as given in (38).*

Proof. a) We take $x = 0$ and adapt the argument in Yor [21] applied therein for continuous martingales. Assume that $Y_t := |X_t|^\gamma$, $\gamma < \alpha - 1$, defines a semimartingale. Then

$$|X_t|^{\alpha-1} = Y_t^\theta$$

with $\theta = \gamma/(\alpha - 1) > 1$, and Itô's formula for semimartingales (notice that $Y^c \equiv 0$) gives

$$Y_t^\theta = \int_0^t \theta \, Y_{s-}^{\theta-1} \, dY_s + \Sigma_t, \tag{42}$$

where

$$\Sigma_t := \sum_{0<s\leqslant t} \left(Y_s^\theta - Y_{s-}^\theta - \theta \, Y_{s-}^{\theta-1} \Delta Y_s \right).$$

The argument of the proof is that under the above assumption the local time

$$L_t^0 \equiv \int_0^t \mathbf{1}_{\{X_{s-}=0\}} \, d|X_s|^{\alpha-1} \tag{43}$$

would be equal to zero. To derive this contradiction notice from (42) and (43) that

$$L_t^0 = \int_0^t \mathbf{1}_{\{Y_{s-}=0\}} \, dY_s^\theta = \int_0^t \mathbf{1}_{\{Y_{s-}=0\}} \, d\Sigma_s.$$

But because Σ is a purely discontinuous increasing process and L^0 is continuous this is possible only if $L^0 \equiv 0$, which cannot be the case; thus proving that Y is not a semimartingale.

b) To prove (40) we consider formula (3) at levels $x+z$ and $x-z$ and write

$$\begin{aligned}
&\int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left(|X_t - (x+z)|^{\alpha-1} - |X_t - (x-z)|^{\alpha-1} \right) \\
&= \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left(N_t^{x+z} - N_t^{x-z} \right) + c_1(\alpha) \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left(L_t^{x+z} - L_t^{x-z} \right). \qquad (44)
\end{aligned}$$

The integral on the left-hand side is well defined since by scaling

$$\int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left(|X_t - (x+z)|^{\alpha-1} - |X_t - (x-z)|^{\alpha-1} \right) = |X_t - x|^\gamma \, r(\alpha,\gamma)$$

with

$$r(\alpha,\gamma) := \int_{\mathbf{R}} \frac{dz}{|z|^{\alpha-\gamma}} \left(|1-z|^{\alpha-1} - |1+z|^{\alpha-1} \right),$$

which is an absolutely convergent integral. Next notice that the principal value integral on the right-hand side of (44) is well defined by the Hölder continuity in x of the local times (cf. (18)). It also follows that the first integral on the right-hand side of (44) is meaningful and, by Fubini's theorem, it is a martingale. In Fitzsimmons and Getoor [8] it is proved that

$$H_t^0 := \int_0^\infty \frac{dz}{z^{\alpha-\gamma}} \left(L_t^{-z} - L_t^0\right).$$

has zero p-variation for $p > p_o := (\alpha - 1)/\gamma$ (notice $1 + \gamma$ in [8] corresponds ours $\alpha - \gamma$). Since $p_o < 2$ it is now easily seen that also $\{A_t^{(\gamma)}\}$ has zero quadratic variation and the claimed Dirichlet process decomposition follows with

$$c_4(\alpha, \gamma) = c_1(\alpha)/r(\alpha, \gamma). \tag{45}$$

□

5 Explicit values of the constants

An important ingredient in the computation of the explicit values of the constants is the formula for absolute moments of symmetric α-stable, $\alpha \in (1, 2)$, random variables due to Shanbhag and Sreehari [18] (see also Sato [17, p. 163] and Chaumont and Yor [4, p. 110]). To discuss this briefly let

- Z be an exponentially distributed r.v. with mean 1.
- U a normally distributed r.v. with mean 0 and variance 1.
- $X^{(\alpha)}$ a symmetric α-stable r.v. with characteristic function $\exp(-|\xi|^\alpha)$.
- $Y^{(\alpha/2)}$ a positive $\alpha/2$-stable r.v. with Laplace transform $\exp(-\xi^{\alpha/2})$.

Assume also that these variables are independent. Then it is easily checked that

$$\left(Z/Y^{(\alpha/2)}\right)^{\alpha/2} \stackrel{d}{=} Z \tag{46}$$

and

$$X^{(\alpha)} \stackrel{d}{=} \sqrt{2}\, U \left(Y^{(\alpha/2)}\right)^{1/2}. \tag{47}$$

From (46) we obtain for $\gamma < \alpha/2$

$$\mathbf{E}\left(\left(Y^{(\alpha/2)}\right)^\gamma\right) = \frac{\Gamma\left(1 - \frac{2\gamma}{\alpha}\right)}{\Gamma(1-\gamma)},$$

and, further, from (47) for $-1 < \gamma < \alpha$

$$m_\gamma := \mathbf{E}\left(|X^{(\alpha)}|^\gamma\right) = 2^\gamma\, \Gamma\left(\frac{1+\gamma}{2}\right) \Gamma\left(\frac{\alpha-\gamma}{\alpha}\right) / \left(\sqrt{\pi}\, \Gamma\left(\frac{2-\gamma}{2}\right)\right). \tag{48}$$

The constants with the associated reference numbers of the formulae where they appear in the paper are summarized in the following table:

Constant	Value	Ref.
$c_1(\alpha)$	$((\alpha-1)\,\pi\, m_{\alpha-1})\,/\Gamma(1/\alpha)$	(3), (56), (57)
$c_2(\alpha)$	$(2(\alpha-1)\,m_{2(\alpha-1)})/(\alpha\, m_{\alpha-2})$	(5)
$c_3(\alpha,\gamma)$	$(\gamma\, m_\gamma)/(\alpha\, m_{\gamma-\alpha})$	(7), (37)
$c_4(\alpha,\gamma)$	$c_1(\alpha)/r(\alpha,\gamma)$	(41), (45)
$c_5(\alpha)$	$\alpha/(2\,\Gamma(1-\alpha)\,\cos(\alpha\pi/2))$	(29)
$c_6(\alpha)$	$(c_1(\alpha))^{-1}=(2\pi c_5(\alpha-1))^{-1}$	(30)
$c_7(\alpha)$	$c_2(\alpha)\,(c_6(\alpha))^{\,2}$	(33)
$c_8(\alpha,\gamma)$	$c_3(\alpha,2\gamma)-2c_3(\alpha,\gamma)$	(38)

We consider first the constant $c_3(\alpha,\gamma)$ and, for clarity, recall formula (36)

$$|X_t-x|^\gamma=|x|^\gamma+N_t^{(\gamma)}+A_t^{(\gamma)}, \tag{49}$$

with $\alpha-1<\gamma<\alpha$ and

$$A_t^{(\gamma)}=c_3(\alpha,\gamma)\int_0^t\frac{ds}{|X_s-x|^{\alpha-\gamma}}.$$

Notice that letting $\gamma\downarrow\alpha-1$ yields, in a sense

$$A_t^{(\alpha-1)}=c_1(\alpha)\,L_t^x, \tag{50}$$

although, using the value in the table, $c_3(\alpha,\gamma)\to 0$. From (49) it is seen that $f(y)=|y-x|^\gamma$ belongs to the domain of the extended generator $\mathcal{G}$, and, by scaling we obtain the following integral representation

$$c_3(\alpha,\gamma)=\int_{\mathbf{R}}\nu(dy)\left(|1+y|^\gamma-1-\gamma y\right).$$

On the other hand, taking $x=0$ in (49), and using scaling again together with (48), we get

$$\mathbf{E}\left(|X_t|^\gamma\right)=c_3(\alpha,\gamma)\int_0^t ds\,\mathbf{E}\left(|X_s|^{\gamma-\alpha}\right),$$

which is equivalent with

$$t^{\gamma/\alpha}\,m_\gamma=c_3(\alpha,\gamma)\,\frac{\alpha\,t^{\gamma/\alpha}}{\gamma}\,m_{\gamma-\alpha}$$

hence,

$$c_3(\alpha,\gamma) = \gamma\, m_\gamma / \alpha\, m_{\gamma-\alpha}.$$

A similar argument leads to an expression for $c_1(\alpha)$. From (50) we get

$$\mathbf{E}\left(|X_t|^{\alpha-1}\right) = c_1(\alpha)\, \mathbf{E}\left(L_t^0\right). \tag{51}$$

We derive from (51) the existence of a constant $c_0(\alpha)$ such that

$$\mathbf{E}\left(d_t L_t^0\right) = c_0(\alpha) dt\, t^{-1/\alpha},$$

and it follows from (50) that

$$m_{\alpha-1} = \alpha\, c_1(\alpha)\, c_0(\alpha)/(\alpha-1). \tag{52}$$

We now compute $c_0(\alpha)$ to obtain $c_1(\alpha)$ from (52). For this consider the identity (20) for $x = y = 0$

$$u^{(p)}(0) = \mathbf{E}_0\left(\int_0^\infty \mathrm{e}^{-p\,s} d_s L_s^0\right),$$

which in terms of $c_0(\alpha)$ reads

$$\frac{1}{\pi}\int_0^\infty \frac{d\xi}{p+\xi^\alpha} = c_0(\alpha) \int_0^\infty \mathrm{e}^{-p\,s} s^{-1/\alpha}\, ds.$$

An elementary computation reveals that

$$c_0(\alpha) = \frac{1}{\pi}\, \Gamma((\alpha+1)/\alpha),$$

hence,

$$c_1(\alpha) = ((\alpha-1)\,\pi\, m_{\alpha-1})\,/\Gamma(1/\alpha).$$

Next we find from formula (31) that

$$c_6(\alpha) = 1/c_1(\alpha). \tag{53}$$

To compute $c_8(\alpha,\gamma)$ for $\alpha - 1 \leqslant \gamma \leqslant \alpha/2$ and the limiting case $c_2(\alpha) = c_8(\alpha, \alpha-1)$ notice from (39) that

$$c_8(\alpha,\gamma) = \int_{\mathbf{R}} \nu(dy)\left(|1+y|^\gamma - 1\right)^2.$$

Comparing the integral representations of c_3 and c_8 it is seen that

$$2c_3(\alpha,\gamma) + c_8(\alpha,\gamma) = c_3(\alpha, 2\gamma) \tag{54}$$

which can also be deduced from the following formulae

$$(i) \qquad \mathbf{E}\left(|X_t|^{2\gamma}\right) = 2\,\mathbf{E}\left(\int_0^t |X_s|^\gamma \, d_s A_s^{(\gamma)}\right) + \mathbf{E}\left(\langle N^{(\gamma)}\rangle_t\right)$$

$$= (2c_3(\alpha,\gamma) + c_8(\alpha,\gamma))\ \mathbf{E}\left(\int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}}\right),$$

$$(ii) \qquad \mathbf{E}\left(|X_t|^{2\gamma}\right) = c_3(\alpha, 2\gamma)\,\mathbf{E}\left(\int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}}\right).$$

The first one of these is an easy application of the Itô formula for semimartingales and the second one follows (49) because $\gamma \leqslant \alpha/2$. From (54) we get

$$c_8(\alpha,\gamma) = c_3(\alpha,2\gamma) - 2c_3(\alpha,\gamma)$$
$$= \frac{2\gamma}{\alpha}\left(\frac{m_{2\gamma}}{m_{2\gamma-\alpha}} - \frac{m_\gamma}{m_{\gamma-\alpha}}\right).$$

The constant c_2 is now obtained by letting here $\gamma \to \alpha - 1$ and using $m_{-1} = +\infty$. Consequently

$$c_2(\alpha) = \frac{2(\alpha-1)}{\alpha}\,\frac{m_{2(\alpha-1)}}{m_{\alpha-2}}.$$

To find the constant $c_5(\alpha)$, we use the relationship (11) between Ψ and ν which yields after substitution $y = \xi z$

$$c_5(\alpha) = \left(2\int_0^\infty \frac{1-\cos y}{y^{\alpha+1}}\,dy\right)^{-1}.$$

Integrating by parts and using the formulae 2.3.(1), p. 68 in Erdelyi et al. [7] lead us to the explicit value of the integral

$$\int_0^\infty \frac{1-\cos y}{y^{\alpha+1}}\,dy = \frac{\Gamma(1-\alpha)}{\alpha}\cos(\alpha\pi/2).$$

The constant $c_6(\alpha)$ can also clearly be expressed in terms of c_5

$$c_6(\alpha) = (2\pi c_5(\alpha-1))^{-1} = \frac{1}{\pi}\int_0^\infty \frac{1-\cos\xi}{\xi^{\,\alpha}}\,d\xi$$
$$= \frac{1}{\pi}\,\frac{\Gamma(2-\alpha)}{\alpha-1}\cos((\alpha-1)\pi/2).$$

It can be verified by the duplication formula for the Gamma function that this agrees with (53). It holds also that $c_6(\alpha) \to 1/2$ as $\alpha \uparrow 2$.

The constant c_7 is obtained by simply comparing the definitions of N^x in (3) and $\tilde{N}^x$ in Proposition 3. We have

$$N_t^x = \frac{1}{c_6(\alpha)}\,\tilde{N}_t^x$$

implying

$$c_7(\alpha) = c_2(\alpha)\,(c_6(\alpha))^2.$$

6 Symmetric principal values of local times

Our previous results may be summarized as follows

1. For $\alpha - 1 \leqslant \gamma < \alpha$ the process $\{|X_t - x|^\gamma\}$ is a submartingale whose Doob-Meyer decomposition is given by (36).
2. For $(\alpha-1)/2 < \gamma < \alpha - 1$ the process $\{|X_t - x|^\gamma\}$ is a Dirichlet process whose canonical decomposition is given by (40).

These results do not discuss whether $\{(X_t - x)^{\gamma,*}\}$, the symmetric power of order γ, i.e.,

$$(X_t - x)^{\gamma,*} := \operatorname{sgn}(X_t - x)\,|X_t - x|^\gamma, \tag{55}$$

is or is not a semimartingale or a Dirichlet process. In the present section it is seen that this question can be answered completely relying on some results in Fitzsimmons and Getoor [8,9], see also K. Yamada [19]. Let $x = 0$ in (55) and introduce the principal value integral (cf. (9))

$$\text{p.v.} \int_0^t \frac{ds}{X_s^{\theta,*}} := \int_0^\infty \frac{dz}{z^\theta} \left(L_t^z - L_t^{-z}\right),$$

where by the Hölder continuity (18) the integral is well defined for $\theta < (\alpha-1)/2$.

Proposition 6. a) *For $\alpha - 1 < \gamma < \alpha$ the process $\{X_t^{\gamma,*}\}$ is a semimartingale.*
b) *For $(\alpha-1)/2 < \gamma \leqslant \alpha - 1$ the process $\{X_t^{\gamma,*}\}$ is a Dirichlet process and not a semimartingale.*
c) *In both cases the unique canonical decomposition of the process can be written as*

$$X_t^{\gamma,*}\, r^+(\alpha,\gamma) = N_t^{\gamma,*} + c_1(\alpha)\ \text{p.v.} \int_0^t \frac{ds}{X_s^{\alpha-\gamma,*}}, \tag{56}$$

where

$$r^+(\alpha,\gamma) = \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(|1-x|^{\alpha-1} - (1+x)^{\alpha-1}\right)$$

and

$$N_t^{\gamma,*} = \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(N_t^x - N_t^{-x}\right).$$

In particular, for $\gamma = \alpha - 1$

$$X_t^{\alpha-1,*}\, r^+(\alpha,1) = N_t^{\alpha-1,*} + c_1(\alpha)\ \text{p.v.} \int_0^t \frac{ds}{X_s}. \tag{57}$$

Proof. Because $\{X_t\}$ is a martingale, it follows from the Itô formula for semimartingales (15) that for $1 \leqslant \gamma \leqslant \alpha$ the process $\{X_t^{\gamma,*}\}$ is a semimartingale. The other statements in (a) and (b) are derived from the decomposition (56) which we now verify similarly as (40) in Proposition 5. Hence, we start again from the identity (3) considered at x and $-x$, and write, informally

$$\int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(|X_t - x|^{\alpha-1} - |X_t + x|^{\alpha-1}\right)$$
$$= \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(N_t^x - N_t^{-x}\right) + c_1(\alpha) \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(L_t^x - L_t^{-x}\right). \quad (58)$$

To analyze the integral on the left-hand side notice that

$$R(a;\alpha,\gamma) = \int_0^\infty \frac{dx}{x^{\alpha-\gamma}} \left(|a - x|^{\alpha-1} - |a + x|^{\alpha-1}\right)$$

is absolutely convergent and

$$R(a;\alpha,\gamma) = a^{\gamma,*}\, r^+(\alpha,\gamma).$$

Now the rest of the proof is very similar to that of Proposition 5b, and is therefore omitted. □

Remark 3. **a)** The increasing process associated with $N^{\gamma,*}$ is given by

$$\langle N_t^{\gamma,*}\rangle_t = (r^+(\alpha,\gamma))^2 \int_0^t ds \int_{\mathbf{R}} \nu(dz) \left((X_s + z)^{\alpha-\gamma,*} - X_s^{\alpha-\gamma,*}\right)^2$$
$$= (r^+(\alpha,\gamma))^2 \int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}} \int_{\mathbf{R}} \nu(dz) \left((1+z)^{\alpha-\gamma,*} - 1\right)^2.$$

We also have by scaling

$$\mathbf{E}\left(\int_0^t \frac{ds}{|X_s|^{\alpha-2\gamma}}\right) = \int_0^t s^{(2\gamma-\alpha)/\alpha}\, ds\, \mathbf{E}\left(|X_1|^{2\gamma-\alpha}\right)$$
$$= \frac{\alpha}{2\gamma}\, t^{2\gamma/\alpha}\, \mathbf{E}\left(|X_1|^{2\gamma-\alpha}\right).$$

b) Since

$$|X_t|^\gamma = (X_t^+)^\gamma + (X_t^-)^\gamma$$

and

$$|X_t|^{\gamma,*} = (X_t^+)^\gamma - (X_t^-)^\gamma$$

it is straightforward to derive the decomposition formulae for $\{(X_t^+)^\gamma\}$ and $\{(X_t^-)^\gamma\}$, and we leave this to the reader.

c) Note how different (57) is in the Brownian case $\alpha = 2$, for which on one hand $\{B_t\}$ is a martingale, and on the other hand

$$\varphi(B_t) = \int_0^t \log|B_s|\, dB_s + \frac{1}{2}\ \text{p.v.} \int_0^t \frac{ds}{B_s}$$

with $\varphi(x) = x\log|x| - x$. For principal values of Brownian motion and extensions of Itô's formula, see Yor [22], [23] and Cherny [5].

References

1. J. Bertoin. *Lévy processes*. Cambridge University Press, Cambridge, UK, 1996
2. A.N. Borodin and I.A. Ibragimov. Limit theorems for functionals of random walks. *Proc. Steklov Inst. Math.*, 195(2):576–590, 1994 (English transl. AMS 1995)
3. E.S. Boylan. Local times for a class of Markov processes. *Illinois J. Math.*, 8:19–39, 1964
4. L. Chaumont and M. Yor. *Exercises in probability; a guided tour from measure theory to random processes via conditioning*. Cambridge University Press, Cambridge, 2003
5. A.S. Cherny. Principal values of the integral functionals of Brownian motion: existence, continuity, and an extension of Itô's formula. In J. Azéma, M Émery, M. Ledoux, and M. Yor, editors, *Séminaire de Probabilités XXXV*, number 1755 in Springer Lecture Notes in Mathematics, pp. 348–370. Springer, Berlin Heidelberg New York, 2001
6. N. Eisenbaum, H. Kaspi, M.B. Marcus, J. Rosen, and Z. Shi. A Ray-Knight theorem for symmetric Markov processes. *Ann. Probab.*, 28:1781–1796, 2000
7. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. *Tables of integral transforms*. McGraw-Hill, New York, 1954
8. P. Fitzsimmons and R.K. Getoor. Limit theorems and variation properties for fractional derivatives of the local time of a stable process. *Annales de l'I.H.P.*, 28:311–333, 1992
9. P. Fitzsimmons and R.K. Getoor. On the distribution of the Hilbert transform of the local time of a symmetric Lévy process. *Ann. Probab.*, 20:1484–1497, 1992
10. H. Föllmer. Dirichlet processes. In D. Williams, editor, *Stochastic integrals*, volume 851 of *Springer Lecture Notes in Mathematics*, pp. 476–478. Springer, Berlin Heidelberg New York, 1981
11. M. Fukushima. *Dirichlet forms and Markov processes*. North-Holland/ Kodansha, Amsterdam/Tokyo, 1980
12. D. Geman and J. Horowitz. Occupation densities. *Ann. Probab.*, 8:1–67, 1980
13. R.K. Getoor and H. Kesten. Continuity of local times for Markov processes. *Compos. Math.*, 24:277–303, 1972
14. N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*, 2nd edition. North-Holland/Kodansha, Amsterdam/Tokyo, 1989
15. T. Jeulin. Ray-Knight's theorem on Brownian local times and Tanaka's formula. In E. Cinlar, K.L. Chung, and R.K. Getoor, editors, *Seminar on stochastic processes, 1983*, Birkhäuser, Basel, 1984
16. P.A. Meyer. Un Cours sur les Intégrales Stochastiques. In C. Dellacherie and P.A. Meyer, editors, *Séminaire de Probabilités X*, number 511 in Springer Lecture Notes in Mathematics. Springer, Berlin Heidelberg New York, 1976
17. K. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, Cambridge, 1999
18. D.N. Shanbhag and M. Sreehari. On certain self-decomposable distributions. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 38:217–222, 1977
19. K. Yamada. Fractional derivatives of local times of α-stable Levy processes as the limits of occupation time problems. In I. Berkes, E. Csáki, and M. Csörgö, editors, *Limit theorems in probability and statistics II*, pp. 553–573. Janos Bolyai Society Publications, Budapest, 2002

20. T. Yamada. Tanaka formula for symmetric stable processes of index α, $1 < \alpha < 2$. *Unpublished manuscript*, 1997
21. M. Yor. Un exemple de processus qui n'est pas une semimartingale. *Astérisque; Temps locaux*, 52–53:219–222, 1978
22. M. Yor. Sur la transformée de Hilbert des temps locaux browniens et une extension de la formule d'Itô. In J. Azéma and M. Yor, editors, *Séminaire de Probabilités XVI*, number 920 in Springer Lecture Notes in Mathematics, pp. 238–247. Springer, Berlin Heidelberg New York, 1982
23. M. Yor. *Some aspects of Brownian motion. Part II: some recent martingale problems*. Birkhäuser, Basel, 1997

An Excursion-Theoretical Approach to Some Boundary Crossing Problems and the Skorokhod Embedding for Reflected Lévy Processes

Martijn R. Pistorius

Department of Mathematics, King's College London
Strand, London WC2R 2LS, United Kingdom
e-mail:pistoriu@mth.kcl.ac.uk

Summary. Let X be a spectrally negative Lévy process, reflect X at its supremum $\overline{X}$ and call this process Y. Let τ_a denote the first time Y crosses the level a. Using excursion theory we solve the *problem of Lehoczky* for a spectrally negative Lévy process, that is, we express the joint law of $(\tau_a, \overline{X}_{\tau_a}, Y_{\tau_a-}, \varDelta X_{\tau_a})$ in terms of so-called *scale functions* that also turn up in the solution of the two-sided exit problem, thereby extending results of Avram et al. [2], who solved for the joint law of (τ_a, Y_{τ_a}). Next we obtain an explicit and non-randomised solution to the *Skorokhod embedding problem of* Y: we find a stopping time T such that $Y_T \sim \nu$ for a measure ν on $(0, \infty)$ without atoms.

Key words: Lévy process, Itô excursion theory, First passage, Skorokhod embedding, Problem of Lehoczky

1 Introduction

A spectrally negative Lévy process is a real-valued random process with stationary independent increments which has no positive jumps. Such processes are frequently encountered in the context of the theories of dams, queues, insurance risk and continuous branching processes; see for example [10, 8, 9, 21]. More recently (e.g. [2]), these processes have also been used to build models in mathematical finance.

In the article [2], Avram et al. studied the first exit from a finite interval of a spectrally negative Lévy process reflected at its supremum. More specifically, using Itô excursion theory they expressed the joint Laplace transform of the first exit time and position in terms of scale functions which also turn up in the solution of the *two-sided exit* problem. The aforementioned stopping time can be identified in the literature of fluid models as the *time to buffer-overflow* (see for example [1,11]). It is worth mentioning that, for the cases of skip-free random walks and Brownian motion, Kennedy [12] used martingale methods to compute the Laplace transform of this buffer overflow time. Kennedy's analysis was extended to the case of diffusions by Azéma and Yor [3] and more recently a martingale proof was given by Nguyen and Yor [14] for the form of the Laplace transform of this stopping time in the case of spectrally negative Lévy processes.

In this note we show how excursion theory can be used to find the joint law of the position of the reflected process just before exit, the overshoot, the time of buffer-overflow and the value of the supremum of the Lévy process at exit, thereby extending the results of Avram et al. [2]. In the literature the same problem has been studied before in a Brownian motion or a diffusion setting by Lehoczky [13] and Azéma and Yor [3], respectively (and was labelled *Lehoczky's problem* by the latter).

It turns out that the solution of Lehoczky's problem is intimately connected to a solution of the *Skorokhod embedding problem* for Y. This problem consists in finding, for a given measure ν, a stopping time T_* such that $Y_{T^*} \sim \nu$ where the stopping time T^* is 'small', e.g. integrable. The Skorokhod embedding problem has received quite some attention in the literature, going back to Skorokhod [24] who posed and solved the embedding problem of a standard Brownian motion. See Obłój [16] for an extensive review of the literature on this topic. The explicit constructions in the literature mainly concern processes with continuous paths such as diffusions and continuous martingales. See, however, Bertoin and Le Jan [7] who construct a new class of embeddings of Hunt processes at a regular point. In the literature at least two different approaches have been followed to tackle the embedding problem.

The first approach, initiated by Azéma and Yor [3], amounts to using the *Kennedy martingale* to construct a simple, explicit solution to the Skorokhod embedding problems of a Brownian motion and of a continuous martingale. Pedersen and Peškir [18] extended their results to the case of diffusions.

In the second approach Itô excursion theory is employed to show that the process stopped at a candidate stopping time has the required distribution. In Rogers [22,23] excursion arguments are given for the construction of Azéma–Yor and for the related embedding problem of the terminal value and the maximum of a continuous martingale. See also Obłój and Yor [17] for work on the embedding of the age of an excursion of Brownian motion and Obłój [15] for embeddings of continuous functionals of Markovian excursions. We will pursue the second approach to find a Skorokhod embedding of Y, as it is in spirit closer to the first part of the paper.

The rest of the paper is organised as follows. In Section 2 we introduce notation and review existing results concerning two-sided exit problems of spectrally negative Lévy processes and prove some preliminary results. In Section 3 an expression is derived for the aforementioned joint law and and for the law of the first passage time of Y over a boundary depending on the running supremum of X. Finally, in Section 4 the Skorokhod embedding of Y is presented.

2 Preliminaries

In this section we set the notation and review standard results fluctuation theory for spectrally negative Lévy processes. For more background, we refer to Bingham [8] or Bertoin [5, Chapter VII].

2.1 Setting

Let $X = (X_t, t \geqslant 0)$ be a spectrally negative Lévy process defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, a filtered probability space where the filtration $\mathbf{F}$ satisfies the usual conditions. To avoid trivialities, we exclude the case that X has monotone paths. Since the jumps of X are all non-positive, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geqslant 0$ and is given by $\psi(\theta) = t^{-1} \log \mathbb{E}[\mathrm{e}^{\theta X_t}]$ for some function $\psi(\theta)$. The function ψ is well defined at least on the positive half-axis where it is strictly convex with the property that $\lim_{\theta\to\infty} \psi(\theta) = +\infty$. Moreover, ψ is strictly increasing on $[\Phi(0), \infty)$, where $\Phi(0)$ is the largest root of $\psi(\theta) = 0$. We shall denote the right-inverse function of ψ by $\Phi : [0, \infty) \to [\Phi(0), \infty)$.

We conclude this subsection by introducing for any Lévy process the family of martingales

$$(\exp\left(cX_t - \psi(c)t\right), t \geqslant 0)$$

defined for any c for which $\psi(c) = \log \mathbb{E}[\exp cX_1]$ is finite, and further the corresponding family of measures $\{\mathbb{P}^c\}$ on $(\Omega, \mathcal{F}, \mathbf{F})$ with Radon-Nikodym derivatives:

$$\left.\frac{\mathrm{d}\mathbb{P}^c}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(c(X_t - X_0) - \psi(c)t\right). \tag{1}$$

For all such c (including $c = 0$) the measure $\mathbb{P}^c_x$ will denote the translation of $\mathbb{P}^c$ under which $X_0 = x$. Under the measure $\mathbb{P}^c$ the process X is still a Lévy process, but with different characteristics (that depend on c).

2.2 Scale functions and exit problems

Bertoin [6] studied two-sided exit problems of spectrally negative Lévy processes and made the connection with a class of functions known as q-scale functions. Here we review the concept of scale functions and the two-sided exit results. For $q \geqslant 0$, there exists a function $W^{(q)} : [0, \infty) \to [0, \infty)$, called the *$q$-scale function*, that is continuous and increasing with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q). \tag{2}$$

See e.g. [8] for a proof. Further, we shall use the notation $W_c^{(q)}(x)$ to mean the q-scale function as defined above for $(X, \mathbb{P}^c)$ and write $\Phi_c(q)$ for the largest root of $\psi_c(\theta) = q$. For every $x \geqslant 0$, we can extend the mapping $q \mapsto W_v^{(q)}(x)$ to the complex plane by the identity

$$W_v^{(q)}(x) = \sum_{k \geqslant 0} q^k W_v^{\star(k+1)}(x), \tag{3}$$

where $W_v^{\star k}$ denotes the kth convolution power of $W_v = W_v^{(0)}$. The convergence of this series is plain from the inequality

$$W_v^{\star k+1}(x) \leqslant x^k W_v(x)^{k+1}/k!, \qquad x \geqslant 0, k \in \mathbb{N},$$

which follows from the monotonicity of W_v. Closely related to $W^{(q)}$ is the function $Z^{(q)}$ given by

$$Z^{(q)}(x) = 1 + q\overline{W}^{(q)}(x),$$

where $\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(z)\mathrm{d}z$. Keeping with our earlier convention, we shall use $Z_c^{(q)}(x)$ in the obvious way. Just like $W^{(q)}$, the function $Z^{(q)}$ may be characterised by its Laplace transform and continuity on $(0, \infty)$. Indeed, one can check that

$$\int_0^\infty \mathrm{e}^{-\theta x} Z^{(q)}(x)\mathrm{d}x = \psi(\theta)/\theta(\psi(\theta) - q), \qquad \theta > \Phi(q). \tag{4}$$

Remark 1 We have the following relationship between scale functions

$$W^{(u)}(x) = \mathrm{e}^{vx} W_v^{(u-\psi(v))}(x)$$

for v such that $\psi(v) < \infty$. To see this, simply take Laplace transforms of both sides. By analytical extension, we see that the identity remains valid for all $u \in \mathbb{C}$.

Example. A stable Lévy process X with index $\alpha \in (1, 2]$ has cumulant $\psi(\theta) = \theta^\alpha$; its scale functions have been shown in [4] to be equal to

$$W^{(q)}(x) = \alpha x^{\alpha-1} E'_\alpha(qx^\alpha), \qquad Z^{(q)}(x) = E_\alpha(qx^\alpha), \tag{5}$$

where E_α is the Mittag-Leffler function with parameter α

$$E_\alpha(y) = \sum_{n=0}^\infty \frac{y^n}{\Gamma(1 + \alpha n)}, \qquad y \in \mathbb{R}. \tag{6}$$

In the case $\alpha = 2$, the process $X/\sqrt{2}$ is a Brownian motion and $W^{(q)}, Z^{(q)}$ reduce to

$$W^{(q)}(x) = q^{-1/2} \sinh(x\sqrt{q}), \qquad Z^{(q)}(x) = \cosh(x\sqrt{q}). \tag{7}$$

Hence for a standard Brownian motion $W^{(q)}, Z^{(q)}$ are found by replacing (x, q) by $(2x, q/2)$ in (7).

The q-scale function turns up in the solution of the two-sided exit problem. Defining $T^-(b)$ and $T^+(b)$ as

$$T^-(b) = \inf\{t \geqslant 0 : X_t < -b\} \text{ and } T^+(b) = \inf\{t \geqslant 0 : X_t > b\}$$

the first times that X down-crosses the level $-b$ and up-crosses the level b, respectively, and setting $T_{0,a} = T^-(0) \wedge T^+(a)$, the first exit time of X from $[0, a]$, the two-sided result reads as

$$\mathbb{E}_x \left[\mathrm{e}^{-qT_{0,a}} I_{\left(X_{T_{0,a}}=a\right)} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \qquad x \in [0, a], \tag{8}$$

$$\mathbb{E}_x \left[\mathrm{e}^{-qT_{0,a}} I_{\left(X_{T_{0,a}}\leqslant 0\right)} \right] = Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)}, \qquad x \in [0, a], \tag{9}$$

where I_A denotes the indicator of the set A. The q-scale function $W^{(q)}$ is also closely connected to the the q-potential measure U^q of X,

$$U^q(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-qt} \mathbb{P}(X_t \in \mathrm{d}x) \mathrm{d}t.$$

Indeed, U^q is absolutely continuous with density u^q, say, and the q-scale function $W^{(q)}$ and the potential density u^q are related by the identity

$$W^{(q)}(x) = \Phi(q) \exp(\Phi(q)x) - u^q(-x) \qquad \text{for a.e. } x > 0,\ q > 0, \tag{10}$$

as shown in e.g. [20] for a proof. Furthermore, in e.g. [19, Lemma 1] it is shown that the q-scale functions $W^{(q)}$ are left- and right-differentiable on $(0, \infty)$; we denote the right- and left-derivative of $W^{(q)}(\cdot)$ by $W_+^{(q)\prime}(\cdot)$ and $W_-^{(q)\prime}(\cdot)$, respectively. If a Gaussian component is present, that is, $\sigma > 0$, these results can be extended as follows:

Lemma 1 *The following hold true:*

(i) If $\sigma > 0$ and $q \geqslant 0$, then $W_+^{(q)\prime}(0) = 2/\sigma^2$.
(ii) If $\sigma > 0$, $\mathbb{P}(X_t \in \mathrm{d}x)$ is absolutely continuous for $t > 0$ and there exists a C^∞ version of its density.
(iii) For $q > 0$, $W^{(q)}(\cdot)$ is $C^\infty((0, \infty))$.

Proof. (i) In [19, Lemma 4(ii)] it is shown that $W'_+(0)$ is equal to $2/\sigma^2$, which combined with the expansion (3), implies the statement.

(ii) Since we may write $X = \sigma B + R$, where B is a Brownian motion and R is a spectrally negative Lévy process, without Gaussian component and independent of B, it follows that for $t > 0$

$$\mathbb{P}(X_t \in \mathrm{d}x) = \left\{ \int_{-\infty}^\infty \frac{1}{\sigma\sqrt{t}} \varphi\left(\frac{x-y}{\sigma\sqrt{t}}\right) \mathbb{P}(R_t \in \mathrm{d}y) \right\} \mathrm{d}x, \tag{11}$$

where φ is the standard normal density function. Since the integral on the right-hand side of (11) is bounded for fixed $t > 0$, it follows that $\mathbb{P}(X_t \in \mathrm{d}x)$ is absolutely continuous for $t > 0$ with bounded Radon-Nikodym derivative

$$p(t,x) = \frac{\mathbb{P}(X_t \in \mathrm{d}x)}{\mathrm{d}x} = \mathbb{E}\left[\varphi\left(\frac{x - R_t}{\sigma\sqrt{t}}\right)\right].$$

Moreover, φ is infinitely differentiable on the real line and all its derivatives are bounded, so that (11), in conjunction with bounded convergence, implies that $p(t,\cdot)$ is $C^\infty(\mathbb{R})$ for $t > 0$.

(iii) We claim that there exists a version of the potential density u^q that is C^∞ on $(-\infty, 0)$. Note that, by (10), assertion (iii) follows once this claim has been proved. Since R_t tends to 0 almost surely as $t \downarrow 0$ by right-continuity of its paths and since $\varphi(\cdot/\sqrt{t})$ converges to the delta distribution in zero (in distributional sense), it follows that $p(t,x)$ and all its x derivatives tend to zero for $x \neq 0$ as $t \downarrow 0$. An application of the bounded convergence theorem yields then that $\int_0^\infty \mathrm{e}^{-qt} p(t,\cdot)\mathrm{d}t$, which is a version of the potential density, is C^∞ on $\mathbb{R}\backslash\{0\}$. □

2.3 Itô excursion theory

In this section we shall review some of the Itô excursion theory using standard notation (see Bertoin [5, Chapter IV]). To this end, we write $Y = \overline{X} - X$ for X reflected at its supremum and we recall the notion of the excursion process $e = (e_t, t \geqslant 0)$ of Y, which takes values in the space of excursions

$$\mathcal{E} = \{f \in D[0,\infty) : f \geqslant 0,\ \exists\, \zeta = \zeta(f) \text{ such that} f(\zeta) = 0\}$$

of càdlàg functions f with a generic lifetime $\zeta = \zeta(f)$ and is given by

$$e_t = (Y_s, L^{-1}(t^-) \leqslant s < L^{-1}(t)) \qquad \text{if } L^{-1}(t^-) < L^{-1}(t),$$

where L^{-1} is the right-inverse of a local time L of Y at zero; else $e_t = \partial$, some isolated point. We take the running supremum $\overline{X}$ to be this local time L (cf. [5, Chapter VII]). The space $\mathcal{E}$ is endowed with the Itô excursion measure n. A famous theorem of Itô states that e is a Poisson point process with characteristic measure n, if Y is recurrent; otherwise $(e_t, t \leqslant L(\infty))$ is a Poisson point process stopped at the first excursion of infinite lifetime. For an excursion $\epsilon \in \mathcal{E}$ its supremum is denoted by $\overline{\epsilon}$. By $\epsilon_g = (Y_{g+t}, t \leqslant \zeta_g)$ is meant the excursion of Y with left-end point g, where ζ_g and $\overline{\epsilon}_g$ denote its lifetime and supremum, respectively.

Using excursion theory the following result, concerning the asymptotic behaviour of the ratio of $W^{(q)}$ and its derivative, can be shown to hold true:

Lemma 2 *For $q \geqslant 0$, $W_+^{(q)\prime}(a)/W^{(q)}(a)$ tends to $\Phi(q)$ as $a \to \infty$.*

Proof. Assume that either $q > 0$ or $q = 0$ and X does not oscillate. Since $W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x)$, the assertion is proved once we show that $W'_{\Phi(q)+}(x) \to 0$ if $x \to \infty$. Itô excursion theory implies that the heights of the excursions e_t form a Poisson point process on the positive half-axis with characteristic measure $n(\overline{\epsilon} > \cdot)$. From [19, Lemma 1], we read off that this characteristic measure is given by $n(\overline{\epsilon} > a) = W'_+(a)/W(a)$. In particular, it holds that

$$\mathbb{P}(\overline{\epsilon} > a + k | \overline{\epsilon} > a) = \frac{n(\overline{\epsilon} > a + k)}{n(\overline{\epsilon} > a)} = \frac{W'_+(a+k)}{W(a+k)} \frac{W(a)}{W'_+(a)}, \qquad k > 0. \quad (12)$$

Since $\psi'_{\Phi(q)}(0) > 0$, X drifts to infinity under the measure $\mathbb{P}^{\Phi(q)}$ and excursions of infinite height do not occur and the probability in (12) (with the original measure $\mathbb{P}$ replaced by $\mathbb{P}^{\Phi(q)}$) converges to zero as $k \to \infty$. Since in this case $W_{\Phi(q)}(\infty) = \lim_{x\to\infty} W_{\Phi(q)}(x)$ is finite it follows that $(W_{\Phi(q)})'_+(x)$ converges to zero if x tends to infinity.

If $q = 0$ and X oscillates, $\sup_{t \geqslant 0} X_t$ is infinite and, as for an infinite excursion $\sup_{t \geqslant 0} X_t$ is finite, we deduce that infinite excursions do not occur in this case, so that $n(\overline{\epsilon} = \infty) = 0$ and $n(\overline{\epsilon} > a) = W'_+(a)/W(a)$ converges to zero as $a \to \infty$ and the proof of the Lemma is done. □

2.4 The renewal measure and creeping

In this subsection we review a few results concerning the creeping of X. For a comprehensive treatment we refer the reader to [5, Chapter VI]. We say that a Lévy process 'creeps downward' (resp., 'creeps upward') over a level $x < 0$ (resp., $x > 0$) if the first time it down-crosses (resp., up-crosses) the level x is not by a jump. A Lévy process can creep both upwards and downwards if and only if its Gaussian component is not zero. This implies in particular that a spectrally negative Lévy process creeps downward if and only if it is either a deterministic negative drift or it has a positive Gaussian coefficient. The creeping of a Lévy process is closely connected to its renewal function. The *q-renewal function* $\widehat{V}^q$ of $-X$ (the renewal function of the process $-X$ killed at an independent exponential time with mean q^{-1}) is characterised by the fact that it is increasing, right-continuous and has Laplace transform given by

$$\lambda \int_0^\infty \mathrm{e}^{-\lambda x} \widehat{V}^q(x) \mathrm{d}x = c \frac{\lambda - \Phi(q)}{\psi(\lambda) - q}$$

where $c > 0$ is some normalising constant. Inverting this transform $\widehat{V}^q$ can be expressed in terms of the q-scale function by

$$\widehat{V}^q(x) = c \left(W^{(q)}(x) - \Phi(q) \overline{W}^{(q)}(x) \right).$$

By Lemma 1(ii) it follows $\widehat{V}^{(q)}(\cdot)$ is continuously differentiable on $(0,\infty)$ if $\sigma > 0$. Denoting by $\widehat{v}^q(x)$ the derivative of $\widehat{V}^q(x)$ in $x > 0$ if $\sigma > 0$, it follows that for $\sigma > 0$

$$\widehat{v}^q(x) = W^{(q)\prime}(x) - \Phi(q)W^{(q)}(x), \qquad x > 0. \tag{13}$$

The precise relationship of the probability of creeping and the renewal function follows then from a result of Miller (1973) (applied to $-X$ killed at an independent exponential time with parameter q)

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-qT^-(a)} I_{(X_{T^-(a)}=-a)}\right] &= \widehat{v}^q(a)/\widehat{v}^q(0^+) \\ &= \frac{\sigma^2}{2}\left(W^{(q)\prime}(a) - \Phi(q)W^{(q)}(a)\right),\end{aligned} \tag{14}$$

where $\widehat{v}^q_+(0) = \lim_{x\downarrow 0}\widehat{v}^q(x)$ and the expression is understood to be equal to zero if $\sigma = 0$. The probability that X leaves the interval $[0,a]$ by hitting 0 can now be determined in terms of scale functions and their derivatives, complementing the two-sided exit results (8) and (9):

Proposition 1 *For $x \in [0,a]$ we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_{0,a}} I_{(X_{T_{0,a}}=0)}\right] = \frac{\sigma^2}{2}\left(W^{(q)\prime}(x) - \frac{W^{(q)\prime}(a)}{W^{(q)}(a)}W^{(q)}(x)\right), \tag{15}$$

where the expression is understood to be equal to 0 *if* $\sigma = 0$.

Proof. By the strong Markov property of X it follows that the left-hand side of (15) is equal to

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-qT^-(0)} I_{(T^-(0)<T^+(a), X_{T^-(0)}=0)}\right] \\ &\quad = \mathbb{E}_x\left[\mathrm{e}^{-qT^-(0)} I_{(X_{T^-(0)}=-0)}\right] \\ &\qquad + \mathbb{E}_x\left[\mathrm{e}^{-qT^+(a)} I_{(T^-(0)>T^+(a))}\right]\mathbb{E}_0\left[\mathrm{e}^{-qT^-(0)} I_{(X_{T^-(0)}=0)}\right].\end{aligned}$$

Inserting the two-sided exit probability (8) and the expression (14) combined with (13) and Lemma 1(i) yields then the expression on the right-hand side of (15). □

3 First passage of reflected Lévy processes

Let

$$\overline{X}_t = \max\left\{s, \sup_{0\leqslant u\leqslant t} X_u\right\}$$

be the non-decreasing process representing the current maximum of X given that at time zero the maximum is equal to s. Further, let $\mathbb{P}_{s,x}$ refer to the

Lévy process X which at time zero is given to have a current maximum s and position x. In this section the focus is on the process Y, X reflected at its supremum $\overline{X}$, first crossing a positive level a and the related stopping time

$$\tau_a := \inf\{t \geqslant 0 : Y_t > a\}$$

defined for $a > 0$. Denoting by δ_s the delta measure in s, by Λ the Lévy measure of X and writing for $u \geqslant 0$

$$F_{q,a}(u) = \exp\left(-u\frac{W_+^{(q)\prime}(a)}{W^{(q)}(a)}\right),$$

we find the following expression for the joint law of τ_a, Y_{τ_a-}, $\overline{X}_{\tau_a}$ and ΔX_{τ_a} in the case that Y crosses the level a by a jump:

Theorem 1 *For $q, z \geqslant 0$, $h < 0$, $x, y \in [0, a)$ it holds that*

$$\begin{aligned}
&\mathbb{E}_{s,x}\left[\mathrm{e}^{-q\tau_a} I_{\left(Y_{\tau_a-}\in \mathrm{d}y, \overline{X}_{\tau_a}\in \mathrm{d}m, \Delta X_{\tau_a}\in \mathrm{d}h\right)}\right] \\
&\quad = I_{(y-h>a)}\Lambda(\mathrm{d}h)\left[\delta_s(\mathrm{d}m)\left(\tfrac{W^{(q)}(a+x-s)}{W^{(q)}(a)}W^{(q)}(y) - W^{(q)}(y+x-s)\right)\mathrm{d}y\right. \\
&\qquad \left. + \tfrac{W^{(q)}(a+x-s)}{W^{(q)}(a)}\left(W^{(q)\prime}(y) - \tfrac{W_+^{(q)\prime}(a)}{W^{(q)}(a)}W^{(q)}(y)\right)F_{q,a}(m-s)\mathrm{d}m\mathrm{d}y\right].
\end{aligned}$$

The next result is a complement to the previous one and considers the case that Y creeps over the level a, $Y_{\tau(a)} = a$.

Theorem 2 *For $a > 0$, $m \geqslant s \geqslant x$,*

$$\begin{aligned}
&\mathbb{E}_{s,x}\left[\mathrm{e}^{-q\tau(a)} I_{(Y_{\tau(a)}=a, \overline{X}_{\tau(a)}\in \mathrm{d}m)}\right] \\
&\quad = \frac{\sigma^2}{2}\delta_s(\mathrm{d}m)\left[W^{(q)\prime}(a-s+x) - \tfrac{W^{(q)\prime}(a)}{W^{(q)}(a)}W^{(q)}(a-s+x)\right] \\
&\qquad + \frac{\sigma^2}{2}\tfrac{W^{(q)}(a-s+x)}{W^{(q)}(a)}F_{q,a}(m-s)\left[\frac{W^{(q)\prime}(a)^2}{W^{(q)}(a)} - W^{(q)\prime\prime}(a)\right]\mathrm{d}m,
\end{aligned}$$

where the expression is understood to be equal to 0 if $\sigma = 0$.

By integrating out the supremum $\overline{X}_{\tau_a}$ and jump size ΔX_{τ_a} at τ_a, we obtain from above result the joint Laplace transform of $(\tau_a, \overline{X}_{\tau_a}, X_{\tau_a})$, which is an extension of [2, Theorem 1]:

Corollary 1 *For $u, v, w \geqslant 0$, one has that*

$$\begin{aligned}
\mathbb{E}_{s,x}\left[\mathrm{e}^{-u\tau_a - v\overline{X}_{\tau_a} + wX_{\tau_a}}\right] = \mathrm{e}^{-vs}\Bigg(& Z_w^{(\tilde{p})}(a+x-s) \\
& -W_w^{(\tilde{p})}(a+x-s)\frac{\tilde{p}W_w^{(\tilde{p})}(a) + vZ_w^{(\tilde{p})}(a)}{(W_w^{(\tilde{p})\prime})_+(a) + vW_w^{(\tilde{p})}(a)}\Bigg),
\end{aligned} \tag{16}$$

where $\tilde{p} = u - \psi(w)$.

In particular, by choosing $w = v$ we find back the identity of [2, Theorem 1] and by choosing $u = w = 0$ and $s = x$ we see that the supremum $\overline{X}_{\tau_a}$ under $\mathbb{P}_{0,0}$ is exponentially distributed with parameter $W'_+(a)/W(a)$. If X is a diffusion or local martingale, $\overline{X}_{\tau_a}$ is also exponentially distributed, as shown in [3, 18].

Proof of Theorem 1. By the compensation formula (see e.g. [5, Section O.5]) applied to the Poisson point process $(\Delta X_t, t \geqslant 0)$ we find that for $h < 0$

$$\begin{aligned}
&\mathbb{E}_{s,x}\left[\mathrm{e}^{-q\tau_a} I_{\left(Y_{\tau_a-}\in \mathrm{d}y,\ \Delta X_{\tau_a}\in \mathrm{d}h,\ \overline{X}_{\tau_a}\in \mathrm{d}m\right)}\right]\\
&\quad = \mathbb{E}_{s,x}\left[\sum_{t\geqslant 0} \mathrm{e}^{-qt} I_{\left(\overline{X}_t\in \mathrm{d}m,\ \Delta X_t\in \mathrm{d}h,\ Y_{t-}\in \mathrm{d}y,\ \sup_{s<t} Y_s<a, Y_t>a\right)}\right]\\
&\quad = \int \Lambda(\mathrm{d}h) I_{(y-h>a)} \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_{s,x}\left[\overline{X}_t \in \mathrm{d}m,\ Y_{t-}\in \mathrm{d}y,\ \sup_{s<t} Y_s < a\right]\mathrm{d}t.
\end{aligned}$$

Applying the strong Markov property of Y at $\tau_0 = \inf\{t \geqslant 0 : X_t = \overline{X}_t\}$ and noting that $\overline{X}_{\tau_a} = s$ on $\{\tau_0 > \tau_a\}$ under the measure $\mathbb{P}_{s,x}$, we can write the inner integral in previous display as

$$\begin{aligned}
&\mathbb{P}_{s,x}\left[\overline{X}_{\eta(q)} \in \mathrm{d}m, Y_{\eta(q)} \in \mathrm{d}y,\ \sup_{s<\eta(q)} Y_s < a\right]\\
&\quad = \mathbb{P}_{s,x}(\overline{X}_{\eta(q)} \in \mathrm{d}m, Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_0 \wedge \tau_a)\\
&\qquad + \mathbb{P}_{s,x}(\overline{X}_{\eta(q)} \in \mathrm{d}m, Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a, \eta(q) \geqslant \tau_0)\\
&\quad = \delta_s(\mathrm{d}m)\mathbb{P}_{0,x-s}(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_0 \wedge \tau_a)\\
&\qquad + \frac{W^{(q)}(a+x-s)}{W^{(q)}(a)}\mathbb{P}_{0,0}(\overline{X}_{\eta(q)} \in \mathrm{d}(m-s), Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a), \quad (17)
\end{aligned}$$

where $\eta(q)$ is an independent exponential time with parameter q and we substituted the two-sided exit probability (8) in the last line. Since $(Y_t; t < \tau_0)$ has the same law as $(-X_t; t < T_0)$, the first quantity in the second equality of (17) is seen to be equal to the resolvent of X killed upon leaving the finite interval; Suprun [25] showed that it can be expressed in terms of scale functions as

$$\begin{aligned}
&q^{-1}\mathbb{P}_{-x}(-X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T_{0,a})\\
&\quad = \left(\frac{W^{(q)}(a-x)}{W^{(q)}(a)}W^{(q)}(y) - I_{(y>x)}W^{(q)}(y-x)\right)\mathrm{d}y. \qquad (18)
\end{aligned}$$

To evaluate the probability in the second quantity in the second equality of (17), we shall make use of the Master formula of excursion theory (e.g. [5, Corollary IV.11]). Using the notation of Section 2.3 and letting $T_a(\epsilon) = \inf\{t \geqslant 0 : \epsilon(t) > a\}$, we first express this probability in terms of the excursion process of Y and then apply the Master formula (see [2, 19] for similar reasonings) to find that

$$
\begin{aligned}
&\mathbb{P}_{0,0}(\overline{X}_{\eta(q)} \in \mathrm{d}m, Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_a) \\
&\quad = \mathbb{E}\left[\int_0^\infty \sum_g q\,\mathrm{e}^{-qt} I_{\left(\sup_{h<g} \overline{\epsilon}_h \leqslant a, g<t, \overline{X}_g \in \mathrm{d}m\right)} I_{(t<T_a(\epsilon), \epsilon_g(t-g)\in \mathrm{d}y,\ g<t<g+\zeta_g)} \mathrm{d}t\right] \\
&\quad = \mathbb{E}\left[\int_0^\infty \mathrm{d}\overline{X}_s \int_0^\infty \mathrm{d}t \mathrm{e}^{-qs} I_{(s<t, \overline{X}_s\in \mathrm{d}m, \sup_{h<s} \overline{\epsilon}_h \leqslant a)} \right. \\
&\qquad \left. \times\ q\,\mathrm{e}^{-q(t-s)} \int n(\mathrm{d}\epsilon) I_{(\epsilon(t-s)\in \mathrm{d}y,\ t-s<\zeta\wedge T_a(\epsilon))}\right] \\
&\quad = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-qs} I_{(\overline{X}_s\in \mathrm{d}m, \sup_{h<s} \overline{\epsilon}_h \leqslant a)} \mathrm{d}\overline{X}_s\right] \times \int n(\mathrm{d}\epsilon) I_{(\epsilon(\eta(q))\in \mathrm{d}y, \eta(q)<\zeta\wedge T_a(\epsilon))}.
\end{aligned}
$$

The last line of the previous display consists of two factors, let us call them C_1 and C_2. The first factor can be inferred from [2] to be equal to

$$
\begin{aligned}
C_1 &= \mathbb{E}\left[\mathrm{e}^{-qL_v^{-1}} I_{(\sup_{h<L_v^{-1}} \overline{\epsilon}_h < a)}\right] \mathrm{d}m \\
&= \exp\left(-m \frac{W_+^{(q)\prime}(a)}{W^{(q)}(a)}\right) \mathrm{d}m
\end{aligned}
\tag{19}
$$

and the second factor was shown in [19, equation (22)] to be equal to

$$
\begin{aligned}
C_2 &= \int n(\mathrm{d}\epsilon) I_{(\epsilon(\eta(q))\in \mathrm{d}y,\ \eta(q)<\zeta\wedge T_a(\epsilon))} \\
&= q\left(W_+^{(q)\prime}(y) - \frac{W^{(q)\prime}(a)}{W^{(q)}(a)} W^{(q)}(y)\right) \mathrm{d}y.
\end{aligned}
$$

Putting the bits together results in the stated formula. □

Proof of Theorem 2. As in (17) it follows by the strong Markov property of Y applied at τ_0 that

$$
\begin{aligned}
&\mathbb{E}_{s,x}\left[\mathrm{e}^{-q\tau(a)} I_{(Y_{\tau(a)}=a, \overline{X}_{\tau(a)}\in \mathrm{d}m)}\right] \\
&\quad = \delta_s(\mathrm{d}m) \mathbb{E}_{0,x-s}\left[\mathrm{e}^{-qT^-(a)} I_{(X_{T^-(a)}=-a, T^-(a)<T^+(0))}\right] \\
&\qquad + \frac{W^{(q)}(a-s+x)}{W^{(q)}(a)} \mathbb{E}_{0,0}\left[\mathrm{e}^{-q\tau(a)} I_{(Y_{\tau(a)}=a, \overline{X}_{\tau(a)}\in \mathrm{d}(m-s))}\right],
\end{aligned}
$$

where the first factor follows from (19). The first term on the right-hand side follows from Proposition 1. We now turn to the computation of the second term. The Master formula applied to the Poisson point process of excursions of Y away from zero leads to

$$\begin{aligned}
&\mathbb{E}_{0,0}\left[\mathrm{e}^{-q\tau(a)} I_{(Y_{\tau(a)}=a,\overline{X}_{\tau(a)}\in\mathrm{d}m)}\right] \\
&\quad = \mathbb{E}_{0,0}\left[\sum_g \mathrm{e}^{-qg} I_{\left(\sup_{h<g}\overline{\epsilon}_h<a,\overline{X}_g\in\mathrm{d}m\right)} \mathrm{e}^{-q(T_a(\epsilon)-g)} I_{(\epsilon_g(T_a(\epsilon))=a,T_a<g+\zeta_g)}\right] \\
&\quad = \exp\left(-m\frac{W^{(q)\prime}(a)}{W^{(q)}(a)}\right)\mathrm{d}m \int n(\mathrm{d}\epsilon)\mathrm{e}^{-qT_a(\epsilon)} I_{(\epsilon(T_a(\epsilon))=a)}. \qquad (20)
\end{aligned}$$

Next step in the computation is the evaluation of the second factor of (20). We claim that the following identity holds true:

$$\int n(\mathrm{d}\epsilon)\mathrm{e}^{-qT_a} I_{(\epsilon(T_a)=a)} = \lim_{x\downarrow 0} x^{-1}\mathbb{E}\left[\mathrm{e}^{-qT^-(a)} I_{\left(X_{T^-(a)}=-a,\overline{X}_{T^-(a)}\leqslant x\right)}\right]. \qquad (21)$$

This identity is reminiscent of the close relation between n and the Doob h-transform of the measure $\mathbb{P}$ with $h(x) = x$ (see e.g. [5, Chapter VII]). Combining (21) with Proposition 1 yields

$$\int n(\mathrm{d}\epsilon)\mathrm{e}^{-qT_a} I_{(\epsilon(T_a)=a)} = \frac{\sigma^2}{2}\left(\frac{W^{(q)\prime}(a)^2}{W^{(q)}(a)} - W^{(q)\prime\prime}(a)\right).$$

Inserting in (20) leads to the statement in Theorem 2. To finish the proof, we thus have to show the validity of (21). Keeping in mind the close connection between the law $\mathbb{P}^\uparrow$ of X conditioned to stay positive, the excursion measure n and the Doob h-transform with $h(x) = x$, we proceed as follows, invoking again the Master formula of excursion theory

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-qT^-(a)} I_{(X_{T^-(a)}=-a,\overline{X}_{T^-(a)}\leqslant x)}\right] \\
&\quad = \mathbb{E}\left[\sum_g \mathrm{e}^{-qg} I_{\left(\overline{X}_g\leqslant x\right)} \mathrm{e}^{-q(T^-(a)-g)} I_{\left(X_{T^-(a)}=-a,g<T_a<g+\zeta_g\right)}\right] \\
&\quad = \mathbb{E}\left[\int_0^x \mathrm{d}s \mathrm{e}^{-qL_s^{-1}} I_{(\sup_{h<L_s^{-1}}\overline{\epsilon}_h<a+s)}\right. \\
&\qquad \left.\times \int n(\mathrm{d}\epsilon)\mathrm{e}^{-qT(a+s)} I_{(T(a+s)<\zeta,\epsilon(T(a+s))=a+s)}\right] \\
&\quad = \int_0^x \exp\left(-s\frac{W^{(q)\prime}(a+s)}{W^{(q)}(a+s)}\right) \\
&\qquad \times \int n(\mathrm{d}\epsilon)\mathrm{e}^{-qT(a+s)} I_{(T(a+s)<\zeta,\epsilon(T(a+s))=a+s)}\mathrm{d}s, \qquad (22)
\end{aligned}$$

where the last line follows as in (19). By right-continuity of the paths of X and the fact that, if $\sigma > 0$, excursions do not start with a jump a.s. and $W^{(q)}$ is smooth on $(0,\infty)$, $T_{a+s}(\epsilon)$ decreases to $T_a(\epsilon)$ as $s \downarrow 0$, so that the integrand in (22) is right-continuous in $s = 0$ and the claim (21) follows. □

Proof of Corollary 1 (Sketch). We only show that $\mathbb{E}_{s,s}[\mathrm{e}^{-q\tau_a}]$ is given by the right-hand side of (16) with $w = v = 0$. The general formula can be reduced to this one by changes of measure and applying the Markov property of Y at the first time Y reaches 0 (in this connection see e.g. the proof of Theorem 1 in [2]).

If X has a non-zero Gaussian component ($\sigma > 0$), $W^{(q)}$ and $Z^{(q)}$ are C^∞ and it is well known (see e.g. [2, 19]) that

$$\varGamma W^{(q)}(x) = qW^{(q)}(x) \text{ and } \varGamma Z^{(q)}(x) = qZ^{(q)}(x) \text{ for } x > 0, \tag{23}$$

where $\varGamma$ denotes the characteristic operator of X. By integrating the measures given in Theorems 1 and 2, we obtain an expression for $\mathbb{E}_{s,s}[\mathrm{e}^{-q\tau_a}]$. Using the identities (23) it is a matter of algebra to verify that this expression coincides with the right-hand side of (16) and the result is proved for $w = v = 0$, $x = s$ and $\sigma > 0$.

In case $\sigma = 0$, the result follows by considering $X_n = X + \frac{1}{n}B$ and letting $n \to \infty$ (where B denotes an independent Brownian motion). Indeed, as the exponent ψ_n of X_n converges point-wise to the exponent ψ of X, the extended continuity theorem and the definition of the q-scale functions implies that the scale functions $Z_n^{(q)}$, $W_n^{(q)}$ corresponding to X_n and the derivative of the latter, $W_n^{(q)\prime}$, converge point-wise in every continuity point to the scale functions $Z^{(q)}$, $W^{(q)}$ of X and the derivative $W^{(q)\prime}$, respectively. Thus, by the form of the Laplace transform (16), which has been shown to be valid for $\sigma > 0$, $\mathbb{E}_{s,s}[\mathrm{e}^{-q\tau_a(X_n)}]$ converges to $\mathbb{E}_{s,s}[\mathrm{e}^{-q\tau_a(X)}]$ as $n \to \infty$ if a is a continuity point of $W_+^{(q)\prime}$ (recalling that $W^{(q)}$ and $Z^{(q)}$ are continuous on $(0, \infty)$). If a is not a continuity point, it follows by approximation that $\mathbb{E}_{s,s}[\mathrm{e}^{-q\tau_a(X)}]$ is equal to (16). Thus the identity (16) also holds true for $\sigma = 0$ and $w = v = 0 = s - x$. □

3.1 First passage over non-constant boundaries

Let $I \subset [0, \infty)$ be some interval of the non-negative half-axis and let $\varphi : I \to \mathbb{R}$ be a function of bounded variation that is right-continuous with left-hand limits. In this subsection we consider the first time the process Y hits or crosses a boundary $\varphi(\overline{X}_\cdot)$ that depends on the running supremum $\overline{X}$, that is, we examine the stopping times $T = T_\varphi$ and $T' = T'_\varphi$ given by

$$T_\varphi = \inf\{t \geqslant 0 : Y_t = \phi(\overline{X}_t)\}, \qquad T'_\varphi = \inf\{t \geqslant 0 : Y_t > \phi(\overline{X}_t)\}.$$

Note that if we take φ to be constant we are back in the setting of the previous section. Below we shall solve the *problem of Lehoczky* for a spectrally negative Lévy process X, that is, we shall employ excursion theory to solve for the law of $(\sigma, Y_\sigma, \overline{X}_\sigma)$ for σ equal to the stopping times T or T'. In the next section it is shown that these results provide the key to a solution of the Skorokhod embedding problem of Y.

Proposition 2 *For $0 \leqslant x < M$, $u, v > 0$ and $p = u - \psi(v)$ the following identities hold true:*

$$\mathbb{E}\left[\mathrm{e}^{-uT' - v(\overline{X}_{T'} - X_{T'})} I_{(x \leqslant \overline{X}_{T'} \leqslant M)}\right] = \int_{[x,M]} \mathrm{d}s \ \exp\left(-\int_0^s \frac{W_+^{(u)\prime}(\varphi(t))}{W^{(u)}(\varphi(t))}\mathrm{d}t\right) \times\left(Z_v^{(p)}(\varphi(s))\frac{W_{v+}^{(p)\prime}(\varphi(s))}{W_v^{(p)}(\varphi(s))} - pW_v^{(p)}(\varphi(s))\right);$$

$$\mathbb{E}\left[\mathrm{e}^{-uT - v(\overline{X}_T - X_T)} I_{(x \leqslant \overline{X}_T \leqslant M)}\right] = \int_{[x,M]} \mathrm{d}s \ \exp\left(-\int_0^s \frac{W_+^{(u)\prime}(\varphi(t))}{W^{(u)}(\varphi(t))}\mathrm{d}t\right) \times \mathrm{e}^{(\Phi(u)-v)\varphi(s)}\left(\frac{W_+^{(u)\prime}(\varphi(s))}{W^{(u)}(\varphi(s))} - \Phi(u)\right).$$

In particular, note that, if X does not drift to $-\infty$ and the function φ is such that

$$\int_0^\infty \frac{W'_+(\varphi(t))}{W(\varphi(t))}\mathrm{d}t = +\infty, \tag{24}$$

the laws of $\overline{X}_{T'}$ and $\overline{X}_T$ coincide and are given by

$$\mathbb{P}(\overline{X}_{T'} \geqslant x, T' < \infty) = \exp\left(-\int_0^x \frac{W'_+(\varphi(t))}{W(\varphi(t))}\mathrm{d}t\right) \tag{25}$$

at every continuity point x of φ. Indeed, since $\Phi(0) = 0$ in this case, equation (25) follows by letting u and v tend to 0 and $M \uparrow \infty$ in Proposition 2. Moreover, from (25) and condition (24) it follows that T and T' are finite a.s. under the mentioned conditions. For future use, we note that Proposition 2 implies the following identities for the first moments of $T = T_\varphi$ and $T' = T'_\varphi$:

Corollary 2 *Write*

$$C(s) = \int_{[0,s]} \left[\frac{(W \star W)'_+(\varphi(u))}{W(\varphi(u))} - \frac{W'_+(\varphi(u))(W \star W)(\varphi(u))}{W(\varphi(u))^2}\right]\mathrm{d}u.$$

(i) Suppose that the limit as s tends to infinity of

$$\mathbb{P}(\overline{X}_{T'} \geqslant s)[\overline{W}(\varphi(s)) + C(s)] \tag{26}$$

exists and denote it by D. If X does not drift to $-\infty$, then $\mathbb{E}[T']$ is given by

$$\mathbb{E}[T'] = \int_0^\infty W(\varphi(s))\mathbb{P}(\overline{X}_{T'} \geqslant s)\mathrm{d}(s - \varphi(s)) + D - \overline{W}(\varphi(0)). \tag{27}$$

(ii) Suppose that the limit as s tends to infinity of

$$\mathbb{P}(\overline{X}_T \geqslant s)[\Phi'(0^+)(\varphi(s)) + C(s)], \tag{28}$$

exists and denote it by E. If X drifts to ∞, then $\mathbb{E}[T]$ *is given by*

$$\mathbb{E}[T] = \Phi'(0^+)\int_0^\infty \mathbb{P}(\overline{X}_T \geqslant s)\mathrm{d}(s-\varphi(s)) + E - \Phi'(0^+)\varphi(0). \tag{29}$$

Proof of Proposition 2. To prove the first identity, we apply again the compensation formula for Poisson point processes to the excursion process of Y and find, by a computation similar to the ones in the previous subsection, that

$$\begin{aligned}
&E\left[\mathrm{e}^{-uT'-v(\overline{X}_{T'}-X_{T'})}I_{(\overline{X}_{T'}\geq x)}\right]\\
&\quad= E\left[\sum_g \mathrm{e}^{-ug}I_{(\overline{X}_g\geqslant x,\overline{\epsilon}_h<\varphi(\overline{X}_h)\forall h<g)}\times \mathrm{e}^{-uT'\circ\theta_g}I_{(\overline{\epsilon}\geqslant\varphi(\overline{X}_g))}\right]\\
&\quad= E\left[\int L(\mathrm{d}s)\mathrm{e}^{-us}I_{(\overline{X}_s\geqslant x,\overline{e}_r<\varphi(\overline{X}_r)\ \forall r<s)}\int n(\mathrm{d}\epsilon)\mathrm{e}^{-uT_{\varphi(\overline{X}_s)}(\epsilon)}I_{(\overline{\epsilon}\geqslant\varphi(\overline{X}_s))}\right]\\
&\quad= \int_x^\infty \mathrm{d}v\ E\left[\mathrm{e}^{-uL^{-1}(v^-)}I_{(\overline{\epsilon}_{L^{-1}(s-)}<\varphi(s)\ \forall s<v)}\right]\\
&\qquad\times\int n(\mathrm{d}\epsilon)\mathrm{e}^{-uT_{\varphi(v)}(\epsilon)}I_{(\overline{\epsilon}\geq\varphi(v))},
\end{aligned} \tag{30}$$

where, as before, we used the notation $T_a(\epsilon) = \inf\{t \geqslant 0 : \epsilon_t > a\}$. Since the heights of the excursions of Y form a Poisson point process h taking values in $(0,\infty)$, the first factor of (30) with $u = 0$ can be interpreted as the probability of h not visiting the set $A_t = \{(s,\epsilon) \in [0,t] \times \mathcal{E} : \overline{e}_s \geqslant \varphi(s)\}$. Denoting by $R_t^{(0)}$ the mass of the set A_t under the characteristic measure of h, $n(h > a) = W'_+(a)/W(a)$, it follows that the aforementioned probability is equal to $\exp(-R_t^{(0)})$, where, for $u \geqslant 0$,

$$R_t^{(u)} = \int_0^t \frac{W_+^{(u)\prime}(\varphi(s))}{W^{(u)}(\varphi(s))}\mathrm{d}s.$$

The case of $u > 0$ can be dealt with by a change of measure, see e.g. [2] for details. The second factor of (30) is computed in [19] and the result found there reads as

$$\int n(\mathrm{d}\epsilon)\left\{\mathrm{e}^{-uT_a-v\epsilon(T_a)}I_{(\overline{\epsilon}\geqslant a)}\right\} = Z_v^{(p)}(a)W_v^{(p)\prime}(a)/W_v^{(p)}(a) - pW_v^{(p)}(a), \tag{31}$$

where $p = u - \psi(v)$. Substituting the formulas into (31) completes the proof of the first identity.

For the second identity, it is straightforward to verify that replacing T' by T and $T_a(\epsilon)$ by $\rho_a(\epsilon) = \inf\{t \geqslant 0 : \epsilon(t) = a\}$, the previous reasoning remains valid up to (30), so that to complete the proof we have to compute the second factor of (30). The strong Markov property of the excursion process implies that

$$\begin{aligned}\int n(\mathrm{d}\epsilon)\mathrm{e}^{-u\rho_a(\epsilon)}I_{(\overline{\epsilon}\geqslant a)} &= \int n(\mathrm{d}\epsilon)\mathrm{e}^{-uT_a(\epsilon)}I_{(\overline{\epsilon}\geqslant a)}\mathbb{E}_{\epsilon(T_a)}[\mathrm{e}^{-uT_a^-}] \\ &= \mathrm{e}^{\Phi(u)a}\int n(\mathrm{d}\epsilon)\mathrm{e}^{-uT_a(\epsilon)-\Phi(u)\epsilon(T_a)}I_{(T_a(\epsilon)<\zeta)} \\ &= \mathrm{e}^{\Phi(u)a}Z^{(0)}_{\Phi(u)}(a)\frac{W^{(0)\prime}_{\Phi(u)}(a)}{W^{(0)}_{\Phi(u)}(a)} \\ &= \mathrm{e}^{\Phi(u)a}\left(\frac{W^{(u)\prime}(a)}{W^{(u)}(a)}-\Phi(u)\right),\end{aligned}$$

where in the second line we substituted (31) with $p = u - \psi(\Phi(u)) = 0$ and in the third line we used Remark 1. □

Proof of Corollary 2. Note first that the series representation (3) implies that

$$\frac{\partial}{\partial u}\frac{W^{(u)\prime}(s)}{W^{(u)}(s)}\bigg|_{u=0} = \frac{(W\star W)'(s)}{W(s)} - \frac{W'(s)(W\star W)(s)}{W(s)^2}. \tag{32}$$

Set v and x equal to zero in the expressions in Proposition 2 and perform then a partial integration. Differentiating the result with respect to u and letting first u tend to zero and then $M\to\infty$, we find the expression as stated where the constants D and E originate from the stock terms of the partial integration. □

4 Skorokhod embedding

In this section we consider the Skorokhod embedding of a probability measure ν on $(0,\infty)$ without atoms in the process Y. Restricting ourselves to the case that X does not drift to $-\infty$, we address the following problem:

Problem. Construct a stopping time T^* with respect to the filtration generated by X such that $Y_{T^*}\sim\nu$.

Below we shall give such a construction and also address the issue of minimality of the constructed stopping time and of the finiteness of its expectation.

The key step in the construction of the embedding stopping time consists in linking the law of $\overline{X}_T$, which was found in the previous section, to that of Y_T. Let $a\geqslant 0$ be the infimum and $b\in(0,\infty]$ be the supremum of the support of ν and write $\bar{\nu}(y)=\nu([y,\infty))$ for the tail of ν and define the function $\psi_\nu:[0,\infty)\to[0,\infty]$ by

$$\psi_\nu(x) = \int_{[a,x]}\frac{W(y)}{W'_+(y)}\frac{\nu(\mathrm{d}y)}{\bar{\nu}(y)} \qquad \text{for } x\in[a,b]$$

and set $\psi_\nu(x) = 0$ for $x \leqslant a$, $\psi_\nu(x) = +\infty$ for $x > b$. If X is a Brownian motion, $W(x) = 2x$ and the form of ψ_ν coincides with the function introduced in Obłój and Yor [17] to solve for the Skorokhod embedding of the length of a Brownian excursion (which was first solved by Vallois [26]). As W/W'_+ is non-negative, the function ψ_ν is non-decreasing, and, moreover, it is continuous as we assumed that ν has no atoms. Denote by

$$\varphi_\nu(u) = \inf\{v \geqslant 0 : \psi_\nu(v) > u\}$$

the right-inverse of ψ_ν (with $\inf \emptyset = +\infty$).

Proposition 3 *If X does not drift to $-\infty$, then*

$$Y_{T^*} \sim \nu \text{ where } T^* = T_{\varphi_\nu}$$

and $T^* < \infty$ a.s. Moreover, (i) If X drifts to $+\infty$ and $c_\nu + d_\nu < \infty$, where

$$c_\nu := \int_0^\infty \frac{W(s)}{W'_+(s)} \nu(\mathrm{d}s), \qquad d_\nu := \int_0^\infty s\nu(\mathrm{d}s), \tag{33}$$

then $\mathbb{E}[T^*]$ is finite and given by $E[T^*] = (c_\nu - d_\nu)\Phi'(0^+)$.
(ii) If X oscillates and has downward jumps, then $E[T^*] = \infty$.
(iii) If X is standard Brownian motion and ν has finite second moment, $E[T^*]$ is finite and given by $\mathbb{E}[T^*] = \int_0^\infty t^2 \nu(\mathrm{d}t)$.

Remark (Minimality). The embedding stopping time T^* is called minimal if the only stopping time S satisfying $S \leqslant T^*$ and $Y_S = Y_T$ is $S = T^*$. In general the stopping time T^* will *not* be minimal for the embedding by the fact that the first passage time is in general strictly smaller than the first hitting time of a positive level by Y. Instead, a weaker form of minimality can be seen to hold true: if a stopping time S satisfies $S \leqslant T^*$ and $Y_S = Y_T$ then $L(S) = L(T)$, where $L = \overline{X}$ is a local time of Y at zero.

Remark (Measures with atoms). Since the distribution under n of the height h of an excursion ϵ is given by $n(h > a) = W'_+(a)/W(a)$, Proposition 3 can be seen to be a special case of Theorem 1 in Obłój [15] on embeddings of functionals of Markovian excursions. Moreover, from Theorem 1 in Obłój [15] it also directly follows that the above result can be extended to include measures ν with atoms (cf. Proposition 10 in Obłój [15]). (Proposition 3 given above and [15, Theorem 1] were obtained independently.)

Proof. If X does not drift to $-\infty$, T^* is finite almost surely (Proposition 2) and we find, by definition of T^* and by (25) since ν has no atoms, that

$$\begin{aligned}
\mathbb{P}(Y_{T^*} \geqslant x) &= \mathbb{P}(\varphi_\nu(\overline{X}_{T^*}) \geqslant x) \\
&= \mathbb{P}(\overline{X}_{T^*} \geqslant \psi_\nu(x)) = \exp\left(-\int_{[0,x]} \frac{W'_+(u)}{W(u)} \mathrm{d}\psi_\nu(u)\right) \\
&= \exp\left(-\int_{[0,x]} \frac{\nu(\mathrm{d}u)}{\overline{\nu}(u)}\right) = \overline{\nu}(x).
\end{aligned}$$

(i) For $\varphi = \varphi_\nu$ and using the change of variables $t = \psi_\nu(s)$, it is straightforward to verify that the limit as $s \to \infty$ of (28) is equal to

$$E = \lim_{t\to\infty} \bar{\nu}(t) \left\{ \Phi'(0)t + \int_0^t \left[\frac{(W \star W)'_+(s)}{W'_+(s)} - \frac{(W \star W)(s)}{W(s)} \right] \frac{\nu(\mathrm{d}s)}{\bar{\nu}(s)} \right\}.$$

Since W is increasing and bounded when X drifts to infinity, it follows that $(W\star W)(s)/W(s)$ is bounded above by a constant times s and $(W\star W)'_+(s)$ by a constant. Thus by the dominated convergence theorem in conjunction with (33), it follows that $E = 0$. The form of $E[T^*]$ then follows by performing the same change of variables $s = \psi_\nu(t)$ in (29), in conjunction with the definition of ψ_ν and a change of order of integration.

(ii) Write O_{φ_ν} for the time needed for Y to hit the boundary after Y first overshot φ_ν (setting O_{φ_ν} equal to 0 if Y crept over the boundary). Then we see that $\mathbb{E}[T^*]$ is equal to $\mathbb{E}[T'_{\varphi_\nu}] + \mathbb{E}[O_{\varphi_\nu}]$. Writing $o(\mathrm{d}x)$ for the overshoot distribution of Y over φ_ν, $Y_{T^*} - \varphi_\nu(\overline{X}_{T^*})$, it follows by the Markov property of Y, the spatial homogeneity of X and the identity $\mathbb{E}[\mathrm{e}^{-qT_b^+}] = \mathrm{e}^{-b\Phi(q)}$ for $b > 0$, that

$$\mathbb{E}[O_{\varphi_\nu}] = \int_0^\infty o(\mathrm{d}x)\mathbb{E}_{-x}[T_0^+] = \Phi'(0^+) \int x o(\mathrm{d}x).$$

The assertion (ii) is proved.

(iii) Since a Brownian motion has no downward jumps, the expectations $E[T_\varphi]$ and $E[T'_\varphi]$ are equal. Recalling that for a Brownian motion $W(x) = 2x$, we note that $\overline{W}(x) = x^2$ and $(W \star W)(x) = 2x^3/3$, so that the right-hand side of (32) is equal to $2x/3$. Taking $\varphi = \varphi_\nu$ and performing the change of variables $s = \psi_\nu(t)$, (26) reads as

$$D = \lim_{t\to\infty} \bar{\nu}(t) \left[t^2 + \frac{2}{3} \int_0^t s^2 \frac{\nu(\mathrm{d}s)}{\bar{\nu}(s)} \right].$$

Since ν has a finite second moment, dominated convergence implies that D is equal to zero. Performing the same change of variables in (27) shows that

$$\begin{aligned} E[T'_{\varphi_\nu}] &= \int_0^\infty W(x)\bar{\nu}(x)\psi_\nu(\mathrm{d}x) - \int_0^\infty W(x)\bar{\nu}(x)\mathrm{d}x \\ &= \int_0^\infty 2x \cdot \bar{\nu}(x) \cdot x \frac{\nu(\mathrm{d}x)}{\bar{\nu}(x)} - \int_0^\infty x^2 \nu(\mathrm{d}x) \\ &= \int_0^\infty x^2 \nu(\mathrm{d}x) \end{aligned}$$

and the proof is finished. □

Examples. Let X be a stable Lévy process with index $\alpha \in (1,2]$ and recall from (5) that $W(x) = \alpha x^{\alpha-1}$ and thus $W(x)/W'(x) = x/(\alpha-1)$ for $x \geqslant 0$. For example, if ν is a Weibull(β) distribution, i.e.

$$\nu([0,x]) = 1 - \exp(-x^{\beta}), \qquad x \geqslant 0,$$

then, for $x > 0$, $\psi_\nu(x)$ is given by

$$\psi_\nu(x) = \frac{\beta}{(\alpha-1)(\beta+1)} x^{\beta+1}.$$

If ν is a uniform distribution on $[0,\theta]$,

$$\nu(\mathrm{d}x)/\mathrm{d}x = 1/\theta, \qquad x \in [0,\theta],$$

then, for $x \in (0,\theta)$, $\psi_\nu(x)$ is equal to

$$\psi_\nu(x) = \frac{1}{\alpha-1}\left[\theta \log\theta - \theta\log(\theta - x) - x\right].$$

If ν has the distribution of the maximum of two independent uniforms on $(0,\theta)$,

$$\nu(\mathrm{d}x)/\mathrm{d}x = 2x/\theta^2, \qquad x \in [0,\theta],$$

then, for $x \in (0,\theta)$, $\psi_\nu(x)$ reads as

$$\psi_\nu(x) = \frac{1}{\alpha-1}\left[\theta \log(\theta + x) - \theta\log(\theta - x) - 2x\right].$$

Remark (Skorokhod embedding in X). The Skorokhod embedding problem of X is closely related to the embedding of Y studied in this section and consists in finding a stopping time T_* such that $X_{T_*} \sim \mu$, where μ is a probability measure on the real line without atoms. We shall now give an outline how to recover the Azéma–Yor embedding of Brownian motion in a compact centered measure from our results in this and the previous section. Write a and b, $a, b \in \mathbb{R}$ for the infimum and supremum of the support of the measure μ, respectively, and consider the fixed point equation for a càdlàg function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \int_{[a,x]} \frac{W(g(v)-v)}{W'_+(g(v)-v)} \frac{\mu(\mathrm{d}v)}{\bar{\mu}(v)}, \qquad a \leqslant x < b, \tag{34}$$

where $g(x) = 0$ for $x < a$ and $g(x) = x$ for $x \geqslant b$ (where $W/W'_+(x)$ is understood to be zero for $x < 0$). The solution, if it exists, is denoted by g_μ and is right-continuous and non-decreasing, as (W/W'_+) is non-negative. In the case of Brownian motion $W(x)/W'(x) = x$ for $x > 0$ as noted before.

Supposing now the measure μ is centered, $\int y\mu(\mathrm{d}y) = 0$, with $\int |y|\mu(\mathrm{d}y) < \infty$, it can then be verified that the unique solution of (34) is given by $g = g_\mu$, where

$$g_\mu(x) = \frac{1}{\bar{\mu}(x)} \int_{[x,\infty)} y\mu(\mathrm{d}y), \qquad a \leqslant x < b,$$

the barycentric function of μ. Note that $g_\mu(x) > x$ for all $x < b$. Denoting by h_μ the right-continuous inverse of g_μ and by $\widetilde{\varphi}_\mu(x) = x - h_\mu(x)$, reasoning as in the proof of Proposition 3 and using the form of the stopping time $T_{\widetilde{\varphi}_\mu}$, it can be verified that $X_{T_*} \sim \mu$, where $T_* = T_{\widetilde{\varphi}_\mu}$. In particular, if μ has finite second moment, one finds back from Corollary 2 that $\mathbb{E}[T_*]$ is finite and given by $\mathbb{E}[T_*] = \int t^2\mu(\mathrm{d}t)$.

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References

1. S. Asmussen (1994) Busy period analysis, rare events and transient behaviour in fluid flow models. *J. Appl. Math. and Stoch. Analysis* **7**, pp. 269–299
2. F. Avram, A.E. Kyprianou and M.R. Pistorius (2004) Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14**, pp. 215–238
3. J. Azéma and M. Yor (1977) Une solution simple au problème de Skorokhod. *Séminaire de Probabilités* **XIII**, pp. 90–115
4. J. Bertoin (1995) On the first exit time of a completely asymmetric Lévy process from a finite interval, *Bull. London Math. Soc.* **28**, pp. 514–520
5. J. Bertoin (1996) *Lévy processes.* Cambridge University Press, Cambridge
6. J. Bertoin (1997) Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.* **7**, pp. 156–169
7. J. Bertoin and Y. Le Jan (1992) Representation of measures by balayage from a regular recurrent point. *Ann. Probab.* **20**, pp. 538–548
8. N.H. Bingham (1975) Fluctuation theory in continuous time. *Adv. Appl. Probab.* **7**, pp. 705–766
9. N.H. Bingham (1976) Continuous branching processes and spectral positivity. *Stoch. Proc. Appl.* **4**, pp. 217–242
10. A.A. Borovkov (1976) *Stochastic processes in queueing theory.* Springer, Berlin Heidelberg New York
11. D. Heath, S. Resnick and G. Samorodnitsky (1998) Heavy tails and long range dependence in on/off processes and associated fluid models. *Math. Oper. Res.* **23**, pp. 145–165
12. D.P. Kennedy (1976) Some martingales related to cumulative sum tests and single-server queues. *Stoch. Proc. Appl.* **4**, pp. 261–269
13. J.P. Lehoczky (1977) Formulas for stopped diffusion processes with stopping times based on the maximum. *Ann. Probab.* **5**, pp. 601–607

14. L. Nguyen and M. Yor (2004) Some martingales associated to reflected Lévy processes. *Séminaire de Probabilités* **XXXVIII**, pp. 42–69
15. J. Obłój (2005) An explicit Skorokhod embedding for functionals of Markovian excursions. *arXiv*: math.PR/0509553
16. J. Obłój (2004) The Skorokhod embedding problem and its offspring. *Probab. Surv.* **1**, pp. 321–392
17. J. Obłój and M. Yor (2004) An explicit Skorokhod embedding for the age of Brownian excursions and Azéma martingale. *Stoch. Proc. Appl.* **110**, pp. 83–110
18. J.L. Pedersen and G. Peškir (2001) The Azéma-Yor embedding in non-singular diffusions. *Stoch. Proc. Appl.* **96**, pp. 305–312
19. M.R. Pistorius (2004) On exit and ergodicity of the completely asymmetric Lévy process reflected at its infimum. *J. Theor. Probab.* **17**, pp. 183–220
20. M.R. Pistorius (2004) A potential theoretical review of some exit problems of spectrally negative Lévy processes. *Séminaire de Probabilités* **XXXVIII**, pp. 30–41
21. N.U. Prabhu (1981) *Stochastic storage processes, queues, insurance risk and dams*. Springer, Berlin Heidelberg New York
22. L.C.G. Rogers (1981) Williams characterisation of the Brownian excursion law: proof and applications. *Séminaire de Probabilités* **XV**, pp. 227–250
23. L.C.G. Rogers (1993) The joint law of the maximum and the terminal value of a martingale. *Probab. Theory Related Fields* **95**, pp. 451–466
24. A. Skorokhod (1965) *Studies in the theory of random processes.* Addison-Wesley, Reading, MA
25. V.N. Suprun (1976) The ruin problem and the resolvent of a killed independent increment process. *Ukranian Math. J.* **28**, pp. 39–45
26. P. Vallois (1983) Le problème de Skorokhod sur **R**: une approche avec le temps local. *Séminaire de Probabilités* **XVII**, pp. 227–239

Note added in proof (May 2007)

Lemma 1 (iii) in the preceding article states that $W^{(q)} \in C^\infty(0,\infty)$ if $\sigma > 0$. This is however not true in general. I am grateful to Ron Doney for pointing out this error. Chan and Kyprianou [1] proved that, if $\sigma > 0$, $W^{(q)} \in C^2(0,\infty)$ for each fixed $q \geq 0$. In Theorem 7 of [1], sufficient conditions are given for $W = W^{(0)} \in C^\infty(0,\infty)$ to hold.

The error does not affect the validity of the rest of the paper. Indeed, Corollary 1, the only result where Lemma 1 (iii) was invoked in the proof, can be directly proved by combining a change of measure with Theorem 1 in Avram et al. (2004).

Reference

[1] Chan, T. and Kyprianou, A.E. (2007) Smoothness of scale functions for spectrally negative Lévy processes. Preprint.

The Maximality Principle Revisited: On Certain Optimal Stopping Problems

Jan Obłój*

Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6
4, pl. Jussieu, Boîte 188, 75252 Paris Cedex 05, France
and
Faculty of Mathematics, Warsaw University
Banacha 2, 02-097 Warszawa, Poland
e-mail: obloj@mimuw.edu.pl

Summary. We investigate in detail works of Peskir [15] and Meilijson [10] and develop a link between them. We consider the following optimal stopping problem: maximize $V_\tau = \mathbb{E}\big[\phi(S_\tau) - \int_0^\tau c(B_s)ds\big]$ over all stopping times with $\mathbb{E}\int_0^\tau c(B_s)ds < \infty$, where $S = (S_t)_{t\geqslant 0}$ is the maximum process associated with real valued Brownian motion B, $\phi \in C^1$ is non-decreasing and $c \geqslant 0$ is continuous. From work of Peskir [15] we deduce that this problem has a unique solution if and only if the differential equation

$$g'(s) = \frac{\phi'(s)}{2c(g(s))(s - g(s))}$$

admits a maximal solution $g_*(s)$ such that $g_*(s) < s$ for all $s \geqslant 0$. The stopping time which yields the highest payoff can be written as $\tau_* = \inf\{t \geqslant 0 : B_t \leqslant g_*(S_t)\}$. The problem is actually solved in a general case of a real-valued, time homogeneous diffusion $X = (X_t : t \geqslant 0)$ instead of B. We then proceed to solve the problem for more general functions ϕ and c. Explicit formulae for payoff are given.

We apply the results to solve the so-called optimal Skorokhod embedding problem. We give also a sample of applications to various inequalities dealing with terminal value and maximum of a process.

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1 Introduction

In this paper we study certain optimal stopping problems. Our interest in these problems originated from the Skorokhod embedding field and a certain knowledge of the latter will be useful for reading (see our survey paper [11]).

We are directly stimulated by an article by Meilijson [10] and our observation that his results can be seen via the tools developed by Dubins, Shepp and Shiryaev [5] and Peskir [15]. In this paper, we manage to establish a link and generalize both the works of Meilijson [10] and Peskir [15]. We try to investigate the nature of the optimal stopping problem (12) through a series of remarks and rely on reasonings found in the above-cited articles together with some approximation techniques and limit passages to develop certain generalizations. Our work has therefore characteristics of an exposition of the subject with aim at unifying and extending known results.

Let ϕ be a non-negative, increasing, continuous function and c a continuous, positive function, and consider the following optimal stopping problem of maximizing

$$V_\tau = \mathbb{E}\left[\phi(S_\tau) - \int_0^\tau c(B_s)ds\right], \tag{1}$$

over all stopping times τ such that

$$\mathbb{E}\left[\int_0^\tau c(B_s)ds\right] < \infty, \tag{2}$$

where $(B_t : t \geqslant 0)$ is a real-valued Brownian motion and S_t is its unilateral maximum, $S_t = \sup_{u \leqslant t} B_u$.

Suppose, in the first moment, that $\phi(x) = x$ and $c(x) = c > 0$ is a constant. In this formulation the problem was solved by Dubins and Schwarz [6] in an article on Doob-like inequalities. The optimal stopping time is just the Azéma–Yor embedding (see Azéma and Yor [2] or Section 5 in Obłój [11]) for a shifted (to be centered) exponential distribution with parameter $2c$. This leads in particular to an optimal inequality (26).

Keeping $\phi(x) = x$, let $c(x)$ be a non-negative, continuous function. The setup was treated by Dubins, Shepp and Shiryaev [5], and by Peskir in a series of articles [15, 16, 18]. Peskir treated the case of real-valued diffusions, which allows to recover the solution for general ϕ as a corollary.

Theorem 1 (Peskir [15]). *The problem of maximizing (1) over all stopping times τ satisfying (2), for $\phi(x) = x$ and $c(x)$ a non-negative, continuous function, has an optimal solution with finite payoff if and only if there exists a maximal solution g_* of*

$$g'(s) = \frac{1}{2c(g(s))(s - g(s))} \tag{3}$$

which stays strictly below the diagonal in $\mathbb{R}^2$, i.e. $g_*(s) < s$. The Azéma–Yor stopping time $\tau_* = \inf\{t \geqslant 0 : B_t \leqslant g_*(S_t)\}$ is then optimal and satisfies (2) whenever there exists a stopping time which satisfies (2).

As pointed out above, this theorem was proved in a setup of any real, regular, time-homogeneous diffusion to which we will come back later. The characterization of existence of a solution to (1) through existence of a solution to the differential equation (3) is called the *maximality principle*. We point out that Dubins, Shepp and Shiryaev, who worked with Bessel processes, had a different way of characterizing the optimal solution to (3), namely they required that $\frac{g_*(s)}{s} \xrightarrow[s\to\infty]{} 1$.

Let now ϕ be any non-negative, non-decreasing, right-continuous function such that $\phi(B_t) - ct$ is a.s. negative on some (t_0, ∞) (with t_0 random) and keep c constant. This optimal stopping problem was solved by Meilijson [10]. Define

$$H(x) = \sup_{\tau} \mathbb{E}\left[\phi(x + S_\tau) - c\tau\right].$$

Theorem 2 (Meilijson [10]). *Suppose that* $\mathbb{E}\sup_t\{\phi(B_t) - ct\} < \infty$. *Then* H *is absolutely continuous and is the minimal solution to the differential equation*

$$H(x) - \frac{1}{4c}(H'(x))^2 = \phi(x) . \tag{4}$$

If ϕ *is constant on* $[x_0, \infty)$ *then* H *is the unique solution to (4) that equals* ϕ *on* $[x_0, \infty)$. *The optimal stopping time* τ_* *which yields* $H(0)$ *is the Azéma–Yor stopping time given by* $\tau_* = \inf\{t \geqslant 0 : B_t \leqslant S_t - \frac{H'(S_t)}{2c}\}$.

Let us examine in more detail the result of Meilijson in order to compare it with the result of Peskir. The Azéma–Yor stopping time is defined as $\tau_* = \inf\{t \geqslant 0 : B_t \leqslant g(S_t)\}$ with $g(x) = x - \frac{H'(x)}{2c}$. Let us determine the differential equation satisfied by g. Note that H is by definition non-decreasing, so we have $H'(x) = \sqrt{4c}\sqrt{(H(x) - \phi(x))}$. For suitable ϕ, this is a differentiable function and differentiating it we obtain

$$H''(x) = \frac{2c(H'(x) - \phi'(x))}{\sqrt{4c(H(x) - \phi(x))}} = \frac{2c(H'(x) - \phi'(x))}{H'(x)}.$$

Therefore

$$g'(x) = 1 - \frac{H''(x)}{2c} = \frac{\phi'(x)}{H'(x)} = \frac{\phi'(x)}{2c(x - g(x))}. \tag{5}$$

We recognize immediately (3) only there $\phi'(s) = 1$ and c was a function and not a constant. This motivated our investigation of the generalization of problem (1), given in (12), which is solved in Theorems 5–7 in Section 4.

Consider now the converse problem. That is, given a centered probability measure μ describe all pairs of functions (ϕ, c) such that the optimal stopping time τ_*, which solves (1) exists and embeds μ, that is $B_{\tau_\mu} \sim \mu$.

This was called the *optimal Skorokhod embedding problem* (term introduced by Peskir in [16]), as we not only specify a method to obtain an embedding for μ but also construct an optimal stopping problem, of which this embedding is a solution.

Again, let us consider two special cases. First, let $\phi(x) = x$. From the Theorem 1 we know that τ_* is the Azéma–Yor stopping time and the function g_* is just the inverse of barycentre function of some measure μ. The problem therefore consists in identifying the dependence between μ and c. Suppose for simplicity that μ has a strictly positive density f and that it satisfies $L\log L$-integrability condition.

Theorem 3 (Peskir [16]). *In the setup above there exists a unique function*

$$c(x) = \frac{1}{2}\frac{f(x)}{\overline{\mu}(x)} \tag{6}$$

such that the optimal solution τ_* of (1) embeds μ. This optimal solution is then the Azéma–Yor stopping time given by

$$\tau_* = \inf\{t \geqslant 0 : S_t \geqslant \Psi_\mu(B_t)\},\ \textit{where} \tag{7}$$

$$\Psi_\mu(x) = \frac{1}{\mu([x,\infty))}\int_{[x,\infty)} y d\mu(y). \tag{8}$$

The function c in (6) is recognized as a half of the hazard function which plays an important role in some studies. Now let $c(x) = c$ be constant. Then we have

Theorem 4 (Meilijson [10]). *In the setup above, there exists a unique function ϕ defined through (4) with $H'(x) = 2c(x - \Psi_\mu^{-1}(x))$, where Ψ_μ is the barycentre function given in (8), such that the optimal solution τ_* of (12) embeds μ. This optimal solution is then the Azéma–Yor stopping time given by (7).*

We will provide a general solution that identifies all the pairs (ϕ, c) with basic regularity properties, in Proposition 8 in Section 5.

2 Notation

In this section we fix the notation that, unless stated otherwise, is used in the rest of the paper.

$X = (X_t)_{t\geqslant 0}$ denotes a real-valued, time-homogeneous, regular diffusion associated with the infinitesimal generator

$$\mathbb{L}_X = \delta(x)\frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2}{\partial x^2}, \tag{9}$$

where the drift coefficient $x \to \delta(x)$ and the diffusion coefficient $x \to \sigma(x) > 0$ are continuous. We assume moreover that there exists a real-valued, standard Brownian motion $B = (B_t)_{t\geqslant 0}$, such that X solves the stochastic differential equation

$$dX_t = \delta(X_t)dt + \sigma(X_t)dB_t, \tag{10}$$

with $X_0 = x$ under $\mathbb{P}_x := \mathbb{P}$ for $x \in \mathbb{R}$. The natural filtration of X is denoted $(\mathcal{F}_t : t \geqslant 0)$ and is taken right-continuous and completed. The scale function and the speed measure of X are denoted, respectively, by L and by m. The state space of X is (a_X, b_X), where $a_X < b_X$ can be finite or infinite. The interval (a_X, b_X) might be also closed from one, or both, sides under some additional conditions, as will be discussed later. We recall, that for Brownian motion $L(x) = x$ and $m(dx) = 2dx$.

The one-sided maximum process associated with X is

$$S_t = \Big(\sup_{0 \leqslant r \leqslant t} X_r\Big) \vee s \tag{11}$$

started at $s \geqslant x$ under $\mathbb{P}_{x,s} := \mathbb{P}$. When several maximum processes will appear for different diffusions, we will add appropriate superscripts, thus writing S_t^X. The first hitting times for S are noted $T_a = \inf\{t \geqslant 0 : S_t \geqslant a\}$.

We make the following assumptions on the function ϕ:

- $\phi : (a_X, b_X) \to \mathbb{R}$ is non-decreasing, right-continuous and its points of discontinuity are isolated;
- ϕ', taken right-continuous, is well defined and its points of discontinuity are isolated;
- there exists r_ϕ, such that on $[r_\phi - 1, +\infty)$, $\phi \in C^1$ and $\phi' > 0$.

The last assumption has technical signification and is there to allow for a convenient identification of a particular solution to a differential equation. It will be clear from our proofs of main theorems, we hope, that one could impose some other condition, which would yield a different description.

We denote D_ϕ, the set of points of discontinuity of ϕ. It is bounded from above by r_ϕ and can write $D_\phi = \{d_0, d_1, \dots\}$ where $d_i \geqslant d_{i+1}$. Similarly, the set of points of discontinuity of ϕ' is also bounded from above by r_ϕ and we can write $D_{\phi'} = D_\phi \cup \{f_0, f_1, \dots\}$, where $f_i \geqslant f_{i+1}$.

It will be convenient to work sometimes with a different function $\tilde{\phi}$, which is of class C^1 with $\tilde{\phi}' > 0$, and satisfies: $\tilde{\phi} \leqslant \phi$, $\tilde{\phi}_{|[r_\phi,\infty)} = \phi_{|[r_\phi,\infty)}$. No reasoning will depend on a particular choice of $\tilde{\phi}$, so we do not present a specific construction.

The function $c : [0, +\infty) \to [0, +\infty)$ has a countable number of discontinuities. In his work Peskir [15] supposed $c > 0$, but we will see (cf. Remark 5) that allowing c to be zero on some intervals can be of great use. In some parts of the work we will make additional assumptions on the regularity of c, yet in others we will allow it to take the value $+\infty$.

We consider the following optimal stopping problem:

$$V_*(x, s) = \sup_\tau \mathbb{E}_{x,s}\Big(\phi(S_\tau) - \int_0^\tau c(X_t)dt\Big), \tag{12}$$

where τ is a $\mathcal{F}_t$-stopping time, which satisfies

$$\mathbb{E}_{x,s}\Big(\int_0^\tau c(X_t)dt\Big) < \infty. \tag{13}$$

Through a solution to “the optimal stopping problem (12)” we understand a stopping time which yields V_* and satisfies the condition (13). Such solution is denoted τ_*, i.e., $V_*(x,s) = \mathbb{E}_{x,s}\big(\phi(S_{\tau_*}) - \int_0^{\tau_*} c(X_t)dt\big)$. Note that of course V_* and τ_* depend on ϕ and c but it will be obvious from the context if we discuss the general setup of arbitrary (ϕ, c) or some special cases.

3 Some remarks on the problem

In the next section we will present a complete solution to the optimal stopping problem (12). However, before we do it, we want to give some relatively simple observations, which help to understand better the nature of the problem. Most of them become nearly evident once a person is acquainted with the problem, yet we think it is worthwhile to gather them here. As a matter of fact, they will help us a great deal, both in formulating and in proving, the main results of this paper.

The general construction of the optimal stopping problem under consideration is such, that we get rewarded according to the record-high value so far, and get punished all the time, proportionally to the time elapsed and depending on the path of our process. This means basically that the situation when the diffusion increases is favorable and what is potentially dangerous, as it might be “too costly”, are excursions far away below the maximum, that is the negative excursions of the process $(X_t - S_t)$. The first observation is then that the process should not be stopped in the support of dS_t.

Remark 1 (Proposition 2.1 [15]). If $\phi' > 0$ and τ yields the solution to the problem (12), then $X_\tau \neq S_\tau$ a.s., that is the process (X_t, S_t) cannot be optimally stopped on the diagonal of $\mathbb{R}^2$.

Let us investigate in more detail the nature of the optimal stopping time. If we follow the process X_t, as noted above, we gain when it increases. In contrast, when it decreases we only get punished all the time. This means that at some point it gets too costly to continue a given negative excursion of $(X_t - S_t)$. Exactly when it becomes too costly to continue the excursion depends only on three factors: the diffusion characteristics, the functions ϕ, and c, and on the value S_t of the maximum so far. Indeed, the choice of the point in an obvious manner depends on what we expect to earn, if the process climbs back to S_t, which in turn is a function of the diffusion characteristics and the functions ϕ, and c. Note however, that through the Markov property, it doesn’t depend on the way the process arrives at S_t. Similarly, thanks to the Markov property, the way the excursion of $(X - S)$ straddling time t develops, doesn’t affect the choice of the stopping value. Clearly also this value is an increasing function of the maximum.

Remark 2. The solution to the optimal stopping problem (12), if it exists, is given by a stopping time of the following form $\tau = \inf\{t \geqslant 0 : X_t \leqslant g_*(S_t)\}$, where the function g_* is non-decreasing and depends on the characteristics of X and the functions ϕ, and c.

Now that we have a feeling of how the stopping time looks like, we will try to learn more about the function g_*. Consider the function $V(x,s)$ given by (12). Since $V(x,s)$ corresponds to the optimal stopping of the process (X_t, S_t) started at (x,s), the values of $\phi(u)$ for $u < s$ never intervene. Furthermore, if the process reaches some value $s_1 > s$ then, thanks to the strong Markov property, we just face the problem $V(s_1, s_1)$. We could then replace the function ϕ with some other function $\tilde{\phi}$ which coincides with ϕ on $[s_1, \infty)$ and this would not affect the stopping rule, that is it would not affect values of the function g_* on the interval $[s_1, \infty)$. Thus the finiteness of V is not affected by any change of ϕ away from infinity. Likewise, if we add a constant δ to ϕ it will only change $V(\cdot,\cdot)$ by δ and leave g_* unchanged. Combining these observations we arrive at the following remark.

Remark 3. Consider two functions ϕ_1 and ϕ_2 which, up to an additive constant, coincide on some half-line: $\phi_1(s) = \phi_2(s) + \delta$ for all $s \geqslant r$ and some $\delta \in \mathbb{R}$. The optimal stopping problem (12) for ϕ_1 has finite payoff if and only if the optimal stopping problem (12) for ϕ_2 has finite payoff. Suppose this the case and denote respectively $\tau_1 = \inf\{t \geqslant 0 : X_t \leqslant g_*^1(S_t)\}$ and $\tau_2 = \inf\{t \geqslant 0 : X_t \leqslant g_*^2(S_t)\}$ the stopping times, which yield these payoffs. Then $g_*^1(s) = g_*^2(s)$ for all $s \geqslant r$.

We turn now to examine in more depth the relation between g_* and ϕ. Consider an interval of constancy of ϕ. Suppose that $\phi(\alpha) = \phi(\beta)$ for some $\alpha < \beta$ and ϕ increases on right of β (or has a jump). Then starting the process at $x \in [\alpha, \beta)$ we always face the same problem: we get punished all the time, but get rewarded only if we attain the level β. Due to the Markov character of the diffusion, reasoning similarly to the derivation of Remark 2, we see that the optimal stopping time is then just an exit time of some interval $[\gamma, \beta]$ and γ is then precisely the value of g_* on (α, β).

Remark 4. The functions ϕ and g_*, which induces the optimal stopping time, have the same intervals of constancy. In other words $g'_*|_{(\alpha,\beta)} \equiv 0 \Leftrightarrow \phi'|_{(\alpha,\beta)} \equiv 0$.

It is important to realize that if an interval of constancy of ϕ is long and the function doesn't grow fast enough afterwards, it may be optimal to stop immediately when coming to such a level of constancy. Thus the function g_* may stay in part under and in part on the diagonal in $\mathbb{R}^2$.

Analysis of jumps of ϕ and g_* is more involved. When ϕ is continuous, since the diffusion itself is continuous too, it's not hard to believe that g_* should be continuous as well. It is maybe a little more tricky to convince yourself that if ϕ jumps, then g_* has a jump too. We will try to argue this here, and moreover we will determine the height of the jump, following the reasoning of Meilijson developed in the proof of Corollary 1 in [10]. Suppose that ϕ has a jump in s_0: $\phi(s_0) - \phi(s_0-) = j > 0$. For simplicity, suppose also that ϕ is constant on some interval before the jump: $\phi(s_0-\epsilon)=\phi(s_0-)$. Then, as noted above, starting the process at some $x\in[s_0-\epsilon,s_0)$ we wait till the first exit time of the interval $[g_*(x),s_0]$. If we exit at the bottom we stop, and if we exit at the top, we have a new, independent diffusion starting, for which the expected payoff is just $V(s_0,s_0)$. For the process X starting at $y\in[a,b]$, denote $\rho^y_{a,b}$ and ρ^y_a, respectively, the first exit time of the interval $[a,b]$ and the first hitting time of the level a. We can then write the payoff $V(x,x)$ as

$$V(x,x)=\mathbb{E}\Bigg[V(s_0,s_0)\mathbf{1}_{\left\{\rho^x_{g_*(x),s_0}=\rho^x_{s_0}\right\}}+\phi(x)\mathbf{1}_{\left\{\rho^x_{g_*(x),s_0}=\rho^x_{g_*(x)}\right\}}-\int_0^{\rho^x_{g_*(x),s_0}}c(X_u)du\Bigg]. \tag{14}$$

We know, by Remark 3, that the value $g_*(s_0)$ is uniquely determined by the diffusion characteristics, the cost function c, and values of the function ϕ on $[s_0,\infty)$. Replace in the above display $g_*(x)$ by a and denote this quantity $V^a(x,x)$. Then $V(x,x)=\sup_{a<s_0}V^a(x,x)$. This determines the value of $a^x_*=g_*(x)$ and therefore the height of the jump of g_* at s_0. We will now show that the value of a^x_* does not depend on x (which is also clear by Remark 4) and give an equation which describes it.

Actually the equation for a^x_* is obvious: a^x_* is the unique real, such that $\frac{\partial V^a(x,x)}{\partial a}\big|_{a=a^x_*}=0$. This can be rewritten in the following equivalent manner, using the fact that $\mathbb{P}(\rho^x_{a,b}=\rho^x_b)=(L(x)-L(a))/(L(b)-L(a))$:

$$L'(a^x_*)\times\frac{L(s_0)-L(x)}{(L(s_0)-L(a^x_*))^2}\left(\phi(x)-V(s_0,s_0)\right)=\frac{\partial}{\partial a}\mathbb{E}\left[\int_0^{\rho^x_{a^x_*,s_0}}c(X_u)du\right]. \tag{15}$$

We just have to show that actually a^x_* does not depend on x. To this end let $a<y<x<s_0$ and write

$$\mathbb{E}\left[\int_0^{\rho^x_{a,s_0}}c(X_u)du\right]=\mathbb{E}\left[\int_0^{\rho^x_{s_0}}c(X_u)du\times\mathbf{1}_{\rho^x_{y,s_0}=\rho^x_{s_0}}+\int_0^{\rho^x_y}c(X_u)du\times\mathbf{1}_{\rho^x_{y,s_0}=\rho^x_y}\right]+\frac{L(s)-L(x)}{L(s)-L(y)}\mathbb{E}\left[\int_0^{\rho^y_{a,s_0}}c(X_u)du\right],$$

where we used the strong Markov property at time ρ^x_y. The first two terms on the right-hand side do not depend on a, and differentiating the above equation with respect to a yields an identity which proves that $a^x_*=a^y_*=:a_*$ for $x,y\in[s_0-\epsilon,s_0)$, since ϕ was supposed constant on this interval. It is also clear that a_* depends only on the characteristics of X, the cost function c and the payoff value $V(s_0, s_0)$. Since the diffusion is fixed, we write $a_* = a_*(c, V(s_0, s_0))$.

So far we only analyzed the dependence between g_* and ϕ, and it is the time to investigate the rôle of the cost function c. Suppose that the function c disappears on some interval, $c_{|(\alpha,\beta)} \equiv 0$. Thus wandering away from the maximum in the interval (α, β) doesn't cost us anything – there is no reason therefore to stop while in this interval. This implies that (α, β) is not in the range of g_*.

Remark 5. If $c_{|(\alpha,\beta)} \equiv 0$ end τ_* yields the optimal payoff to the problem (12), then $(\alpha, \beta) \cap g_*(\mathbb{R}) = \emptyset$ and $X_{\tau_*} \notin (\alpha, \beta)$ a.s.

Thus, the impacts of the intervals of disappearance of c, and of the jumps of ϕ, on the function g_*, are similar, only the former is much easier to describe. One could predict that the jumps of c would provoke a similar behavior of g_* as do the intervals of constancy of ϕ. This however proves to be untrue. It stems from the fact that the values of c are averaged through integrating them. Thus, jumps of c will only produce discontinuities of g'_*. We will come back to this matter in Theorem 6.

4 Maximality principle revisited

In this section we describe the main result of our paper, namely the complete solution to the optimal stopping problem (12). The solution is described in a sequence of three theorems with increasing generality of the form of the function ϕ. In the first theorem we suppose that ϕ is of class C^1 and strictly increasing. Our theorem is basically a re-writing of the Theorem 3.1 found in Peskir [15]. Theorem 6 treats the case of a continuous function ϕ, and is obtained form the previous one through an approximation procedure. Theorem 7 which deals with the general setup described in Section 2. It relays on Theorem 6 and on the work of Meilijson [10], and is less explicit then Theorems 5 and 6, as the treatment of jumps of ϕ is harder.

Theorem 5 (Peskir). *Let $\phi \in C^1$ with $\phi' > 0$, and $c > 0$ be continuous. The problem (12), has an optimal solution with finite payoff if and only if there exists a maximal solution g_* of*

$$g'(s) = \frac{\phi'(s)\sigma^2\left(g(s)\right)L'\left(g(s)\right)}{2c\left(g(s)\right)\left(L(s) - L(g(s))\right)}, \quad a_X < s < b_X, \tag{16}$$

which stays strictly below the diagonal in $(a_X, b_X)^2$ (i.e., $g_*(s) < s$ for $a_X < s < b_X$). More precisely, let s_* be such that $g_*(s_*) = a_X$. Then g_* satisfies (16) on (s_*, b_X) and it is maximal such function (where the functions are compared on the interval, where both of them are superior to a_X), which stays below the diagonal.

In this case the payoff is given by

$$V_*(x,s) = \phi(s) + \int_{x \wedge g_*(s)}^{x} \big(L(x) - L(u)\big) c(u) m(du), \tag{17}$$

The stopping time $\tau_* = \inf\{t \geqslant 0 : X_t \leqslant g_*(S_t)\}$ is then optimal whenever it satisfies (13), otherwise it is "approximately" optimal[2]*.*
Furthermore if there exists a solution ρ of the optimal stopping problem (12) then $\mathbb{P}_{x,s}(\tau_* \leqslant \rho) = 1$ and τ_* satisfies (13).
If there is no maximal solution to (16), which stays strictly below the diagonal in $\mathbb{R}^2$, then $V_*(x,s) = \infty$ for all $x \leqslant s$.

The last property for τ_* says it's pointwise the smallest solution, providing a uniqueness result. This remains true in more general setups and we will not repeat it below. This property follows from Peskir's [15] arguments but is also closely linked with properties of the Azéma–Yor stopping times, or any solutions to the Skorokhod embedding in general, we refer to Obłój [11], chapter 8, for details. Note that for a general ϕ, we defined in Section 2 the function $\tilde{\phi}$, which coincides with ϕ on the interval $[r_\phi, \infty)$ and is of class C^1. In particular, we can apply the above theorem to the optimal stopping problem (12) with ϕ replaced by $\tilde{\phi}$. We denote $\tilde{g}_*$ the function which gives the optimal stopping time $\tilde{\tau} = \inf\{t \geqslant 0 : X_t \leqslant \tilde{g}_*(S_t)\}$, which solves this problem.

Theorem 6. *Let ϕ be as described in Section 2, but continuous, and $c > 0$ be continuous. Then the optimal stopping problem (12) has a finite payoff if and only if the optimal stopping problem with ϕ replaced by $\tilde{\phi}$ has a finite payoff. In this case there exists a continuous function g_*, which satisfies the differential equation (16) on the interior of the set $\mathbb{R} \setminus D_{\phi'}$, and coincides with $\tilde{g}_*$ on the interval $[r_\phi, \infty)$. The payoff is given by the formula (17) and is obtained for the stopping time $\tau = \inf\{t \geqslant 0 : X_t \leqslant g_*(S_t)\}$, if it satisfies (13). Otherwise this stopping time is approximately optimal.*

It is important to note, that the function g_* may not satisfy anymore $g_*(s) < s$, since if the constancy intervals of ϕ are too long it might be optimal to stop immediately (see also discussion around Remark 4).

The formulation for the case of ϕ with discontinuities is somewhat more technical, as we have to define g_* through an iteration procedure.

Theorem 7. *Let ϕ be as described in Section 2, and $c > 0$ be continuous. The optimal stopping problem (12) has finite payoff if and only if the optimal*

[2] In the limit sense as in Peskir [15].

stopping problem with ϕ replaced by $\tilde{\phi}$ has a finite payoff. In this case there exists a function g_*, continuous on the set $\mathbb{R}\backslash D_\phi$, which satisfies the following:

- *$g_*(s) = \tilde{g}_*(s)$ for all $s \in [r_\phi, \infty)$;*
- *g_* is continuous on the interior of the set $\mathbb{R} \setminus D_\phi$ and differentiable on the interior of the set $\mathbb{R} \setminus D_{\phi'}$, where it satisfies the equation (16);*
- *for all $s \in D_\phi$, $a_* = g_*(s-)$ satisfies (15) with $s_0 = s$.*

The payoff is described via (17).

It is important to note, that even though the optimal payoff V appears in (15) the above construction of g_* is feasible. Recall, that the jumps points were denoted $\cdots < d_1 < d_0$. When determining the value of $g_*(d_i-)$ we already know the values of g_* on $[d_i, \infty)$ and through the formula (17) also the payoff $V(x,s)$ for $d_i \leqslant x \leqslant s$.

Proof. We now prove, in subsequent paragraphs, Theorems 5, 6 and 7.

Theorem 5. Step 1. We start with the case when the state space of the diffusion X is the whole real line, $(a_X, b_X) = \mathbb{R}$, and the function ϕ is strictly increasing and of class C^1, and its image is the real line $\phi : \mathbb{R} \overset{on}{\to} \mathbb{R}$. Let $Y_t = \phi(X_t)$. It is a diffusion with the scale function $L^Y(s) = L(\phi^{-1}(s))$, the drift coefficient $\delta^Y(y) = \phi'\left(\phi^{-1}(y)\right)\delta\left(\phi^{-1}(y)\right) + \frac{1}{2}\phi''\left(\phi^{-1}(y)\right)\sigma^2\left(\phi^{-1}(y)\right)$, the diffusion coefficient $\sigma^Y(y) = \phi'\left(\phi^{-1}(y)\right)\sigma\left(\phi^{-1}(y)\right)$ and the speed measure $m^Y(dy) = \frac{2dy}{L'(\phi^{-1}(y))\phi'(\phi^{-1}(y))\sigma^2(\phi^{-1}(y))}$. The state space of Y is the whole real line. We can rewrite the optimal stopping problem for X in terms of Y (where we add superscripts X and Y to denote the quantities corresponding to these two processes):

$$\begin{aligned} V_*^X(x,s) &= \sup_\tau \mathbb{E}_{x,s}\left(\phi(S_\tau^X) - \int_0^\tau c(X_t)dt\right) \\ &= \sup_\tau \mathbb{E}_{x,s}\left(S_\tau^Y - \int_0^\tau c(\phi^{-1}(Y_t))dt\right) \\ &= V_*^Y(\phi(x),\phi(s)) \text{ under } \tilde{\phi} = Id \text{ and } \tilde{c} = c \circ \phi^{-1}. \end{aligned} \tag{18}$$

We see therefore that the optimal solution for X coincides with the optimal solution for Y for a different problem. We can apply Theorem 3.1 in Peskir [15] for this new problem for Y to obtain that the optimal solution, if it exists, is given by

$$\begin{aligned} \tau_* &= \inf\left\{t \geqslant 0 : Y_t \leqslant g_*^Y(S_t^Y)\right\} \\ &= \inf\left\{t \geqslant 0 : X_t \leqslant \phi^{-1}\left(g_*^Y(\phi(S_t^X))\right)\right\}, \end{aligned} \tag{19}$$

where g_*^Y is the maximal solution which stays below the diagonal of the equation

$$g_*^{Y'}(s) = \frac{\phi'\left(\phi^{-1}(g_*^Y(s))\right)^2 \phi^{-1'}\left(g_*^Y(s)\right)\sigma^2\left(\phi^{-1}(g_*^Y(s))\right)L'\left(\phi^{-1}(g_*^Y(s))\right)}{2c\left(\phi^{-1}(g_*^Y(s))\right)\left(L(\phi^{-1}(s)) - L\left(\phi^{-1}(g_*^Y(s))\right)\right)}. \tag{20}$$

We put $g_*^X(s)=\phi^{-1}\big(g_*^Y(\phi(s))\big)$, so that the existence of g_*^X is equivalent to the existence of g_*^Y. We will now show that g_*^X satisfies (16). We will use the following simple observations: ${\phi^{-1}}'(\phi(s))\phi'(s)=1$ and $\frac{1}{{\phi^{-1}}'(g_*^Y(\phi(s)))}=\phi'(g^X(s))$. Then

$$\begin{aligned} {g_*^X}'(s) &= {\phi^{-1}}'\left(g_*^Y\left(\phi(s)\right)\right) \times {g_*^Y}'(\phi(s)) \times \phi'(s) \\ &= \frac{{\phi^{-1}}'\left(g_*^Y(\phi(s))\right)^2 \left(\phi'(g_*^X(s))\right)^2 \sigma^2(g_*^X(s)) L'(g_*^X(s))\phi'(s)}{2c\big(g_*^X(s)\big)\big(L(s)-L(g_*^X(s))\big)} \\ &= \frac{\phi'(s)\sigma^2(g_*^X(s))L'(g_*^X(s))}{2c\big(g_*^X(s)\big)\big(L(s)-L(g_*^X(s))\big)}\ , \end{aligned}$$

in which we recognize (16). Note that $\big(g_*^X(s) < s$ for $s \in \mathbb{R}\big) \Leftrightarrow \big(g_*^Y(s) < s$ for $s \in \mathbb{R}\big)$ as $\big(g_*^X(s) < s\big) \Leftrightarrow \big(g_*^Y(\phi(s)) < \phi(s)\big)$ and ϕ is a one to one map from $\mathbb{R}$ to $\mathbb{R}$. This yields the description of g_*^X as the maximal solution to (16), which stays strictly below the diagonal in $\mathbb{R}^2$.

The expression for payoff follows also from Peskir [15] by a change of variables.

Step 2. In this step we extend the previous results to the case, when the diffusion X or the function ϕ are not unbounded. First, observe that the results of Peskir work just as well for a diffusion with an arbitrary state space (a_X, b_X), $-\infty \leqslant a_X < b_X \leqslant \infty$. Indeed, if we take $\phi = Id$, the solution is given by the maximal solution to (16) for $s \in (a_X, b_X)$ and which stays strictly below the diagonal in $(a_X, b_X)^2$. Formally, we could repeat the proof in of Theorem 3.1 in [15], in about the same manner that Peskir treats non-negative diffusions in Section 3.11 in the same article. Moreover, in this section, Peskir also discusses the boundary behavior, to which we will come below (notice that so far we considered unattainable boundaries, see Karlin and Taylor [9, pp. 226–236]).

We conclude therefore that the assumption on the state space of the diffusion, as well as on the range of ϕ, were superficial. We need to keep $\phi' > 0$, but we can have $\phi(\mathbb{R}) \subsetneq \mathbb{R}$. Note that $\phi(\mathbb{R})$ is then an open interval $\phi(\mathbb{R}) = (a, b)$. The equation (20) is valid in (a, b), but the original one (16) is valid in (a_X, b_X) as asserted, due to the fact that $g_*^X = \phi^{-1} \circ g_*^Y \circ \phi$. The assumption, that ϕ is strictly increasing is needed for the new process to be a diffusion. We now turn to more general cases, which will be treated by approximation.

Theorem 6. We extend the previous results to functions ϕ, which are just non-decreasing and continuous. This is done through an approximation procedure. Let ϕ_n be an increasing sequence of strictly increasing C^1-functions, converging to ϕ, such that $\phi_n' \to \phi'$, and $\phi_n|_{[r_\phi,\infty)} \equiv \phi|_{[r_\phi,\infty)}$. Note V_n and V the payoffs given in (12), under the condition (13), for functions ϕ_n and ϕ respectively. Obviously $V_n \leqslant V_{n+1} \leqslant V$. Furthermore, from Remark 3 it follows that the payoff V is finite if and only if all V_n are finite. Suppose this is the case. We know, by Theorem 5 proved above, that V_n are given by a sequence of stopping times τ_*^n, $\tau_*^n = \inf\{t \geqslant 0 : X_t \leqslant g_*^n(S_t)\}$, where g_*^n are defined as the maximal solutions of (16), with ϕ replaced by ϕ_n, which stay below the diagonal. It is easy to see that $\tau_*^n \leqslant \tau_*^{n+1}$ and therefore $g_*^n \geqslant g_*^{n+1}$. This is a direct consequence of the fact that in the problem (12) the reward $(\phi_n(S_\tau))$ increases and the punishment $(\int_0^\tau c(X_s)ds)$ stays the same. It can also be seen from the differential equation (16) itself. We have therefore $\tau_*^n \nearrow \tau_*$ a.s., for some stopping time τ_*, which is finite since the payoff associated with it is finite, $V < \infty$. Moreover, $\tau_* = \inf\{t \geqslant 0 : X_t \leqslant g_*(S_t)\}$, where $g_*^n \searrow g_*$. Note that the limit is finite and satisfies $g_*(s) < s$. Furthermore, we can pass to the limit in the differential equations describing g_*^n to see that g_* satisfies (16). Note that this agrees with Remark 4. Suppose that the stopping time τ_* satisfies (13). We then see in a straightforward manner, that τ_* solves the optimal stopping problem for ϕ. It suffices to write:

$$\begin{aligned}
\mathbb{E}_{x,s}\left(\phi(S_{\tau_*}) - \int_0^{\tau_*} c(X_t)dt\right) &\leqslant V_*(x,s) \\
&= \sup_\tau \mathbb{E}_{x,s}\left(\phi(S_\tau) - \int_0^\tau c(X_t)dt\right) \\
&\leqslant \lim_{n\to\infty} \sup_\tau \mathbb{E}_{x,s}\left(\phi_n(S_\tau) - \int_0^\tau c(X_t)dt\right) \\
&= \lim_{n\to\infty} \mathbb{E}_{x,s}\left(\phi_n(S_{\tau_*^n}) - \int_0^{\tau_*^n} c(X_t)dt\right) \\
&= \mathbb{E}_{x,s}\left(\phi(S_{\tau_*}) - \int_0^{\tau_*} c(X_t)dt\right), \qquad (21)
\end{aligned}$$

where passing to the limit in both cases is justified by the monotone convergence of $\phi_n \nearrow \phi$ and $\tau_*^n \nearrow \tau_*$ a.s. The expression for payoff given in (17) follows also upon taking the limit.

If the stopping time τ_* fails to satisfy (13), we proceed as Peskir [15, p. 1626], to see that the expression (17) for payoff remains true (and we say τ^* is approximately optimal as it is in fact a limit of stopping times satisfying (13)).

Theorem 7. All that has to be verified is the expression for $g_*(d_i-)$, $i \geqslant 0$, where $D_\phi = \{d_0, d_1, \dots\}$. Obviously, it is enough to discuss one jump, say d_0. The proposed value for $g(d_0-)$ is just the value we obtained at the end of Section 3. We had however an assumption that ϕ was constant on some interval $[d_0-\epsilon, d_0)$ and we need to argue that it can be omitted. This is done by approximating ϕ. Define ϕ_n through $\phi_n(s) = \phi(s)$ for all $s \in \mathbb{R}\setminus[d_0-\frac{1}{n}, d_0)$ and ϕ_n constant and equal to $\phi(d_0-)$ on $[d_0-\frac{1}{n}, d_0)$. The theorem then applies to functions ϕ_n, as by Remark 3 the finiteness of the payoff for ϕ, $\tilde{\phi}$ and ϕ_n are equivalent. Passing to the limit, as in the proof of Theorem 6 above, yields the result. Seemingly, taking a sequence of continuous functions $\tilde{\phi}_n$ increasing to ϕ and repeating the reasoning in (21) we see that the payoff equation (17) is still valid. □

Note that $g_*(b_X) = b_X$. It is evident when the state space is $(a_X, b_X]$, since we have to stop upon achieving the upper bound for the maximum S, as continuation will only decrease the payoff. Generally, if $g_*(b_X) = \beta < b_X$ (understood as limit if necessary), then as $S_t \to b_X$, the possible increase of payoff goes to zero but the cost till stopping doesn't.

The value of g_* at a_X depends on the boundary behavior of X at a_X. Peskir [15] dealt with this subject in detail. He considered the case $a_X = 0$, as it was motivated by applications, but this has no impact on the result. We state briefly the results and refer to his work for more details. When a_X is a natural or exit boundary point, then $g_*(a_X) = \lim_{s\downarrow a_X} g_*(s) = a_X$. In contrast, when a_X is an entrance or regular instantaneously reflecting boundary then $g_*(a_X) = \lim_{s\downarrow a_X} g_*(s) < a_X$.

5 General optimal Skorokhod embedding problem

In this section we consider the so-called optimal Skorokhod embedding problem. The general idea is to design, for a given measure μ, an optimal stopping problem with finite payoff, such that the stopping time which solves it embeds μ, i.e., $X_\tau \sim \mu$. We choose here to restrain ourselves to the classical setup of Brownian motion, $X = B$. Study for general diffusions is equally possible, only the formulae would be more complicated and less-intuitive. In principle however it is not harder, since we know very well how to rephrase the classical Azéma–Yor embedding for the setup of any real diffusion (Azéma and Yor [2], Cox and Hobson [3], see Obłój [11] for a complete description of the subject).

This problem was first introduced in the context of classification of contingent claims, where the measure μ had an interpretation in terms of risk associated with an option (see Peskir [17]). Peskir in his article on the optimal Skorokhod embedding [16] gave an explicit solution, but he assumed for simplicity that the measure μ had positive density on $\mathbb{R}$. One could think that this will generalize easily to arbitrary measures. Unfortunately this proves not to be true. In the setup when $\phi(x) = x$, it is not possible to solve the optimal embedding problem for a measure, which has an atom in the interior of its support. To see this, note that the presence of an atom in the interior of the support implies that the barycentre function Ψ_μ has a jump, which in turn means that the function $g_* = \Psi_\mu^{-1}$ has to be constant on some interval. However, the derivative of g is given by $g'(s) = [2c(g(s))(s - g(s))]^{-1}$ which is strictly positive. In this sense, it is the presence of the function ϕ which proves fundamental to solve the problem. Meilijson [10], who had $c(x) = c$ but arbitrary ϕ, solved this problem for any centered measure μ with finite variance, but his solution is not really explicit. We will see that actually to obtain a general explicit solution, both function c and ϕ are needed, as the former allows to treat the regular (absolutely continuous) part of a measure, and the latter serves to obtain atoms.

We set ourselves two goals. First, for a given measure μ, we want to identify all the pairs (ϕ, c) such that the optimal stopping time τ_* which solves (12) embeds μ in B, $B_{\tau_*} \sim \mu$. We are interested in stopping times τ_* such that $(B_{t\wedge\tau_*} : t \geqslant 0)$ is a uniformly integrable martingale. This limits the class of measures which are admissible to μ such that $\int_{\mathbb{R}} |x| d\mu(x) < \infty$ and $\int_{\mathbb{R}} x d\mu(x) = 0$. Our second goal is to give a particular explicit description of a particular pair (ϕ, c).

Let us start with the case of a measure μ with a positive density $f > 0$ on $\mathbb{R}$. Then, we know that for $\tau_* = \inf\{t \geqslant 0 : B_t \leqslant g_*(S_t)\}$ we have $B_{\tau_*} \sim \mu$ if and only if $g_*(s) = \Psi_\mu^{-1}(s)$, where Ψ_μ^{-1} denotes the inverse of Ψ_μ. This is a direct consequence of the Azéma–Yor embedding [2] and the fact that g is increasing with $g(s) < s$. Let us investigate the differential equation satisfied by Ψ_μ^{-1}. Using the explicit formula (8), we obtain

$$\left(\Psi_\mu^{-1}(s)\right)' = \frac{\overline{\mu}\left(\Psi_\mu^{-1}(s)\right)}{\left(s - \Psi_\mu^{-1}(s)\right) f\left(\Psi_\mu^{-1}(s)\right)}. \tag{22}$$

Comparing this with the differential equation for g_* given by (16), we see that we need to have

$$\frac{\phi'(s)}{2c\left(\Psi_\mu^{-1}(s)\right)} = \frac{\overline{\mu}(\Psi_\mu^{-1}(s))}{f(\Psi_\mu^{-1}(s))}. \tag{23}$$

Equivalently, we have

$$\frac{\phi'\left(\Psi_\mu(u)\right)}{2c(u)} = \frac{\overline{\mu}(u)}{f(u)}, \quad \text{for } u \in \mathbb{R}. \tag{24}$$

Finally, we have to ensure that the payoff is finite. This is easy since we want $\mathbb{E}\phi(S_{\tau_*})$ to be finite and we know that $S_{\tau_*} = \Psi_\mu(B_{\tau_*})$, which implies that we need to have

$$\int \phi\left(\Psi_\mu(x)\right) d\mu(x) < \infty. \tag{25}$$

We have thus obtained the following proposition:

Proposition 8 *Let μ be a probability measure on $\mathbb{R}$ with a strictly positive density f. Then the optimal stopping problem (12) has finite payoff and the stopping time τ_*, which yields it embeds μ in B, i.e., $B_{\tau_*} \sim \mu$ if and only if (ϕ, c) satisfy (24) and (25) for all $s \geqslant 0$. The stopping time τ_* is then just the Azéma–Yor stopping time given by (7).*

Note that, if we take $\phi(s) = s$ so that $\phi'(s) = 1$ we obtain a half of the hazard function for c, which is the result of Peskir, stated in Theorem 3. The integrability condition (25) then reads $\int \Psi_\mu(x) d\mu(x) < \infty$, which is known (see Azéma and Yor [1]) to be equivalent to the $L \log L$ integrability condition on μ: $\int_1^\infty x \log x d\mu(x) < \infty$.

Proof. In light of the reasoning that led to (24) all we have to comment on is the continuity of the function c. Theorems 5–7 were formulated for a continuous function $c > 0$. Here, keeping the condition $c > 0$, we drop the assumption on continuity. Still, this is not a problem. It suffices to take a sequence of functions $c_n \searrow c$ a.e. and proceed as in the proof of Theorem 6 around (21), to see that the function g_*, which yields the solution $\tau_* = \inf\{t \geqslant 0 : B_t \leqslant g_*(S_t)\}$ to the optimal stopping problem (12), satisfies locally the equation (16) and is continuous. This in turn implies that it is indeed the inverse of the barycentre function associated with μ. □

Identifying all pairs (ϕ, c), which solve the optimal Skorokhod embedding problem for an arbitrary measure is harder. More precisely it's just not explicit any more. The reason is hidden in Theorem 7 – the description of jumps of g_*, which correspond to intervals not charged by the target measure, is done through an iteration procedure and is not explicit. This is exactly the reason why Meilijson was only able to prove the existence of the solution to the optimal Skorokhod embedding without giving explicit formulae. In our approach we will use the duality between ϕ and c to encode explicitly both the jumps and the regular part of the target measure. Still, we will not be able to cover all probability measures.

Let μ be a centered probability measure $\int_{\mathbb{R}} |x| d\mu(x) < \infty$, $\int_{\mathbb{R}} x d\mu(x) = 0$ and note $-\infty \leqslant a_\mu < b_\mu \leqslant +\infty$ respectively the lower, and the upper, bound of its support. Suppose μ is a sum of its regular and atomic parts: $\mu = \mu_r + \mu_a$, $d\mu_r(x) = f(x)dx$ and $\mu_a = \sum_{i\in\mathbb{Z}} p_i \delta_{j_i}$. In other words f is the density of the absolutely continuous (with respect to the Lebesgue measure) part and $\cdots < j_{-1} < j_0 < j_1 < \cdots$ are the jump points of μ, which are also the jump points of the barycentre function Ψ_μ, so that $\Psi_\mu(\mathbb{R}) = \mathbb{R}_+ \setminus \left(\bigcup_{i\in\mathbb{Z}} \left[\Psi_\mu(j_i), \Psi_\mu(j_i+)\right]\right)$. We may note that $\Psi\mu(j_i+) = \Psi_\mu(j_i) + \frac{p_i(\Psi_\mu(j_i)-j_i)}{\mu((j_i,+\infty))}$.

Theorem 9. *In the above setup, in the case when $f > 0$ or $f \geqslant 0$ but $\mu_a = 0$, the optimal stopping problem (12) with*

$$c(x) = \begin{cases} \frac{f(x)}{2\overline{\mu}(x)} & , \textit{for } x \in [a_\mu, b_\mu] \\ +\infty & , \textit{for } x \in \mathbb{R} \setminus [a_\mu, b_\mu] \end{cases} \qquad \textit{and} \qquad \begin{cases} \phi'(x) = \mathbf{1}_{\Psi_\mu(\mathbb{R})}(x), \\ \phi \textit{ continuous, } \phi(0) = 0 \end{cases}$$

has finite payoff under (25). The payoff is realized by the Azéma–Yor stopping time $\tau_* = \inf\{t \geqslant 0 : S_t \geqslant \Psi_\mu(B_t)\}$ and $B_{\tau_*} \sim \mu$.

Note that the condition $\phi(0) = 0$ is just a convention, as adding a constant to ϕ doesn't affect the solution τ_* of (12) (cf. Remark 3). The restriction (25), thanks to the definition of ϕ, is satisfied in particular when $\int \Psi_\mu(x) d\mu(x) < \infty$, that is if μ satisfies the $L \log L$ integrability condition.

Proof. The problem arising from discontinuity of c is treated exactly as in the proof of Proposition 8 above. In the case of $f > 0$ on $\mathbb{R}$ the above theorem follows immediately from Theorem 6. If μ has no atoms, then $\Psi_\mu(\mathbb{R}) = \mathbb{R}_+$ and so $\phi = Id$. From the differential equation (16) we can derive a differential equation for the inverse of g_*, from which it is clear that we can treat intervals where c is zero by passing to the limit. Thus, our theorem is also valid in the case of absolutely continuous measure but with a density which can be zero at some intervals. Finally, the case of atoms with $\limsup_{n\to\infty} j_n = +\infty$ is also handled upon approximating and taking the limit. Indeed this does not bother us here as we do not need anymore to describe the limiting solution through a differential equation, as was the case for Theorems 5–7. □

Unfortunately, our method does not work for an arbitrary measure μ. If one tries to apply it for a purely atomic measure, he would end up with $\phi \equiv c \equiv 0$. In fact, when there is no absolutely continuous part we need to take ϕ discontinuous, as Meilijson does. We are not able to solve the problem explicitly then. We cannot treat measures with singular non-atomic component.

6 Important inequalities

In the above section we saw how the solution to the optimal stopping problem yields a solution to the so-called optimal Skorokhod embedding problem. This actually motivated our research but is by no means a canonical application of the maximality principle. Probably the main and the most important applications are found among stopping inequalities. We will try to present some of them here. A specialist in the optimal stopping theory will find nothing new in this section and can probably skip it, yet we think it is useful to put it in this note, as it completes our study and allows us to give some references for further research. The main interest of the method presented here, is that the constants obtained are always optimal. Some of the inequalities below, as (30), are easy to obtain with "some" constant. It was however the question of the optimal constant which stimulated researchers for a certain time.

Consider a continuous local martingale $X = (X_t : t \geqslant 0)$ and a stopping time T such that $(X_{t\wedge T} : t \geqslant 0)$ is a uniformly integrable martingale. We have then

$$\mathbb{E}S_T^X \leqslant \sqrt{\mathbb{E}X_T^2}, \quad \text{where } S_t^X = \sup_{s\leqslant t} X_s, \tag{26}$$

and this is optimal. This simple inequality was first observed by Dubins and Schwarz [6]. It can be easily seen in the following manner: $\mathbb{E}S_T = \mathbb{E}[S_T - X_T] \leqslant \sqrt{\mathbb{E}[S_T - X_T]^2} = \sqrt{\mathbb{E}[X_T^2]}$, where we used the fact that $(S_t - X_t)^2 - X_t^2$ is a local martingale (see Obłój and Yor [12] for various applications of these martingales). This inequality is optimal as the equality is attained for $T = \inf\{t \geqslant 0 : S_t^X - X_t = a\}$ for any $a > 0$ (X_T has then shifted exponential distribution with parameter $\frac{1}{a}$). If we want to establish an analogous inequality for $|X|$, that is for a submartingale, some more care is needed. We propose to follow Peskir [14, 15] in order to obtain more general inequalities.

Consider the optimal stopping problem (12), with $X = |B|$ the absolute value of a Brownian motion, $\phi(s) = s^p$ and $c(x) = cx^{p-2}$, where $p > 1$.

We solve the problem applying Theorem 5. The differential equation (16) takes the form

$$g'(s) = \frac{ps^{p-1}}{2cg(s)^{p-2}\left(s - g(s)\right)}, \tag{27}$$

which is solved by $g(s) = \alpha s$, where α is the larger root of the equation $\alpha^{p-1} - \alpha^p = \frac{p}{2c}$, which is seen to have solutions for $c \geqslant p^{p+1}/2(p-1)^{(p-1)}$. Taking $c \searrow p^{p+1}/2(p-1)^{(p-1)}$ yields

$$\mathbb{E}\left(\max_{0\leqslant t\leqslant\tau} |B_t|^p\right) \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}|B_\tau|^p, \tag{28}$$

where τ is any stopping time such that $\mathbb{E}\tau^{p/2} < \infty$. Note that to obtain the right-hand side of the above inequality we used the fact that $\mathbb{E}\big(\int_0^\tau |B_t|^{p-2}dt\big) = \frac{2}{p(p-1)}\mathbb{E}|B_\tau|^p$. More generally we could consider the optimal stopping problem (12) with $\phi(s) = s^p$ and $c(x) = cx^{q-1}$ for $0 < p < q+1$, $q > 0$. Optimizing upon c, Peskir [14] obtains

$$\mathbb{E}\left(\max_{0\leqslant t\leqslant\tau} |B_t|^p\right) \leqslant \gamma^*_{p,q}\mathbb{E}\left(\int_0^\tau |B_t|^{q-1}dt\right)^{p/(q-1)}, \tag{29}$$

for all stopping times τ of $|B|$. The optimal constant $\gamma^*_{p,q}$ is given implicitly as a solution to an equation. In the case $p = 1$ it can be written explicitly, $\gamma^*_{1,q} = \big(q(1+q)/2\big)^{1/(1+q)}\big(\Gamma(2+1/q)\big)^{q/(1+q)}$. This was obtained independently by Jacka [8] and Gilat [7]. In particular we find $\gamma^*_{1,1} = \sqrt{2}$ which is exactly the value found by Dubins and Schwarz [6]. If we consider the well-known bounds

$$c_p\mathbb{E}\left(\int_0^\tau |B_t|^{p-2}dt\right) \leqslant \mathbb{E}\left(\max_{0\leqslant t\leqslant\tau} |B_t|^p\right) \leqslant C_p\mathbb{E}\left(\int_0^\tau |B_t|^{p-2}dt\right), \tag{30}$$

where $p > 1$ and c_p, C_p are some universal constants, we see that the inequality (29) complements them for $0 < p \leqslant 1$, where (30) doesn't have much sense. Also, as pointed out above, the method presented here allows to recover the optimal constants.

The inequalities given above are just a sample of applications of the method presented in this note. Peskir [15], for example, develops also a $L\log L$-type inequalities and some inequalities for Geometric Brownian motion. Although the method presented is very general it has of course its limits. For example it seems one could not recover optimal constants[3] a_p and A_p, obtained by Davis [4], such that for any stopping time T

$$\mathbb{E}|B_T|^p \leqslant A_p\mathbb{E}T^{p/2}, \text{ for } 0 < p < \infty, \text{ and} \tag{31}$$

$$a_p\mathbb{E}T^{p/2} \leqslant \mathbb{E}|B_T|^p, \text{ for } 1 < p < \infty, \ \mathbb{E}T^{p/2} < \infty. \tag{32}$$

[3] These can be recovered using different methodology involving determinist time changing (see Pedersen and Peskir [13]).

The maximality principle deals with the maximum process and is an essential tool in its study. However the passage from the maximum to the terminal value is quite complicated and in general we should not hope that optimal inequalities for the terminal value could be obtained from the ones for maximum.

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References

1. J. Azéma and M. Yor. Le problème de Skorokhod: compléments à "Une solution simple au problème de Skorokhod". In *Séminaire de Probabilités, XIII*, volume 721 of *Lecture Notes in Math.*, pages 625–633. Springer, Berlin Heidelberg New York, 1979
2. J. Azéma and M. Yor. Une solution simple au problème de Skorokhod. In *Séminaire de Probabilités, XIII*, volume 721 of *Lecture Notes in Math.*, pages 90–115. Springer, Berlin Heidelberg New York, 1979
3. A.M.G. Cox and D.G. Hobson. An optimal Skorokhod embedding for diffusions. *Stochastic Process. Appl.*, 111(1):17–39, 2004.
4. B. Davis. On the L^p norms of stochastic integrals and other martingales. *Duke Math. J.*, 43(4):697–704, 1976.
5. L. E. Dubins, L. A. Shepp, and A. N. Shiryaev. Optimal stopping rules and maximal inequalities for Bessel processes. *Teor. Veroyatnost. i Primenen.*, 38(2):288–330, 1993.
6. L. E. Dubins and Gideon Schwarz. A sharp inequality for sub-martingales and stopping-times. *Astérisque*, (157–158):129–145, 1988. Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987)
7. D. Gilat. On the ratio of the expected maximum of a martingale and the L_p-norm of its last term. *Israel J. Math.*, 63(3):270–280, 1988
8. S. D. Jacka. Doob's inequalities revisited: a maximal H^1-embedding. *Stochastic Process. Appl.*, 29(2):281–290, 1988
9. S. Karlin and H. M. Taylor. *A second course in stochastic processes.* Academic Press [Harcourt Brace Jovanovich], New York, 1981.
10. I. Meilijson. The time to a given drawdown in Brownian motion. In *Séminaire de Probabilités, XXXVII*, volume 1832 of *Lecture Notes in Math.*, pages 94–108. Springer, Berlin Heidelberg New York, 2003
11. J. Obłój. The Skorokhod embedding problem and its offspring. *Probability Surveys*, 1:321–392, 2004
12. J. Obłój and M. Yor. On local martingale and its supremum: harmonic functions and beyond. In: Y. Kabanov, R. Liptser, J. Stoyanov (Eds), From stochastic calculus to mathematical finance, 517–533, Springer, Berlin Heidelberg New York, 2006
13. J. L. Pedersen and G. Peskir. Solving non-linear optimal stopping problems by the method of time-change. *Stochastic Anal. Appl.*, 18(5):811–835, 2000
14. G. Peskir. The best Doob-type bounds for the maximum of Brownian paths. In *High dimensional probability (Oberwolfach, 1996)*, volume 43 of *Progr. Probab.*, pages 287–296. Birkhäuser, Basel, 1998

15. G. Peskir. Optimal stopping of the maximum process: the maximality principle. *Ann. Probab.*, 26(4):1614–1640, 1998
16. G. Peskir. Designing options given the risk: the optimal Skorokhod-embedding problem. *Stochastic Process. Appl.*, 81(1):25–38, 1999
17. G. Peskir. A true Buyer's risk and classification of options. *ITEM Inform. Technol. Econ. Manag.*, (1):Paper 1, 20 pp. (electronic), 2001
18. G. Peskir. Maximum process problems in optimal control theory. *J. Appl. Math. Stoch. Anal.*, (1):77–88, 2005

Correlated Processes and the Composition of Generators

Nathanaël Enriquez

Laboratoire de Probabilités, Université Paris 6
4, place Jussieu, 75252 Paris Cedex 05, France
e-mail: enriquez@ccr.jussieu.fr

Summary. We show how correlated processes give a probabilistic significance to the composition of generators. In particular, we give some new probabilistic representations of the solution of linear differential equations with initial boundary conditions, of the linear Klein–Gordon equation and of some biharmonic equations in the presence of a potential.

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Key words: Correlated processes, Feynman–Kac formula

1 Introduction

Discrete correlated processes can be viewed as processes with a finite memory. In the simple case of a nearest neighbour random process on $\mathbb{Z}$ with memory one, the law of a step is determined by the sign of the preceding step and we are led to the study of a Markov process on $\mathbb{Z} \times \{1, 2\}$. Such random walks have been extensively studied even in higher dimension (e.g., [3], [6], [12] and the references therein).

An analogous continuous time process on $\mathbb{R} \times \{1, 2\}$ has been considered by Goldstein [4] and later by Kac [8]. The process has speed 1 on $(\mathbb{R}, 1)$ and speed -1 on $(\mathbb{R}, 2)$, and at exponential times, it switches from one line to the other. Goldstein and Kac remarked that the Chapman-Kolmogorov equation of this process gives rise to a probabilistic representation of the damped wave equation, also called the telegraph equation (the fact was noticed first by Goldstein [4] using a passage from a discrete to a continuous time process, but the first direct treatment of that equation was done by Kac [8] and was later reformulated by Kabanov [7]). Later, several authors considered one-dimensional generalizations of this fact (e.g., [10], [11] and the references therein).

In Section 2, we generalize Goldstein's process by considering an open domain D of $\mathbb{R}^d$, and a process on $D \times \{1, 2, \ldots, N\}$ that follows different Markovian regimes on each (D, i) $(1 \leqslant i \leqslant N)$ and jumps from (D, i) to $(D, i+1)$ for $i \neq N$, and from (D, N) to $(D, 1)$, spending exponential time on each domain (D, i). A Feynman–Kac formula for this process gives a probabilistic representation of differential equations involving the *composition* of the generators of the Markov processes together with a linear *potential*. In this way, we generalize a result of Khasminskii [9] which appears to correspond to the special case where the different regimes have the same generator and the process does not jump from (D, N).

We give some applications of this fact in Section 3. When D is $\mathbb{R}_+$, taking the Markovian regimes deterministic leads to a simple probabilistic representation of the solution to the initial value problem of a rather general class of linear differential equations. In this case, the necessary and sufficient condition for the existence and uniqueness of a solution is equivalent to an accessibility condition for the corresponding correlated process.

The case $D = \mathbb{R} \times \mathbb{R}_+$ with two suitable uniform motions as Markov processes gives a probabilistic representation of the solution of some variations of the one-dimensional wave equation including the telegraph equation, treated in [4,8], and the linear one-dimensional Klein–Gordon equation which can not be interpreted by Goldstein's one-dimensional process.

Back to the case where D is an open domain of $\mathbb{R}^d$, when $N = 2$ and both Markov processes are Brownian motions, we get a probabilistic representation for the solution of the biharmonic equation with linear potential and boundary conditions on u and Δu. In the special case where the potential is null, we recover a result of Khasminskii (special case of [9]) and Helms [5] expressing the solution of the biharmonic equation as a functional of the hitting time and hitting position on ∂D of the usual Brownian motion. Moreover, we notice that their formula cannot be generalized in the presence of a potential and that this case requires explicitly the use of correlated processes. Finally, using the reflecting Brownian motion on $(D, 1)$, one can treat the biharmonic equation with boundary conditions involving $\frac{\partial u}{\partial n}$.

In Section 4, we mention possible generalizations of this model and indicate how correlated processes may also represent the solution of *delay* differential equations.

2 Definition and main result

Let D be an open subset of $\mathbb{R}^d$ with a continuous boundary ∂D, and N a strictly positive integer.

We consider, on D, a family of strong Markov processes $(X_t^i)_{1 \leqslant i \leqslant N}$ defined by their respective infinitesimal generator L_i.

Fix $\lambda_i \geqslant 0$ for $i = 1, \dots, N$. We introduce now the process $Y_t := (y_t, \varepsilon_t)$ on $D \times \{1, 2, \dots, N\}$ (that can be seen as the union of N "copies" of D) in the following way:

- ε_t equals successively $1, 2, \dots, N, 1, 2 \dots$ during independent random periods and ε_t stays at i during a time having the exponential distribution with parameter λ_i.
- during the time Y_t belongs to (D, i), y_t follows the law of X^i.
- for all "switching" time T (i.e., $\varepsilon_T \equiv \lim_{t\to T-} \varepsilon_t + 1 \bmod[N]$), $y_T = \lim_{t\to T-} y_t$.

Proposition 1. *The process Y_t admits an infinitesimal generator L, defined on the functions*

$$\begin{aligned} f : D \times \{1, 2 \dots, N\} &\longrightarrow \mathbb{R} \\ (x, i) &\longmapsto f_i(x) \end{aligned}$$

where f_i belongs to the domain of L_i. Moreover,

$$\forall i \in \{1, \dots, N-1\}, \quad Lf(x, i) = L_i f_i(x) + \lambda_i (f_{i+1}(x) - f_i(x))$$

$$Lf(x, N) = L_N f_N(x) + \lambda_N (f_1(x) - f_N(x)) \,.$$

Proof. Denote by T the first switching time.

$$\begin{aligned} \forall t > 0, \quad E_{x,1}[f(Y_t)] &= E_x[f(X_t^1) 1_{T>t}] + E_{x,1}[f(Y_t) 1_{T \leqslant t}] \\ &= e^{-\lambda_1 t} E_x[f(X_t^1)] + f_2(x)(1 - e^{-\lambda_1 t}) + o(t) \,. \end{aligned}$$

Taking $\lim_{t\to 0} \dfrac{E_{x,1}[f(Y_t)] - f(x,1)}{t}$, we get the equality for $i = 1$. Idem for the other equalities. □

From this proposition, we deduce the following corollary:

Corollary 1. *Let Y_t be the process of Proposition 1.*

Denote, for all $i = 1, \dots, N$,

$$T_t^i := \int_0^t 1_{\{\varepsilon_s = i\}} ds \,.$$

For all f in the domain of L and for all sequence of real numbers $c_1, \dots, c_N$,

$$\exp\left(\sum_{i=1}^N c_i T_t^i \right) f(Y_t) - \int_0^t (c_{\varepsilon_s} f(Y_s) + Lf(Y_s)) ds$$

is a martingale.

A direct consequence of this corollary is the probabilistic representation of solutions of higher order linear differential equations, generalizing the result of [9]:

Theorem 1. *Given $\mu_1, \dots, \mu_N \in \mathbb{R}$, $\alpha \geqslant 0$, let $g_0, \dots, g_{N-1}$ be continuous real functions on ∂D, and F a continuous function on D. Consider the equation*

$$(-1)^N (L_N + \mu_N)(L_{N-1} + \mu_{N-1}) \dots (L_1 + \mu_1)u - \alpha u = F(x)$$

with boundary conditions $\lim_{x \to x_0} u(x) = g_0(x_0)$ and

$$\lim_{x \to x_0} (L_k + \mu_k)(L_{k-1} + \mu_{k-1}) \dots (L_1 + \mu_1)u(x) = (-1)^k g_k(x_0)$$

for all $x_0 \in \partial D$, and $k = 1, \dots, N-1$.
Consider the process Y_t with parameters $(\lambda_1, \lambda_2, \dots, \lambda_{N-1}, \lambda_N) = (1, 1, \dots, 1, \alpha)$, and introduce the stopping time $\tau := \inf\{s > 0, y_s \in \partial D\}$.

If the expectation

$$E_{(x,1)} \left[\exp \left(\sum_{i=1}^{N-1} (1 + \mu_i) T_\tau^i + (\alpha + \mu_N) T_\tau^N \right) \right]$$

is finite, the solution of the equation is unique and is given by

$$\begin{aligned} u(x) = {} & E_{(x,1)} \left[\exp \left(\sum_{i=1}^{N-1} (1 + \mu_i) T_\tau^i + (\alpha + \mu_N) T_\tau^N \right) \times g(Y_\tau) \right] \\ & + E_{(x,1)} \left[\int_0^\tau \exp \left(\sum_{i=1}^{N-1} (1 + \mu_i) T_\tau^i + (\alpha + \mu_N) T_s^N \right) \right. \\ & \left. \qquad \times F(y_s) \times 1_{\{\varepsilon_s = N\}} ds \right] \end{aligned}$$

where g is the function defined on $\partial D \times \{1, \dots, N\}$, by $g|_{(\partial D, i)} = g_{i-1}$.

Remark 1. The result can be generalized to the case where μ_i are continuous functions on D. The functional $(1 + \mu_i)T_t^i$ has then to be replaced by $\int_0^t (1 + \mu_i(y_s)) 1_{\{\varepsilon_s = i\}} ds$ and $\mu_N T_t^N$ is replaced by $\int_0^t \mu_N(y_s) 1_{\{\varepsilon_s = i\}} ds$.

Remark 2. The constant α may be replaced by a nonconstant *positive* "potential" V: for this purpose, one has to generalize the notion of correlated processes and to allow the switching operation on (D, N) to have some rate depending on the position (in this case, the length of a stay on (D, N) does not follow an exponential law). We give an example of such a situation in Section 3.3.

Remark 3. The possible existence of non-null potentials in the equations we consider makes them nontrivial since a chain resolution is impossible in that case. This improves seriously Theorem 2 below of Khasminskii. In Section 3, the examples of linear differential equations, of the Klein–Gordon equation and of the biharmonic equation with potential illustrate this situation.

Proof. For any function f in the domain of L, we denote by f_i the restriction of f to the domain (D,i), and apply Corollary 1 with $c_i = 1+\mu_i$ for $i = 1,\ldots,N-1$ and $c_N = \alpha + \mu_N$.

We deduce that

$$\exp\left(\sum_{i=1}^{N-1}(1+\mu_i)T_t^i + (\alpha+\mu_N)T_t^N\right) f(Y_t)$$
$$-\int_0^t \exp\left(\sum_{i=1}^{N-1}(1+\mu_i)T_s^i + (\alpha+\mu_N)T_s^N\right)$$
$$\times (L_1 f_1 + (f_2 - f_1) + (1+\mu_1)f_1)(y_s) \times 1_{\{\varepsilon_s=1\}} ds$$
$$\cdots$$
$$-\int_0^t \exp\left(\sum_{i=1}^{N-1}(1+\mu_i)T_s^i + (\alpha+\mu_N)T_s^N\right)$$
$$\times (L_N f_N + \alpha(f_1 - f_N) + (\alpha+\mu_N)f_N)(y_s) \times 1_{\{\varepsilon_s=N\}} ds$$

is a martingale.

Consider the function f defined on $D \times \{1,\ldots,N\}$ by:

$$f(x,1) = u(x)$$
$$f(x,2) = -(L_1+\mu_1)u$$
$$f(x,3) = -(L_2+\mu_2)f(x,2)$$
$$\cdots$$
$$f(x,N) = -(L_{N-1}+\mu_{N-1})f(x,N-1).$$

We deduce that

$$\exp\left(\sum_{i=1}^{N-1}(1+\mu_i)T_t^i + (\alpha+\mu_N)T_t^N\right) f(Y_t)$$
$$+\int_0^t \exp\left(\sum_{i=1}^{N-1}(1+\mu_i)T_s^i + (\alpha+\mu_N)T_s^N\right) \times F(y_s) \times 1_{\{\varepsilon_s=N\}} ds$$

is a martingale.

Now, one notices that on the event $\{\varepsilon_s = N\}$, for all $i \in \{1, \ldots, N-1\}$, $T_s^i = T_\tau^i$ almost surely. The optional sampling theorem applied at times $\tau \wedge n$ $(n \to \infty)$, allied to the bounded convergence theorem allows to conclude. □

In the special case where the generators L_i are equal to the same L and $\alpha = 0$, we find Khasminskii's result [9] which gives a probabilistic representation of the solution of an equation of the type $P(L) = F$ where P is a polynomial. We recall here the result:

Theorem 2. *[9] Let $\mu_1, \ldots, \mu_N \in \mathbb{R}$, $g_0, \ldots, g_{N-1}$ be continuous real functions on ∂D, and F a continuous function on D. Let $P(X) := (X + \mu_N)$ $(X + \mu_{N-1}) \ldots (X + \mu_1)$. Let (X_t) be a Markov process with generator L.*

Consider the equation $(L + \mu_N)(L + \mu_{N-1}) \ldots (L + \mu_1)u = (-1)^N F(x)$ with boundary conditions $\lim_{x \to x_0} L^k u(x) = g_k(x_0)$ for all $x_0 \in \partial D$, and $k = 0, \ldots, N-1$.

Let $y_k(t), k = 1, \ldots, N-1$ be the solutions of $P(-d/dt)y_k = 0$, $y_k^{(k)}(0) = (-1)^k$, $y_k^{(j)}(0) = 0$, $j \neq k$. If the mathematical expectations $E_x[\exp \mu_k \tau]$, where τ denotes the first passage time of X_t on the boundary, are finite, then the solution u is unique and is given by

$$u(x) = \sum_{i=0}^{N-1} E_x[y_i(\tau) g_i(X_\tau)] + E_x\left[\int_0^\tau y_{N-1}(t) F(X_t) dt\right] .$$

We want to show here how Theorem 2 which does not make use of correlated processes can be deduced from Theorem 1.

For simplicity, we restrict our proof to the case $N = 2$ and $\mu_1 \neq \mu_2$.

Let us consider the process Y_t on $D \times \{1, 2\}$ with parameters $\lambda_1 = 1$ and $\lambda_2 = 0$, so that when Y_t enters $(D, 2)$, it never goes back to $(D, 1)$.

By Theorem 1,

$$\begin{aligned} u(x) &= E_{(x,1)}[\exp((1+\mu_1)T_\tau^1 + \mu_2 T_\tau^2) g(Y_\tau)] \\ &\quad + E_{(x,1)}\left[\int_0^\tau \exp((1+\mu_1)T_s^1 + \mu_2 T_s^2) \times F(y_s) \times 1_{\{\varepsilon_s = 2\}} ds\right] \end{aligned}$$

where $g|_{(D,1)} = g_0$ and $g|_{(D,2)} = \mu_1 g_0 + g_1$.

But this last quantity can also be expressed in terms of the process X_t and of an independent exponential variable T representing the time when the correlated process jumps from $(D, 1)$ to $(D, 2)$:

$$\begin{aligned} u(x) = E_x \big[&\exp((1+\mu_1)\tau) g_0(X_\tau) 1_{\{\tau < T\}} \\ &+ \exp((1+\mu_1)T + \mu_2(\tau - T))(\mu_1 g_0 + g_1)(X_\tau) 1_{\{T < \tau\}}\big] \\ &+ E_x\left[1_{\{\tau > T\}} \int_T^\tau \exp((1+\mu_1)T + \mu_2(s - T)) F(X_s) ds\right] \end{aligned}$$

Then, we can disintegrate the expectation with respect to T and obtain the desired expression.

3 Some applications

3.1 Linear differential equations

Suppose v_1, v_2, ..., v_n are positive continuous functions and consider the equation

$$v_n \times (v_{n-1} \times (\dots v_3 \times (v_2 \times (v_1 \times u')')' \dots)')'(x) - u(x) = 0, \qquad x \geqslant 0\,, \quad (1)$$

with initial conditions expressed in the form $u(0) = \gamma_0$, $(v_1 \times u)'(0) = \gamma_1$, $v_2 \times (v_1 \times u')'(0) = \gamma_2$, ..., $v_{n-1} \times (\dots v_3 \times (v_2 \times (v_1 \times u')')' \dots)'(0) = \gamma_{n-1}$.

Let Y_t be the correlated process on $\mathbb{R}_+ \times \{1, \dots, N\}$, whose speed is $-v_i$ on the line $(\mathbb{R}_+, i)$, and whose parameters are $(\lambda_1, \dots, \lambda_N) = (1, 1, \dots, 1)$.

Proposition 2. *Set* $\tau = \inf\{s > 0,\, y_s = 0\}$.

(i) The variable $\exp(\tau)$ *is integrable if and only if all the functions* $\frac{1}{v_i}$ *are integrable.*

(ii) In this case, (1) has a unique solution given by

$$u(x) = E_{(x,1)} \left[\exp(\tau) \times \left(\sum_{i=0}^{n-1} \gamma_i 1_{Y_\tau = (0,i+1)} \right) \right].$$

Proof. (i) The fact that $\frac{1}{v_i}$ is not integrable makes $(0, i)$ inaccessible. Indeed, it takes forever for the i-th regime to reach 0. Moreover, the fact that one of the endpoints $(0, i)$ is inaccessible implies that, conditional on the fact that the process reaches (D, i) before $0 \times \{1, \dots, N\}$, τ is stochastically bigger than the interval of time between two changes of regime which has, in that case, exponential distribution with parameter 1, so its exponential is *not* integrable.

Conversely, if the functions $\frac{1}{u_i}$ are integrable,

$$\tau \leqslant \int_0^x \left(\max_{1 \leqslant i \leqslant N} \frac{1}{u_i} \right)(v) dv.$$

(ii) It is well known (see [14] for a well written proof) that integrability of (i) is a necessary and sufficient condition for the existence and uniqueness of the solution of (1). Then, we proceed like in the proof of Theorem 1. □

3.2 Wave equations and Zig-Zag processes

We treat in this section some variations around the one-dimensional wave equation. The domain D is $\mathbb{R} \times \mathbb{R}_+$. In the two cases treated below, the trajectory of y_t has alternatively two velocities $v_1 = (1, -1)$ and $v_2 = (-1, -1)$. It draws some "zigzags," falling on the line $(\mathbb{R}, 0)$.

The telegraph equation

Let $\alpha > 0$. Consider the following equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) u(x,t) - 2\alpha \frac{\partial u}{\partial t} = 0 \tag{2}$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, with boundary conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, where $f, g \in C^2(\mathbb{R})$. This equation is known as the telegraph equation or the damped wave equation, the term $\frac{\partial u}{\partial t}$ corresponding to a damping.

Consider now the correlated process with velocities v_1 and v_2 and with $(\lambda_1, \lambda_2) = (\alpha, \alpha)$, so that the trajectory of y_s "zigzags" at exponential times with parameter α.

Proposition 3. *The solution of (2) is given by*

$$u(x,t) = E_{(x,t)}\left[f(x_t)1_{\{\varepsilon_t=1\}} + (f(x_t) + \frac{1}{\alpha}(g(x_t) - f'(x_t))1_{\{\varepsilon_t=2\}}\right]$$

where x_s denotes the first coordinate of y_s.

Proof. This is a consequence of Theorem 1 and of the fact that

$$\tau := \inf\{s > 0, y_s \in (\mathbb{R}, 0)\} = t\,. \qquad \square$$

This proposition is an equivalent formulation of the result of [4, 8] which makes use of the Chapman-Kolmogorov equation applied to the correlated process on $\mathbb{R}$ whose speed alternates between 1 and -1 at exponential times.

The linear Klein–Gordon equation

Given $\alpha \geqslant 0$, consider the following equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) u(x,t) + \alpha u(x,t) = 0 \tag{3}$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, with boundary conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, where $f, g \in C^2(\mathbb{R})$. For $\alpha = 0$, we deal with the classical wave equation.

We consider the correlated process with velocities v_1 and v_2 and with $(\lambda_1, \lambda_2) = (1, \alpha)$.

Proposition 4. *The solution of (3) is given by*

$$u(x,t) = E_{(x,t)}[\exp(T_t^1 + \alpha T_t^2)(f(x_t)1_{\{\varepsilon_t=1\}} + (g(x_t) - f'(x_t))1_{\{\varepsilon_t=2\}})]$$

where x_s denotes the first coordinate of y_s and T_t^1 (resp. T_t^2) denote the time spent by the process Y_s in $(D,1)$ (resp. $(D,2)$) before time t.

We mention here that the one-dimensional correlated process of Goldstein and Kac does not give any equivalent representation for solutions of this equation.

Remark 1. In the case $\alpha = 0$, we deal with the classical wave equation. One can easily check that the expression of Proposition 4 coincides with the explicit solution of (3), namely $u(x,t) = \frac{1}{2}(f(x+t) + f(x-t) + G(x+t) - G(x-t))$, where G denotes an inverse derivative of g.

Remark 2. The above "zigzag" processes can be generalized by choosing instead of the two velocities $(1,-1)$ and $(-1,-1)$, two arbitrary *vector fields* falling on the line $(\mathbb{R}, 0)$. This leads to a probabilistic representation of *"generalized wave equations."*

Remark 3. There is no difficulty to adapt the result to the case where α is replaced by a positive continuous function V, using a generalization of correlated processes (see Section 3.3 below).

3.3 The biharmonic equation

The biharmonic equation appears in the theory of elasticity and describes the shape of a plate subjected to some constraints at its boundary.

Conditions on u and Δu

Consider an open bounded domain D of $\mathbb{R}^k$, and the equation

$$\Delta^2 u = 0 \quad \text{on} \quad D \tag{4}$$

with boundary conditions $\lim_{x \to x_0} u(x) = f(x_0)$ and $\lim_{x \to x_0} \Delta u(x) = g(x_0)$ for all $x_0 \in \partial D$. This equation exactly fits in the framework of Khasminskii's result [9] and was also treated with some further precisions by Helms [5].

Proposition 5. *Let B_t be the Brownian motion on D. Let $\theta := \inf\{s > 0, B_s \in \partial D\}$. Suppose θ has finite expectation, then the solution of (4) is given by*

$$u(x) = E_x \left[f(B_\theta) - \frac{\theta}{2} g(B_\theta) \right] .$$

The direct proof of this fact relies on the fact that $u(B_t) - \frac{t}{2}\Delta u(B_t)$ is a martingale [5].

But one can also consider the process Y_t on $D \times \{1,2\}$, where both Markovian regimes are Brownian motions and the parameters are $(\lambda_1, \lambda_2) = (1, 0)$.

Put $\tau := \inf\{s > 0, y_s \in \partial D\}$, $T_t^1 := \int_0^t 1_{\{\varepsilon_s = 1\}} ds$.

If the expectation $E_{(x,1)}[\exp(T_\tau^1)]$ is finite, the solution of (4) is given by

$$u(x) = E_{(x,1)} \left[\exp(T_\tau^1)(f(Y_\tau) 1_{\{\varepsilon_\tau = 1\}} - \frac{1}{2} g(Y_\tau) 1_{\{\varepsilon_\tau = 2\}}) \right] .$$

If we denote by T the switching time of the process from $(D,1)$ to $(D,2)$, the last expression of u can be rewritten in the form

$$u(x) = E_x\left[\exp(\theta)f(B_\theta)1_{\{T>\theta\}} - \frac{1}{2}\exp(T)g(B_\theta)1_{\{T<\theta\}}\right].$$

where (B_t) is a Brownian motion and T a variable independent of (B_t) having exponential law with parameter 1. Disintegrating this expression with respect to T gives the result. □

The interest of this point of view comes out in the next section, when we introduce a nonconstant potential in this equation.

The presence of a potential

Let V be a strictly positive continuous function on D. We are interested in the equation

$$\Delta^2 u - 4Vu = 0 \quad \text{on} \quad D \tag{5}$$

with boundary conditions $\lim_{x\to x_0} u(x) = f(x_0)$ and $\lim_{x\to x_0} \Delta u(x) = g(x_0)$ for all $x_0 \in \partial D$. It describes the shape of an elastic plate subjected to a linear potential. This equation does not enter the frame of Khasminskii's result. Indeed, except in the case where V is a strictly positive constant, the operator $\Delta^2 - V$ is not the product of two operators $(\Delta - V_1)(\Delta - V_2)$.

Let us introduce, as suggested in Remark 2 of Theorem 1, the process $Y_t^V := (y_t^V, \varepsilon_t^V)$ on $D \times \{1,2\}$ defined as follows: y_t^V is a Brownian motion on D and

$$P(\varepsilon_{t+\delta}^V = 2 \,|\, \varepsilon_t^V = 1) = \delta + o(\delta):$$
$$P(\varepsilon_{t+\delta}^V = 1 \,|\, \varepsilon_t^V = 2) = V(y_t^V)\delta + o(\delta)\,.$$

Y_t admits an infinitesimal generator L^V, defined on the functions f on $D \times \{1,2\}$ whose restrictions f_1 and f_2 respectively to $(D,1)$ and $(D,2)$ are C^2-functions.

One can easily see, as in Proposition 1, that

$$L^V f(x,1) = \frac{1}{2}\Delta f_1(x) + (f_2(x) - f_1(x))\,;$$

$$L^V f(x,2) = \frac{1}{2}\Delta f_2(x) + V(x)(f_1(x) - f_2(x))\,.$$

An analogous result to Theorem 1 yields, in this modified situation, a simple probabilistic representation for the solution:

Proposition 6. *We introduce the stopping time $\tau := \inf\{s > 0, y_s^V \in \partial D\}$, and the two processes $T_t^1 := \int_0^t 1_{\{\varepsilon_s^V = 1\}} ds$, and $T_t^2 := \int_0^t V(y_s^V) 1_{\{\varepsilon_s^V = 2\}} ds$.*

If the expectation $E_{(x,1)}[\exp(T_\tau^1 + T_\tau^2)]$ is finite, the solution of (5) is unique and is given by

$$u(x) = E_{(x,1)}\left[\exp\left(T_\tau^1 + T_\tau^2\right) \times \left(f(y_\tau^V)1_{\varepsilon_\tau^V = 1} - \frac{1}{2}g(y_\tau^V)1_{\varepsilon_\tau^V = 2}\right)\right].$$

We want to stress here that a formula in terms of the hitting time and hitting position of ∂D by the usual Brownian motion, like in Proposition 5, is *not available* here and the use of correlated processes turns out to be quite necessary to get a probabilistic representation of the solutions of this kind of equation.

Toward conditions on $\frac{\partial u}{\partial n}$ and Δu

Set $\alpha > 0$. Denote by $\Delta^\alpha := \Delta - \alpha \mathrm{Id}$. Consider an open bounded smooth (C^3 is sufficient) domain D of $\mathbb{R}^k$, and the equation

$$\Delta \Delta^\alpha u = 0 \quad \text{on} \quad D \tag{6}$$

with boundary conditions $\frac{\partial u}{\partial n}(x_0) = f(x_0)$ and $\lim_{x \to x_0} \Delta^\alpha u(x) = g(x_0)$ for all $x_0 \in \partial D$.

The "method" of correlated processes appears to be flexible enough to give a representation of the solution of this equation, that we can express after simplification without using any correlated process.

Introduce the correlated process Y_t on $D \times \{1, 2\}$, where the first Markovian regime is the *reflecting* Brownian motion and the second one is the standard Brownian motion. Finally, take $(\lambda_1, \lambda_2) = (\alpha/2, 0)$.

Let $\tau := \inf\{s > 0, y_s \in \partial D\}$, and let $L_t^{(\partial D,1)}$ be the local time of Y_t on $(\partial D, 1)$. By definition of τ, $\varepsilon_\tau = 2$ almost surely, so that the process Y_t is a reflecting Brownian motion on $(D, 1)$ up to an independent random time having exponential law with parameter 1, and is afterwards a Brownian motion on $(D, 2)$ until it is killed at $(\partial D, 2)$.

We refer to [1] and the references therein for a well-written presentation of all the objects and properties relative to the reflecting Brownian motion and its local time at the boundary.

Proposition 7. *The solution of (6) can be expressed in the following two ways:*

(i)

$$u(x) = E_{(x,1)} \left[-\frac{1}{\alpha} g(Y_\tau) + \frac{1}{2} \int_0^\tau f(y_s) dL_s^{(\partial D,1)} \right].$$

(ii) Let (B_t) be the Brownian motion on D reflecting at ∂D. For all $u > 0$, let $\theta_u := \inf\{s > u,\ B_s \in \partial D\}$. We denote by $l_t^{\partial D}$ the local time of B_t on ∂D.

$$u(x) = E_x \left[-\frac{1}{2} \int_0^{+\infty} e^{-\frac{\alpha}{2} u} g(B_{\theta_u}) du + \frac{1}{2} \int_0^{+\infty} e^{-\frac{\alpha}{2} u} f(B_u) dl_u^{\partial D} \right] \tag{$\star$}$$

or equivalently,

$$u(x) = E_x \left[-\left(\frac{1 - e^{-\frac{\alpha}{2}\tau}}{\frac{\alpha}{2}} \right) \frac{g(B_\tau)}{2} + \frac{1}{2} \int_0^{+\infty} e^{-\frac{\alpha}{2} u} (f - G_\alpha)(B_u) dl_u^{\partial D} \right] \tag{$\star\star$}$$

where $\tau := \inf\{s > 0,\, B_s \in \partial D\}$ *and* G_α *is the function defined on* ∂D *by*

$$G_\alpha(x) = \hat{E}_x \left[\left(\frac{1 - e^{-\frac{\alpha}{2}T}}{\frac{\alpha}{2}} \right) g(d) \right]$$

where $\hat{E}_x$ *denotes the excursion measure of the reflecting Brownian motion starting at* x*,* T *is the duration of the excursion and "*d*" is the final point of the excursion.*

Remark. The second term in the expectation $(\star)$ is classical and corresponds naturally to the solution of the equation $\Delta u - \alpha u = 0$ with boundary condition $\frac{\partial u}{\partial n}(x_0) = f(x_0)$. The original part of the result lies in the first term.

Proof. For any function $\varphi \in C^2(D \times \{1, 2\}) \cap C^1((\partial D, 1))$, we denote $\varphi_1(x) := \varphi(x, 1)$ and $\varphi_2(x) = \varphi(x, 2)$,

$$\varphi(Y_t) - \int_0^t \frac{1}{2}\Delta\varphi_1(y_s) + \frac{\alpha}{2}(\varphi_2 - \varphi_1)(y_s) 1_{\{\varepsilon_s = 1\}} ds$$
$$+ \frac{1}{2}\int_0^t \frac{\partial \varphi_1}{\partial n}(y_s) dL_s^{(\partial D, 1)} - \frac{1}{2}\int_0^t \Delta\varphi_2(y_s) 1_{\{\varepsilon_s = 2\}} ds$$

is a martingale.

We apply this property to the function φ defined by $\varphi_1 = u$ and $\varphi_2 = -\dfrac{\Delta u - \alpha u}{\alpha}$.

We get that $\varphi(Y_t) + \frac{1}{2}\int_0^t f(y_s) dL_s^{(\partial D, 1)}$ is a martingale. We apply the optional sampling theorem at time $\tau \wedge n$, and the bounded convergence theorem applies as the expectation of the local time of $(\partial D, 1)$ at an independent exponential time is finite. We get (i).

The statement (ii) of the proposition is nothing but the translation of the expression of (i) in terms of the reflecting Brownian motion B_t: let $T_{\alpha/2}$ be an independent random variable having an exponential law with parameter $\alpha/2$, and $\theta := \inf\{s > T_{\alpha/2},\, B_s \in \partial D\}$. We denote by $l_t^{\partial D}$ the local time of B_t on ∂D.

$$u(x) = E_x \left[-\frac{1}{\alpha} g(B_\theta) + \frac{1}{2}\int_0^{T_{\alpha/2}} f(B_s) dl_s^{\partial D} \right]$$

We get $(\star)$ by disintegrating this formula with respect to $T_{\alpha/2}$.

The expression $(\star)$ can now be expressed in terms of the excursions in D of the reflecting Brownian motion. Let e_i be these excursions, $[S_{e_i}, T_{e_i}]$ the time interval of the excursion e_i, the integral $\int_0^{+\infty} e^{-\frac{\alpha}{2}u} g(B_{\theta_u}) du$ separates according to the first "incomplete excursion" and the other standard ones e_i in the following sum

$$E_x\left[\int_0^{+\infty} e^{-\frac{\alpha}{2}u} g(B_{\theta_u})du\right]$$
$$= E_x\left[\int_0^{\tau} e^{-\frac{\alpha}{2}u} g(B_\tau)du + \sum_{S(e_i)>0}\int_{S_{e_i}}^{T_{e_i}} e^{-\frac{\alpha}{2}u} g(B_{T_{e_i}})du\right]$$
$$= E_x\left[\left(\frac{1-e^{-\frac{\alpha}{2}\tau}}{\frac{\alpha}{2}}\right) g(B_\tau) + \sum_{S(e_i)>0} e^{-\frac{\alpha}{2}S_{e_i}}\left(\frac{1-e^{-\frac{\alpha}{2}(T_{e_i}-S_{e_i})}}{\frac{\alpha}{2}}\right) g(B_{T_{e_i}})\right]$$

The expression ($\star\star$) is then a direct consequence of the excursion formula. $\square$

The problem of making α converge to 0 remains here open, but having in mind the well-known probabilistic solution of the Neumann problem concerning harmonic functions (see Brosamler [2]), one would conjecture that, up to an additive constant, the probabilistic solution of the biharmonic equation with conditions $\frac{\partial u}{\partial n} = f$ and $\Delta u = g$ is given by

$$u(x) = \lim_{t\to+\infty} E_x\left[-\tau\frac{g(B_\tau)}{2} + \frac{1}{2}\int_0^t (f(B_u) - \hat{E}_{B_u}[Tg(d)])dl_u^{\partial D}\right]$$

$$\text{(or equivalently, } u(x) = \lim_{t\to+\infty} E_x\left[\frac{1}{2}\int_0^t f(B_u)dl_u^{\partial D} - \frac{1}{2}\int_0^t g(B_{\theta_u})du\right]\text{)}$$

assuming that the integral of $f(x) - \hat{E}_x[Tg(d)]$ along ∂D equals zero.

We hope we will devote a further work to this case and to a probabilistic representation of the biharmonic equation subjected to the classical Dirichlet conditions on u and $\frac{\partial u}{\partial n}$, that would give a continuous analogue to Vanderbei's result [13], but our feeling is that correlated processes cannot treat these problems directly.

4 Conclusion and generalizations

We wanted to show in this paper how correlated processes can help find a probabilistic representation of solutions to linear differential equations dealing with the composition of differential operators. Some special cases among our examples could be treated using usual random processes. However, as soon as one considers the presence of a *potential* or the composition of *different* generators, correlated processes turn out to be quite necessary.

Here are two possible generalizations of the processes we considered:

- In the whole text, we dealt with processes that can only jump from one domain to the domain next to it. Now, if we allow jumps from any domain to any other domain according to some prescribed rates, we will get a richer class of differential equations.

– It is interesting to note that the notion of correlated processes can be generalized if the condition that $y_T = \lim_{t \to T^-} y_t$ for all "switching" time T is replaced by the condition that y_T is a prescribed translation of $\lim_{t \to T^-} y_t$. This can lead to a probabilistic representation of some *delayed* differential equations.

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References

1. Abraham, R., (2000) Reflecting Brownian snake and a Neumann-Dirichlet problem. Stoch. Process. Appl. Vol. 89, no. 2, 239–260
2. Brosamler, G.A., (1976) A probabilistic solution of the Neumann problem. Math. Scand. Vol. 38, no. 1, 137–147
3. Chen, A-Y., Renshaw, E., (1994) The general correlated random walk. J. Appl. Probab. Vol. 31 , no. 4, 869–884
4. Goldstein, S., (1951) On diffusion by discontinuous movements and on telegraph equation. Q. J. Mech. Appl. Math. Vol. 4, 129–156
5. Helms, L.L., (1967) Biharmonic functions and Brownian motion. J. Appl. Probab. Vol. 4, 130–136
6. Henderson, R., Renshaw, E., Ford, D., (1984) A correlated random walk model for two-dimensional diffusion. J. Appl. Probab. Vol. 21 no. 2, 233–246
7. Kabanov, Yu.M. Probabilistic representation of a solution of the telegraph equation. (Russian) Teor. Veroyatnost. i Primenen Vol 37, no. 2, 425–426. English translation: Theory Probab. Appl. Vol 37, no. 2, 379–380
8. Kac, M., (1974) A stochastic model related to the telegrapher's equation. (reprinted from a 1956 lecture "Some stochastic problems in physics and mathematics," Magnolia Petroleum Co. Dallas, Texas) Rocky Mountain J. Math. Vol. 4, no. 3, 497–509
9. Khasminskii, R., (1962) Probability representation of solutions of some differential eqn. (Russian) Proc. Sixth All-Union Conf. Theory Probab. Math. Statist. (Vilnius, 1960) (Russian) pp. 177–182
10. Orsingher, E., (1995) Motions with reflecting and absorbing barriers driven by the telegraph equation. Random Oper. Stoch. Equations. Vol. 3, no. 1, 9–21
11. Ratanov, N.E. (1999) Telegraph evolutions in inhomogeneous media. Markov Process. Related Field. Vol. 5, no. 1, 53–68
12. Renshaw, E., Henderson, R., (1981) The correlated random walk. J. Appl. Probab. Vol. 18, no. 2, 403–414
13. Vanderbei, R.J., (1984) Probabilistic solution of the Dirichlet problem for biharmonic functions in discrete space. Ann. Probab. Vol. 12 , no. 2, 311–324
14. Zettl, A., (1997) Sturm-Liouville problems. Spectral theory and computational methods of Sturm-Liouville problems (Knoxville, TN, 1996), 1–104, Lecture Notes in Pure and Applied Mathematics, 191, Dekker, New York

Representation of the Martingales for the Brownian Snake

Laurent Serlet

Laboratoire de Mathématiques, Université Blaise Pascal
Campus Universitaire des Cézeaux, 63177 Aubière cedex, France
e-mail: Laurent.Serlet@math.univ-bpclermont.fr

Summary. We prove the previsible representation property for the filtration of the Brownian snake and give a representation of the martingales in the filtration associated to the historical Brownian motion. We deduce a representation of the martingale measure of the historical Brownian motion.

Key words: Super-Brownian motion, Brownian snake, Previsible representation property, Martingale measure

1 Introduction and statement of results

In this paper we deal with the Brownian snake introduced by Le Gall which can be seen as a continuous-time parametrization of a tree of branching trajectories. Abundant literature has shown the interest of this process to prove results on super-processes, to express solutions of certain semilinear pde or to describe the limit behavior of important interacting particle systems. See [Lg] for a comprehensive treatment of the subject. The definitions of the basic objects and the terminology used in the present introduction are however recalled in the next section.

The Brownian snake (W_s) is a simple example of process taking its values in the space $\mathcal{W}$ of stopped paths in $\mathbf{R}^d$. In [DS] we have developed some tools of stochastic calculus for this process, in particular an Itô formula; a simplified form of this statement is as follows. Suppose $F^{(2)} : \mathcal{W} \to \mathbf{R}$ is a continuous function and $F^{(1)}, F : \mathcal{W} \to \mathbf{R}$ are defined by

$$\begin{aligned} F^{(1)}(w) &= \int_0^\zeta F^{(2)}(w_{\leqslant r})\, dr \\ F(w) &= \int_0^\zeta F^{(1)}(w_{\leqslant r})\, dr = \int_0^\zeta \int_0^r F^{(2)}(w_{\leqslant u})\, du \\ &= \int_0^\zeta (\zeta - u)\, F^{(2)}(w_{\leqslant u})\, du \end{aligned}$$

then, for $0 \leqslant r < t$:

$$F(W_t) = F(W_r) + \int_r^t F^{(1)}(W_s)\, d\zeta_s + \frac{1}{2}\int_r^t F^{(2)}(W_s)\, ds \tag{1}$$

where (ζ_s) denotes the lifetime process of (W_s). The present paper is devoted to some applications of this formula concerning the representation of the martingales associated to the Brownian snake. First, the filtration $(\mathcal{F}_t^W)$ of the Brownian snake has a surprising previsible representation property:

Theorem 1. *For every random variable $X \in \mathrm{L}^2(\mathcal{F}_\infty^W)$, there exists a $(\mathcal{F}_s^W)$-previsible process (H_s) vanishing a.s. on $\{s,\ \zeta_s = 0\}$ such that*

$$\mathrm{E}\left(\int_0^\infty H_s^2\, ds\right) < +\infty \text{ and } X = \mathrm{E}(X) + \int_0^\infty H_s\, d\zeta_s\,. \tag{2}$$

The proof is given in Section 3. By standard arguments (see [RY] V.3.4), we quickly deduce the following corollary

Proposition 2. *For every local martingale (M_s) with respect to the filtration $(\mathcal{F}_s^W)$, there exists a $(\mathcal{F}_s^W)$-previsible process (H_s) locally in L^2 such that*

$$M_s = M_0 + \int_0^s H_r\, d\zeta_r. \tag{3}$$

Note that the above stochastic integral is effectively a local martingale because the integrand H_r vanishes on $\{s,\ \zeta_s = 0\}$ and we could write the integral with respect to the martingale part of the reflecting Brownian motion (ζ_s).

One of the interests of the Brownian snake lies in its connections with super-Brownian motion. More precisely let us consider a Brownian snake starting from $\tilde{x}$ and set

$$\forall t \geqslant 0,\ \ X_t = \int_0^{\tau_1} d_{(s)} L_s^t(\zeta)\, \delta_{\hat{W}_s} \text{ and } H_t = \int_0^{\tau_1} d_{(s)} L_s^t(\zeta)\, \delta_{W_s}$$

where $L_s^t(\zeta)$ is the local time of the lifetime process $(\zeta_\cdot)$ at level t and time s, and $\tau_1 = \inf\left\{s \geqslant 0; L_s^0(\zeta) > 1\right\}$ is the hitting time of 1 by the local time of ζ at level 0. The process (X_t) [resp. (H_t)] takes its values in the space $\mathcal{M}_F(\mathbf{R}^d)$ [resp. $\mathcal{M}_F(\mathcal{W})$] and is called super-Brownian motion starting from δ_x (resp. historical Brownian motion starting from $\delta_{\tilde{x}}$). To be honest, the usual definitions include a factor 1/4 that we have dropped here to simplify notations, as we did in [DS]. From this definition emerges a new filtration to be considered. Let

$$\tau_s^t = \inf\left\{u;\ \ \int_0^{u\wedge\tau_1} \mathbf{1}_{\{\zeta_v \leqslant t\}}\, dv > s\right\}$$

be the inverse of the time spent by the lifetime (ζ_s) under level t with the convention $\inf\{\emptyset\} = \tau_1$. We see that at least the following σ-algebras naturally arise:

$$\begin{aligned}
\mathcal{G}_t &= \sigma(W_{\tau_s^t},\ s \geqslant 0)\\
\mathcal{G}_t^\zeta &= \sigma(\zeta_{\tau_s^t},\ s \geqslant 0)\\
\mathcal{G}_t^X &= \sigma(X_r,\ r \leqslant t)\\
\mathcal{G}_t^H &= \sigma(H_r,\ r \leqslant t)\\
\mathcal{F}_s &= \sigma(W_{r\wedge\tau_1},\ r \leqslant s)\\
\mathcal{F}_s^\zeta &= \sigma(\zeta_{r\wedge\tau_1},\ r \leqslant s).
\end{aligned}$$

We note the following obvious relations

$$\mathcal{G}_t^\zeta = \mathcal{G}_t \cap \mathcal{F}_\infty^\zeta,\ \ \mathcal{G}_t^X \subset \mathcal{G}_t^H \subset \mathcal{G}_t,\ \ \mathcal{G}_\infty = \mathcal{F}_\infty.$$

A question now arises concerning the representation of the martingales in the "vertical" filtration $(\mathcal{G}_t)$. Since super-Brownian motion (and historical Brownian motion) has often been studied via martingale problems we already know a class of $(\mathcal{G}_t)$-martingale: for every bounded ϕ in the domain of the generator A,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s(A\phi)\, ds$$

defines a martingale and moreover the quadratic variation of this martingale is known to be

$$\langle M(\phi)\rangle_t = 4 \int_0^t X_s(\phi^2)\, ds\,. \tag{4}$$

An interpretation with the Brownian snake can be given.

Proposition 3. ([DS] Lemma 10 and Theorem 7) *We have*

$$M_t(\phi) = 2 \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_s)\, \phi(\hat{W}_s)\, d\zeta_s \tag{5}$$

and this process is a $(\mathcal{G}_t)$-martingale with quadratic variation given by (4).

A natural question is then to ask if any $(\mathcal{G}_t)$-martingale can be expressed in a way that generalizes the above expression.

Theorem 4. *For every $(\mathcal{G}_t)$-martingale (M_t) which is bounded in L^2, there exists a $(\mathcal{F}_s)$-previsible process (H_s) such that*

$$\mathrm{E}\left(\int_0^{\tau_1} H_s^2\, ds\right) < +\infty \text{ and } \forall t \geqslant 0,\ \ M_t = M_0 + \int_0^{\tau_1} H_s\, \mathbf{1}_{(0,t]}(\zeta_s)\, d\zeta_s\,.$$

By [Je] this result is known when it is restricted to the filtration $(\mathcal{G}_t^\zeta)$, i.e., considering only the lifetime. In that case it is essentially seen on Tanaka's formula interpreted as a reflection equation. Our proof is given in Section 4. We deduce the corollary:

Proposition 5. *For every $Y \in \mathrm{L}^2(\mathcal{G}_t)$, there exists a $(\mathcal{F}_s)$-previsible process (H_s) such that*

$$Y = \mathrm{E}(Y) + \int_0^{\tau_1} H_s \, \mathbf{1}_{(0,t]}(\zeta_s) \, d\zeta_s \text{ and } \mathrm{E}\left(\int_0^{\tau_1} H_s^2 \, \mathbf{1}_{(0,t]}(\zeta_s) \, ds \right) < +\infty \, .$$

The representation given in (5) can be pushed a little further into the terminology of martingale measure as the notation already suggests. The notion of martingale measure of super-processes is explained for instance in [Da] Chapter 7; Example 7.1.3 covers the case of super-Brownian motion as it is defined here. In our setting the martingale measure of super-Brownian motion or even historical Brownian motion is easily described. It is stated in the following proposition where L denotes the generator of the so-called A-path process which is the process in $\mathcal{W}$ whose lifetime increases at constant speed 1 and consists in a trajectory of the diffusion governed by A.

Proposition 6. *Let us set, for $t \geqslant 0$ and $\Omega \in \mathcal{B}(\mathcal{W})$,*

$$M_t(\Omega) = 2 \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r) \, \mathbf{1}_{\Omega}(W_r) \, d\zeta_r \, . \tag{6}$$

Then, $(M_t(\Omega),\ t \geqslant 0,\ \Omega \in \mathcal{B}(\mathcal{W}))$ defines an L^2-martingale measure $M(ds\,dw)$. It is associated to the historical Brownian motion, that is, for every $\phi : \mathcal{W} \to \mathbf{R}$ in the domain of L and bounded,

$$H_t(\phi) = H_0(\phi) + \int_0^t H_s(L\phi) \, ds + \int_0^t \int_{\mathcal{W}} \phi(w) \, M(ds\,dw) \, . \tag{7}$$

This martingale measure is orthogonal and its intensity is the random measure ν on $\mathbf{R}_+ \times \mathcal{W}$ given by

$$\int_{\mathbf{R}_+ \times \mathcal{W}} \psi(t,w) \, \nu(dt\,dw) = 4 \int_0^{\tau_1} \psi(\zeta_s, W_s) \, ds \, . \tag{8}$$

We recall that the intensity of a martingale measure is defined so that $\nu([0,t] \times \Omega)$, for Ω Borel subset of W, is the quadratic variation of the martingale $(M_t(\Omega))$.

2 Basic objects and notations

We will use the following common notations:
$\mathbf{N} = \{1, 2, 3, \ldots\}$, $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbf{R}_+ = [0, +\infty)$.
$\mathcal{C}(X, Y)$: set of continuous functions from metric space X to metric space Y.
$\sigma(X_i, i \in I)$: σ-algebra generated by the random variables $X_i,\ i \in I$ in a fixed probability space, completed with all negligible sets.
$\mathcal{B}(X)$: Borel σ-algebra of the metric space X.

$\mathcal{M}_F(X)$: set of all finite measures on the metric space X equipped with the (metrizable) topology of weak convergence and its Borel σ-algebra $\mathcal{B}(\mathcal{M}_F(X))$.

A stopped path is a couple (w, ζ), where $\zeta \geqslant 0$ is called the lifetime of the path, and $w : \mathbf{R}_+ \to \mathbf{R}^d$ is a continuous mapping, which is constant on $[\zeta, +\infty)$. We denote by $\mathcal{W}$ the set of all stopped paths. We sometimes abbreviate (w, ζ) into w and denote $\zeta(w)$ the lifetime. The distance on $\mathcal{W}$ is $d(w, w') = \sup_{t \geqslant 0} |w(t) - w'(t)| + |\zeta(w) - \zeta(w')|$, making $\mathcal{W}$ a Polish space. We denote by $\hat{w} = w(\zeta)$ the endpoint of w, and $\tilde{x}$ the path of lifetime 0 started at $x \in \mathbf{R}^d$. Finally, we denote by $w_{\leqslant r}$ the path of lifetime $\zeta(w) \wedge r$ such that for $u \geqslant 0$, $w_{\leqslant r}(u) = w(u \wedge r)$.

Let us fix a diffusion in $\mathbf{R}^d$ with generator A. The Brownian snake started at x with spatial motion governed by A is the strong Markov continuous process $W = (W_s, s \geqslant 0)$ with values in $\mathcal{W}$ characterized by the following properties:

1. $W_s(0) = x$ for every s;
2. The lifetime process $\zeta_s = \zeta(W_s)$ is a reflecting Brownian motion in $\mathbf{R}_+$;
3. Conditionally on $(\zeta_s, s \geqslant 0)$, the distribution of $(W_s, s \geqslant 0)$ is that of an inhomogeneous Markov process whose transition kernels are described as follows: for every $s < s'$,
   - $W_{s'}^{\leqslant m} = W_s^{\leqslant m}$ where $m = \inf_{r \in [s, s']} \zeta_r$;
   - $(W_{s'}(m + t), 0 \leqslant t \leqslant \zeta_{s'} - m)$ is independent of W_s conditionally on $W_s(m)$ and has the law of a diffusion in $\mathbf{R}^d$ with generator A, starting from $W_s(m)$ and stopped at time $\zeta_{s'} - m$.

The filtration $(\mathcal{F}_t^W)$ used in the introduction is the filtration associated to (W_s), completed the usual way (see [RY] p. 45 and 93 for precisions on completion) and

$$\mathcal{F}_\infty^W = \sigma\left(\bigcup_{t \geqslant 0} \mathcal{F}_t^W \right).$$

3 Proof of the previsible representation property (Theorem 1.)

This proof is inspired by Exercise 3.15 of [RY], dealing with the classical Brownian filtration. In that case a step of the proof is to solve a linear differential equation. This is replaced in our path space setting by an integral equation that we first discuss.

Lemma 7. *Let* $\alpha > 0$ *and*

$$\mathcal{E}_\alpha = \left\{ \psi \in \mathcal{C}(\mathcal{W}, \mathbf{R});\ \sup_{w \in \mathcal{W}} e^{-\alpha\, \zeta} \, |\psi(w)| < +\infty \right\}.$$

For all $\lambda > 0$, if $\alpha > \sqrt{2\lambda}$ and $f \in \mathcal{E}_\alpha$ then there exists $\varphi \in \mathcal{E}_\alpha$ such that

$$\forall w \in \mathcal{W},\ \ \frac{1}{2}\varphi(w) - \lambda \int_0^\zeta (\zeta - u)\, \varphi(w_{\leqslant u})\, du = f(w)\,.$$

Proof. It is easy to see that the formula $||\psi||_\alpha = \sup_{w\in\mathcal{W}} e^{-\alpha\,\zeta}\,|\psi(w)|$ defines a norm on the vector space $\mathcal{E}_\alpha$ which makes this space complete. The result of the lemma consists in finding a fixed point for the map

$$\theta\ :\ \ \varphi \to \left(w \to 2f(w) + 2\lambda \int_0^\zeta (\zeta - u)\, \varphi(w_{\leqslant u})\, du \right).$$

It is easy to verify that θ maps $\mathcal{E}_\alpha$ into itself. In order to apply the classical Lipschitz fixed point theorem to θ in the Banach space $\mathcal{E}_\alpha$, it remains to check that θ satisfies the Lipschitz condition. For $\varphi_1, \varphi_2 \in \mathcal{E}_\alpha$,

$$\begin{aligned}
||\theta(\varphi_1) - \theta(\varphi_2)||_\alpha &= \sup_{w\in\mathcal{W}} e^{-\alpha\,\zeta} \left| 2\lambda \int_0^\zeta (\zeta - u)\, (\varphi_1 - \varphi_2)(w_{\leqslant u})\, du \right| \\
&\leqslant \sup_{w\in\mathcal{W}} e^{-\alpha\,\zeta} \left| 2\lambda \int_0^\zeta (\zeta - u)\, e^{\alpha u} ||\varphi_1 - \varphi_2||_\alpha\, du \right| \\
&\leqslant 2\lambda\, ||\varphi_1 - \varphi_2||_\alpha\ \sup_{\zeta > 0} \left\{ e^{-\alpha\,\zeta} \int_0^\zeta (\zeta - u)\, e^{\alpha\, u}\, du \right\} \\
&\leqslant 2\lambda\, ||\varphi_1 - \varphi_2||_\alpha \int_0^{+\infty} v\, e^{\alpha\, v} dv = \frac{2\lambda}{\alpha^2}\, ||\varphi_1 - \varphi_2||_\alpha\,.
\end{aligned}$$

Since the last ratio is by assumption smaller than 1, the proof of the lemma is complete. □

We now denote $\mathcal{E}$ the increasing limit of the sets $\mathcal{E}_\alpha,\ \ \alpha > 0$, that is $\mathcal{E} = \bigcup_{\alpha>0} \mathcal{E}_\alpha$. We are now ready to give a previsible representation for certain variables, namely, the type on the left-hand side of the following equality.

Lemma 8. *For every $f \in \mathcal{E}$ and every $\lambda > 0$, there exist $g_0, g_1 \in \mathcal{E}$ such that, for every $r > 0$,*

$$\int_r^\infty e^{-\lambda s}\, f(W_s)\, ds = e^{-\lambda r}\, g_0(W_r) + \int_r^\infty e^{-\lambda s}\, g_1(W_s)\, d\zeta_s\,. \tag{9}$$

Moreover g_0 vanishes at 0 in the following sense: $g_0(w) = 0$ if $\zeta(w) = 0$ and identically for g_1.

Proof. Let us first remark that, since f belongs to a certain $\mathcal{E}_\alpha$, we have

$$\int_r^\infty e^{-\lambda s}\, |f(W_s)|\, ds \leqslant ||f||_\alpha\ \int_r^\infty e^{-\lambda s}\, e^{\alpha\zeta_s}\, ds$$

and the integral on the right-hand side is finite because the reflecting Brownian motion (ζ_s) satisfies the law of the iterated logarithm. Hence the integral appearing on the left-hand side of (9) is defined almost surely. So is the integral on the right-hand side using a similar argument and [RY] IV.1.26.

By increasing α if necessary, we may suppose that $\alpha > \sqrt{2\lambda}$. By Lemma 7, we can associate to $f \in \mathcal{E}_\alpha$ a continuous function $\varphi \in \mathcal{E}_\alpha$ as specified. Let $F^{(2)} = \varphi$ and $F^{(1)}, F$ be defined as in the assumptions of formula (1). Note that $F^{(1)}$ and F vanish at 0, in the sense defined in the statement of the lemma. It is easy to check that $F^{(1)}, F \in \mathcal{E}_\alpha$ and more precisely,

$$|F(w)| \leqslant ||\varphi||_\alpha \, \frac{e^{\alpha\,\zeta}}{\alpha^2} \,. \tag{10}$$

We obtain, by formula (1) and the classical Itô formula for a product, for $0 \leqslant r < t$:

$$\begin{aligned} e^{-\lambda t}\, F(W_t) - e^{-\lambda r}\, F(W_r) = & \int_r^t e^{-\lambda s} \left(\frac{1}{2}\varphi - \lambda\, F\right)(W_s)\, ds \\ & + \int_r^t e^{-\lambda s} F^{(1)}(W_s)\, d\zeta_s. \end{aligned}$$

We recall that $(1/2)\,\varphi - \lambda F = f$. Using the bound (10), the law of the iterated logarithm for (ζ_s) entails that $\lim_{t\to+\infty} e^{-\lambda\, t}\; F(W_t) = 0$, almost surely. Therefore we get

$$\int_r^\infty e^{-\lambda s}\; f(W_s)\, ds = -e^{-\lambda r}\, F(W_r) - \int_0^\infty e^{-\lambda s}\; F^{(1)}(W_s)\, d\zeta_s$$

We obtain the sought after representation, up to a change of notations. □

Lemma 9. *For all $n \in \mathbf{N}$, $\lambda_1, \ldots, \lambda_n > 0$, $f_1, \ldots, f_n \in \mathcal{E}$, there exist $\mu_0, \mu_1, \mu_j^k > 0$, $(1 \leqslant k \leqslant n-1, 0 \leqslant j \leqslant k)$, $g_0, g_1, g_j^k \in \mathcal{E}$ $(1 \leqslant k \leqslant n-1$, $0 \leqslant j \leqslant k)$ with g_0, g_1, g_0^k vanishing at 0, such that, for all $r \geqslant 0$,*

$$\begin{aligned} & \int_{\{r<s_1<\cdots<s_n\}} \left(\prod_{i=1}^n e^{-\lambda_i\, s_i}\; f_i(W_{s_i})\right)\; ds_1 \ldots ds_n \\ & = e^{-\mu_0\, r}\; g_0(W_r) + \int_r^{+\infty} \left\{ e^{-\mu_1\, s}\; g_1(W_s) + \sum_{k=1}^{n-1} e^{-\mu_0^k\, s}\; g_0^k(W_s) \right. \\ & \qquad \left. \int_{\{r<s_1<\cdots<s_k<s\}} \left(\prod_{j=1}^k e^{-\mu_j^k\, s_j}\; g_j^k(W_{s_j})\right)\; ds_1 \ldots ds_k \right\}\; d\zeta_s \end{aligned}$$

with the convention that the sum over k disappears if $n = 1$.

Proof. By (9), we know that the lemma is true for $n = 1$. Then we proceed by induction. Admitting the result at rank $n \geqslant 1$, we examine the case of rank $n+1$:

$$\begin{aligned}
&\int_{\{r<s_1<\dots<s_{n+1}\}} \left(\prod_{i=1}^{n+1} e^{-\lambda_i s_i} f_i(W_{s_i})\right) ds_1 \dots ds_{n+1} \\
&= \int_r^{+\infty} ds_1 e^{-\lambda_1 s_1} f_1(W_{s_1}) \int_{\{s_1<s_2<\dots<s_{n+1}\}} \left(\prod_{i=2}^{n+1} e^{-\lambda_i s_i} f_i(W_{s_i})\right) ds_2 \dots ds_{n+1} \\
&= \int_r^{+\infty} ds_1 \; e^{-\lambda_1 s_1} f_1(W_{s_1}) \; \Bigg[e^{-\mu_0 s_1} g_0(W_{s_1}) + \int_{s_1}^{+\infty} \Bigg\{ e^{-\mu_1 s} g_1(W_s) \\
&\quad + \sum_{k=2}^{n} e^{-\mu_0^k s} g_0^k(W_s) \int_{\{s_1<s_2<\dots<s_k<s\}} \prod_{j=2}^{k} e^{-\mu_j^k s_j} g_j^k(W_{s_j})\, ds_2 \dots ds_k \Bigg\} d\zeta_s \Bigg] \\
&= \int_r^{+\infty} e^{-(\lambda_1+\mu_0) s_1} (f_1 g_0)(W_{s_1})\, ds_1 \\
&\quad + \int_r^{+\infty} e^{-\mu_1 s} g_1(W_s) \left(\int_r^{s} e^{-\lambda_1 s_1} f_1(W_{s_1}) ds_1\right) d\zeta_s \\
&\quad + \sum_{k=2}^{n} \int_r^{+\infty} e^{-\mu_0^k s} g_0^k(W_s) \Bigg(\int_r^{s} ds_1 e^{-\lambda_1 s_1} f_1(W_{s_1}) \\
&\qquad \int_{\{s_1<s_2<\dots<s_k<s\}} \left(\prod_{j=2}^{k} e^{-\mu_j^k s_j} g_j^k(W_{s_j})\right) ds_2 \dots ds_k \Bigg) d\zeta_s .
\end{aligned}$$

The first equality is simply Fubini's formula; then we use the induction hypothesis for the integral with respect to $s_2, \dots, s_{n+1}$; and for the last equality a stochastic version of Fubini's theorem. To the first term obtained at the last equality we can apply the result at rank 1, i.e., (9); the second and third term are the desired quantities to obtain the sought-after formula, up to a change of notations of course. ◻

Lemma 10. *For every $\hat{s} > 0$, there exist, for every $i \in \mathbf{N}$, coefficients $m_i \in \mathbf{N}$, $\alpha_j^i \in \mathbf{R}$, $\lambda_j^i > 0$ for $1 \leqslant j \leqslant m_i$ such that, for every continuous function $\gamma : \mathbf{R}_+ \to \mathbf{R}$ (absolutely) integrable over $\mathbf{R}_+$,*

$$\int_0^{+\infty} \left(\sum_{j=1}^{m_i} \alpha_j^i \, e^{-\lambda_j^i s}\right) \gamma(s)\, ds \overset{i\to+\infty}{\longrightarrow} \gamma(\hat{s}) \, .$$

Proof. We first consider an approximation $p_i(s)\, ds$ of the Dirac measure $\delta_{\hat{s}}$ with continuous density whose support is contained in $(0, 2\hat{s})$ so that we have

$$\int_0^{+\infty} p_i(s)\, \gamma(s)\, ds \overset{i\to+\infty}{\longrightarrow} \gamma(\hat{s})$$

for every continuous γ. The set of functions over $\mathbf{R}_+$

$$\Lambda = \left\{ \psi : s \to \sum_{j=1}^{m} \alpha_j \, e^{-\lambda_j \, s}; \;\; m \in \mathbf{N}, \; \alpha_j \in \mathbf{R}, \; \lambda_j > 0, \;\; \psi(0) = 0 \right\}$$

is a linear subspace, closed under multiplication. Let $\arg z \in [-\pi, \pi)$ denote the value of the argument of $z \in \mathbf{U} = \{z \in \mathbf{C}; \;\; |z| = 1\}$. On the compact $\mathbf{U}$, equipped with uniform topology, we can apply the the classical Stone–Weierstrass approximation theorem to the set of continuous functions:

$$\left\{ z \to a + \psi \left(\tan \frac{1}{4} (\arg z + \pi) \right); \; a \in \mathbf{R}, \; \psi \in \Lambda \right\}$$

in order to approximate by functions of this set, the continuous function $z \to p_i(\tan[(\arg z + \pi)/4])$. It is thus possible to find $a_i \in \mathbf{R}, m_i \in \mathbf{N}, \; \alpha_j^i \in \mathbf{R}$, $\lambda_j^i > 0$ such that

$$\sup_{s \in \mathbf{R}_+} |p_i(s) - \psi_i(s)| \overset{i \to +\infty}{\longrightarrow} 0 \text{ with } \psi_i \; : \; s \to a_i + \sum_{j=1}^{m_i} \alpha_j^i \, e^{-\lambda_j^i \, s} \;\; \in \Lambda \, .$$

By considering the value at 0 we may suppose that $a_i = 0$. We have found the desired sequence of functions. □

Proof of Theorem 1. We denote by R the linear subspace of $\mathrm{L}^2(\mathcal{F}_\infty^W)$ consisting of variables X admitting the specified representation (2) with (H_s) a previsible process such that $H_s = 0$ a.s. on $\{s; \; \zeta_s = 0\}$. With such a representation we obtain

$$\mathrm{E}(X^2) = (\mathrm{E}X)^2 + \mathrm{E} \left(\int_0^{+\infty} H_s^2 \, ds \right) .$$

By a classical argument (cf [RY] p. 199) we deduce that R is complete hence closed in $\mathrm{L}^2(\mathcal{F}_\infty^W)$.

Lemma 9 implies that for all $n \in \mathbf{N}$, $\lambda_1, \ldots, \lambda_n > 0$, $f_1, \ldots, f_n$ bounded and continuous on $\mathcal{W}$ (hence in $\mathcal{E}$), the set R contains the variable

$$\int_{\{0 < s_1 < \cdots < s_n\}} \prod_{i=1}^{n} e^{-\lambda_i \, s_i} \, f_i(W_{s_i}) \, ds_1 \ldots ds_n$$

hence also the variable

$$\prod_{i=1}^{n} \int_0^\infty e^{-\lambda_i s} \, f_i(W_s) \, ds$$

and, by linear combination, R contains also the variable:

$$\prod_{i=1}^{n} \int_0^\infty \left(\sum_{j=1}^{m_i} \alpha_j^i \, e^{-\lambda_j^i s} \right) \; e^{-s} \, f_i(W_s) \, ds$$

where the coefficients $\alpha_j^i \in \mathbf{R}, \lambda_j^i > 0, m_i \in \mathbf{N}$ are arbitrary.

We deduce from Lemma 10 that, for every $n \in \mathbf{N}$, for all $f_1, \ldots, f_n$ bounded and continuous and all $\hat{s}_1, \ldots, \hat{s}_n > 0$, the set R contains the variable

$$\prod_{i=1}^{n} f_i(W_{\hat{s}_i})$$

(dropping useless constant exponential factors) and this is clearly sufficient to claim that $R = \mathrm{L}^2(\mathcal{F}_\infty^W)$.

4 Representation in filtration $(\mathcal{G}_t)$

4.1 Proof of Theorem 4.

We first establish that a process (M_t) given as in the statement of the theorem is effectively a martingale, that is, for all $t, h > 0$, for every $\mathcal{G}_t$-mesurable U,

$$\mathrm{E}\left[(M_{t+h} - M_t)\, U\right] = \mathrm{E}\left[\left(\int_0^{\tau_1} H_s\, \mathbf{1}_{\{t<\zeta_s\leqslant t+h\}}\, d\zeta_s\right) U\right] = 0 \qquad (11)$$

We fix $\epsilon > 0$ and introduce the successive time intervals of descent from $t+\epsilon$ down to t, that is we consider the $(\mathcal{F}_s)$-stopping times $(S_k, k \geqslant 0)$ and $(T_k, k \geqslant 0)$ defined by $S_0 = T_0 = 0$, and if $k \geqslant 1$,

$$\begin{aligned} S_k &= \inf\left\{s \in (T_{k-1}, \tau_1); \zeta_s = t+\epsilon\right\}\ , \\ T_k &= \inf\left\{s \in (S_k, \tau_1); \zeta_s = t\right\}\ , \end{aligned}$$

with the convention $\inf \emptyset = \tau_1$. Equation (11) will be proved, by letting $\epsilon \downarrow 0$, as soon as we can show that, for every $k \in \mathbf{N}$,

$$\mathrm{E}\left[\left(\int_{S_k}^{T_k} H_s\, \mathbf{1}_{\{t<\zeta_s\leqslant t+h\}}\, d\zeta_s\right) U\right] = 0\,.$$

By the definition of $\mathcal{G}_t$, it is sufficient to prove that

$$\mathrm{E}\left[\left(\int_{S_k}^{T_k} H_s\, \mathbf{1}_{\{t<\zeta_s\leqslant t+h\}}\, d\zeta_s\right) X\ G(W_{T_k+\cdot})\right] = 0$$

where X is $\mathcal{F}_{S_k}$-measurable and bounded and G is a bounded measurable function. By applying the Markov property at time T_k the left-hand side of the above expression reduces to

$$\mathrm{E}\left[X\left(\int_{S_k}^{T_k} H_s\, \mathbf{1}_{\{t<\zeta_s\leqslant t+h\}}\, d\zeta_s\right) \mathrm{E}_{W_{T_k}}[G]\right].$$

But $W_{T_k} = W_{S_k}^{\leqslant t}$. The variable $X\, \mathrm{E}_{W_{S_k}^{\leqslant t}}[G]$ is $\mathcal{F}_{S_k}$-measurable and bounded and we can represent it under the following form:

$$X\, \mathrm{E}_{W_{S_k}^{\leqslant t}}[G] = c + \int_0^{S_k} K_s\, d\zeta_s$$

with a $(\mathcal{F}_s)$-previsible process (K_s). Therefore we have finally to prove that

$$\mathrm{E}\left[\left(\int_{S_k}^{T_k} H_s\ \mathbf{1}_{\{t<\zeta_s\leqslant t+h\}}\ d\zeta_s\right)\left(c+\int_0^{S_k} K_s\ d\zeta_s\right)\right]=0\,.$$

The contribution coming from the multiplication by c is null, by applying the stopping Theorem for martingale $\int_0^{\cdot} H_s\ \mathbf{1}_{\{t<\zeta_s\leqslant t+h\}}\ d\zeta_s$. The remaining term is equal to

$$\mathrm{E}\left[\int_0^{\tau_1} \mathbf{1}_{[S_k,T_k]}(s)\ H_s\ \mathbf{1}_{\{t<\zeta_s\leqslant t+h\}}\mathbf{1}_{[0,S_k]}(s)\ K_s\ \ ds\right]$$

which is clearly zero.

Now we denote by (M_t) any $(\mathcal{G}_t)$-martingale bounded in L^2. Let M_∞ be the almost sure and L^2 limit of M_t. This variable of $\mathcal{G}_\infty=\mathcal{F}_\infty$ can be repesented as

$$M_\infty=\mathrm{E}(M_\infty)+\int_0^{\tau_1} H_s\ d\zeta_s$$

with a $(\mathcal{F}_s)$-previsible process (H_s). Then

$$M_t=\mathrm{E}[M_\infty|\mathcal{G}_t]=\mathrm{E}(M_\infty)+\int_0^{\tau_1} H_s\ \mathbf{1}_{\{0<\zeta_s\leqslant t\}}\ d\zeta_s\,,$$

the last equality resulting from the first part of the proof.

4.2 Comments on Proposition 6.

It is straightforward that Formula (6) defines for every $t>0$, an L^2-valued finite measure. Firstly, it is finitely additive. Secondly we have, for every $\Omega\in\mathcal{B}(\mathcal{W})$,

$$\begin{aligned}||M_t(\Omega)||_2^2 &= 4\ \mathrm{E}\left[\int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\ \mathbf{1}_{\Omega}(W_r)\ dr\right]\\ &\leqslant 4\ \mathrm{E}\left[\int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\, dr\right]\\ &= 4\ \mathrm{E}\left[\int_0^{t} L^a_{\tau_1}(\zeta)\, da\right] = 4\ \int_0^t \mathrm{E}\left[L^a_{\tau_1}(\zeta)\right] da = 4t.\end{aligned}$$

The last equality follows from the Ray-Knight Theorem (or can been seen as the first moment of super-Brownian motion). Moreover it is clear by the dominated convergence Theorem that $||M_t(\Omega)||_2$ converges to 0 if Ω decreases to $\emptyset$ and this proves the L^2 countable additivity. Thus we are in the classical setting of martingale measures as described in [Da] Chapter 7. We have

$$\int_0^t\int_{\mathcal{W}} \phi(w)\ M(ds\,dw)=2\ \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\ \phi(W_r)\ d\zeta_r\,.$$

Then Formulas (7) and (8) are essentially a reformulation of Proposition 3 for historical Brownian motion instead of super-Brownian motion, but this extension was also covered by [DS]. In particular, the quadratic variation of $\int_0^t \int_{\mathcal{W}} \phi(w)\, M(ds\, dw)$ is:

$$4 \int_0^t H_s(\phi^2)\, ds = 4 \int_0^t ds \int_0^{\tau_1} \phi^2(W_r)\, d_{(r)} L_r^s(\zeta)$$
$$= 4 \int_0^{\tau_1} \mathbf{1}_{(0,t]}(\zeta_r)\, \phi^2(W_r)\, dr$$

and this leads to (8).

References

[Da] Dawson D.A.: Measure-valued Markov processes. In: École d'été de probabilités de Saint-Flour 1991. Lecture Notes in Mathematics 1541, 1–260. Springer, Berlin Heidelberg New York (1993)

[DS] Dhersin J.S., Serlet L.: A stochastic calculus approach for the Brownian snake. Can. J. Math. 52 (1), 92–118 (2000)

[Je] Jeulin T.: Applications de la théorie du grossissement à l'étude des temps locaux browniens. In: Grossissement de filtrations. Lecture Notes in Mathematics 1118. Springer, Berlin Heidelberg New York (1985)

[Lg] Le Gall J.F.: Spatial branching processes, random snakes and partial differential equations. Lectures in Mathematics ETH Zurich, Birkhauser, Basel (1999)

[RY] Revuz D., Yor M.: Continuous martingales and Brownian motion, third edition. Springer, Berlin Heidelberg New York (1999)

Discrete Sampling of Functionals of Itô Processes

Emmanuel Gobet[1] **and Stéphane Menozzi**[2] *

Centre de Mathématiques Appliquées, Ecole Polytechnique
91128 Palaiseau Cedex, France
[1] *e-mail: gobet@cmapx.polytechnique.fr*
[2] *e-mail: menozzi@cmapx.polytechnique.fr*

Summary. For a multidimensional Itô process $(X_t)_{t\geqslant 0}$ driven by a Brownian motion, we are interested in approximating the law of $\psi\left((X_s)_{s\in[0,T]}\right)$, $T > 0$ deterministic, for a given functional ψ using a discrete sample of the process X. For various functionals (related to the maximum, to the integral of the process, or to the killed/stopped path) we extend to the non-Markovian framework of Itô processes, the results available in the diffusion case. We thus prove that the order of convergence is more specifically linked to the Brownian driver and not to the Markov property of SDEs.

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1 Introduction: statement of the problem

Let $(X_t)_{t\in[0,T]}$ be a d-dimensional Itô process, whose dynamics is given by

$$X_t = x + \int_0^t b_s ds + \int_0^t \sigma_s dW_s \tag{1}$$

with a fixed initial data x and a fixed terminal time T. Here, W is a d'-dimensional standard Brownian motion (BM) defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural completed filtration of W. The progressively measurable coefficients $(b_s)_{s\geqslant 0}$ and $(\sigma_s)_{s\geqslant 0}$ are bounded. In this work, we are mainly interested in approximating the law of $\psi\left((X_s)_{s\in[0,T]}\right)$, where ψ is a real valued functional

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defined on the space of càdlàg functions, using a discrete sample of the process X. For this latter, we use the stepwise constant counterpart of X defined by $(X_{\phi(s)})_{s\in[0,T]}$ where $\phi(s) = t_i$ if $t_i := ih \leqslant s < t_{i+1}$ ($h = T/N$ being the step size). The main problem consists in controlling the difference

$$\mathrm{Err}(T,h,\psi,x) := \mathbb{E}[\psi\left((X_s)_{s\in[0,T]}\right)] - \mathbb{E}[\psi\left((X_{\phi(s)})_{s\in[0,T]}\right)] \tag{2}$$

for a certain class of functionals ψ w.r.t. the time step h. This kind of problem has been widely studied in the Markovian setting (i.e., when X is a solution of a SDE) for a large class of functionals ψ, see the short list and references below. What we want to emphasize in this paper is that the rates of convergence obtained in the Markovian case, through proofs relying on an associated PDE, are still valid in the non-Markovian framework of Itô processes. Hence, it is not the Markov property that gives the order of convergence, but actually the Brownian stochastic integral. Here are some controls of $\mathrm{Err}(T,h,\psi,x)$ in the Markovian setting for some specific functionals ψ.

1. Integral of the process.
   This case corresponds to $\psi_1(y) := \varphi\left(\int_0^T y(s)ds\right)$, where φ is a Lipschitz continuous function from $\mathbb{R}^d$ into $\mathbb{R}$. We know from Temam [Tem01] that $\mathrm{Err}(T,h,\psi_1,x) = O(h)$.
2. Maximum of the drifted BM when $d = 1$.
   This case corresponds to $\psi_2(y) := \max_{s\in[0,T]} y(s)$. For $X_s = x+\mu s+\sigma W_s$, we derive from Lemma 6 in Asmussen et al. [AGP95] that there exists a constant $C > 0$ s.t. $0 \leqslant \mathrm{Err}(T,h,\psi_2,x) \leqslant Ch^{1/2}$.
3. Killed/stopped processes.
   For the killed case, the functional writes $\psi_3(y) := f(y(T))\mathbb{I}_{\forall s\in[0,T],\ y(s)\in D}$ where f is a measurable function and D a given open set of $\mathbb{R}^d$. In the Markovian setting of uniformly elliptic diffusion processes, the first author showed in [Gob00], Theorem 2.4, that for a smooth domain D and bounded f satisfying a support condition w.r.t. D,

$$\exists C > 0,\ |\mathbb{E}[f(X_T)\mathbb{I}_{\tau^N>T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau>T}]| \leqslant C\sqrt{h}\,, \tag{3}$$

   where τ and τ^N are respectively the continuous and discrete exit times. Let us mention that the above result remains valid if we additionally replace the discretely killed diffusion by its discretely killed Euler scheme, see [Gob00] and [GM04] for an extension to a hypoelliptic framework. Anyhow, (3) emphasizes that, for killed processes, the order 1/2 is intrinsic to the discrete time killing.

In this work, we show that under suitable assumptions, the previous bounds still hold when X follows the dynamics (1).

In terms of financial applications, the above results concerning the discretely sampled integral and maximum, can respectively be seen as preliminary controls to deal with the impact of a time discretization for Asian and look-back options. The estimate associated to the killed path gives an upper bound for the error associated to a discrete time observation for barrier options.

We first detail how standard stochastic analysis arguments provide the necessary tools to control (2) in the case of a discretely sampled integral or maximum (cases 1. and 2. of the former list).

Proposition 1.1 *Let X be an Itô process following the dynamics of (1). Assume the coefficients b and σ are bounded and that φ is a Lipschitz continuous function from $\mathbb{R}^d$ into $\mathbb{R}$. For $p \geqslant 1$ one has*

$$\varphi\Big(\int_0^T X_s\,ds\Big) - \varphi\Big(\int_0^T X_{\phi(s)}\,ds\Big) \underset{\mathbf{L}_p(\mathbb{P})}{=} O(h)\,.$$

Note that a direct use of $\|X_s - X_{\phi(s)}\|_{\mathbf{L}_p} = O(\sqrt{h})$ leads to a suboptimal rate of convergence.

Proof. Because φ is Lipschitz continuous, it is enough to prove that $\Delta I := \int_0^T X_s\,ds - \int_0^T X_{\phi(s)}\,ds \underset{\mathbf{L}_p}{=} O(h)$. Using Fubini's theorem for stochastic integrals, see [RY99] Chapter IV.5, we get

$$\Delta I = \int_0^T \left(\int_0^T \mathbb{I}_{t\in[\phi(s),s]} dX_t\right) ds = \int_0^T (\phi(t) + h - t)\,dX_t\,.$$

We complete the proof using standard BDG inequalities combined with $|\phi(t) + h - t| \leqslant h$. □

Concerning the discretely sampled maximum we state the following

Proposition 1.2 *Assume $(X_s)_{s\in[0,T]}$ follows the dynamics of (1), where $(b_u)_{u\geqslant 0}$ is a bounded progressively measurable coefficient and $\sigma_s = \sigma(X_s)$ where σ is bounded in $C^1(\mathbb{R})$ and s.t. $\exists \sigma_0 > 0,\ \forall y \in \mathbb{R},\ \sigma(y) \geqslant \sigma_0$. There exists a constant $C > 0$ s.t.*

$$0 \leqslant \mathrm{Err}(T, h, \psi_2, x) \leqslant C\sqrt{h}\,.$$

Proof. Define $\Delta M := \psi_2\left((X_s)_{s\in[0,T]}\right) - \psi_2\left((X_{\phi(s)})_{s\in[0,T]}\right) = \max_{s\in[0,T]} X_s - \max_{s\in[0,T]} X_{\phi(s)}$. If X is a BM, as a consequence of Lemma 6 in [AGP95], we have $\mathbb{E}[\Delta M^2]^{1/2} = O(\sqrt{h})$. This estimate is still valid if X is solution of the one dimensional SDE $X_t = x + \int_0^t \frac{1}{2}(\sigma\sigma')(X_s)ds + \int_0^t \sigma(X_s)dW_s$ with the above assumptions on σ. Indeed, introducing the Lamperti transform $(Y_t)_{t\geqslant 0} = (\varphi(X_t))_{t\geqslant 0}$, $\forall y \in \mathbb{R},\ \varphi(y) = \int_0^y \frac{dz}{\sigma(z)}$, we derive that Y is a standard one dimensional BM with starting point $\varphi(x)$. By construction, the inverse of φ is uniformly Lipschitz continuous. This gives the result. To obtain the statement of the proposition, we finally apply a Girsanov transformation, exploiting that the associated Radon–Nikodym density belongs to any $\mathbf{L}_p$ because of the drift's boundedness, and the previous result. □

The limiting factor in our approach is the use of Lamperti's transformation that imposes to have a Markovian diffusion term.

Propositions 1.1 and 1.2 extend the results stated for ψ_1 and ψ_2 in our initial list to a wider non-Markovian framework without major difficulties. Hence, in the sequel we consider the more difficult cases of discretely killed or stopped processes for which the corresponding functionals are not Lipschitz continuous anymore. We denote the discretization error associated to the killed case by

$$\begin{aligned}\mathrm{Err}(T,h,f,x) &= \mathbb{E}[\psi_3\left((X_{\phi(s)})_{s\in[0,T]}\right)] - \mathbb{E}[\psi_3\left((X_s)_{s\in[0,T]}\right)] \\ &= \mathbb{E}[f(X_T)\mathbb{I}_{\tau^N>T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau>T}]\end{aligned} \tag{4}$$

where, from now on, $\tau := \inf\{t \geqslant 0 : X_t \notin D\},\ \tau^N := \inf\{t_i \geqslant 0 : X_{t_i} \notin D\}$. For the stopped case, and a smooth domain D, for a given real valued bounded function g defined on $[0,T) \times \partial D \cup \{T\} \times \bar{D}$, we introduce

$$\mathrm{Err}(T,h,g,x) := \mathbb{E}[g(T\wedge\tau^N, \pi_{\bar{D}}(X_{T\wedge\tau^N}))] - \mathbb{E}[g(T\wedge\tau, X_{T\wedge\tau})]\,. \tag{5}$$

The careful reader can object that without further assumptions on the domain (like convexity for instance) the projection on $\bar{D}$ is only locally uniquely defined. By convention, for $y \in \mathbb{R}^d$ s.t. $\pi_{\bar{D}}(y)$ is not unique, we arbitrarily set $\pi_{\bar{D}}(y) = x_0 \in \partial D$. This can seem awkward. Anyhow, we should always keep in mind that, because of the boundedness of the coefficients in (1), for h small enough, the events for which the process exits the domain where $\pi_{\bar{D}}$ is uniquely defined, before being discretely stopped are of exponentially small probability. For such events, we derive from the boundedness of g that the definition of the projection has no relevant impact on the convergence analysis. We refer to Section 3.2 for details.

In this work, we extend the result of Theorem 2.4 in [Gob00] to a possibly degenerate non-Markovian framework and to a more general class of functions. For the reader familiar with error decomposition techniques, we guess it is interesting to present below an analogy between standard PDE methods employed in the Markovian setting [TL90] and ours.

Note first that the killed case can be seen as a special case of the stopped one with $\forall t \in [0,T],\ g(t,.)|_{\partial D} = 0,\ g(T,.)|_D = f(.)|_D$. Introducing $\forall t \in [0,T]$, $V_t := \mathbb{E}[g(T\wedge\tau_t, \pi_{\bar{D}}(X_{T\wedge\tau_t}))|\mathcal{F}_t] := \mathbb{E}[\tilde{g}(T\wedge\tau_t, X_{T\wedge\tau_t})|\mathcal{F}_t]$ where $\tau_t := \inf\{s \geqslant t : X_s \notin D\}$, the error writes

$$\mathrm{Err}(T,h,g,x) = \mathbb{E}[V_{T\wedge\tau^N}] - V_0\,. \tag{6}$$

In a Markovian framework, for all $t \leqslant T \wedge \tau,\ V_t = v(t, X_t)$ where, under suitable assumptions, v is a smooth function satisfying the mixed Cauchy–Dirichlet problem

$$\begin{cases}(\partial_t + L)v(t,x) = 0,\ (t,x) \in [0,T) \times D,\\ v(t,x) = g(t,x), \forall (t,x) \in [0,T) \times \partial D \bigcup \{T\} \times \bar{D}\,,\end{cases} \tag{7}$$

L being the infinitesimal generator of the diffusion X. The process $(V_{t\wedge\tau})_{t\in[0,T]}$ is associated to the standard Feynman–Kac representation of the solution of (7). In our case, we can not rely on a PDE, but on a martingale property that is one of the main ingredients needed for the proof. Namely, one has the following

Proposition 1.3 *Let X be an Itô process that follows the dynamics of (1). Assume the function g of (7) is bounded. Then, $\forall t \in [0,T)$, the process $(V_{s\wedge\tau_t})_{s\in[t,T]}$ is a martingale.*

Observe that in the Markovian case, one can derive this martingale property from the PDE (7) using Itô's formula.

Proof. Note that $\forall s \in [t,T]$, on $\{s < \tau_t\}$, $V_{s\wedge\tau_t} = V_s = \mathbb{E}[\tilde{g}(T\wedge\tau_s, X_{T\wedge\tau_s})|\mathcal{F}_s]$, and on $\{s \geqslant \tau_t\}$, $V_{s\wedge\tau_t} = V_{\tau_t} = \tilde{g}(\tau_t, X_{\tau_t})$. Turning to the former definition of V it comes

$$\begin{aligned}\mathbb{E}[V_{s\wedge\tau_t} - V_t|\mathcal{F}_t] &= \mathbb{E}[\tilde{g}(T\wedge\tau_{s\wedge\tau_t}, X_{T\wedge\tau_{s\wedge\tau_t}}) - \tilde{g}(T\wedge\tau_t, X_{T\wedge\tau_t})|\mathcal{F}_t]\\ &= \mathbb{E}[\mathbb{I}_{s<\tau_t}(\tilde{g}(T\wedge\tau_s, X_{T\wedge\tau_s}) - \tilde{g}(T\wedge\tau_t, X_{T\wedge\tau_t}))|\mathcal{F}_t]\\ &\quad+\mathbb{E}[\mathbb{I}_{s\geqslant\tau_t}(\tilde{g}(\tau_t, X_{\tau_t}) - \tilde{g}(\tau_t, X_{\tau_t}))|\mathcal{F}_t] = 0\end{aligned}$$

since on the event $\{s < \tau_t\}$ one has $\tau_t = \tau_s$. □

From (6), the strategy in the Markovian setting consists in writing Itô like expansions in order to isolate the leading term of the error (see [Gob00]). The above martingale property is crucial for our error decomposition. Namely, it replaces the use of Itô's formula on v in the Markovian case.

Outline of the paper

In Section 2 we state our working assumptions as well as our main results. Section 3 is dedicated to the common decomposition of the errors $\mathrm{Err}(T,h,f,x), \mathrm{Err}(T,h,g,x)$. We give in Section 4 the auxiliary results needed to obtain the bound of the error in the killed and stopped case. In Section 5, we show how our previous techniques can be employed to extend the previous control on $\mathrm{Err}(T,h,f,x)$ to the case of an intersection of smooth domains. We conclude in Section 6 giving some possible extensions and evoking some remaining open problems.

2 Assumptions and main results

2.1 About the process

We assume the coefficients $(b_s)_{s\in[0,T]}, (\sigma_s)_{s\in[0,T]}$ of (1) are bounded. Some mild smoothness property on σ (some continuity in probability) will be also needed: the condition stated below is not restrictive at all and is fulfilled for instance as soon as $(\sigma_s)_{0\leqslant s\leqslant T}$ satisfies a Hölder-continuity property in $\mathbf{L}_p$-norm.

(S) For any $\delta > 0$, there is some function η_δ with $\lim_{h\to 0^+} \eta_\delta(h) = 0$ such that $a.s$, for $s \in]t_i, t_{i+1}[$ with $X_s \in \partial D$, one has $\mathbb{P}(|\int_s^{t_{i+1}}(\sigma_u - \sigma_s)dW_u| \geqslant \delta\sqrt{t_{i+1}-s}\,|\,\mathcal{F}_s) \leqslant \eta_\delta(h)$.

2.2 About the domain

In this section we assume the domain D satisfies assumption

(D) The domain D is of class C^2 with bounded boundary ∂D, $X_0 = x \in \bar{D}$.

Additional notations and assumptions concerning the intersection of domains satisfying **(D)** are specified in Section 5. For $x \in \partial D$, denote by $n(x)$ the unit inward normal vector at x. For $r \geqslant 0$, set $V_{\partial D}(r) := \{z \in \mathbb{R}^d : d(z, \partial D) \leqslant r\}$ and $D(r) := \{z \in \mathbb{R}^d : d(z, D) \leqslant r\}$. $B(z, r)$ stands for the closed ball with center z and radius r. We now recall standard facts on the distance to the boundary and the orthogonal projection on ∂D (see Lemma 1 and its Proof from [GT77] page 382).

Proposition 2.1 *Assume* **(D)**. *There is a constant $R > 0$ such that:*

i) for any $x \in V_{\partial D}(R)$, there are unique $s = \pi_{\partial D}(x) \in \partial D$ and $F(x) \in \mathbb{R}$ such that $x = \pi_{\partial D}(x) + F(x)n(\pi_{\partial D}(x))$.
ii) The function $x \longmapsto F(x)$ is the signed normal distance of x to ∂D: this is a C^2-function on $V_{\partial D}(R)$, which can be extended to a C^2 function on $\mathbb{R}^d$ with bounded derivatives. This extension satisfies $F(x) \geqslant d(x, \partial D) \wedge R$ on D, $F(x) \leqslant -[d(x, \partial D) \wedge R]$ on D^c and $F = 0$ on ∂D.
iii) For $x \in V_{\partial D}(R)$, one has $\nabla F(x) = n(\pi_{\partial D}(x))$.

Assume D satisfies **(D)**. Following the notations of Proposition 2.1, we now introduce the non characteristic boundary condition

(C) $\exists a_0 > 0$ such that

$$a.s.\ \big(X_s \in V_{\partial D}(R), s \in [0,T] \Longrightarrow \alpha_s := \nabla F(X_s).\sigma_s\sigma_s^*\nabla F(X_s) \geqslant a_0\big)$$

which enforces the process to exit the domain in a nontangential manner.

2.3 Main results

We are now in a position to state our main results for killed and stopped processes in the case of smooth domains.

Theorem 2.2 Upper bound in the smooth domain case for a killed process.
Assume **(C)**, **(D)**, **(S)** *and suppose* f *is a borelian and bounded function s.t.* $\exists\varepsilon > 0,\ d(\text{supp}(f), \partial D) \geqslant 2\varepsilon$. *For some constant* C, *one has*

$$|\text{Err}(T, h, f, x)| = |\mathbb{E}[f(X_T)\mathbb{I}_{\tau^N > T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau > T}]| \leqslant C\frac{\|f\|_\infty}{1 \wedge \varepsilon}\sqrt{h}\,.$$

Remark 2.1 *Note that if* f *is non-negative one also has* $\text{Err}(T, h, f, x) \geqslant 0$. *This readily derives from the inequality* $\tau^N \geqslant \tau$ *a.s.*

Theorem 2.3 Upper bound in the smooth domain case for a stopped process.
Assume **(C)**, **(D)**, **(S)** *and suppose* g *is bounded in* $C^{1,2}([0, T] \times \mathbb{R}^d)$. *For some constant* C, *one has*

$$|\text{Err}(T, h, g, x)| = |\mathbb{E}[g(T \wedge \tau^N, \pi_{\bar{D}}(X_{T\wedge\tau^N})) - g(T \wedge \tau, X_{T\wedge\tau})]| \leqslant C\sqrt{h}\,.$$

Remark 2.2 *Let us first mention that we cannot improve the above rate in our framework, since in the Brownian case, one has an expansion w.r.t.* $\sqrt{h}$ *(cf. Siegmund and Yuh [SY82] and [Men04]).*

Remark 2.3 *To study the impact of the time discretization, few assumptions are needed to get, as indicated in the previous remark, the expected rate of convergence. To obtain the same upper bound with the discretely killed Euler scheme of a diffusion process, an additional hypoellepticity condition is necessary (see [GM04]).*

Note also that Assumptions **(D)** *and* **(S)** *could possibly be weakened. On the other hand, Assumption* **(C)** *is somehow a minimal condition to ensure a convergent approximation. Indeed, it easy to imagine a deterministic path which hits* ∂D *only at time* $\tau = \chi T$ *where* χ *is an irrational number in* $[0, 1]$*: for this,* $\tau^N > T$ *for any* $N \geqslant 1$ *and* $\text{Err}(T, h, f, x) = f(X_T)$ *is constant.*

Remark 2.4 *Recall also that the results of Theorems 2.2 and 2.3 concern respectively the impact of a discretization time in the quantities* $\mathbb{E}[f(X_T)\mathbb{I}_{\tau>T}]$ *and* $\mathbb{E}[g(T \wedge \tau, X_{T\wedge\tau})]$. *They can therefore not be directly compared to the results of Theorem 2.3 in [Gob00] or Section 6.4 Chapter I in [Men04] except in the special case of Brownian motion. Note anyhow that in that case we obtain the upper bound of the weak error with a much simpler proof. The next natural question, in the killed case and when* $f \geqslant 0$, *concerns a possible lower bound of the same order for* $\text{Err}(T, h, f, x)$ *as stated in Theorem 5 in [GM04] in a Markovian framework. We give a counter example that illustrates this property can fail under the sole assumption* **(C)**. *Define for all* $t \geqslant 0$, *the one* *dimensional diffusion process* $X_t = \pi/2 + \int_0^t \cos(X_s)ds + \int_0^t \sin(X_s)dW_s$ *and put* $D :=]-\pi/2, 3\pi/2[$. **(C)** *is readily satisfied and by construction one has* $X_s \in [0,\pi]$ *a.s. Hence,* $\mathbb{I}_{\tau^N > T} = \mathbb{I}_{\tau > T} = 1$ *and* $\mathrm{Err}(T,h,f,x) = 0$. *A minimal necessary condition to have a lower bound of order* $1/2$ *w.r.t* h *is to reach the boundary on the interval* $[0,T]$ *with positive probability.*

3 Common decomposition of the error

In this section we assume **(D)** is in force. The constant R is the one of Proposition 2.1. In particular, on $D(R)$ the projection on $\bar{D}$ is uniquely defined.

3.1 Miscellaneous

We will keep the same notation C (or C') for all finite, non-negative constants which will appear in our computations: they may depend on D, T, b, σ, f or g, but they will not depend on the number of time steps N and the initial value x. We reserve the notation c and c' for constants also independent of x, T, f, or g.

3.2 Localization of X in $D(R)$

In this subsection we justify that for studying $\mathrm{Err}(T,h,g,x)$, we can assume w.l.o.g. that $\forall t \in [0,T]$, $X_t \in D(R)$ *a.s.* Indeed, if it is not the case, we introduce $\tau_R := \inf\{s \geqslant 0 : X_s \notin D(R)\}$, $\bar{X}_t = X_{t\wedge\tau_R}$, $\bar{\tau}^N := \inf\{t_i \geqslant 0 : \bar{X}_{t_i} \notin D\}$, $\bar{\tau} := \inf\{t \geqslant 0 : \bar{X}_t \notin D\} = \tau$. Note that

$$\begin{aligned}&|\mathrm{Err}(T,h,g,x) - \big(\mathbb{E}[g(T\wedge\bar{\tau}^N, \pi_{\bar{D}}(\bar{X}_{T\wedge\bar{\tau}^N}))] - \mathbb{E}[g(T\wedge\bar{\tau}, \bar{X}_{T\wedge\bar{\tau}})]\big)|\\ &\quad := |\mathrm{Err}(T,h,g,x) - \mathrm{Err}_2(T,h,g,x)| \leqslant 2|g|_\infty \mathbb{P}[\tau_R < \tau^N].\end{aligned}$$

The process $\bar{X}$ satisfies **(C)**, **(S)** and is $D(R)$ valued. Hence, from Assumption **(D)**, the projection on $\bar{D}$ is uniquely defined in the term $\mathrm{Err}_2(T,h,g,x)$. It therefore remains to control the probability $\mathbb{P}[\tau_R < \tau^N]$. To this end, a key tool is the following

Lemma 3.1 (Bernstein's type inequality) *Consider two stopping times* S, S' *upper bounded by* T *with* $0 \leqslant S' - S \leqslant \Delta \leqslant T$. *Then for any* $p \geqslant 1$ *and* $c' > 0$, *there are some constants* $c > 0$ *and* C, *such that for any* $\eta \geqslant 0$, *one has a.s:*

$$\mathbb{P}[\sup_{t\in[S,S']} \|X_t - X_S\| \geqslant \eta \mid \mathcal{F}_S] \leqslant C\exp\left(-c\frac{\eta^2}{\Delta}\right),$$

$$\mathbb{E}[\sup_{t\in[S,S']} \|X_t - X_S\|^p \mid \mathcal{F}_S] \leqslant C\Delta^{p/2}.$$

Proof. We omit the proof of the first inequality which is standard and refer the reader to Lemma 4.1 in [Gob00] for instance. The other one easily follows from the first one. □

Lemma 3.1 readily gives $\mathbb{P}[\tau_R < \tau^N] \leqslant C \exp\big(-c\frac{R^2}{h}\big)$. Thus, taking $(\bar{X}, \bar{\tau}^N)$ instead of (X, τ^N) has no significant impact. This has however the advantage to keep the projection on $\bar{D}$ well defined. Hence, in the following we assume

$$(X_t)_{t\in[0,T]} \in D(R) \ a.s.$$

3.3 Error decomposition and proof of the main results

The error decomposition is common to both the killed and stopped cases. Put $\forall (t,z) \in [0,T] \times D(R)$,

$$\tilde{g}(t,z) := \begin{cases} \mathbb{I}_{t=T} f(z) \text{ in the killed case}, \\ g(t, \pi_{\bar{D}}(z)) \text{ in the stopped case}. \end{cases}$$

We denote by $\mathrm{Err}(T,h,\tilde{g},x)$ the error corresponding to $\mathrm{Err}(T,h,f,x)$ in the killed case (resp. $\mathrm{Err}(T,h,g,x)$ in the stopped case). It comes

$$\begin{aligned} \mathrm{Err}(T,h,\tilde{g},x) &= \mathbb{E}[\tilde{g}(T\wedge\tau^N, X_{T\wedge\tau^N}) - \tilde{g}(T\wedge\tau, X_{T\wedge\tau})] \\ &= \mathbb{E}[\mathbb{I}_{\tau<T}\mathbb{E}[\tilde{g}(T\wedge\tau^N, X_{T\wedge\tau^N}) - \tilde{g}(\tau, X_\tau)|\mathcal{F}_\tau]]\,. \end{aligned}$$

Hence, to show Theorems 2.2 and 2.3, it is enough to derive

$$|\mathcal{E}| := |\mathbb{E}[\tilde{g}(T'\wedge\tau^{N'}, X_{T'\wedge\tau^{N'}}) - \tilde{g}(t,x)]| \leqslant C\sqrt{h}\,, \tag{8}$$

for an initial point $x \in \partial D,\ t \in [0,T)$, for a shifted time mesh defined by $\{t_i : 0 \leqslant i \leqslant N'\}$ with $t_0 = 0,\ 0 < t_1 \leqslant h,\ t_{i+1} = t_i + h\ (i \geqslant 1)$, for a new terminal time $T' = t_{N'}$ and a modified exit time $\tau^{N'} = \inf\{t_i \geqslant t_1 : X_{t_i} \notin D\}$. The constant C in (8) has to be uniform in T' in a compact set, in N', in x and in t. For the sake of simplicity, we still write N for N', T for T' and take $t = 0$. Introduce now for all $s \in [0,T]$, $V_s := \mathbb{E}[\tilde{g}(T\wedge\tau_s, X_{T\wedge\tau_s})|\mathcal{F}_s]$ where $\tau_s := \inf\{u \geqslant s : X_u \notin D\}$ and recall from Proposition 1.3 that $(V_{u\wedge\tau_s})_{u\in[s,T]}$ is a martingale. For $x \in \partial D,\ \tau_0 = 0$ so $V_0 = \tilde{g}(0,x)$. On the other hand $V_{T\wedge\tau^N} = \tilde{g}(T\wedge\tau^N, X_{T\wedge\tau^N})$. Thus,

$$\begin{aligned} \mathcal{E} = \mathbb{E}[V_{T\wedge\tau^N}] - V_0 &= \sum_{i=0}^{N-1} \mathbb{E}[V_{t_{i+1}\wedge\tau^N} - V_{t_i\wedge\tau^N}] = \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\big(V_{t_{i+1}} - V_{t_i}\big)] \\ &= \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\big(V_{t_{i+1}} - V_{t_{i+1}\wedge\tau_{t_i}}\big)] + \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\big(V_{t_{i+1}\wedge\tau_{t_i}} - V_{t_i}\big)]\,. \end{aligned}$$

It readily follows from the martingale property of $(V_{u\wedge\tau_{t_i}})_{u\in[t_i,T]}$ (see Proposition 1.3) that $\mathbb{E}[\mathbb{I}_{\tau^N>t_i}(V_{t_{i+1}\wedge\tau_{t_i}} - V_{t_i})] = 0$. Therefore we have

$$\mathcal{E} = \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\mathbb{I}_{\tau_{t_i}<t_{i+1}}(V_{t_{i+1}} - V_{\tau_{t_i}})]\,. \tag{9}$$

Remark 3.1 *Note that to obtain (9) we did not use any smoothness properties of $\tilde{g}$.*

To control $\mathcal{E}$ we state two auxiliary Lemmas whose proofs are postponed to Section 4.

Lemma 3.2 *Assume* **(C)**, **(D)**, **(S)** *and that in the killed case f satisfies the assumptions of Theorem 2.2 (resp. in the stopped case g satisfies the assumptions of Theorem 2.3). For all $i \in [\![0, N-1]\!]$, on the set $\{\tau^N > t_i, \tau_{t_i} < t_{i+1}\}$ one has*

$$|\mathbb{E}[V_{t_{i+1}} - V_{\tau_{t_i}} | \mathcal{F}_{\tau_{t_i}}]| \leqslant C\sqrt{h}\,.$$

Lemma 3.3 *Assume* **(C)**, **(D)**, *and* **(S)**. *There are some positive constants C and N_0 such that for $N \geqslant N_0$, for any $i \in [\![0, N-1]\!]$, one has for $X_{t_i} \in D$*

$$\mathbb{P}[\exists t \in [t_i, t_{i+1}] : X_t \notin D \mid \mathcal{F}_{t_i}] \leqslant C\,\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{t_i}]\,.$$

Plugging the control of Lemma 3.2 into (9) we obtain

$$|\mathcal{E}| \leqslant C\sqrt{h} \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N > t_i} \mathbb{I}_{\tau_{t_i} < t_{i+1}}]\,.$$

Using now Lemma 3.3 it comes

$$|\mathcal{E}| \leqslant C\sqrt{h} \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N > t_i} \mathbb{I}_{X_{t_{i+1}} \notin D}] = C\sqrt{h} \sum_{i=0}^{N-1} \mathbb{P}[\tau^N = t_{i+1}] \leqslant C\sqrt{h}$$

which completes the proof of Theorems 2.2 and 2.3. □

4 Proof of the technical Lemmas

This section is devoted to the proof of Lemmas 3.2 and 3.3. For smooth functions $g(t,x)$, we denote by $\partial_t g(t,x)$ its time derivative, by $\nabla g(t,x)$ its gradient w.r.t. x and by $H_g(t,x)$ its Hessian matrix w.r.t. x. The notation $\frac{\partial g}{\partial n}(t,x) = \nabla g(t,x).n(x)$ stands for the normal derivative on the boundary.

Using the results of Proposition 2.1 and Lemma 3.1, we prove the following Lemma that will be repeatedly used.

Lemma 4.1 *Assume* **(D)**. *For all $i \in [\![0, N-1]\!]$, on the set $\{\tau_{t_i} \leqslant t_{i+1}\}$, one has*

$$\mathbb{E}[\big|F(X_{t_{i+1}})\big| \; |\mathcal{F}_{\tau_{t_i}}] = \mathbb{E}[\big|F(X_{t_{i+1}}) - F(X_{\tau_{t_i}})\big| \; |\mathcal{F}_{\tau_{t_i}}] \leqslant C\sqrt{h}\,.$$

4.1 Proof of Lemma 3.2

For this proof we distinguish the killed and stopped cases.

Proof in the killed case

In that case Lemma 3.2 is a direct consequence of the following

Lemma 4.2 *Assume* **(C)**, **(D)**, **(S)** *and let the function* f *be as in Theorem 2.2. There is some constant* C *such that for any* $t \in [0,T]$, *one has a.s*

$$|V_t| \leqslant C\frac{\|f\|_\infty}{1\wedge\varepsilon}[F(X_t)]_+ .$$

Indeed, we deduce from Lemma 4.1 that $\forall i \in [\![0, N-1]\!]$, on $\{\tau^N > t_i, \tau_{t_i} \leqslant t_{i+1}\}$ one has

$$|\mathbb{E}[V_{t_{i+1}}|\mathcal{F}_{\tau_{t_i}}]| \leqslant C\frac{\|f\|_\infty}{1\wedge\varepsilon}\mathbb{E}[[F(X_{t_{i+1}})]_+|\mathcal{F}_{\tau_{t_i}}] \leqslant C\sqrt{h}\frac{\|f\|_\infty}{1\wedge\varepsilon} .$$

Proof of Lemma 4.2 W.l.o.g. we assume $f \geqslant 0$. Since $V_t = 0$ for $X_t \notin D$, it is enough to prove the estimate for $X_t \in D \cap V_{\partial D}(R\wedge\varepsilon/2)$ for which $0 < F(X_t) \leqslant R\wedge\varepsilon/2$. Denote $\tau_t^R = \inf\{s \geqslant t : F(X_s) \geqslant R\}$ and split V into two parts $V_t = V_t^1 + V_t^2$ with $V_t^1 = \mathbb{E}\big[\mathbb{I}_{T<\tau_t}\mathbb{I}_{T<\tau_t^R}f(X_T) \mid \mathcal{F}_t\big]$ and $V_t^2 = \mathbb{E}\big[\mathbb{I}_{T<\tau_t}\mathbb{I}_{T\geqslant\tau_t^R}f(X_T) \mid \mathcal{F}_t\big]$.

Before estimating separately each contribution, we set some standard notations related to time-changed Brownian martingales. Define the increasing continuous process $\mathcal{A}_s = \int_t^s \alpha_u du$ (from $[t,+\infty[$ into $\mathbb{R}^+$) and its increasing right-continuous inverse $\mathcal{C}_s = \inf\{u \geqslant t : \mathcal{A}_u > s\}$ (from $\mathbb{R}^+$ into $[t,+\infty[$) (see Section V.1 in Revuz-Yor [RY99]) and put $M_s = \int_t^{\mathcal{C}_s} \nabla F(X_u).\sigma_u dW_u$, $Z_s = F(X_{\mathcal{C}_s})$. From the Dambis–Dubins–Schwarz theorem, M coincides with a standard BM β (defined on a possibly enlarged probability space) for $s < \int_t^\infty \alpha_u du$ and it is easy to check that β is independent of $\mathcal{F}_t$ (see the arguments in the proof of Theorem V.1.7 in [RY99]).

Owing to the assumption **(C)**, $\mathcal{A}$ and $\mathcal{C}$ are strictly increasing on $[t,\tau_t^R]$ and $[0,\int_t^{\tau_t^R}\alpha_u du]$. Thus, for $s \in [0,\int_t^{\tau_t^R}\alpha_u du]$, one easily obtains

$$Z_s = F(X_t) + \beta_s + \int_0^s \lambda_v dv$$

where $\lambda_v = \{[\nabla F(X_u).b_u + \frac{1}{2}\mathrm{tr}(H_F(X_u)\sigma_u\sigma_u^*)]|_{u=\mathcal{C}_v}\}\frac{1}{\alpha_{\mathcal{C}_v}}$ is bounded by $\|\lambda\|_\infty$. Define

$$Z'_s = F(X_t) + \beta_s + \|\lambda\|_\infty s \geqslant Z_s. \tag{10}$$

Finally, put $\tau_0^Z = \inf\{s \geqslant 0 : Z_s \leqslant 0\}$, $\tau_R^Z = \inf\{s \geqslant 0 : Z_s \geqslant R\}$ and analogously $\tau_0^{Z'}$, $\tau_R^{Z'}$ for Z'.

Estimation of V^1. Let us first prove that for any stopping time $S \in [t,T]$, one has

$$\begin{aligned}\mathbb{E}\big[f(X_T) \mid \mathcal{F}_S\big] \leqslant & \|f\|_\infty \mathbb{P}[F(X_T) \geqslant 2\varepsilon \mid \mathcal{F}_S] \\ \leqslant & C\|f\|_\infty \exp\left(-c\frac{(2\varepsilon - F(X_S))_+^2}{T-S}\right) \ a.s. \qquad (11)\end{aligned}$$

The first inequality simply results from the support of f included in $D\backslash V_{\partial D}(2\varepsilon)$. To justify the second one, note that $\{F(X_T) \geqslant 2\varepsilon\} \subset \{|F(X_T) - F(X_S)| \geqslant 2\varepsilon - F(X_S)\} \subset \{|F(X_T) - F(X_S)| \geqslant (2\varepsilon - F(X_S))_+\}$ and the proof of (11) is complete using Lemma 3.1 applied to the Itô process $(F(X_s))_{s\geqslant 0}$ with bounded coefficients. We now turn to the evaluation of V_t^1. On $\{T < \tau_t^R\}$, using the notation with the time change above, one has $T = \mathcal{C}_{\mathcal{A}_T} \geqslant \mathcal{C}_{a_0(T-t)}$ and $a_0(T - \mathcal{C}_{a_0(T-t)}) \leqslant \int_{\mathcal{C}_{a_0(T-t)}}^T \alpha_u du = \mathcal{A}_T - \mathcal{A}_{\mathcal{C}_{a_0(T-t)}}$. Hence, $T - \mathcal{C}_{a_0(T-t)} \leqslant \frac{1}{a_0}(\mathcal{A}_T - a_0(T-t)) \leqslant \frac{\|\alpha\|_\infty}{a_0}(T-t)$. Thus, one obtains

$$\begin{aligned} & V_t^1 \\ & \leqslant \mathbb{E}\left[\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t}\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R}\mathbb{I}_{T-\mathcal{C}_{a_0(T-t)}\leqslant\frac{\|\alpha\|_\infty}{a_0}(T-t)}\mathbb{E}\big[f(X_T)\mid\mathcal{F}_{\mathcal{C}_{a_0(T-t)}}\big]\mid\mathcal{F}_t\right] \\ & \leqslant C\|f\|_\infty\mathbb{E}\left[\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t}\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R}\exp\left(-c'\frac{(2\varepsilon - F(X_{\mathcal{C}_{a_0(T-t)}}))_+^2}{T-t}\right)\mid\mathcal{F}_t\right] \\ & \leqslant C\|f\|_\infty\mathbb{E}\left[\mathbb{I}_{a_0(T-t)<\tau_0^{Z'}}\mathbb{I}_{\mathcal{C}_{a_0(T-t)}<\tau_t^R}\exp\left(-c'\frac{(2\varepsilon - Z'_{a_0(T-t)})_+^2}{T-t}\right)\mid\mathcal{F}_t\right]\end{aligned}$$

where one has applied at the second line the estimate (11) with $S = \mathcal{C}_{a_0(T-t)}$ (here $c' = c\frac{a_0}{\|\alpha\|_\infty}$), at the third one

$$\begin{aligned}\{\mathcal{C}_{a_0(T-t)} < \tau_t\} &= \{\forall s \in [t, \mathcal{C}_{a_0(T-t)}] : F(X_s) > 0\} \\ &= \{\forall u \in [0, a_0(T-t)] : Z_u > 0\} \\ &= \{a_0(T-t) < \tau_0^Z\} \subset \{a_0(T-t) < \tau_0^{Z'}\}\end{aligned}$$

and $(2\varepsilon - F(X_{\mathcal{C}_{a_0(T-t)}}))_+ = (2\varepsilon - Z_{a_0(T-t)})_+ \geqslant (2\varepsilon - Z'_{a_0(T-t)})_+$. Reminding the law of β, one finally gets that $V_t^1 \leqslant C\|f\|_\infty \Phi_1(a_0(T-t), F(X_t))$ with $\Phi_1(r,z) = \mathbb{E}\left[\mathbb{I}_{\forall u\in[0,r]:z+\beta_u+\|\lambda\|_\infty u>0}\exp\left(-a_0c'\frac{(2\varepsilon-z-\beta_r-\|\lambda\|_\infty r)_+^2}{r}\right)\right]$. With clear notations involving the smooth transition density of the killed drifted BM and Gaussian type estimates of its gradient (see [LSU68] Chapter IV Theorem 16.3), one has $\Phi_1(r,z) = \int_0^\infty q_r(z,y)\exp\left(-a_0c'\frac{(2\varepsilon-y)_+^2}{r}\right)dy$ and

$$|\partial_z \Phi_1(r,z)| \leqslant C\int_0^\infty \frac{1}{r}\exp\left(-c\frac{(z-y)^2}{r}\right)\exp\left(-a_0c'\frac{(2\varepsilon-y)_+^2}{r}\right)dy.$$

We now justify that $|\partial_z \Phi_1(r,z)| \leqslant \frac{C}{1\wedge\varepsilon}$ for $0 \leqslant z \leqslant \varepsilon/2$ and for this, we may split the domain of integration into two parts. For $y < \varepsilon$, $(2\varepsilon - y)_+^2 \geqslant \varepsilon^2$ and the corresponding contribution for the integral is bounded by $\int_0^\infty \frac{1}{\sqrt{r}} \exp\left(-c\frac{(z-y)^2}{r}\right)\left[\frac{1}{\sqrt{r}}\exp\left(-a_0 c' \frac{\varepsilon^2}{r}\right)\right] dy \leqslant \frac{C}{1\wedge\varepsilon}$. For $y \geqslant \varepsilon$ and $0 \leqslant z \leqslant \varepsilon/2$, $(z-y)^2 \geqslant \varepsilon^2/4$ and the integral is bounded by $\int_0^\infty \frac{1}{\sqrt{r}} \exp\left(-\frac{c}{2}\frac{(z-y)^2}{r}\right)\frac{1}{\sqrt{r}}\exp\left(-\frac{c}{2}\frac{\varepsilon^2}{4r}\right) dy \leqslant \frac{C}{1\wedge\varepsilon}$.
Since $\Phi_1(r,0) = 0$, one gets $\Phi_1(r,z) \leqslant \frac{C}{1\wedge\varepsilon} z$ for $z \in [0,\varepsilon/2]$ and this proves that $V_t^1 \leqslant C\frac{\|f\|_\infty}{1\wedge\varepsilon} F(X_t)$.
Estimation of V^2. Clearly, one has $V_t^2 \leqslant \|f\|_\infty \mathbb{P}\big[\tau_t^R < \tau_t \mid \mathcal{F}_t\big]$. Note that $\{\tau_t^R < \tau_t\} = \{\tau_R^Z < \tau_0^Z\} \subset \{\tau_R^{Z'} < \tau_0^{Z'}\}$ because of (10). Hence, one has $V_t^2 \leqslant \|f\|_\infty \Phi_2(F(X_t))$ where $\Phi_2(z) = \mathbb{P}[(z+\beta_u+\|\lambda\|_\infty u)_{u\geqslant 0}$ hits R before $0]$. It is well-known that $\Phi_2(z) = \frac{1-\exp(-2\|\lambda\|_\infty z)}{1-\exp(-2\|\lambda\|_\infty R)} \leqslant Cz$ (see Section 5.5 in [KS91]) and this proves that $V_t^2 \leqslant C\|f\|_\infty F(X_t)$. Combining estimates for V^1 and V^2 gives the result of Lemma 4.2. □

Proof in the stopped case

Assume the function g is as in Theorem 2.3. In this case, we use the smoothness of g. Since we also assumed X_t is $D(R)$ valued, the semi-martingale decomposition stated in Proposition 3.1 in [Gob00] remains valid for $(\pi_{\bar{D}}(X_t))_{t\geqslant 0}$. Hence, $\forall i \in [\![0, N-1]\!]$, on the set $\{\tau_{t_i} \leqslant t_{i+1}\}$ we write

$$\begin{aligned}
&\tilde{g}(T\wedge\tau_{t_{i+1}}, X_{T\wedge\tau_{t_{i+1}}}) - \tilde{g}(\tau_{t_i}, X_{\tau_{t_i}}) \\
&\quad = \int_{\tau_{t_i}}^{T\wedge\tau_{t_{i+1}}} \partial_u g(u, \pi_{\bar{D}}(X_u))du + \nabla g(u, \pi_{\bar{D}}(X_u))\cdot d(\pi_{\bar{D}}(X_u)) \\
&\qquad\quad + \frac{1}{2}\mathrm{tr}(H_g(u,\pi_{\bar{D}}(X_u))d\langle \pi_{\bar{D}}(X)\rangle_u) \\
&\quad := (M_{T\wedge\tau_{t_{i+1}}} - M_{\tau_{t_i}}) + (V_{T\wedge\tau_{t_{i+1}}} - V_{\tau_{t_i}}) + \int_{\tau_{t_i}}^{T\wedge\tau_{t_{i+1}}} \frac{\partial g}{\partial n}(u, X_u) dL_u^0(F(X))
\end{aligned}$$

where M is a local martingale and V an Itô process with finite variations. From the boundedness of the derivatives of g and of the coefficients b_s, σ_s, we derive that M is a true martingale and that $a.s$ $|V_{T\wedge\tau_{t_{i+1}}} - V_{\tau_{t_i}}| \leqslant C(T\wedge\tau_{t_{i+1}} - \tau_{t_i})$. It comes

$$\begin{aligned}
&|\mathbb{E}[\tilde{g}(T\wedge\tau_{t_{i+1}}, X_{T\wedge\tau_{t_{i+1}}}) - \tilde{g}(\tau_{t_i}, X_{\tau_{t_i}})|\mathcal{F}_{\tau_{t_i}}]| \\
&\quad \leqslant C\left\{\mathbb{E}[L^0_{T\wedge\tau_{t_{i+1}}}(F(X)) - L^0_{\tau_{t_i}}(F(X))|\mathcal{F}_{\tau_{t_i}}] + \mathbb{E}[(T\wedge\tau_{t_{i+1}} - \tau_{t_i})|\mathcal{F}_{\tau_{t_i}}]\right\} \\
&\quad := C\left(A^1_{\tau_{t_i}} + A^2_{\tau_{t_i}}\right).
\end{aligned}$$

Term $A^1_{\tau_{t_i}}$: control of the local time.
Since the measure $dL^0_t(F(X))$ is $a.s$ carried by the set $\{t : F(X_t) = 0\}$ we write

$$\begin{aligned} A^1_{\tau_{t_i}} &= \mathbb{E}[L^0_{t_{i+1}}(F(X)) - L^0_{\tau_{t_i}}(F(X))|\mathcal{F}_{\tau_{t_i}}] \\ &= 2\mathbb{E}[[F(X_{t_{i+1}})]_- - [F(X_{\tau_{t_i}})]_- + \int_{\tau_{t_i}}^{t_{i+1}} \mathbb{I}_{F(X_s)<0} dF(X_s)|\mathcal{F}_{\tau_{t_i}}] \leqslant C\sqrt{h}\,. \end{aligned} \tag{12}$$

The last equality follows from Tanaka's formula. The last inequality is a consequence of the boundedness of F and its derivatives, the boundedness of the coefficients of X and Lemma 4.1.

Term $A^2_{\tau_{t_i}}$: time-change techniques.
Write

$$\begin{aligned} A^2_{\tau_{t_i}} &= (T-\tau_{t_i})\mathbb{P}[\tau_{t_{i+1}} > T|\mathcal{F}_{\tau_{t_i}}] + \mathbb{E}[(\tau_{t_{i+1}} - \tau_{t_i})\mathbb{I}_{\tau_{t_{i+1}}\leqslant T}|\mathcal{F}_{\tau_{t_i}}] \\ &:= A^{21}_{\tau_{t_i}} + A^{22}_{\tau_{t_i}}\,. \end{aligned}$$

The key idea is now, as in the proof of Lemma 4.2, to use time-changes in order to apply well known results for hitting times in a Brownian framework. We rewrite

$$A^{21}_{\tau_{t_i}} = (T-\tau_{t_i})\mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D}\mathbb{E}[\mathbb{I}_{\tau_{t_{i+1}}>T}|\mathcal{F}_{t_{i+1}}]|\mathcal{F}_{\tau_{t_i}}]\,.$$

Put $C_{t_{i+1}} := \mathbb{P}[\tau_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}]$ and define $\tau^R_t := \inf\{s \geqslant t : F(X_s) \geqslant R\}$. We decompose $C_{t_{i+1}} = \mathbb{P}[\tau_{t_{i+1}} > T, \tau^R_{t_{i+1}} \leqslant T|\mathcal{F}_{t_{i+1}}] + \mathbb{P}[\tau_{t_{i+1}} > T, \tau^R_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}] := C^1_{t_{i+1}} + C^2_{t_{i+1}}$. Since $C^1_{t_{i+1}} \leqslant \mathbb{P}[\tau_{t_{i+1}} > \tau^R_{t_{i+1}}|\mathcal{F}_{t_{i+1}}]$, we can control this term in the same way we did for V^2 in the proof of Lemma 4.2. Namely, we get

$$\mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D} C^1_{t_{i+1}}|\mathcal{F}_{\tau_{t_i}}] \leqslant C\mathbb{E}[[F(X_{t_{i+1}})]_+|\mathcal{F}_{\tau_{t_i}}]. \tag{13}$$

In the following we use the notation introduced in the proof of Lemma 4.2 for time-changed martingales with $t = t_{i+1}$. For all $i \in [\![0, N-2]\!]$, on the set $\{X_{t_{i+1}} \in D\}$ we write

$$\begin{aligned} C^2_{t_{i+1}} &= \mathbb{P}[\inf_{s\in[t_{i+1},T]} F(X_{t_{i+1}}) + \beta_{\mathcal{A}_s} + \int_0^{\mathcal{A}_s} \lambda_v\, dv > 0, \tau^R_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}] \\ &\leqslant \mathbb{P}[\inf_{s\in[0,\mathcal{A}_T]} F(X_{t_{i+1}}) + \beta_s + \|\lambda\|_\infty s > 0, \tau^R_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}] \\ &\leqslant \mathbb{P}[\inf_{s\in[0,a_0(T-t_{i+1})]} F(X_{t_{i+1}}) + \beta_s + \|\lambda\|_\infty s > 0, \tau^R_{t_{i+1}} > T|\mathcal{F}_{t_{i+1}}] \\ &\leqslant \int_{a_0(T-t_{i+1})}^\infty dt \frac{F(X_{t_{i+1}})}{(2\pi t^3)^{1/2}} \exp\Big(-\frac{(F(X_{t_{i+1}}) + \|\lambda\|_\infty t)^2}{2t}\Big) \leqslant \frac{CF(X_{t_{i+1}})}{(T-t_{i+1})^{1/2}} \end{aligned} \tag{14}$$

exploiting the explicit density for the hitting times of the drifted BM, see [KS91] Section 3.5.C, for the last but one inequality. From (13) and (14) we derive that $\forall i \in [\![0, N-2]\!]$

$$A^{21}_{\tau_{t_i}} \leqslant C(T-\tau_{t_i})\mathbb{E}[[F(X_{t_{i+1}})]_+(1+\frac{1}{(T-t_{i+1})^{1/2}})|\mathcal{F}_{\tau_{t_i}}]\,.$$

Observing that $\forall i \in [\![0, N-2]\!], T-t_{i+1} \geqslant \dfrac{T-t_i}{2} \geqslant \dfrac{T-\tau_{t_i}}{2}$ we derive from Lemma 4.1

$$A^{21}_{\tau_{t_i}} \leqslant C\mathbb{E}[[F(X_{t_{i+1}})]_+|\mathcal{F}_{\tau_{t_i}}] \leqslant C\sqrt{h}. \tag{15}$$

Since for $i = N-1$ we also have $A^{21}_{\tau_{t_i}} \leqslant (T-\tau_{t_i}) \leqslant h$, we finally obtain that (15) is valid for all $i \in [\![0, N-1]\!]$. We now turn to the control of $A^{22}_{\tau_{t_i}}$ reintroducing the events $\{\tau^R_{t_{i+1}} > \tau_{t_{i+1}}\}, \{\tau^R_{t_{i+1}} < \tau_{t_{i+1}}\}$. It comes

$$\begin{aligned} A^{22}_{\tau_{t_i}} &= \mathbb{E}[(\tau_{t_{i+1}}-\tau_{t_i})\mathbb{I}_{\tau_{t_{i+1}}\leqslant T}\mathbb{I}_{X_{t_{i+1}}\in D}(\mathbb{I}_{\tau^R_{t_{i+1}}>\tau_{t_{i+1}}}+\mathbb{I}_{\tau^R_{t_{i+1}}<\tau_{t_{i+1}}})|\mathcal{F}_{\tau_{t_i}}] \\ &\quad +O(h) := A^{221}_{\tau_{t_i}} + A^{222}_{\tau_{t_i}} + O(h). \end{aligned}$$

Conditioning w.r.t. $\mathcal{F}_{t_{i+1}}$ and using the same arguments as for $C^1_{t_{i+1}}$ we readily get $A^{222}_{\tau_{t_i}} \leqslant C\mathbb{E}[[F(X_{t_{i+1}})]_+|\mathcal{F}_{\tau_{t_i}}] \leqslant C\sqrt{h}$. For $A^{221}_{\tau_{t_i}}$ write

$$\begin{aligned} A^{221}_{\tau_{t_i}} &\leqslant h + \mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D}\mathbb{E}[(\tau_{t_{i+1}}-t_{i+1})\mathbb{I}_{\tau_{t_{i+1}}\leqslant T}\mathbb{I}_{\tau^R_{t_{i+1}}>\tau_{t_{i+1}}}|\mathcal{F}_{t_{i+1}}]|\mathcal{F}_{\tau_{t_i}}] \\ &:= h + \mathbb{E}[\mathbb{I}_{X_{t_{i+1}}\in D}Q_{t_{i+1}}|\mathcal{F}_{\tau_{t_i}}]\,. \end{aligned}$$

Regarding $Q_{t_{i+1}}$, one has

$$\begin{aligned} Q_{t_{i+1}} &\leqslant \int_0^{T-t_{i+1}} ds\mathbb{P}[\tau_{t_{i+1}}-t_{i+1} \geqslant s, \tau^R_{t_{i+1}} > \tau_{t_{i+1}}|\mathcal{F}_{t_{i+1}}] \\ &\leqslant \int_0^{T-t_{i+1}} ds\mathbb{P}[\inf_{u\in[0,\mathcal{A}_{s+t_{i+1}}]} F(X_{t_{i+1}}) + \beta_u + \|\lambda\|_\infty u > 0, \\ &\qquad \tau^R_{t_{i+1}} > \tau_{t_{i+1}}|\mathcal{F}_{t_{i+1}}] \leqslant \int_0^{T-t_{i+1}} ds\mathbb{P}_y[\tau_0^{\tilde{\beta}} \geqslant a_0 s] \end{aligned}$$

where we denote $y = F(X_{t_{i+1}}),\ \tilde{\beta}_u = y + \beta_u + \|\lambda\|_\infty u,\ \tau_0^{\tilde{\beta}} := \inf\{s \geqslant 0 : \tilde{\beta}_s = 0\}$. Thus, recalling that $y > 0$ on the set $\{X_{t_{i+1}} \in D\}$, it comes

$$\begin{aligned} Q_{t_{i+1}} &\leqslant a_0^{-1}\int_0^{(T-t_{i+1})a_0} ds\mathbb{P}_y[\tau_0^{\tilde{\beta}} \geqslant s] = a_0^{-1}\mathbb{E}_y\left[\tau_0^{\tilde{\beta}} \wedge a_0(T-t_{i+1})\right] \\ &\leqslant a_0^{-1}\int_0^\infty dt\frac{\big(t\wedge a_0(T-t_{i+1})\big)y}{(2\pi t^3)^{1/2}}\exp\left(-\frac{(y+\|\lambda\|_\infty t)^2}{2t}\right) \leqslant Cy\,. \end{aligned}$$

From this last estimate and the previous controls we derive

$$A^{221}_{\tau_{t_i}} \leqslant h + C\mathbb{E}[\mathbb{I}_{X_{t_{i+1}} \in D} F(X_{t_{i+1}}) | \mathcal{F}_{\tau_{t_i}}] \leqslant C\sqrt{h}\,.$$

Hence, for all $i \in [\![0, N-1]\!]$,

$$A^{22}_{\tau_{t_i}} \leqslant C\sqrt{h}\,. \tag{16}$$

We conclude the proof of Lemma 3.2 in the stopped case putting together the controls (12), (15), and (16). □

4.2 Proof of Lemma 3.3

We adapt some ideas from [Gob00]: in the cited paper, a uniform ellipticity condition was assumed, and this enabled to use a Gaussian type lower bound for the conditional density of $X_{t_{i+1}}$ w.r.t. the Lebesgue measure, together with some computations related to a cone exterior to D. Here, under **(C)**, the conditional law of $X_{t_{i+1}}$ may be degenerate and our proof rather exploits the scaling invariance of the cone and of the Brownian increments.
It is enough to prove that $a.s$ on $\{t_i < \tau_{t_i} < t_{i+1}\}$, one has

$$\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \geqslant \frac{1}{C}\,. \tag{17}$$

Indeed, it follows that $\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{t_i}] = \mathbb{E}[\mathbb{I}_{\tau_{t_i} \leqslant t_{i+1}} \mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \mid \mathcal{F}_{t_i}] \geqslant \frac{\mathbb{P}[\tau_{t_i} \leqslant t_{i+1} | \mathcal{F}_{t_i}]}{C}$ and Lemma 3.3 is proved.

To get (17), write $X_{t_{i+1}} = X_{\tau_{t_i}} + \sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}}) + R_i$ where $R_i = \int_{\tau_{t_i}}^{t_{i+1}} b_u du + \int_{\tau_{t_i}}^{t_{i+1}} (\sigma_u - \sigma_{\tau_{t_i}}) dW_u$. The domain D is of class C^2, and thus satisfies a uniform exterior sphere condition with radius $R/2$ (R defined in Proposition 2.1): for any $z \in \partial D$, $B(z - \frac{R}{2}n(z), \frac{R}{2}) \subset D^c$. In particular, if we define for $\theta \in]0, \pi/2[$ the cone $\mathcal{K}(\theta, z) := \{y \in \mathbb{R}^d : (y - z).[-n(z)] \geqslant \|y - z\| \cos(\theta)\}$, then one has $\mathcal{K}(\theta, z) \cap B(z, R(\theta)) \subset B(z - \frac{R}{2}n(z), \frac{R}{2}) \subset D^c$ for some appropriate choice of the *positive* function $R(.)$. Then, it follows that

$$\begin{aligned}
&\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \geqslant \mathbb{P}[X_{t_{i+1}} \in \mathcal{K}(\theta, X_{\tau_{t_i}}) \cap B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}] \\
&\quad \geqslant \mathbb{P}[X_{t_{i+1}} \in \mathcal{K}(\theta, X_{\tau_{t_i}}) \mid \mathcal{F}_{\tau_{t_i}}] - \mathbb{P}[X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}] \\
&\quad \geqslant \mathbb{P}[(X_{\tau_{t_i}} - X_{t_{i+1}}).n(X_{\tau_{t_i}}) \geqslant (\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i}))^{1/2} \geqslant \|X_{t_{i+1}} - X_{\tau_{t_i}}\| \cos(\theta) \\
&\qquad \mid \mathcal{F}_{\tau_{t_i}}] - \mathbb{P}[X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}] \geqslant A_1 - A_2(\theta) - A_3(\theta),
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
A_1 &= \mathbb{P}[(X_{t_{i+1}} - X_{\tau_{t_i}}).(-n(X_{\tau_{t_i}})) \geqslant \sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}], \\
A_2(\theta) &= \mathbb{P}[\sqrt{\alpha_{\tau_{t_i}}(t_{i+1} - \tau_{t_i})} < \|X_{t_{i+1}} - X_{\tau_{t_i}}\| \cos(\theta) \mid \mathcal{F}_{\tau_{t_i}}], \\
A_3(\theta) &= \mathbb{P}[X_{t_{i+1}} \notin B(X_{\tau_{t_i}}, R(\theta)) \mid \mathcal{F}_{\tau_{t_i}}]\,.
\end{aligned}$$

Term A_1. Clearly, one has $A_1 \geqslant \mathbb{P}[(-n(X_{\tau_{t_i}})).\sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}}) \geqslant 2\sqrt{\alpha_{\tau_{t_i}}(t_{i+1}-\tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}] - \mathbb{P}[|n(X_{\tau_{t_i}}).R_i| \geqslant \sqrt{\alpha_{\tau_{t_i}}(t_{i+1}-\tau_{t_i})} \mid \mathcal{F}_{\tau_{t_i}}] := A_{11} - A_{12}$. The random variable $(-n(X_{\tau_{t_i}})).\sigma_{\tau_{t_i}}(W_{t_{i+1}} - W_{\tau_{t_i}})$ is conditionally to $\mathcal{F}_{\tau_{t_i}}$ a centered Gaussian variable with variance $\alpha_{\tau_{t_i}}(t_{i+1}-\tau_{t_i})$, and thus $A_{11} = \Phi(-2) > 0$, where Φ denotes the distribution function of the standard normal law. Owing to the condition **(S)** and since $\alpha_{\tau_{t_i}} \geqslant a_0$ *a.s*, it is easy to see that the contribution A_{12} converges uniformly to 0 when h goes to 0, and thus for $h = T/N$ small enough, one has $A_1 \geqslant \frac{A_{11}}{2} > 0$.

Term $A_2(\theta)$. From Markov's inequality, $A_2(\theta) \leqslant \frac{\mathbb{E}[\|X_{t_{i+1}} - X_{\tau_{t_i}}\|^2 \cos^2(\theta) \mid \mathcal{F}_{\tau_{t_i}}]}{\alpha_{\tau_{t_i}}(t_{i+1}-\tau_{t_i})} \leqslant C\cos^2(\theta)$ using **(C)** and estimates of Lemma 3.1. In particular, taking θ close to $\pi/2$ ensures that $A_2(\theta) \leqslant \frac{A_{11}}{6}$.

Term $A_3(\theta)$. Using Lemma 3.1, one readily gets $A_3(\theta) \leqslant C\exp\big(-c\frac{R^2(\theta)}{h}\big) \leqslant \frac{A_{11}}{6}$ for h small enough ($R(\theta) > 0$). Putting together estimates for $A_1, A_2(\theta)$ and $A_3(\theta)$ into (18) gives $\mathbb{P}[X_{t_{i+1}} \notin D \mid \mathcal{F}_{\tau_{t_i}}] \geqslant \frac{A_{11}}{6}$. This proves (17). □

4.3 A simple extension in the stopped case

From the previous controls we easily derive the following

Theorem 4.3 *Assume* **(C)**, **(D)**, **(S)** *and that g is bounded, uniformly Hölder continuous with index $\alpha \in (0, 1/2]$ in time and Hölder continuous with index 2α in space. For some constant C, one has*

$$|\mathrm{Err}(T, h, g, x)| \leqslant Ch^{\alpha/2}.$$

Proof. Starting from (9) we write

$$\begin{aligned}|\mathcal{E}| &\leqslant C\sum_{i=0}^{N-1}\mathbb{E}[\mathbb{I}_{\tau^N > t_i}\mathbb{I}_{\tau_{t_i}\leqslant t_{i+1}}\mathbb{E}[(T\wedge\tau_{t_{i+1}} - \tau_{t_i})^\alpha + \|X_{T\wedge\tau_{t_{i+1}}} - X_{\tau_{t_i}}\|^{2\alpha}|\mathcal{F}_{\tau_{t_i}}]]\\ &\leqslant C\sum_{i=0}^{N-1}\mathbb{E}[\mathbb{I}_{\tau^N > t_i}\mathbb{I}_{\tau_{t_i}\leqslant t_{i+1}}\mathbb{E}[(T\wedge\tau_{t_{i+1}} - \tau_{t_i})^\alpha|\mathcal{F}_{\tau_{t_i}}]]\end{aligned}$$

using the BDG inequalities for the last inequality. We controlled the term $\mathbb{E}[(T\wedge\tau_{t_{i+1}} - \tau_{t_i})|\mathcal{F}_{\tau_{t_i}}] := A^2_{\tau_{t_i}} \leqslant C\sqrt{h}$ in the proof of Lemma 3.2 in the stopped case. Hence, the result is a consequence of Hölder's inequality and Lemma 3.3. □

5 Extension to an intersection of smooth domains

5.1 Additional notations and assumptions

In this section we allow the domain to be singular in the sense of the following Assumption

(D') The domain $D = \bigcap_{j=1}^{m} D_j,\ m \geqslant 2$. For all $j \in [\![1,m]\!]$, D_j satisfies **(D)**. We denote its boundary by $\Gamma_j := \partial D_j$.

For $r \geqslant 0$, we set $\forall j \in [\![1,m]\!],\ V_{\Gamma_j}(r) := \{z \in \mathbb{R}^d : d(z,\Gamma_j) \leqslant r\},\ V_{\partial D}(r) := \{z \in \mathbb{R}^d : d(z,\partial D) \leqslant r\},\ D(r) := D \cup V_{\partial D}(r)$. Since the Γ_j are C^2, we recall from Proposition 2.1 that $\exists R_j > 0$ s.t. on $V_{\Gamma_j}(R_j)$ the projection on Γ_j is uniquely defined. For all $x \in \Gamma_j$, the notation $n_j(x)$ stands for the inner normal unit of D_j. In the following, F_j denotes the signed distance to Γ_j which is C^2 on $V_{\Gamma_j}(R_j)$ and can be extended into a C^2 function on $\mathbb{R}^d$ with bounded derivatives (see once again Proposition 2.1 for details). Set $R := \wedge_{j=1}^{m} R_j$. Our nondegeneracy assumption on the domain D is stated as follows:

(C') $\exists a_0 > 0$ such that *a.s.* $\big(X_s \in V_{\Gamma_j}(R) \cap V_{\partial D}(R), s \in [0,T], j \in [\![1,m]\!] \Longrightarrow \nabla F_j(X_s).\sigma_s\sigma_s^*\nabla F_j(X_s) \geqslant a_0\big)$.

This corresponds to a non characteristic boundary condition w.r.t. every hypersurface in a neighbourhood of the domain D.

5.2 Main result

We are now in a position to state the main result of the section.

Theorem 5.1 (Upper Bound for an intersection of smooth domains in the killed case)
Assume **(C')**, **(D')**, **(S)** *and let f be as in Theorem 2.2. For some constant $C := C(m)$, one has*

$$|\mathrm{Err}(T,h,f,x)| = |\mathbb{E}[f(X_T)\mathbb{I}_{\tau^N > T}] - \mathbb{E}[f(X_T)\mathbb{I}_{\tau > T}]| \leqslant C\frac{\|f\|_\infty}{1 \wedge \varepsilon}\sqrt{h}.$$

We restrict ourselves to the killed case for simplicity because we do not need to project X_{τ^N} on the boundary to define our approximation.

Remark 5.1 *The result of Theorem 5.1 is very interesting even in the Markovian setting of Brownian Motion. Indeed, for non smooth domains it is a hard task to use the traditional error analysis techniques that require the smoothness of the derivatives of the solution of the underlying PDE (7) up to the boundary, see also [Men04]. We thus provide an alternative technique that points out that the main difficulty to upper-bound the weak error in the Brownian context does not lie in the lack of regularity of the domain.*

5.3 Proof of Theorem 5.1

Without modifying the rate of convergence, see Section 3.2 for details, we can assume $X_t \in D(R)$ $a.s.$

Using the above definition of $(V_t)_{t\in[0,T]}$, i.e. $\forall t \in [0,T]$, $V_t = \mathbb{E}[f(X_T)\mathbb{I}_{\tau_t>T}|\mathcal{F}_t]$, and for an initial point $x \in \partial D$, we derive in a similar way than for the proof of Theorem 2.2

$$\mathcal{E} := \mathbb{E}[f(X_T)\mathbb{I}_{\tau^N>T}] = \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i}\mathbb{I}_{\tau_{t_i}<t_{i+1}}V_{t_{i+1}}].$$

Recall that, to prove Theorem 5.1, it is enough to show $|\mathcal{E}| \leqslant C\sqrt{h}$ controlling that C is uniform w.r.t. $x \in \partial D$.

Put $\tau_t^j := \inf\{s > t : X_s \notin D_j\}$ and note that $\tau_t = \wedge_{j=1}^m \tau_t^j$. From **(C')**, we then derive that X satisfies our previous assumption **(C)** w.r.t. $D_j, \forall j \in [\![1,m]\!]$. Hence, as a consequence of Lemma 4.2, we have

$$\begin{aligned}|V_{t_{i+1}}| = |\mathbb{E}[f(X_T)\mathbb{I}_{\tau_{t_{i+1}}>T}|\mathcal{F}_{t_{i+1}}]| &\leqslant \mathbb{E}[|f(X_T)|\mathbb{I}_{\tau^j_{t_{i+1}}>T}|\mathcal{F}_{t_{i+1}}] \\ &\leqslant \frac{C\|f\|_\infty}{1\wedge\varepsilon}[F_j(X_{t_{i+1}})]_+,\ \forall j \in [\![1,m]\!]\,.\end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{E}| &\leqslant \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i,\tau_{t_i}<t_{i+1}}|V_{t_{i+1}}|] = \sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^N>t_i,\cup_{j=1}^m\{\tau^j_{t_i}<t_{i+1}\}}|V_{t_{i+1}}|] \\ &\leqslant \frac{C\|f\|_\infty}{1\wedge\varepsilon}\sum_{j=1}^m\sum_{i=0}^{N-1} \mathbb{E}[\mathbb{I}_{\tau^{N,j}>t_i,\tau^j_{t_i}<t_{i+1}}[F_j(X_{t_{i+1}})]_+]\end{aligned}$$

where $\tau^{N,j} := \inf\{s_i \geqslant 0 : X_{s_i} \notin D_j\}$. Applying Lemma 4.1 we derive that

$$|\mathcal{E}| \leqslant \sqrt{h}\frac{C\|f\|_\infty}{1\wedge\varepsilon}\sum_{j=1}^m\sum_{i=0}^{N-1}\mathbb{P}[\tau^{N,j} > t_i, \tau^j_{t_i} < t_{i+1}]\,.$$

We conclude the proof using Lemma 3.3 for all $j \in [\![1,m]\!]$. □

6 Conclusion

In this paper, we first emphasized that, under suitable assumptions, the error $\text{Err}(T,h,\psi,x)$ associated to the discrete sampling of X for a given set of functionals ψ, is not given by the Markov property of SDEs but actually only depends on the Brownian stochastic integral in the dynamics (1). For a discretely sampled maximum or integral we used standard arguments to get this result. For killed/stopped processes, we introduced some martingale techniques that allow to go beyond the Markovian framework and also to control $\mathrm{Err}(T, h, f, x)$ at the expected rate for a certain class of non smooth domains. In the killed/stopped case, as a matter of fact, few technical tools are needed for the error analysis we present. This is promising since even in a Brownian setting, for non-smooth domains the PDE approach for the error analysis is rather tedious or fails. The next natural question concerns the possible extension of our techniques when the stochastic integral in (1) is driven by a stable process.

References

[AGP95] S. Asmussen, P. Glynn, and J. Pitman. Discretization error in simulation of one-dimensional reflecting brownian motion. *Ann. Appl. Probab.* 5(4): 875–896, 1995

[GM04] E. Gobet and S. Menozzi. Exact approximation rate of killed hypoelliptic diffusions using the discrete euler scheme. *Stoch. Proc. Appl.* 112(2): 201–223, 2004

[Gob00] E. Gobet. Euler schemes for the weak approximation of killed diffusion. *Stoch. Proc. Appl.* 87: 167–197, 2000

[GT77] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer, Berlin Heidelberg New York 1977

[KS91] I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus.* Second Edition, Springer, Berlin Heidelberg New York 1991

[LSU68] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and quasi-linear equations of parabolic type.* Vol. 23 Trans. Math. Monog. AMS, Providence, 1968

[Men04] S. Menozzi. *Discrétisations associées à un processus dans un domaine et schémas numériques probabilistes pour les EDP paraboliques quasi-linéaires.* Ph.D. Thesis, University Paris VI, 2004

[RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Third Edition, Grundlehren der Mathematischen Wissenschaften. 293. Springer, Berlin Heidelberg New York 1999

[SY82] D. Siegmund and Y.S. Yuh. Brownian approximations for first passage probabilities. *Z. Wahrsch. verw. Gebiete* 59: 239–248, 1982

[Tem01] E. Temam. *Couverture Approchée d'Options Exotiques, Pricing des Options Asiatiques.* Ph.D. Thesis, University Paris VI, 2001

[TL90] D. Talay and L.Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stoch. Anal. Appl.* 8(4): 94–120, 1990

Itô's Integrated Formula for Strict Local Martingales with Jumps

Oleksandr Chybiryakov

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie
175 rue du Chevaleret, 75013 Paris, France
e-mail: chyb@ccr.jussieu.fr

Summary. This note presents some properties of positive càdlàg local martingales which are not martingales – strict local martingales – extending the results from [MY06] to local martingales with jumps. Some new examples of strict local martingales are given. The construction relies on absolute continuity relationships between Dunkl processes and absolute continuity relationships between semi-stable Markov processes.

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1 Main results

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. On $\Omega \times \mathbb{R}_+$ we denote by $\mathcal{O}$ and $\mathcal{P}$ respectively – the optional and predictable sigma fields and by $\mathcal{B}(\mathbb{R})$ the Borel sigma field. Consider $(S_t)_{t\geqslant 0}$ – an $\mathbb{R}_+$ valued local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. For the definitions of local time for discontinuous local martingales we follow ([TL78], pages 17–22; see also [Mey76] and [Pro05]). For each $a \in \mathbb{R}$ there exists a continuous increasing process $(L_t^a,\ t \geqslant 0)$, such that Tanaka's formula holds:

$$(S_t - a)^+ = (S_0 - a)^+ + \int_{0+}^{t} \mathbf{1}_{\{S_{u-} > a\}} dS_u + \sum_{0<u\leqslant t} \left[\mathbf{1}_{\{S_{u-} > a\}} (S_u - a)^- + \mathbf{1}_{\{S_{u-} \leqslant a\}} (S_u - a)^+ \right] + \frac{1}{2} L_t^a, \quad (1)$$

which we write equivalently:

$$(S_t - a)^+ = (S_0 - a)^+ + \int_{0+}^t \mathbf{1}_{\{S_{u-}>a\}} dS_u + \frac{1}{2}\mathcal{L}_t^a.$$

Furthermore, there exists a $\mathcal{B}(\mathbb{R}) \times \mathcal{O}$ measurable version of $\mathcal{L}$, a.s. càdlàg in t, and a $\mathcal{B}(\mathbb{R}) \times \mathcal{P}$ measurable version of L, which is a.s. continuous in t. We will only consider such versions. Also note that for any $f \geqslant 0$, Borel,

$$\int_0^t f(S_u)\, d\langle S^c, S^c\rangle_u = \int_{-\infty}^{+\infty} f(a)\, L_t^a da.$$

We shall say that T is a $(\mathcal{F}_t)$ stopping time which reduces the local martingale S if $(S_{t\wedge T})$ is a uniformly integrable martingale. We shall say that a process X is in class (D) if the family $\{X_\tau,\ \tau$ - a.s. finite $(\mathcal{F}_t)$ stopping time$\}$ is uniformly integrable.

The following Theorem is a straightforward generalization of Theorem 1 in [MY06].

Theorem 1. *Let τ be an $(\mathcal{F}_t)$ stopping time such that $\tau < +\infty$ a.s. and $K \geqslant 0$. Then there is the following identity*

$$\mathbb{E}(S_\tau - K)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J_\tau^K + \frac{1}{2}\mathbb{E}L_\tau^K - c_S(\tau), \tag{2}$$

where $c_S(\tau) := \mathbb{E}(S_0 - S_\tau)$,

$$\begin{aligned} J_\tau^K &:= \sum_{0<u\leqslant\tau} \mathbf{1}_{\{S_{u-}>K\}} (S_u - K)^- + \sum_{0<u\leqslant\tau} \mathbf{1}_{\{S_{u-}\leqslant K\}} (S_u - K)^+ \\ &= \frac{1}{2}\left(\mathcal{L}_\tau^K - L_\tau^K\right) \end{aligned} \tag{3}$$

and $\left(L_t^K\right)_{t\geqslant 0}$ is the (continuous) local time at K of S.

Proof. Taking $a = K$ in Tanaka's formula (1), one has

$$\begin{aligned} (S_t - K)^+ - (S_0 - K)^+ = \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u + \sum_{0<u\leqslant t} \mathbf{1}_{\{S_{u-}>K\}} (S_u - K)^- \\ + \sum_{0<u\leqslant t} \mathbf{1}_{\{S_{u-}\leqslant K\}} (S_u - K)^+ + \frac{1}{2}L_t^K, \end{aligned}$$

introducing J_t^K as in (3) one obtains

$$\begin{aligned} N_t^K &:= \left[(S_t - K)^+ - S_t\right] - \left[(S_0 - K)^+ - S_0 + J_t^K + \frac{1}{2}L_t^K\right] \\ &= \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u + S_0 - S_t \end{aligned}$$

and $\left(N_t^K\right)_{t\geqslant 0}$ is a local martingale. Since $(S_t-K)^+-S_t=-\left(S_t\wedge K\right)$, one has

$$-N_t^K=S_t\wedge K-S_0\wedge K+J_t^K+\frac{1}{2}L_t^K.$$

In order to get (2) it is enough to prove that N_t^K is in class (D), i.e., the family $N_\tau^K\mathbf{1}_{\{\tau<+\infty\}}$, where τ ranges all $(\mathcal{F}_t)$ stopping times, is uniformly integrable. Indeed, again from Tanaka's formula

$$(S_t-K)^+-(S_0-K)^+-J_t^K-\frac{1}{2}L_t^K=\int_{0+}^t\mathbf{1}_{\{S_{u-}>K\}}dS_u.$$

Let $(\tau_n)_{n\geqslant 1}$ ($\tau_n\to+\infty$ *a.s.*) be a sequence of $(\mathcal{F}_t)$ stopping times which reduces both $(S_t)_{t\geqslant 0}$ and $\left(\int_{0+}^t\mathbf{1}_{\{S_{u-}>K\}}dS_u\right)_{t\geqslant 0}$. Then one gets

$$\mathbb{E}J_{t\wedge\tau_n}^K+\frac{1}{2}\mathbb{E}L_{t\wedge\tau_n}^K=\mathbb{E}\left[(S_{t\wedge\tau_n}-K)^+-(S_0-K)^+\right]\leqslant\mathbb{E}S_{t\wedge\tau_n}=\mathbb{E}S_0.$$

Finally, by Beppo-Levi:

$$\mathbb{E}J_t^K+\frac{1}{2}\mathbb{E}L_t^K\leqslant\mathbb{E}S_0$$

and

$$\mathbb{E}J_\infty^K+\frac{1}{2}\mathbb{E}L_\infty^K\leqslant\mathbb{E}S_0,$$

then for any $(\mathcal{F}_t)$ stopping time τ

$$\left|N_\tau^K\mathbf{1}_{\{\tau<+\infty\}}\right|\leqslant 2K+J_\infty^K+\frac{1}{2}L_\infty^K\text{ a.s.},$$

which ensures that $\left(N_t^K\right)_{t\geqslant 0}$ is in class (D). Therefore $\left(N_t^K\right)_{t\geqslant 0}$ is a uniformly integrable martingale and the result follows. □

Let τ be an $(\mathcal{F}_t)$ stopping time which is a.s. finite. With notations from [LN06] suppose that $S\in\mathcal{M}_{loc}^2$, and moreover that: $\langle S\rangle_\infty<\infty$ a.s., S^+ is in class (D), $|\Delta S|\leqslant C$ and

$$\mathbb{E}e^{\varepsilon S_\tau}<\infty$$

for some positive constants C and ε. Then from Theorem 1.1 in [LN06] the term $c_S(\tau):=\mathbb{E}\left(S_0-S_\tau\right)$ in (2) can be characterized as

$$c_S(\tau)=\lim_{\lambda\to\infty}\lambda\sqrt{\frac{\pi}{2}}\mathbb{P}\left(\langle S\rangle_\tau^{1/2}>\lambda\right)=\lim_{\lambda\to\infty}\lambda\sqrt{\frac{\pi}{2}}\mathbb{P}\left([S,S]_\tau^{1/2}>\lambda\right).$$

Besides as a consequence of (2) one obtains

$$c_S(\tau)=\lim_{K\to\infty}\left(\mathbb{E}J_\tau^K+\frac{1}{2}\mathbb{E}L_\tau^K\right).$$

Let τ be an a.s. finite $(\mathcal{F}_t)$ stopping time. Define

$$C^{\text{strict}}(K,\tau) := \lim_{n\to\infty} \mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+, \tag{4}$$

where $T_n \to \infty$ a.s., T_n reduces $(S_t)_{t\geqslant 0}$. The following proposition shows that this limit exists and does not depend on the reducing sequence T_n, $n \geqslant 1$.

Proposition 1. *Let τ be an a.s. finite $(\mathcal{F}_t)$ stopping time. Then*

$$C^{\text{strict}}(K,\tau) = \mathbb{E}\left(S_0 - K\right)^+ + \mathbb{E}J_\tau^K + \frac{1}{2}\mathbb{E}L_\tau^K. \tag{5}$$

Furthermore, if the process $(\Delta S_t)_{t\geqslant 0}$ is in class (D), then

$$C^{\text{strict}}(K,\tau) = \mathbb{E}\left[\left(S_\tau - K\right)^+\right] + \lim_{n\to\infty} n\mathbb{P}\left(S_\tau^* > n\right),$$

where $S_t^* := \sup_{0\leqslant u\leqslant t} S_u$.

Proof. By Tanaka's formula

$$\left(S_t - K\right)^+ - \left(S_0 - K\right)^+ = \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u + J_t^K + \frac{1}{2}L_t^K,$$

where J_t^K is defined by (3). Since $(S_t)_{t\geqslant 0}$ is a local martingale,

$$S_t^K := \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u$$

is also a local martingale. We have seen in the proof of Theorem 1 that

$$N_t^K = \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u + S_0 - S_t$$

is a uniformly integrable martingale, then

$$S_t^K := N_t^K + S_t - S_0$$

is the sum of a uniformly integrable martingale and a local martingale $(S_t)_{t\geqslant 0}$. Therefore a stopping time which reduces $(S_t)_{t\geqslant 0}$ reduces $\left(S_t^K\right)_{t\geqslant 0}$ as well.

Let T be an $(\mathcal{F}_t)$ stopping time which reduces $(S_t)_{t\geqslant 0}$. Then $S_{t\wedge T}$ and $S_{t\wedge T}^K$ are uniformly integrable martingales. For any τ – an a.s. finite $(\mathcal{F}_t)$ stopping time one gets

$$\mathbb{E}\left(S_{\tau\wedge T} - K\right)^+ = \mathbb{E}\left(S_0 - K\right)^+ + \mathbb{E}J_{\tau\wedge T}^K + \frac{1}{2}\mathbb{E}L_{\tau\wedge T}^K, \tag{6}$$

now taking for T a stopping time T_n, such that $T_n \to \infty$ a.s. and T_n reduces $(S_t)_{t\geqslant 0}$, one obtains that

$$C^{\text{strict}}(K,\tau) := \lim_{n\to\infty} \mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+$$

exists and does not depend on the sequence of stopping times reducing $(S_t)_{t\geqslant 0}$. Furthermore,

$$C^{\text{strict}}(K,\tau) = \mathbb{E}\left(S_0 - K\right)^+ + \mathbb{E}J_\tau^K + \frac{1}{2}\mathbb{E}L_\tau^K .$$

In order to move further suppose that the process $(\Delta S_t)_{t\geqslant 0}$ is in class (D). Take

$$T_n := \inf\left\{u > 0 \,|S_u > n\right\}.$$

Since $(S_t)_{t\geqslant 0}$ is an adapted càdlàg process, T_n is an $(\mathcal{F}_t)$ stopping time and $T_n \to \infty$ a.s. Since

$$|S_{t\wedge T_n}| \leqslant n + \Delta S_{T_n},$$

$(S_{t\wedge T_n})_{t\geqslant 0}$ is in class (D) and subsequently is a uniformly integrable martingale. In particular T_n is an $(\mathcal{F}_t)$ stopping time which reduces $(S_t)_{t\geqslant 0}$. Now one can get for any τ - an a.s. finite $(\mathcal{F}_t)$ stopping time

$$\mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+ = \mathbb{E}\left[\left(S_\tau - K\right)^+ \mathbf{1}_{\{\tau\leqslant T_n\}}\right] + \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right].$$

The left hand side converges and equals $C^{\text{strict}}(K,\tau)$. The first expression on the right hand side converges as well (by Beppo-Levi) to $\mathbb{E}\big[\left(S_\tau - K\right)^+\big]$. Hence $\mathbb{E}\big[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\big]$ converges as well. Besides one has for $n > K$

$$\begin{aligned}(n-K)\,\mathbb{P}\left(\tau > T_n\right) &\leqslant \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right] \\ &\leqslant (n-K)\,\mathbb{P}\left(\tau > T_n\right) + \mathbb{E}\left[\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right] - \mathbb{E}\left[\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right] &\leqslant (n-K)\,\mathbb{P}\left(\tau > T_n\right) \\ &\leqslant \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right].\end{aligned}$$

Since $\left(\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right)_{n\geqslant 1}$ is a uniformly integrable family $\mathbb{E}\left[\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right] \to 0$, as $n \to \infty$, and

$$\lim_{n\to\infty} \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right] = \lim_{n\to\infty} (n-K)\,\mathbb{P}\left(\tau > T_n\right) = \lim_{n\to\infty} n\mathbb{P}\left(S_\tau^* > n\right).$$

Finally

$$\lim_{n\to\infty} \mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+ = \mathbb{E}\left[\left(S_\tau - K\right)^+\right] + \lim_{n\to\infty} n\mathbb{P}\left(S_\tau^* > n\right),$$

where $S_t^* := \sup_{0\leqslant u\leqslant t} S_u$. □

Remark 1. Note that from Theorem 1 and Proposition 1 for any positive local martingale S, such that $(\Delta S_t)_{t\geqslant 0}$ is in class (D), and for any a.s. finite $(\mathcal{F}_t)$ stopping time τ

$$c_S(\tau) = \lim_{n\to\infty} n\mathbb{P}(S^*_\tau > n).$$

Remark 2. Under the conditions of Proposition 1

$$C^{\text{strict}}(K,\ \tau) = \sup_{\sigma\text{-}(\mathcal{F}_t)\text{ stopping time}} \mathbb{E}(S_{\sigma\wedge\tau} - K)^+. \tag{7}$$

Proof. From (6) one obtains that for any pair of $(\mathcal{F}_t)$ stopping times τ, σ and a sequence of stopping times R_n, such that $R_n \to \infty$ a.s. and R_n reduces $(S_t)_{t\geqslant 0}$

$$\mathbb{E}(S_{\tau\wedge\sigma\wedge R_n} - K)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge\sigma\wedge R_n} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge\sigma\wedge R_n}.$$

Then by Fatou's Lemma

$$\begin{aligned}\mathbb{E}(S_{\tau\wedge\sigma} - K)^+ &\leqslant \mathbb{E}(S_0 - K)^+ + \liminf_{n\to\infty}\left[\mathbb{E}J^K_{\tau\wedge\sigma\wedge R_n} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge\sigma\wedge R_n}\right]\\ &= \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge\sigma} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge\sigma}\\ &\leqslant \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau} + \frac{1}{2}\mathbb{E}L^K_{\tau}.\end{aligned}$$

Now (7) follows from (5) and (4). □

Remark 3. The original proof of Proposition 2 in [MY06] differs a little from ours. In order to obtain (6), the fact that the stopping time which reduces $(S_t)_{t\geqslant 0}$ reduces as well $\left(S^K_t\right)_{t\geqslant 0}$, is not used. Let us go through this other proof and see that there is no contradiction.

Proof. Let T be an $(\mathcal{F}_t)$ stopping time which reduces $(S_t)_{t\geqslant 0}$ and T^K_n, $n \geqslant 1$, $T^K_n \to \infty$ be a sequence of stopping times that reduce $\left(S^K_t\right)_{t\geqslant 0}$. Then for any τ – an a.s. finite $(\mathcal{F}_t)$ stopping time – one gets

$$\mathbb{E}\left(S_{\tau\wedge T\wedge T^K_n} - K\right)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge T\wedge T^K_n} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge T\wedge T^K_n}.$$

On the right hand side one can pass to the limit as $T^K_n \to \infty$ by Beppo-Levi and get a finite limit as soon as we already know from the proof of Theorem 1 that

$$\mathbb{E}J^K_\infty + \frac{1}{2}\mathbb{E}L^K_\infty \leqslant \mathbb{E}S_0.$$

On the left hand side, $\left(S_{\tau\wedge T\wedge T^K_n}\right)_{n\geqslant 1}$ is a uniformly integrable martingale, thus it converges in L^1 to $S_{\tau\wedge T}$. Finally one gets

$$\mathbb{E}(S_{\tau\wedge T} - K)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge T} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge T}, \tag{8}$$

which is the same as (6). □

For any μ – a finite measure on $\mathbb{R}_+$ define

$$F_\mu(x) := \int_0^{+\infty} \mu(dK)(x-K)^+$$

and $\bar{\mu} := \int_0^{+\infty} \mu(dK)$. As in [MY06] we have the following Proposition and Corollary (the proofs are the same as in the continuous case).

Proposition 2. *Under the notations and assumptions of Theorem 1*

$$\mathbb{E}\left[F_\mu(S_\tau)\right] = F_\mu(S_0) + \mathbb{E}\left[\int_0^{+\infty} \mu(dK)\left(J_\tau^K + \frac{1}{2}L_\tau^K\right)\right] - \bar{\mu} c_S(\tau).$$

Corollary 1. *The process*

$$F_\mu(S_t) - F_\mu(S_0) - \int_0^{+\infty} \mu(dK)\left(J_t^K + \frac{1}{2}L_t^K\right) - \bar{\mu}(S_t - S_0),\ t \geqslant 0$$

is a martingale.

2 Examples

One can trivially construct strict local martingales from continuous strict local martingales: indeed, $M_t := M_t^{(c)} + M_t^{(d)}$ and $\left(M_t^{(c)}\right)$ is a strict local martingale and $\left(M_t^{(d)}\right)$ is a uniformly integrable martingale, then (M_t) is a strict local martingale.

We now obtain strict local martingales with jumps which are generalizations of the strict local martingale $\left(1/R_t^{(3)}\right)$, where $\left(R_t^{(3)}\right)$ is a Bessel process of dimension 3. As in the case of $\left(1/R_t^{(3)}\right)$, such strict local martingales can be obtained from absolute continuity relationships between two Dunkl Markov processes instead of Bessel processes. For simplicity, we consider here only one dimensional Dunkl Markov processes (see [GY06]).

The Dunkl Markov process (X_t) with parameter k is a Feller process with extended generator given for $f \in C^2(\mathbb{R})$ by

$$\mathcal{L}_k f(x) = \frac{1}{2} f''(x) + k\left(\frac{1}{x} f'(x) - \frac{f(x) - f(-x)}{2x^2}\right),$$

where $k \geqslant 0$. Note that $|X|$ is a Bessel process with index $\nu := k - \frac{1}{2}$. Denote by $P_x^{(k)}$ the law of (X_t) started at $x \in \mathbb{R}$, and by $\left(\mathcal{F}_t^X\right)$ the natural filtration of X.

Proposition 3. *Let* $0 \leqslant k < \frac{1}{2} \leqslant k'$ *and* $x > 0$. *Define*

$$T_0 := \inf\left\{s \geqslant 0 \,|\, X_{s-} = 0 \text{ or } X_s = 0\right\}.$$

Then $P_x^{(k)}(T_0 < +\infty) = 1$ *and there is the following absolute continuity relationship:*

$$P_x^{(k')}\Big|_{\mathcal{F}_t^X} = \left(\frac{|X_{t\wedge T_0}|}{|x|}\right)^{k'-k}\left(\frac{k'}{k}\right)^{N_{t\wedge T_0}}\exp\left(-\frac{(k')^2-k^2}{2}\int_0^{t\wedge T_0}\frac{ds}{X_s^2}\right)P_x^{(k)}\Big|_{\mathcal{F}_t^X}, \quad (9)$$

where N_t *denotes the number of jumps of* X *on* $[0,\ t]$*. Furthermore*

$$M_t := \left(\frac{|x|}{|X_t|}\right)^{k'-k}\left(\frac{k}{k'}\right)^{N_t}\exp\left(\frac{(k')^2-k^2}{2}\int_0^t\frac{ds}{X_s^2}\right) \quad (10)$$

is a strict local martingale under $P_x^{(k')}$*, and*

$$P_x^{(k)}(T_0 > t) = \mathbb{E}_x^{(k')}M_t, \quad (11)$$

where $\mathbb{E}_x^{(k')}$ *is the expectation under* $P_x^{(k')}$*.*

Remark 4. Note that the law of T_0 under $P_x^{(k)}$ is that of $x^2/\left(2Z_{\left(\frac{1}{2}-k\right)}\right)$, where $Z_{\left(\frac{1}{2}-k\right)}$ is a gamma variable of a parameter $\frac{1}{2}-k$ (see page 98 in [Yor01]).

Proof. Let X be a Dunkl Markov process. Note that $\Delta X_s = X_s - X_{s-} = -2X_{s-}$, when $\Delta X_s \neq 0$. Hence if $X_s = 0$, then $X_{s-} = 0$ and

$$T_0 = \inf\{s \geqslant 0\,|X_{s-} = 0\} = \inf\{s \geqslant 0\,||X_s| = 0\}.$$

In order to prove (9) we proceed as in the proof of Proposition 4 in [GY06]. First we need to extend Theorem 3 in [GY06] for $k < \frac{1}{2}$. Since $|X|$ is a Bessel process with index $\left(k-\frac{1}{2}\right)$, for $k < \frac{1}{2}$, $T_0 < +\infty$ a.s., and, for $k \geqslant \frac{1}{2}$, $T_0 = +\infty$ a.s. Denote

$$\tau_t := \inf\left\{s \geqslant 0\,\middle|\int_0^s\frac{du}{X_u^2} = t\right\},$$

then τ_t is a continuous strictly increasing time change and $\tau_\infty = T_0$. Denote $Y_u := X_{\tau_u}$. Since for any $f \in C^2(\mathbb{R})$

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}_k f(X_s)\,ds$$

is a local martingale,

$$f(Y_u) - f(Y_0) - \int_0^t Y_s^2\mathcal{L}_k f(Y_s)\,ds$$

is a local martingale. Then, as in the proof of Theorem 4 in [GY06], one obtains that Y is of the form

$$Y_u = \exp\left(\beta_u^{(\nu)} + i\pi N_u^{(k/2)}\right),$$

where $\nu := k - \frac{1}{2}$, $\left(\beta_u^{(\nu)}\right)$ is a Brownian motion with drift ν, $\left(N_u^{(k/2)}\right)$ is a Poisson process with parameter $k/2$ independent from $\left(\beta_u^{(\nu)}\right)$. Denote

$$A_t := \int_0^t \frac{du}{X_u^2},$$

then $\tau_{A_t} = t$, for $t < T_0$. Hence

$$X_t = Y_{A_t},\ t < T_0. \tag{12}$$

Note also that differentiating the equality $A_{\tau_t} = t$ with respect to time one gets

$$\frac{d}{dt}\tau_t = Y_t^2$$

and $A_t = \inf\left\{s \geqslant 0 \,\middle|\, \int_0^s Y_u^2 du = t\right\}$, $t < T_0$. Note that (9) is equivalent to

$$P_x^{(k)}\Big|_{\mathcal{F}_t^X \cap \{t<T_0\}} = \left(\frac{|x|}{|X_t|}\right)^{k'-k} \left(\frac{k}{k'}\right)^{N_t} \exp\left(\frac{(k')^2 - k^2}{2}\int_0^t \frac{ds}{X_s^2}\right) P_x^{(k')}\Big|_{\mathcal{F}_t^X}. \tag{13}$$

Indeed (9) is equivalent to

$$\begin{aligned}\mathbb{E}_x^{(k')}\left(F\left(X_s,\ s \leqslant t\right)\right) &= \mathbb{E}_x^{(k)}\left(F\left(X_s,\ s \leqslant t\right)\frac{1}{M_{t\wedge T_0}}\right) \\ &= \mathbb{E}_x^{(k)}\left(F\left(X_s,\ s \leqslant t\right)\frac{1}{M_t}\mathbf{1}_{\{t<T_0\}}\right),\end{aligned}$$

for any bounded measurable F, (M_t) is given by (10). Then

$$\mathbb{E}_x^{(k')}\left(M_t \mathbf{1}_{\{t<T_0\}}\hat{F}\left(X_s,\ s \leqslant t\right)\right) = \mathbb{E}_x^{(k)}\left(\hat{F}\left(X_s,\ s \leqslant t\right)\mathbf{1}_{\{t<T_0\}}\right),$$

which is equivalent to (13). By (12) X is associated to the pair $\left(\beta^{(\nu)}, N^{(k/2)}\right)$ under $P^{(k)}$, and to the pair $\left(\beta^{(\nu')}, N^{(k'/2)}\right)$ under $P^{(k')}$. Both pairs consist of a Brownian motion with drift and a Poisson process which are mutually independent, and $\nu := k - \frac{1}{2}$, $\nu' := k' - \frac{1}{2}$. Now in the same way as in the proof of Proposition 4 in [GY06], for any bounded measurable F,

$$\mathbb{E}_x^{(k)}\left(F\left(\beta_s^{(\nu)}, N_s^{(k)},\ s \leqslant t\right)\right) = \mathbb{E}_x^{(k')}\left(D_t F\left(\beta_s^{(\nu')}, N_s^{(k')},\ s \leqslant t\right)\right),$$

where

$$D_t := \exp\left(\left(\nu - \nu'\right)\beta_t^{(\nu')} - \frac{1}{2}\left(\nu^2 - \left(\nu'\right)^2\right)t\right)\left(\frac{k}{k'}\right)^{N_t^{(k'/2)}}\exp\left(-\frac{1}{2}\left(k - k'\right)t\right)$$

$$= \exp\left(\left(k - k'\right)\beta_t^{(\nu')} - \frac{1}{2}\left(k^2 - \left(k'\right)^2\right)t\right)\left(\frac{k}{k'}\right)^{N_t^{(k'/2)}}$$

and $D_{A_t} = M_t$, $t < T_0$. Denote $\mathcal{G}_t := \sigma\left\{\beta_s^{(\nu')}, N_s^{(k'/2)},\ s \leqslant t\right\}$, then

$$F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)}\right)\mathbf{1}_{\{A_s \leqslant u\}}$$

is $\mathcal{G}_{A_s \wedge u}$ measurable and

$$\begin{aligned}
&\mathbb{E}_x^{(k)}\left(F\left(\beta_{A_s}^{(\nu)}, N_{A_s}^{(k/2)}\right)\mathbf{1}_{\{A_s \leqslant u\}}\right)\\
&= \mathbb{E}_x^{(k')}\left(\mathbb{E}^{(k')}\left(D_t\,|\mathcal{G}_{A_s \wedge u}\right)F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)}\right)\mathbf{1}_{\{A_s \leqslant u\}}\right)\\
&= \mathbb{E}_x^{(k')}\left(D_{A_s}F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)}\right)\mathbf{1}_{\{A_s \leqslant u\}}\right).
\end{aligned}$$

As $u \to +\infty$ one gets

$$\begin{aligned}
&\mathbb{E}_x^{(k)}\left(F\left(\beta_{A_s}^{(\nu)}, N_{A_s}^{(k/2)}\right)\mathbf{1}_{\{A_s < +\infty\}}\right)\\
&\quad = \mathbb{E}_x^{(k')}\left(D_{A_s}F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)}\right)\mathbf{1}_{\{A_s < +\infty\}}\right). \qquad (14)
\end{aligned}$$

Noting that $(A_s < +\infty) = (s < T_0)$, (14) leads to (13). From (13) one easily obtains (11). Suppose that (M_t) is a martingale then from (11) for any $t \geqslant 0$ $P_x^{(k)}(T_0 > t) = 1$ and $P_x^{(k)}(T_0 = +\infty) = 1$, which is impossible because $k < \frac{1}{2}$. Hence (M_t) is a strict local martingale. □

Other examples of strict local martingales with jumps can be obtained from absolute continuity relationships between two non-negative semi-stable Markov processes. We shortly recall the definition of a semi-stable Markov process (see [Lam72]):

A semi-stable Markov process (with index of stability $\alpha = 1$) on $\mathbb{R}_+ := [0, +\infty)$ is a Markov process (X_t) with the following scaling property: for any $c > 0$

$$\left(\frac{1}{c}X_{ct}^{(x)}\right)_{t \geqslant 0} \overset{(d)}{=} \left(X_t^{(xc^{-1})}\right)_{t \geqslant 0},$$

where $\left(X_t^{(x)}\right)$ denotes a semi-stable Markov process started at $x > 0$. Denote

$$T_0 := \inf\{s \geqslant 0 \,|\, X_{s-} = 0 \text{ or } X_s = 0\}, \tag{15}$$

then Lamperti in [Lam72] showed that: either $T_0 = +\infty$ a.s., or $T_0 < +\infty$ a.s. and $X_{T_0-} = 0$ a.s., or $T_0 < +\infty$ a.s. and $X_{T_0-} > 0$ a.s. Furthermore this does not depend on the starting point $x > 0$.

Note that for a semi-stable Markov process the following Lamperti relation is true. We suppose that there is no killing inside $(0, \infty)$.

Proposition 4. *Let (ξ_t) be a one-dimensional Lévy process, starting at 0. Define*

$$A_t^{(x)} := \int_0^t x \exp(\xi_s)\, ds,$$

for any $x > 0$. Then the process (X_u), defined implicitly by

$$x \exp \xi_t = X_{A_t^{(x)}},\ t < T_0, \tag{16}$$

is a semi-stable Markov process, starting at x, and

$$A_\infty^{(x)} = T_0, \tag{17}$$

where T_0 is defined by (15). The converse is also true.

Denote

$$\tau_t^{(x)} := \inf\left\{s \geqslant 0 \,\middle|\, A_s^{(x)} = t\right\}.$$

Let $\left(\mathcal{F}_t^\xi\right)$ be the natural filtration of (ξ_t) and $\left(\mathcal{F}_t^X\right)$ be the natural filtration of (X_t). As in [CPY94], using Proposition 4, one obtains the following absolute continuity relationship between two semi-stable Markov processes.

Proposition 5. *Suppose that (X_t) is a semi-stable Markov process associated with Lévy process (ξ_t) via Lamperti relation (16) and $\mathbb{E}_P e^{b\xi_t} = e^{t\rho(b)} < \infty$. Define Q by*

$$Q|_{\mathcal{F}_t^X \cap \{t<T_0\}} = \left(\frac{X_t}{x}\right)^b \exp\left(-\rho(b) \int_0^t \frac{ds}{X_s^2}\right) P|_{\mathcal{F}_t^X \cap \{t<T_0\}},$$

where T_0 is defined by (15). Then, under Q, (X_t) is still a semi-stable Markov process associated with Lévy process (ξ_t) via Lamperti relation (16) and

$$\tilde{\Psi}(u) = \Psi(u - ib) - \Psi(-ib),$$

where Ψ, $\tilde{\Psi}$ are the characteristic exponents of (ξ_t) under P and Q respectively.

Proof. Let us consider the change of measure given by the Esscher transform:

$$Q|_{\mathcal{F}_t^\xi} = \exp(b\xi_t - \rho(b)\, t)\ P|_{\mathcal{F}_t^\xi}.$$

Since $\mathbb{E}e^{b\xi_t} = e^{t\rho(b)} < \infty$

$$M_t := \exp\left(b\xi_t - \rho\left(b\right)t\right)$$

is a martingale. Furthermore (ξ_t) is still a Lévy process under Q. Note that $\left\{\tau_t^{(x)} < +\infty\right\} = \{t < T_0\}$ and for any $t < T_0$

$$A^{(x)}_{\tau_t^{(x)}} = t. \tag{18}$$

Denote $\mathcal{G}_t := \mathcal{F}^{\xi}_{\tau_t^{(x)}}$, then for any $A \in \mathcal{G}_t$

$$Q\left(A \cap \left\{\tau_t^{(x)} \leqslant u\right\}\right) = \mathbb{E}_P\left(\mathbf{1}_{A\cap\left\{\tau_t^{(x)}\leqslant u\right\}} \exp\left(b\xi_{\tau_t^{(x)}} - \rho\left(b\right)\tau_t^{(x)}\right)\right). \tag{19}$$

Note that $\left(X_t/x\right)^b = \exp\left(b\xi_{\tau_t^{(x)}}\right)$ on $\{t < T_0\}$. Differentiating (18) one gets that

$$\frac{d}{dt}\tau_t^{(x)} = \frac{1}{X_t^2}.$$

Letting u tend to infinity, from (19) one gets

$$Q\left(A \cap \left\{\tau_t^{(x)} < +\infty\right\}\right)$$
$$= \mathbb{E}_P\left(\mathbf{1}_{A\cap\left\{\tau_t^{(x)}<+\infty\right\}}\left(\frac{X_t}{x}\right)^b \exp\left(-\rho\left(b\right)\int_0^t \frac{ds}{X_s^2}\right)\right).$$

But from (17) $\left\{\tau_t^{(x)} < +\infty\right\} = \{t < T_0\}$. Hence

$$Q\left(A \cap \{t < T_0\}\right) = \mathbb{E}_P\left(\mathbf{1}_{A\cap\{t<T_0\}}\left(\frac{X_t}{x}\right)^b \exp\left(-\rho\left(b\right)\int_0^t \frac{ds}{X_s^2}\right)\right). \quad \square$$

Let us find the range of the parameter b such that

$$M_t := \left(\frac{X_t}{x}\right)^b \exp\left(-\rho\left(b\right)\int_0^t \frac{ds}{X_s^2}\right)$$

is a strict local martingale. Note that it is sufficient to find b such that $Q(T_0 < +\infty) = 1$ and $P\left(T_0 = +\infty\right) = 1$. Indeed, given such a parameter

$$Q\left(t < T_0\right) = \mathbb{E}_P\left(M_t\right)$$

and as in the proof of Proposition 3 (M_t) is a strict local P-martingale.

Now let ξ be a Lévy process under P with the characteristic exponent Ψ, given by Lévy–Khintchine formula

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}}\right) \pi(dx),$$

where $a \in \mathbb{R}$, $\sigma^2 \geqslant 0$ and π is a positive measure on $\mathbb{R}\setminus\{0\}$ such that

$$\int \left(1 \wedge |x|^2\right) \pi(dx) < \infty.$$

Let us suppose that π has compact support. Then $\mathbb{E}_P e^{b\xi_t} = e^{t\rho(b)} < \infty$ for any t and b. Let the semi-stable Markov process X be associated to ξ via Lamperti relation. Conditions for $P(T_0 = +\infty) = 1$ or $P(T_0 < +\infty) = 1$ bearing on $\left(a, \sigma^2, \pi\right)$ can be deduced from Theorem 1 in [BY05]. Note that $T_0 < +\infty$ if and only if $\xi_t \to -\infty$. Since π has compact support, from the Central Limit Theorem for a Lévy process, $\xi_t \to -\infty$ if and only if $\mathbb{E}_P \xi_1 < 0$ i.e.,

$$-a + \int_{|x|>1} x\pi(dx) < 0.$$

Let Q be given by Proposition 5. Denote by $\tilde{\Psi}$ the characteristic exponent of ξ under Q, then

$$\tilde{\Psi}(\lambda) = i\lambda \left[a - b\sigma^2 + \int_{|x|<1} x\left(1 - e^{bx}\right) \pi(dx)\right] + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}}\right) \tilde{\pi}(dx),$$

where $\tilde{\pi}(dx) = e^{bx}\pi(dx)$. Hence, in order to have $Q(T_0 < +\infty) = 1$ and $P(T_0 = +\infty) = 1$ one can choose b such that

$$-a + \int_{|x|>1} x\pi(dx) \geqslant 0 \tag{20}$$

and

$$-a + b\sigma^2 - \int_{|x|<1} x\left(1 - e^{bx}\right) \pi(dx) + \int_{|x|>1} xe^{bx}\pi(dx) < 0. \tag{21}$$

It is easy to see that (20) and (21) imply that $b < 0$. For example, for any given a and π, such that (20) is true, one can always choose $b < 0$ such that

$$b\sigma^2 - e^{-b} \int_{x<-1} |x|\, \pi(dx) < a - \int_{x>1} x\pi(dx), \tag{22}$$

which implies (21). Note that condition (22) is more restrictive than (21).

References

[BY05] J. Bertoin and M. Yor, *Exponential functionals of Lévy processes*, Probab. Surveys **2** (2005), 191–212

[CPY94] P. Carmona, F. Petit, and M. Yor, *Sur les fonctionnelles exponentielles de certains processus de Lévy*, Stoch. Stoch. Rep. **47** (1994), no. 1–2, 71–101

[GY06] L. Gallardo and M. Yor, *Some remarkable properties of the Dunkl martingales*, Séminaire de Probabilités XXXIX. Dedicated to Paul-André Meyer, Lecture Notes in Mathematics 1874, 337–356, Springer, Berlin Heidelberg New York 2006

[Lam72] J. Lamperti, *Semi-stable Markov processes I*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **22** (1972), 205–225

[LN06] R. Liptser and A. Novikov, *On tail distributions of supremum and quadratic variation of local martingales.* In: From stochastic calculus to mathematical finance, 421–432, Springer, Berlin Heidelberg New York 2006

[Mey76] P.A. Meyer, *Un cours sur les intégrales stochastiques*, Séminaire de Probabilités, X (Seconde partie: Théorie des intégrales stochastiques, University of Strasbourg, Strasbourg, année universitaire 1974/1975), Springer, Berlin Heidelberg New York, 1976, pp. 245–400. Lecture Notes in Mathematics Vol. 511

[MY06] D.B. Madan and M. Yor, *Itô's integrated formula for strict local martingales*, Séminaire de Probabilités XXXIX. Dedicated to Paul-André Meyer, Lecture Notes in Mathematics 1874, 157–170, Springer, Berlin Heidelberg New York 2006

[Pro05] P.E. Protter, *Stochastic integration and differential equations*, second edition, Version 2.1, Applications of Mathematics (New York), vol. 21, Springer, Berlin Heidelberg New York 2005, Stoch. Model. Appl. Probab.

[TL78] *Temps locaux*, Astérisque, vol. 52, Société Mathématique de France, Paris, 1978, Exposés du Séminaire J. Azéma-M. Yor, Held at the Université Pierre et Marie Curie, Paris, 1976–1977, With an English summary

[Yor01] M. Yor, *Exponential functionals of Brownian motion and related processes*, Springer Finance, Springer, Berlin Heidelberg New York 2001. With an introductory chapter by Hélyette Geman, Chapters 1, 3, 4, 8 translated from the French by Stephen S. Wilson

Enlargement of Filtrations and Continuous Girsanov-Type Embeddings

Stefan Ankirchner[1], Steffen Dereich[2] and Peter Imkeller[1]

[1] Institut für Mathematik, Humboldt-Universität zu Berlin
Unter den Linden 6, 10099 Berlin, Germany
e-mail: ankirch@math.tu-berlin.de and *imkeller@mathematik.hu-berlin.de*
[2] Fachbereich Mathematik, Technische Universität Berlin
Strasse des 17. Juni 136, 10623 Berlin, Germany
e-mail: dereich@math.tu-berlin.de

Summary. Let $(\mathcal{G}_t)$ be an enlargement of the filtration $(\mathcal{F}_t)$. Jeulin and Jacod discussed a sufficient criterion for the inheritance of the semimartingale property when passing to the larger filtration. We provide alternative proofs of their results in a more general setting by using decoupling measures and Girsanov's changes of measure. We derive necessary and sufficient conditions for the embedding of vector spaces of $(\mathcal{F}_t)$-semimartingales into spaces of $(\mathcal{G}_t)$-semimartingales to be continuous in terms of generalized entropies of the information increment.

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Introduction

On a probability space, let $(\mathcal{G}_t)$ be a filtration containing a smaller filtration $(\mathcal{F}_t)$. The basic question of the well known theory of enlargement of filtrations (see [JY85]) with some relevance in simple models of financial markets with asymmetric information (see for instance [Imk03]) is this: under which conditions does every $(\mathcal{F}_t)$-semimartingale remain a semimartingale relative to $(\mathcal{G}_t)$? In the pioneering papers of [JY85] this inheritance property has been called "Hypothèse (H′)." Jacod [Jac85] gives a sufficient criterion for it to hold and studies Doob–Meyer decompositions of semimartingales relative to $(\mathcal{G}_t)$. With respect to vector space topologies on the set of $(\mathcal{F}_t)$- and $(\mathcal{G}_t)$-semimartingales Yor [Yor85] investigates continuity properties of the associated mapping of $(\mathcal{F}_t)$-semimartingales into the space of $(\mathcal{G}_t)$-semimartingales.

In this paper we reconsider the problem of the inheritance of the semimartingale property from a different and more general perspective. In fact, in Section 1 we derive inheritance results generalizing Jacod's [Jac85], which were proved in the setting of initial enlargements by the information stored in random elements with values in Lusin spaces. Our proofs are based on the concept of the *decoupling measure*, which allows an independent view on the additional information contained in the enlarged filtration, specified in σ-fields $\mathcal{H}_t$ enlarging $\mathcal{F}_t$ to obtain $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. The key observation is that under the decoupling measure every $(\mathcal{F}_t)$-martingale is a $(\mathcal{G}_t)$-martingale. Hence, enlarging the filtration can be seen as stepping from a view of processes through the decoupling measure to a view by the original measure. In particular, the associated Girsanov transform can be used to obtain explicit representations of the Doob–Meyer decomposition w.r.t. the larger filtration. This idea goes back to [FI93], where this method was used to analyze initial enlargements of the Wiener filtration by some random variable G. Later [AIS98] and [GP98] extended these techniques to more general stochastic bases and semimartingales. In more recent approaches it was rediscovered in terms of a Bayesian interpretation of simple models of insider trading by Gasbarra and Valkeila [GV03]. Of course, the cost of this approach consists in the very assumption of the existence of the decoupling measure. It restricts generality to a nontrivial extent, as is seen if compared for example to the setting of [ADI06]. For instance, if the *information drift* to be deducted from a martingale in the larger filtration does not generate an equivalent martingale measure capturing the change of views from the small to the large filtration, then there will be no decoupling measure. In order to tackle the problem, as Yoeurp [Yoe85] for the analysis of progressive enlargements, we choose a formulation a product space: the first marginal contains the original information, while the second describes the additional information. Under the product measure both marginals are independent. Therefore it will be the appropriate candidate for our decoupling measure.

Here is an outline of the structure of the material presented. Our main occupation in Section 1 consists in showing how objects are transferred from the original space into the artificial product space and vice versa. Once this is handled, an application of the Girsanov transform leads to explicit Doob–Meyer decompositions. In Section 2 we provide estimates of the strength of the information drift by appropriate generalized entropies. These are used in Section 3 in order to prove continuity properties of the embedding of the $(\mathcal{F}_t)$-semimartingales into the set of $(\mathcal{G}_t)$-semimartingales with respect to well known vector space topologies. These results generalize continuity results obtained by Yor [Yor85].

1 Enlargement of filtrations and Girsanov's theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with right-continuous filtrations $(\mathcal{F}_t)_{t\geqslant 0}$ and $(\mathcal{H}_t)_{t\geqslant 0}$. Moreover, let $\mathcal{F}_\infty = \bigvee_{t\geqslant 0} \mathcal{F}_t$ and $\mathcal{H}_\infty = \bigvee_{t\geqslant 0} \mathcal{H}_t$.

Our objective is to study the enlarged filtration

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_s)\,, \qquad t \geqslant 0.$$

We relate this enlargement to a measure change on the product space

$$\bar{\Omega} = \Omega \times \Omega$$

equipped with the σ-field

$$\bar{\mathcal{F}} = \mathcal{F}_\infty \otimes \mathcal{H}_\infty\,.$$

We endow $\bar{\Omega}$ with the filtration

$$\bar{\mathcal{F}}_t = \bigcap_{s>t} (\mathcal{F}_s \otimes \mathcal{H}_s), \qquad t \geqslant 0\,.$$

Ω will be embedded into $\bar{\Omega}$ by the map

$$\psi : (\Omega, \mathcal{F}) \to (\bar{\Omega}, \bar{\mathcal{F}}),\ \omega \mapsto (\omega, \omega)\,.$$

We denote by $\bar{P}$ the image of the measure P under ψ, i.e.,

$$\bar{P} = P_\psi\,.$$

Hence for all $\bar{\mathcal{F}}$-measurable functions $f : \bar{\Omega} \to \mathbb{R}$ we have

$$\int f(\omega, \omega') d\bar{P}(\omega, \omega') = \int f(\omega, \omega) dP(\omega)\,. \tag{1}$$

In the following measure the two components in $\bar{\Omega}$ are decoupled, and weighted according to P:

$$\bar{Q} = P\big|_{\mathcal{F}_\infty} \otimes P\big|_{\mathcal{H}_\infty}\,.$$

We use notations and concepts of stochastic analysis as explained in the book by Protter [Pro04]. Our results will be stated for completed filtrations. We remark that due to our general *Assumption 1* below, all possible probability measures on the enlarged space we will consider possess systems of null sets that are at least bigger than the one related to $\bar{Q}$. So we could refer to the same completion throughout, and working with completions will not reduce generality. We shall use the following notation. Let $(\mathcal{K}_t)$ be a filtration and R a probability measure. We denote by $(\mathcal{K}_t^R)$ the filtration $(\mathcal{K}_t)$ completed by the R-negligible sets.

The map ψ will be used to translate processes $\bar{X} = (\bar{X}_t)_{t \geqslant 0}$ defined on $(\bar{\Omega}, \bar{\mathcal{F}}^{\bar{P}})$ into processes $X = \bar{X} \circ \psi$ defined on $(\Omega, \mathcal{F}^P)$. The following Proposition shows that structural properties are preserved by this embedding.

Proposition 1. *Let $\bar{X} = (\bar{X}_t)_{t \in [0,\infty)}$ and $\bar{Y}$ denote stochastic processes and $\bar{T}$ a random time all defined on the measurable space $(\bar{\Omega}, \bar{\mathcal{F}}^{\bar{P}})$. We set $X = \bar{X} \circ \psi$, $Y = \bar{Y} \circ \psi$ and $T = \bar{T} \circ \psi$. Then adaptedness, predictability, the local martingale or semimartingale properties are transferred from $\bar{X}$ with respect to $(\bar{\mathcal{F}}_t^{\bar{P}})$ and $\bar{P}$ to $\bar{Y}$ with respect to $(\mathcal{G}_t^P)$ and P. If $\bar{T}$ is a $(\bar{\mathcal{F}}_t^{\bar{P}})$-stopping time, then T is* *a* $(\mathcal{G}_t^P)$*-stopping time. Moreover, if* $\bar{X}$ *is a* $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$*-semimartingale and* $\bar{Y}$ *is a càglàd* $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$*-adapted process, then*

$$\left(\int \bar{Y} \, d\bar{X} \right) \circ \psi = \int Y \, dX$$

up to indistinguishability. If $\bar{X}$ *and* $\bar{Y}$ *are* $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$*-semimartingales, then*

$$[\bar{X}, \bar{Y}] \circ \psi = [X, Y]$$

up to indistinguishability.

Proof. Just observe that

$$\begin{aligned} \mathcal{G}_t &= \bigcap_{s>t} \sigma\left(A \cap B : A \in \mathcal{F}_s, B \in \mathcal{H}_s\right) = \bigcap_{s>t} \sigma\left(\psi^{-1}(A \times B) : A \in \mathcal{F}_s, B \in \mathcal{H}_s\right) \\ &= \psi^{-1}\left(\bigcap_{s>t} (\mathcal{F}_s \otimes \mathcal{H}_s) \right) = \psi^{-1}(\bar{\mathcal{F}}_t) \end{aligned}$$

so that

$$\psi^{-1}(\bar{\mathcal{F}}_t^{\bar{P}}) \subset \mathcal{G}_t^P . \tag{2}$$

Now the properties follow by straightforward arguments. □

In the reverse direction, structural properties are transferred quite as easily.

Lemma 1. *Let* M *be a right-continuous* $(\mathcal{F}_t^P, P)$*-local martingale. Then the process* $\bar{M}(\omega, \omega') = M(\omega)$ *is a* $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$*-local martingale.*

Proof. Modulo localization, the martingale property follows readily by employing a monotone class argument to pass from indicators of rectangles to general bounded measurable functions in the smaller σ-field. □

In the sequel we will always assume that $\bar{P}$ is absolutely continuous with respect to $\bar{Q}$, i.e.,

Assumption 1

$$\bar{P} \ll \bar{Q} \ \textit{on} \ \bar{\mathcal{F}} .$$

Note that this assumption is always satisfied if $(\mathcal{G}_t)$ is obtained by an initial enlargement by some discrete random variable G, i.e., $\mathcal{H}_t = \sigma(G)$ for all $t \geqslant 0$. In particular, this holds true for any progressive enlargement by $\mathcal{H}_t = \bigvee_{u \leqslant t} \{L \leqslant u\}$ where L is a discrete random time with values in $[0, \infty]$ (see also the end of the section).

Now let M be a $(\mathcal{F}_t^P, P)$-local martingale and $\bar{M}$ its extension to $\bar{\Omega}$ as in Lemma 1. Since $\bar{P} \ll \bar{Q}$, $\bar{M}$ is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-semimartingale and hence, by Proposition 1, M is a $(\mathcal{G}_t^P, P)$-semimartingale. Thus, clearly hypothesis (H') is satisfied. But what is its Doob–Meyer decomposition relative to $(\mathcal{G}_t^P, P)$?

Essentially the change of filtrations corresponds to changing the measure from $\bar{Q}$ to $\bar{P}$ on the product space $\bar{\Omega}$. Girsanov's theorem applies on $\bar{\Omega}$, since the measure $\bar{P}$ is absolutely continuous with respect to $\bar{Q}$. As a consequence we obtain a Girsanov-type result for the corresponding change of filtrations. For its explicit description we introduce the density process. Let $(\bar{Z}_t)$ denote a cadlag $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-martingale satisfying

$$\bar{Z}_t = \left.\frac{d\bar{P}}{d\bar{Q}}\right|_{\bar{\mathcal{F}}_t^{\bar{Q}}}.$$

We are now in a position to state the main Girsanov-type result.

Theorem 1. *If M is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$, then*

$$M - \frac{1}{Z_-} \cdot [M, Z]$$

is a $(\mathcal{G}_t^P, P)$-local martingale.

Proof. Let M be a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$. Lemma 1 implies that the process defined by $\bar{M}(\omega, \omega') = M(\omega)$ is a $(\bar{\mathcal{F}}_t^{\bar{Q}})$-local martingale and the Girsanov Theorem (see for instance [Pro04], page 136) yields that

$$\bar{M} - \frac{1}{\bar{Z}_-} \cdot [\bar{M}, \bar{Z}]$$

is a $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-local martingale. It remains to appeal to simple transfer properties stated in Proposition 1. □

Remark 1. Similar results as in Theorem 1 may of course be derived for *non-continuous* martingales M (see [Ank05] for details).

The preceding may also be formulated in terms of the stochastic logarithm of the density process $\bar{Z}$. To this end set $\bar{S}' = \inf\{t > 0 : \bar{Z}_t = 0\}$ and

$$\bar{S} = \begin{cases} \bar{S}' & \text{if } \bar{S}' < \infty \text{ and } \Delta \bar{Z}_{\bar{S}'} = 0 \\ \infty & \text{otherwise}. \end{cases}$$

$\bar{S}$ is a $(\bar{\mathcal{F}}_t^{\bar{Q}})$-predictable stopping time and we define

$$\bar{L} = \int_{0+}^{\cdot} \frac{1}{\bar{Z}_-} \, d\bar{Z} \quad \text{on} \quad [0, \bar{S}[\tag{3}$$

with the convention that $\bar{L}_t = \bar{L}_{\bar{S}'}$ for $t \in [\bar{S}', \bar{S}[$. So far, the process $\bar{L}$ is determined $\bar{P}$-, but *not* $\bar{Q}$-almost everywhere. (In order to define it everywhere we may put $\bar{L} = 0$ on $[\bar{S}, \infty[$.) Then $\bar{L}$ is an $(\bar{\mathcal{F}}_t^{\bar{P}}, \bar{P})$-semimartingale but not necessarily an $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-semimartingale. However, restricted to the time interval $[0, \bar{S}[$ it is an $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale. As usual we write $L = \bar{L} \circ \psi$. Alternatively, one can define L through the stochastic integral $L = \int_{0+}^{\cdot} \frac{1}{Z_-}\, dZ$.

Since the process $\bar{L}$ is a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale on the interval $[0, \bar{S}[$, it can be decomposed into a unique continuous local-martingale part $\bar{L}^c$ starting in 0 and a sum of compensated jumps $\bar{L}^d$. As before, we consider the processes $L^c = \bar{L}^c \circ \psi$ and $L^d = \bar{L}^d \circ \psi$. Theorem 1 can now be reformulated as follows.

Theorem 2. *If M is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$, then $M - [M, L]$ is a $(\mathcal{G}_t^P, P)$-local martingale.*

Proof. The definition of L implies that $\frac{1}{Z_-} \cdot [M, Z] = [M, L]$, P-a.s. Now apply Theorem 1. □

Finally, we will need the following formula, in which the subtracted drift is represented in terms of the quadration variation of the given local martingale.

Theorem 3. *If M is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$, then there is a $(\mathcal{G}_t^P)$-predictable process α, called* information drift*, such that $M - \alpha \cdot [M, M]$ is a $(\mathcal{G}_t^P)$-local martingale satisfying P-a.e.*

$$\int_0^\infty \alpha_t^2 \; d[M, M]_t \leqslant [L, L]_\infty^c < \infty.$$

Proof. Let M be a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$. As a consequence of the Kunita-Watanabe inequality (see for instance Lemme 1.36 in [Jac79] or page 136 of [Pro04]), there exists a $(\mathcal{G}_t^P)$-predictable process (α_t) such that

$$\alpha \cdot [M, M] = [M, L] = [M, L^c]\,.$$

The processes M and $O = L^c - \alpha \cdot M$ are orthogonal w.r.t. $[\cdot, \cdot]$ so that

$$\alpha^2 \cdot [M, M] = [\alpha \cdot M, \alpha \cdot M] \leqslant [L^c, L^c] = [L, L]^c\,.$$

Recall that $[L, L] = \left(\frac{1}{\bar{Z}_-^2} \cdot [\bar{Z}, \bar{Z}]\right) \circ \psi$ and that $\bar{Z}$ is a uniformly integrable non-negative $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-martingale. Since $\bar{P}$-a.s. $\bar{Z}_\infty > 0$, one has also $\inf_{t \geqslant 0} \bar{Z}_t > 0$, $\bar{P}$-a.s. Moreover, $[\bar{Z}, \bar{Z}]_\infty < \infty$, $\bar{Q}$-a.s. Therefore, $[\bar{L}, \bar{L}]$ is $\bar{P}$-a.s. bounded and consequently $[L, L]_t^c$ converges as $t \to \infty$ P-a.s. to some real value which we denote by $[L, L]_\infty^c$. □

Remark 2. Due to the previous theorem the information drift obtained via the Girsanov approach under Assumption 1 is always locally square integrable. It was shown in [ADI06] that in case Ω is standard Borel and each $(\mathcal{F}_t)$-martingale has a continuous modification, square integrability on the product space $\Omega \times [0,T]$ with respect to the measure $d[M,M] \otimes P$ implies the absolute continuity of the kernels $k_t(\cdot, d\omega')$ with respect to the conditional laws $P_t(\cdot, d\omega')$ of $\mathcal{G}_{t-}$ with respect to $\mathcal{F}_t$, where

$$P_t(\cdot, A) = P(A) + \int_0^t k_s(\cdot, A) dM_s + L_t^A, \quad A \in \mathcal{G}_{t-}, t \in [0,T],$$

L^A being orthogonal to M. In this case the R–N density process

$$\gamma_t(\omega, \omega') = \frac{k_t(\cdot, d\omega')}{P_t(\cdot, d\omega')}$$

is identical to α if restricted to the diagonal $\omega = \omega'$. Hence this absolute continuity condition (ACL) is implied by Assumption 1. Enlargements with locally integrable but not square integrable information drifts are beyond the scope of this article. But they provide examples for which (ACL) does not imply Assumption 1. One example is obtained for instance by enlarging the Wiener filtration by the maximum of the Wiener process over some finite time interval. In this case Malliavin's calculus can be applied and an explicit representation of the information drift is obtained via the Clark-Ocone formula (see [IPW01] and [Imk03]). In case Ω is not standard Borel we do not know at the moment whether Assumption 1 is more restrictive than (ACL). The methods of [ADI06] allow in a more general setting the description of information drifts which are not necessarily locally square integrable.

Comparison with Jacod's condition

In Jacod's paper (see [Jac85]) the filtration $(\mathcal{F}_t)$ is supposed to be enlarged by some random variable G taking values in a Lusin space $(E, \mathcal{E})$. As a consequence, for $t \in [0,T]$ regular conditional distributions Q_t of G relative to $\mathcal{F}_t$ exist. The following condition is assumed to be satisfied:

(A′) For every $t \geqslant 0$ and P-a.a. ω the measure $Q_t(\omega, \cdot)$ is absolutely continuous with respect to the law η of G.

We will show that in this setting condition (A') is equivalent to our Assumption 1. More precisely, with $\mathcal{H}_t = \sigma(G)$, we have the following.

Lemma 2. *(A′) is satisfied if and only if* $\bar{P} \ll \bar{Q}$ *on* $\bar{\mathcal{F}}_t$ *for all* $t \geqslant 0$.

Proof. First assume property (A′). Let $t \geqslant 0$ and $C \in \bar{\mathcal{F}}_t$ with $\bar{Q}(C) = 0$. We choose $\tilde{C} \in \mathcal{F}_t \otimes \mathcal{E}$ such that

$$1_C(\omega, \omega') = 1_{\tilde{C}}(\omega, G(\omega')),$$

and observe that

$$\bar{Q}(C) = \int_{\bar{\Omega}} 1_C(\omega, \omega')\, d\bar{Q}(\omega, \omega') = \int_{\Omega} \left(\int_E 1_{\tilde{C}}(\omega, g)\, d\eta(g) \right) dP(\omega)\,.$$

Hence for P-a.a. ω the set $C_\omega = \{g \in E : (\omega, g) \in \tilde{C}\}$ is a η-nullset. Consequently,

$$\bar{P}(C) = \int_{\Omega} 1_C(\omega, \omega)\, dP(\omega) = \int Q_t(\omega, C_\omega)\, dP(\omega)$$

is equal to 0 due to (A′).

Now fix $t \geqslant 0$ and assume that $\bar{P} \ll \bar{Q}$ on $\bar{\mathcal{F}}_s$. Then there exists a $\bar{\mathcal{F}}_t$-measurable density φ which can be represented in the form

$$\varphi(\omega, \tilde{\omega}) = \tilde{\varphi}(\omega, G(\tilde{\omega}))$$

where $\tilde{\varphi}$ is an appropriate $\mathcal{F}_t \otimes \mathcal{E}$-measurable function. Now integrating in $\tilde{\omega}$ will, by using Fubini's theorem in a similar manner as above, yield the conditional law of G relative to $\mathcal{F}_t$ which is absolutely continuous with respect to η. This entails property (A′). □

Jacod does not use Girsanov's theorem in his paper [Jac85]. However, he points out that his results could also be deduced by applying it to the conditional measures $P^x = P(\cdot|G = x)$, $x \in E$. Condition (A′) implies that the conditional measures P^x are absolutely continuous with respect to P. Hence, by Girsanov, for a given $(\mathcal{F}_t, P)$-local martingale there is a drift A^x such that $M - A^x$ is a $(\mathcal{F}_t, P^x)$-local martingale. By combining the processes A^x we obtain that

$$M - A^G$$

is a $(\mathcal{G}_t, P)$-local martingale. The main work consists in proving that the processes A^x can be combined in a meaningful way. As far as we know, Jacod's sketch has never been worked out rigorously.

In our approach we embed every local martingale into the product space $\bar{\Omega}$. We apply Girsanov's theorem on the product space and then translate our results back into the original space. One of the advantages of our approach is that we do not have to assume regular conditional distributions to exist. And we do not need to show how processes can be combined. Instead we have to show how one can transfer objects from Ω to $\bar{\Omega}$ and vice versa. Moreover we are not restricted to initial enlargements, but only to enlargements of the form

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{H}_s), \quad t \in [0, T]\,.$$

Starting with Jacod's results one can obtain decompositions for filtrations of this kind by using predictable projections. For this suppose A to be a bounded variation process such that $M - A$ is a local martingale with respect to the initially enlarged filtration $(\mathcal{F}_t \vee \mathcal{H}_\infty)$. If B is the predictable projection of A onto $(\mathcal{G}_t)$, then $M - B$ is a $(\mathcal{G}_t)$-local martingale.

2 Estimates for the drift

Suppose M is a continuous $(\mathcal{F}_t^P, P)$-local martingale with $M_0 = 0$. Under the assumptions of the previous section we know that there is a $(\mathcal{G}_t^P)$-predictable process α such that $M - \alpha \cdot [M, M]$ is $(\mathcal{G}_t^P, P)$-local martingale. Moreover, the information drift α satisfies

$$(\alpha^2 \cdot [M, M])_\infty \leqslant [L, L]^c_\infty. \tag{4}$$

In this section we provide bounds for

$$E\left[(\alpha^2 \cdot [M, M])^p_\infty\right]$$

for various moments $p \geqslant 1$ based on inequality (4).

Throughout this section we suppose the assumptions of the previous section and maintain the notation. More precisely, we assume that $\bar{P} \ll \bar{Q}$, denote by $\bar{Z}_t = \frac{d\bar{P}}{d\bar{Q}}\Big|_{\bar{\mathcal{F}}_t^{\bar{Q}}}$ the density process, and by $\bar{L}$ the stochastic logarithm of $\bar{Z}$. We use again the decomposition of $\bar{L}$ into a continuous part $\bar{L}^c$ and a part $\bar{L}^d$ consisting of compensated jumps. As earlier we denote by Z, L, and L^c the corresponding $(\mathcal{G}_t)$-adapted processes obtained by a right side application of ψ.

2.1 Moment $p = 1$

Recall that the relative entropy of two probability measures P and Q on some σ-algebra $\mathcal{M}$ is defined by

$$H_{\mathcal{M}}(P\|Q) = \begin{cases} E^P\left(\log \frac{dP}{dQ}\Big|_{\mathcal{M}}\right), & \text{if } P \ll Q \text{ on } \mathcal{M} \\ \infty, & \text{if not } P \ll Q \text{ on } \mathcal{M}\,. \end{cases}$$

In our situation, the relative entropy $H_{\bar{\mathcal{F}}}(\bar{P}\|\bar{Q})$ provides an upper bound for the first moment of $[L, L]^c$:

Lemma 3.

$$\frac{1}{2} E^P [L, L]^c_\infty \leqslant H_{\bar{\mathcal{F}}}(\bar{P}\|\bar{Q})\,.$$

If $(\bar{Z}_t)_{t \geqslant 0}$ is continuous and $\bar{Z}_0 = 1$, then one even has

$$\frac{1}{2} E^P [L, L]_\infty = H_{\bar{\mathcal{F}}}(\bar{P}\|\bar{Q})\,.$$

Remark 3. If the σ-field $\mathcal{F}_0$ is trivial, then the measures $\bar{P}$ and $\bar{Q}$ coincide on $\mathcal{F}_0 \otimes \mathcal{H}_0$, and hence in this case $\bar{Z}_0 = 1$.

Proof. Let $(\bar{T}_n)$ denote an increasing sequence of stopping times with $\lim_{n\to\infty} \bar{T}_n \geqslant \bar{S}'$. Since $(\bar{Z}_t)$ is a uniformly integrable $(\bar{\mathcal{F}}_t^{\bar{Q}})$-martingale Jensen's inequality implies that

$$E^{\bar{Q}} \bar{Z}_{\bar{T}_n} \log \bar{Z}_{\bar{T}_n} \leqslant E^{\bar{Q}} \bar{Z}_\infty \log \bar{Z}_\infty$$

so that Fatou's lemma leads to

$$\lim_{n\to\infty} E^{\bar{Q}} \bar{Z}_{\bar{T}_n} \log \bar{Z}_{\bar{T}_n} = E^{\bar{Q}} \bar{Z}_\infty \log \bar{Z}_\infty = H_{\bar{\mathcal{F}}}(\bar{P}\|\bar{Q})\,. \tag{5}$$

On $[0, \bar{S}[$ we decompose $\bar{L}$ into its continuous and discontinuous part $\bar{L} = \bar{L}^c + \bar{L}^d$ and let $\bar{Z}_t^c = \mathcal{E}(\bar{L}^c)_t$ and $\bar{Z}_t^d = \bar{Z}_0\,\mathcal{E}(\bar{L}^d)_t$. Then $\bar{Z}_t = \bar{Z}_t^c\,\bar{Z}_t^d$ and

$$\bar{Z}_t \log \bar{Z}_t = \bar{Z}_t \log \bar{Z}_t^c + \bar{Z}_t \log \bar{Z}_t^d\,.$$

Now Itô's formula implies that $\bar{A}_t = \bar{Z}_t \log \bar{Z}_t^d$ is a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local submartingale on $[0, \bar{S}[$. In fact, with $\xi(x) = x \log x$ for $x > 0$ and $\xi(0) = 0$ one obtains on $[0, \bar{S}[$

$$\begin{aligned} \bar{A}_t = \xi(\bar{Z}_0) + \int_{0+}^t \xi(\bar{Z}_{s-}^d)\, d\bar{Z}_s^c + \int_{0+}^t \bar{Z}_{s-}^c \xi'(\bar{Z}_{s-}^d)\, d\bar{Z}_s^d \\ + \sum_{0<s\leqslant t} \bar{Z}_{s-}^c \big(\xi(\bar{Z}_s^d) - \xi(\bar{Z}_{s-}^d) - \xi'(\bar{Z}_{s-}^d)\, \Delta\bar{Z}_s^d\big), \end{aligned}$$

where all summands in the previous line are non-negative due to the convexity of ξ.

Next, note that due to the Girsanov transform

$$\bar{L}^c - [\bar{L}, \bar{L}]^c$$

is a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{P})$-local martingale. Now choose an increasing sequence of bounded stopping times $(\bar{T}_n)_{n\in\mathbb{N}}$ such that $(\bar{L}^c - [\bar{L}, \bar{L}]^c)^{\bar{T}_n}$ is a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{P})$-martingale, $\bar{A}^{\bar{T}_n}$ is a $\bar{Q}$-submartingale and $\lim_{n\to\infty} \bar{T}_n \geqslant \bar{S}'$. Such a sequence exists, and combining the above results gives

$$\begin{aligned} E^{\bar{Q}} \bar{Z}_{\bar{T}_n} \log \bar{Z}_{\bar{T}_n} &\geqslant E^{\bar{Q}} \bar{Z}_{\bar{T}_n} \log \bar{Z}_{\bar{T}_n}^c + E^{\bar{Q}} \bar{Z}_0 \log \bar{Z}_0 \\ &\geqslant E^{\bar{P}} \log \bar{Z}_{\bar{T}_n}^c = \frac{1}{2} E^{\bar{P}} [\bar{L}, \bar{L}]_{\bar{T}_n}^c\,. \end{aligned}$$

The first assertion follows by (5).

If $\bar{Z}$ is continuous and $\bar{Z}_0 = 1$, then $\bar{Z}_t = \bar{Z}_t^c$ which implies that

$$E^{\bar{Q}} \bar{Z}_{\bar{T}_n} \log \bar{Z}_{\bar{T}_n} = \frac{1}{2} E^{\bar{P}} [\bar{L}, \bar{L}]_{\bar{T}_n}^c\,.$$

The second assertion is an immediate consequence of (5). □

2.2 Moments $p > 1$

Now we consider moments of order $p > 1$. In this case the pth moment of $[L, L]_\infty$ can be compared to some generalized relative entropy. See [Imk96] for elementary versions of the inequalities to be derived.

Our analysis requires some additional assumption. We suppose that $(\mathcal{G}_t)$ is an initial enlargement of $(\mathcal{F}_t)$, i.e.

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{A}), \qquad t \geqslant 0,$$

where $\mathcal{A}$ is some fixed sub-σ-algebra of $\mathcal{F}$. Moreover, we assume that $\mathcal{F}_0$ is trivial. As in [Yor85], we need to impose the following additional assumption.

Assumption 2 (C) *Every $(\mathcal{F}_t^P, P)$-martingale has a continuous modification.*

We shall see that under this condition $\bar{L}$ is a continuous $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-local martingale. We begin with the definition of the generalized relative entropy.

Definition 1. *For $p > 1$, and probability measures $P \ll Q$ on a σ-algebra $\mathcal{M}$, let*

$$H^p_{\mathcal{M}}(P\|Q) := E^P \left(\log_+ \left. \frac{dP}{dQ} \right|_{\mathcal{M}} \right)^p .$$

We provide now an upper bound of $E[L, L]^p_\infty$ with the help of the generalized entropy of $\bar{P}$ with respect to $\bar{Q}$ on the set $\bar{\mathcal{F}}_\infty$. To simplify notations, we omit the σ-algebra $\bar{\mathcal{F}}_\infty$, and write only $H^p(\bar{P}\|\bar{Q})$ and $H(\bar{P}\|\bar{Q})$. The aim of this section is to prove

Theorem 4. *For any $p \geqslant 1$ there exists a universal constant $C = C(p) < \infty$ such that under the above assumptions one has*

$$E[L, L]^p_\infty \leqslant C \left[H(\bar{P}\|\bar{Q}) + H^p(\bar{P}\|\bar{Q}) \right] .$$

For the proof we need some auxiliary results. We start by showing that there exists a continuous modification for $\bar{Z}$.

Lemma 4. *Let $\bar{M}$ be a uniformly integrable $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-martingale. If Assumption (C) is satisfied, then for P-a.a. ω' the process $\bar{M}^{\omega'} = \bar{M}(\cdot, \omega')$ is a $(\mathcal{F}_t^P)$-local martingale.*

Proof. Choose a modification such that every path of $\bar{M}$ is cadlag. Now let $\hat{M}$ be an $\mathcal{A} \otimes \mathcal{O}(\mathcal{F})$-measurable process such that for all ω' and $s \geqslant 0$

$$\hat{M}^{\omega'}_s = E^P[\bar{M}^{\omega'}_\infty | \mathcal{F}_s] .$$

For the existence of such a process we refer to [SY78], Proposition 3. Put $C_t = \{\hat{M}_t > \bar{M}_t\}$. Clearly $C_t \in \bar{\mathcal{F}}_t^{\bar{Q}}$ and $C_t(\cdot, \omega') \in \mathcal{F}_t^P$ for all P-a.a. ω' (recall that $(\mathcal{F}_t)$ is right-continuous). Moreover for $t \geqslant 0$

$$\begin{aligned}
\int\int 1_{C_t}(\omega,\omega')(\hat{M}_t^{\omega'} - \bar{M}_t^{\omega'})\ dP(\omega)\ dP(\omega')& \\
&= E^{\bar{Q}}[1_{C_t}(\hat{M}_t - \bar{M}_t)] \\
&= E^{\bar{Q}}[1_{C_t}(\hat{M}_t - \bar{M}_\infty)] \\
&= \int\int 1_{C_t}(\omega,\omega')(\hat{M}_t^{\omega'} - \bar{M}_\infty^{\omega'})\ dP(\omega)\ dP(\omega') \\
&= \int 0\ dP(\omega') = 0,
\end{aligned}$$

A similar result holds true on the set $\{\hat{M}_t < \bar{M}_t\}$, and as a consequence we have for P-a.a. ω'

$$\hat{M}_t(\cdot,\omega') = \bar{M}_t(\cdot,\omega'),\ P\text{-a.s.}$$

Hence for P-a.a. ω' the process $(\bar{M}_q^{\omega'})_{q\in\mathbb{Q}^+}$ is a $(\mathcal{F}_t^P)$-martingale. Since $\bar{M}_t$ is cadlag and uniformly integrable we obtain that also

$$(\bar{M}_t^{\omega'})_{t\geqslant 0}$$

is a $(\mathcal{F}_t^P)$-martingale for P-a.a. ω'. □

Lemma 5. *If (C) is satisfied, then every uniformly integrable $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-martingale has a continuous modification.*

Proof. Let $\bar{M}$ be a $(\bar{\mathcal{F}}_t^{\bar{Q}}, \bar{Q})$-martingale. We may suppose that $\bar{M}$ is cadlag everywhere, and hence, the set

$$N = \{(\omega,\omega') : t \mapsto \bar{M}_t(\omega,\omega') \text{ is not continuous}\}$$

is measurable. Fix ω' and suppose that $\bar{M}^{\omega'}$ is a $(\mathcal{F}_t^P)$-martingale. Then Assumption (C) implies that for P-a.a. ω the paths $t \mapsto \bar{M}_t^{\omega'}(\omega)$ are continuous, i.e., $P(N^{\omega'}) = 0$. Now Fubini's theorem yields with Lemma 4

$$\begin{aligned}
E^{\bar{Q}}(N) &= \int\int 1_{N^{\omega'}}(\omega)\ dP(\omega)\ dP(\omega') \\
&= \int 0\ dP(\omega') = 0,
\end{aligned}$$

and hence the result. □

For the rest of the section we will suppose that $\bar{Z}$ is a continuous modification of our density process $\frac{d\bar{P}}{d\bar{Q}}\Big|_{\bar{\mathcal{F}}_t^{\bar{Q}}}$. Similarly, $\bar{L}$ will be assumed to be continuous.

Proof (of Theorem 4). We assume that $H(\bar{P}\|\bar{Q})$ and $H^p(\bar{P}\|\bar{Q})$ are finite. Then $X_t := \bar{L}_t - [\bar{L},\bar{L}]_t$ is a continuous L^2-bounded $\bar{P}$-martingale by Lemma 3 and we write $\log \bar{Z}_t = X_t + \frac{1}{2}A_t$ with $A_t := [\bar{L},\bar{L}]_t = [X,X]_t$. Next, observe that

$$\begin{aligned} H^p(\bar P\|\bar Q)^{1/p} &= E^{\bar P}\big[(X_\infty + \frac{1}{2}A_\infty)_+^p\big]^{1/p} \\ &\geqslant E^{\bar P}\Big[\Big(\frac{1}{2}A_\infty - (|X_\infty| \wedge \frac{1}{2}A_\infty)\Big)^p\Big]^{1/p} \\ &\geqslant \frac{1}{2}E^{\bar P}\big[A_\infty^p\big]^{1/p} - E^{\bar P}\big[|X_\infty|^p\big]^{1/p} \\ &\geqslant \frac{1}{2}E^{\bar P}[A_\infty^p]^{1/p} - C\,E^{\bar P}\big[A_\infty^{p/2}\big]^{1/p}, \end{aligned} \tag{6}$$

where the last inequality holds for some constant $C > 0$ due to the Burkholder–Davis–Gundy inequality. Now choose $\xi > 0$ such that for all $x \geqslant 0$

$$C^p x^{p/2} \leqslant \xi^p x + \frac{1}{4^p}x^p\,.$$

This leads to

$$C^p E^{\bar P} A_\infty^{p/2} \leqslant \xi^p E^{\bar P} A_\infty + \frac{1}{4^p}E^{\bar P} A_\infty^p$$

and hence to

$$C\,E^{\bar P}\big[A_\infty^{p/2}\big]^{1/p} \leqslant \xi E^{\bar P}\big[A_\infty\big]^{1/p} + \frac{1}{4}E^{\bar P}\big[A_\infty^p\big]^{1/p}.$$

With (6) we conclude that

$$H^p(\bar P\|\bar Q)^{1/p} \geqslant \frac{1}{4}E^{\bar P}\big[A_\infty^p\big]^{1/p} - \xi E^{\bar P}\big[A_\infty\big]^{1/p} = \frac{1}{4}E^{\bar P}\big[A_\infty^p\big]^{1/p} - \xi H(\bar P\|\bar Q)^{1/p}.$$

Consequently,

$$\begin{aligned} E^{\bar P}\big[A_\infty^p\big]^{1/p} &\leqslant 4\xi H(\bar P\|\bar Q)^{1/p} + 4H^p(\bar P\|\bar Q)^{1/p} \\ &\leqslant 8(\xi^p H(\bar P\|\bar Q) + H^p(\bar P\|\bar Q))^{1/p}\,, \end{aligned}$$

where the last step follows from the elementary inequality $a+b \leqslant 2(a^p+b^p)^{1/p}$, $a, b \geqslant 0$. □

Remark 4. The above proof is based on the fact that there exists a constant C_p such that for any continuous L^2-bounded $\bar P$-martingale (X_t) with $X_0 = 0$ and quadratic variation process (A_t) one has

$$E^{\bar P} A_\infty^p \leqslant C_p\, E^{\bar P}\Big[X_\infty + \frac{1}{2}A_\infty + \big(X_\infty + \frac{1}{2}A_\infty\big)_+^p\Big]\,.$$

Improving the estimate to

$$E^{\bar P} A_\infty^p \leqslant C_p\, E^{\bar P}\big(X_\infty + \frac{1}{2}A_\infty\big)_+^p \tag{7}$$

would lead to the better estimate $E^P[L, L]^p \leqslant C_p\, H^p(\bar P\|\bar Q)$. However, an estimate stating (7) is not valid, as the following example shows.

Example 1. Let W be a Wiener process and for fixed $\varepsilon > 0$, let T denote the first hitting time of the slope $t \mapsto \varepsilon - t/2$. We consider $X_t := W_t^T$ and $A_t := [X, X]_t$. Then by the Lévy-Bachelier formula the law of $T = A_\infty$ has density

$$1_{(0,\infty)}(t)\, \frac{\varepsilon}{t^{3/2}}\, \phi\left(\frac{\varepsilon - t/2}{\sqrt{t}}\right),$$

where ϕ is the density of the standard normal law. Hence,

$$E[A_\infty^p] = \varepsilon \int_0^\infty t^{p-3/2}\, \phi\left(\frac{\varepsilon - t/2}{\sqrt{t}}\right)\, dt\,.$$

In particular, for $\varepsilon \downarrow 0$, one has $E[A_\infty^p] \approx \varepsilon$. On the other hand,

$$E\Big[\big(X_\infty + \frac{1}{2} A_\infty\big)_+^p\Big] = E[(W_T + T/2)^p] = \varepsilon^p$$

such that one can always find a sufficiently small $\varepsilon > 0$ for which the inequality (7) is not valid.

We next show a result which in a sense contains the inverse statement to Theorem 4.

Lemma 6. *For $p \geqslant 1$ there exists a universal constant $C = C(p) < \infty$ such that*

$$H^p(\bar{P} \| \bar{Q}) \leqslant C\left[E^{\bar{P}}[\bar{L}, \bar{L}]_\infty^p + 1\right].$$

In particular finiteness of $E^{\bar{P}}\left([\bar{L}, \bar{L}]_\infty^p\right)$ implies finiteness of the entropy $H^p(\bar{P} \| \bar{Q})$.

Proof. We have, by Burkholder–Davis–Gundy, with a universal constant C_1

$$\begin{aligned}
H^p(\bar{P} \| \bar{Q})^{1/p} &\leqslant E\left(|\bar{L}_\infty - \frac{1}{2}[\bar{L}, \bar{L}]_\infty|^p\right)^{1/p} \\
&\leqslant E\left(|\bar{L}_\infty|^p\right)^{1/p} + E\left(\frac{1}{2}[\bar{L}, \bar{L}]_\infty^p\right)^{1/p} \\
&\leqslant C_1 E\left([\bar{L}, \bar{L}]_\infty^{p/2}\right)^{1/p} + E\left(\frac{1}{2}[\bar{L}, \bar{L}]_\infty^p\right)^{1/p} \\
&\leqslant C_1\left(1 + E[\bar{L}, \bar{L}]_\infty^p\right)^{1/p} + E\left(\frac{1}{2}[\bar{L}, \bar{L}]_\infty^p\right)^{1/p} \\
&\leqslant C_2\left(1 + E\left([\bar{L}, \bar{L}]_\infty^p\right)\right)^{1/p},
\end{aligned}$$

and thus the result. □

Suppose now that the enlargement $\mathcal{A}$ is induced by some discrete random variable G, i.e., $\mathcal{A} = \sigma(G)$. In that case one can estimate the moments of $[L, L]_\infty$ against some generalized *absolute* entropy of G.

Definition 2. *Let (q_g) denote the probability weights of G. We denote by*

$$H^p(G) = \sum_g q_g (\log 1/q_g)^p$$

the generalized absolute entropy of order p.

Lemma 7. *One has*

$$H^p(\bar{P}\|\bar{Q}) \leqslant H^p(G)\,,$$

and if G is $\mathcal{F}_\infty$-measurable, then

$$H^p(\bar{P}\|\bar{Q}) = H^p(G)\,.$$

Proof. For the proof we need a monotonicity property of f-divergences. Due to Corollary 1.29 in [LV87] one has

$$\begin{aligned} H^p(\bar{P}\|\bar{Q}) &= H^p(P_{\mathrm{id}_{\mathcal{F}_\infty},\mathrm{id}_{\mathcal{A}}}\|P_{\mathrm{id}_{\mathcal{F}_\infty}} \otimes P_{\mathrm{id}_{\mathcal{A}}}) \\ &\leqslant H^p(P_{\mathrm{id}_{\mathcal{F}_\infty},G,\mathrm{id}_{\mathcal{A}}}\|P_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_{\mathcal{A}}})\,. \end{aligned}$$

Moreover, if G is $\mathcal{F}_\infty$-measurable, then one even has equality in the previous line. We denote by (q_g) the probability weights of G. One easily verifies that

$$\frac{dP_{\mathrm{id}_{\mathcal{F}_\infty},G,\mathrm{id}_{\mathcal{A}}}}{dP_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_{\mathcal{A}}}}(\omega, g, \omega') = 1_{\{g=G(\omega')\}}\,\frac{1}{q_g}\,.$$

Set $f(g,g') = 1_{\{g=g'\}}\,\frac{1}{q_g}$. Then

$$\begin{aligned} &H^p(P_{\mathrm{id}_{\mathcal{F}_\infty},G,\mathrm{id}_{\mathcal{A}}}\|P_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_{\mathcal{A}}}) \\ &\quad = \int f(g,G(\omega'))\,(\log_+ f(g,G(\omega')))^p\, d(P_{\mathrm{id}_{\mathcal{F}_\infty},G} \otimes P_{\mathrm{id}_{\mathcal{A}}})(\omega,g,\omega') \\ &\quad = \int_{\{(g,\omega'):g=G(\omega')\}} \frac{1}{q_g}\left(\log_+ \frac{1}{q_g}\right)^p d(P_G \otimes P_{\mathrm{id}_{\mathcal{A}}})(g,\omega')\,, \end{aligned}$$

since $f(g,G(\omega')) = 0$ if $g \neq G(\omega')$ and the integrand does not depend on ω. Altogether, we arrive at

$$H^p(\bar{P}\|\bar{Q}) \leqslant \sum_g q_g \left(\log \frac{1}{q_g}\right)^p = H^p(G)$$

and equality holds if G is $\mathcal{F}_\infty$-measurable. □

Example 2. Let $M_t = W_t$ denote a Wiener process and consider the completed filtration $(\mathcal{F}_t) = (\mathcal{F}_t^W)$ generated by the Wiener process. We now consider an initial enlargement of the filtration $(\mathcal{F}_t)$ by some arbitrary σ-field $\mathcal{A}$, i.e., $\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \wedge \mathcal{A})$. Supposing that $\bar{P} \ll \bar{Q}$, the Doob–Meyer decomposition for W with respect to $(\mathcal{G}_t)$ is of the form

$$W_t = \tilde{W}_t + \int_0^t \alpha_s \, ds \,,$$

where $\tilde{W}$ is a $(\mathcal{G}_t)$-Wiener process and α is a $(\mathcal{G}_t)$-adapted process. In fact, $\tilde{W}$ is continuous with quadratic variation process $[\tilde{W}, \tilde{W}]_t = t$. Moreover, since $\mathcal{F}_0$ is trivial and all $(\mathcal{F}_t)$-martingales have continuous modifications, the results of this section lead to the estimate

$$E\left(\int_0^t \alpha_s^2 \, ds\right)^p \leqslant C_p \left[H(\bar{P}\|\bar{Q}) + H^p(\bar{P}\|\bar{Q})\right] \,.$$

If in addition $\mathcal{A} = \sigma(G)$ is generated by some discrete random variable G, then

$$E\left(\int_0^t \alpha_s^2 \, ds\right)^p \leqslant C_p \left[H(G) + H^p(G)\right] \,.$$

3 Continuity of initial enlargements

In Section 1 we have seen that every $(\mathcal{F}_t^P)$-semimartingale is also a semimartingale relative to a bigger filtration $(\mathcal{G}_t^P)$ if the measure $\bar{P}$ is absolutely continuous with respect to $\bar{Q}$. In this section we analyze to which extent this embedding of $(\mathcal{F}_t^P)$-semimartingales into some space of $(\mathcal{G}_t^P)$-semimartingales is continuous. For simplicity we restrict to initial enlargements. It turns out that the embedding is continuous if and only if some generalized entropy of the measures $\bar{P}$ and $\bar{Q}$ is finite.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space as in the previous section. Throughout this section we assume that $\mathcal{F}_0$ is trivial and we let

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s \vee \mathcal{A}) \,, \quad t \geqslant 0 \,,$$

where $\mathcal{A}$ is some fixed sub-σ-algebra of $\mathcal{F}$. The measures $\bar{P}$ and $\bar{Q}$ are defined as in the previous section and we assume again that $\bar{P}$ is absolutely continuous with respect to $\bar{Q}$. As before we will abbreviate $\bar{Z}_t = \frac{d\bar{P}}{d\bar{Q}}\big|_{\bar{\mathcal{F}}_t}, \quad t \geqslant 0$. For a treatment of basic questions and ideas of this section in the setting of initial enlargements by random variables see [Imk96].

3.1 Preliminaries

We now recall the definition of some basic norms on the set of semimartingales. For this let X be a $(\mathcal{F}_t^P)$-semimartingale. Given a decomposition $X = M + A$ we define for all $1 \leqslant p < \infty$,

$$j_p(M,A) = \left\| [M,M]_\infty^{\frac{1}{2}} + \int_{[0,\infty[} |dA_s| \right\|_{L^p}$$

and

$$\|X\|_{\mathcal{S}^p} = \inf_{X=M+A} j_p(M,A)\,.$$

We denote by $\mathcal{S}^p$ the set of all $(\mathcal{F}_t^P)$-semimartingales X such that $\|X\|_{\mathcal{S}^p} < \infty$. If we want to emphasize the filtration we are referring to we write $\mathcal{S}^p(\mathcal{F}_t)$. The space $\mathcal{S}^p$ is a Banach space with the following properties (see e.g., [DM82]):

- Any $X \in \mathcal{S}^p$ is a special semimartingale.
- Let $X \in \mathcal{S}^p$ and $X = \bar{M} + \bar{A}$ be the unique decomposition such that $\bar{A}$ is predictable and $\bar{A}_0 = 0$. There is a constant $c > 0$, depending only on p, such that $j_p(\bar{M},\bar{A}) \leqslant c\|X\|_{\mathcal{S}^p}$.
- The space of all martingales in $\mathcal{S}^p$, denoted by $\mathcal{H}^p$, is a closed subspace.
- The set of all continuous semimartingales in $\mathcal{S}^p$, denoted by $\mathcal{S}^p_c$, and the set of all continuous martingales in $\mathcal{S}^p$, denoted by $\mathcal{H}^p_c$, are closed subspaces.
- The set of all predictable processes with integrable variation, vanishing in 0 and with norm $A \mapsto \| \int |dA_s| \|_{L^p}$ is a closed subspace of $\mathcal{S}^p$.

We will see that under suitable conditions every semimartingale in $\mathcal{S}^2(\mathcal{F}_t)$ belongs to $\mathcal{S}^1(\mathcal{G}_t)$.

3.2 Continuity and relative entropy

We are now in a position to prove the first main result.

Theorem 5. *Suppose $H(\bar{P}\|\bar{Q}) = C < \infty$. Then the embedding*

$$\mathcal{H}^2_c(\mathcal{F}_t) \to \mathcal{S}^1(\mathcal{G}_t), X \mapsto X,$$

is a continuous linear mapping with norm $\leqslant 1 + \sqrt{2C}$.

Proof. Let $M \in \mathcal{H}^2(\mathcal{F}_t)$. By Theorem 2, $(M - [M,L]) + [M,L]$ is a decomposition relative to $(\mathcal{G}_t)$. The Kunita–Watanabe inequality implies

$$\left\| \int_0^\infty |d[M,L]_t| \right\|_1 \leqslant \|[L,L]_\infty^{\frac{1}{2}}\|_2 \|[M,M]_\infty^{\frac{1}{2}}\|_2.$$

Hence by Lemma 3

$$\begin{aligned}
\|M\|_{\mathcal{S}^1(\mathcal{G}_t)} &\leqslant \left\| [M,M]_\infty^{\frac{1}{2}} + \int_0^\infty |d[M,L]_t| \right\|_1 \\
&\leqslant \left(1 + \|[L,L]_\infty^{\frac{1}{2}}\|_2\right) \|[M,M]_\infty^{\frac{1}{2}}\|_2 \\
&\leqslant \left(1 + (E[L,L]_\infty)^{\frac{1}{2}}\right) \|M\|_{\mathcal{H}^2(\mathcal{F}_t)} \\
&\leqslant (1 + \sqrt{2C})\|M\|_{\mathcal{H}^2(\mathcal{F}_t)},
\end{aligned}$$

and the proof is complete. □

As an immediate consequence we get the following

Corollary 1. *Suppose $H(\bar{P}\|\bar{Q}) < \infty$. Then the embedding*

$$\mathcal{S}^2_c(\mathcal{F}_t) \to \mathcal{S}^1(\mathcal{G}_t), X \mapsto X,$$

is a continuous linear mapping.

3.3 Continuity and generalized entropy

We aim at generalizing Theorem 5 and Corollary 1. Starting from the Banach space $\mathcal{S}^r(\mathcal{F}_t)$ with $r > 1$, what are sufficient criteria for the embedding into the space of $(\mathcal{G}_t)$-semimartingales to be continuous?

Throughout this section we assume Assumption (C). In other words, we will assume that $\mathcal{H}^p_c(\mathcal{F}_t) = \mathcal{H}^p(\mathcal{F}_t)$ for $p > 1$.

We begin by stating a result obtained by Yor.

Lemma 8. *(see Lemme 2 in [Yor85]) Let $r \geqslant 1$ and $p, q > 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$. Then the following conditions are equivalent:*

1) There is a constant $C > 0$ such that every continuous $(\mathcal{G}_t)$-local martingale satisfies

$$\left\| \int_0^\infty |d[M,L]_t| \right\|_r \leqslant C \| [M,M]^{\frac{1}{2}}_\infty \|_q \,.$$

2) $E[[L,L]^p_\infty] < \infty$.

We are now ready to state the main theorem.

Theorem 6. *Suppose Assumption (C) is satisfied and let $p \geqslant 1$ and $q, r \geqslant 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$. The generalized entropy $H^p(\bar{P}\|\bar{Q})$ is finite if and only if the embedding*

$$\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X,$$

is a continuous linear mapping.

Proof. Suppose $H^p(\bar{P}\|\bar{Q}) < \infty$. Theorem 4 implies that $[L,L]_\infty$ is L^p-integrable. Thus, by Lemma 8, there is a constant $C > 0$ such that for all continuous $(\mathcal{G}_t)$-local martingales we have

$$\left\| \int_0^\infty |d[M,L]_s| \right\|_{L^r} \leqslant C \| [M,M]^{\frac{1}{2}}_\infty \|_{L^q} \,.$$

Hence, for a martingale M in $\mathcal{S}^q(\mathcal{F}_t)$ with decomposition $M = (M - [M,L]) + [M,L]$ relative to $(\mathcal{G}_t)$, we have

$$
\begin{aligned}
\|M\|_{\mathcal{S}^r(\mathcal{G}_t)} &= \left\| [M,M]_\infty^{\frac{1}{2}} + \int_0^\infty |d[M,L]_s| \right\|_{L^r} \\
&\leqslant \|[M,M]_\infty^{\frac{1}{2}}\|_{L^r} + \left\| \int_0^\infty |d[M,L]_s| \right\|_{L^r} \\
&\leqslant \|[M,M]_\infty^{\frac{1}{2}}\|_{L^r} + C\|[M,M]_\infty^{\frac{1}{2}}\|_{L^q} \\
&\leqslant (1+C)\|[M,M]_\infty^{\frac{1}{2}}\|_{L^q} \\
&\leqslant (1+C)\|M\|_{\mathcal{S}^q(\mathcal{F}_t)} .
\end{aligned}
$$

Therefore the map $\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X$, is continuous.

Now suppose the embedding to be continuous. Then Lemma 8 implies

$$E[[\bar{L},\bar{L}]_\infty^p] < \infty .$$

So by Lemma 6 the proof is complete. □

Example 3. Suppose $\mathcal{A}$ is generated by a countable partition $\mathcal{P} = \{A_1, A_2, \ldots\}$ of Ω into $\mathcal{F}_\infty$-measurable sets. Then the corresponding initial enlargement can be viewed as enlargement by the discrete random variable $G(\omega) := \sum_n n\, 1_{A_n}(\omega)$. Hence, for $p \geqslant 1$, we have by Lemma 7

$$H^p_{\bar{\mathcal{F}}_\infty}(\bar{P}\|\bar{Q}) = \sum_{i\geqslant 1} P(A_i)\left(\log \frac{1}{P(A_i)}\right)^p .$$

Now let $q, r \geqslant 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$. Theorem 6 implies that the embedding $\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X$, is continuous if and only if

$$\sum_{i\geqslant 1} P(A_i)\left(\log \frac{1}{P(A_i)}\right)^p < \infty .$$

This result was already shown by Marc Yor, using different arguments (see Théorème 2 in [Yor85]).

3.4 Continuity and Shannon information

If the filtration $(\mathcal{F}_t)$ is generated by a fixed martingale M with cadlag paths, then the relative entropy of $\bar{P}$ with respect to $\bar{Q}$ is equal to the so-called mutual information between M and the enlarging σ-algebra $\mathcal{A}$. We recall this notion.

Definition 3. *Let X and Y be two random variables with values in the measure spaces $(M, \mathcal{M})$ and $(K, \mathcal{K})$, respectively. The* mutual information *between X and Y is defined by*

$$I(X,Y) = H_{\mathcal{M}\otimes\mathcal{K}}(P_{(X,Y)}\|P_X \otimes P_Y) .$$

Similarly, one can define the generalized mutual information to be

$$I^p(X,Y) = H^p_{\mathcal{M}\otimes\mathcal{K}}(P_{(X,Y)} \| P_X \otimes P_Y), \quad p > 1\,.$$

For a given σ*-algebra* $\mathcal{J} \subset \mathcal{F}$ *let* $\mathrm{id}_{\mathcal{J}}$ *denote the map* $(\Omega, \mathcal{F}) \to (\Omega, \mathcal{J}), \omega \mapsto \omega$. *The mutual information between* X *and* $\mathcal{J}$ *is defined by*

$$I(X, \mathcal{J}) = I(X, \mathrm{id}_{\mathcal{J}})\,.$$

We start with the following observation.

Lemma 9. *If* $(\mathcal{F}_t)$ *equals the filtration generated by* M*, then*

$$I(M, \mathcal{A}) = H(\bar{P}\|\bar{Q})\,,$$

and for $p > 1$*,*

$$I^p(M, \mathcal{A}) = H^p(\bar{P}\|\bar{Q})\,.$$

Proof. First observe that $\bar{\mathcal{F}} = \mathcal{F}_\infty \otimes \mathcal{A}$, because

$$\bar{\mathcal{F}} = \bigvee_t \bar{\mathcal{F}}_t \subset \bigvee_t (\mathcal{F}_t \otimes \mathcal{A}) \subset \mathcal{F}_\infty \otimes \mathcal{A} \subset \bar{\mathcal{F}}\,.$$

Now let $\mathbb{D}$ denote the Skorokhod space. We define a map ϕ by

$$\Omega \times \Omega \to \mathbb{D} \times \Omega, (\omega, \omega') \mapsto (M_\cdot(\omega), \omega')\,.$$

Since $\mathcal{F}_\infty$ is generated by M, we have

$$\phi^{-1}(\mathcal{B}(\mathbb{D}) \otimes \mathcal{A}) = M^{-1}(\mathcal{B}(\mathbb{D})) \otimes \mathcal{A} = \mathcal{F}_\infty \otimes \mathcal{A}\,,$$

and hence

$$H_{\bar{\mathcal{F}}}(\bar{P}\|\bar{Q}) = H_{\mathcal{B}(\mathbb{D})\otimes\mathcal{A}}(\bar{P}_\phi\|\bar{Q}_\phi)\,.$$

Now observe

$$\bar{P}_\phi = P_{\phi\circ\psi} = P_{(M,\mathrm{id}_{\mathcal{A}})}$$

and

$$\bar{Q}_\phi = P_M \otimes P_{\mathrm{id}_{\mathcal{A}}}\,,$$

which yields the first claim. The second follows by similar arguments. □

As a consequence we obtain the following.

Theorem 7. *Suppose Assumption (C) is satisfied and let* $p \geqslant 1$ *and* $q, r \geqslant 0$ *such that* $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$*. If* $(\mathcal{F}_t)$ *equals the filtration generated by* M*, then the generalized mutual information* $I^p(M, \mathcal{A})$ *is finite if and only if the embedding*

$$\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X\,,$$

is a continuous linear mapping.

Proof. This follows by combining Theorem 6 with Lemma 9. □

Example 4. Let W be the standard Wiener process and $(\mathcal{F}_t)$ the filtration generated by W and completed by the negligible sets relative to the Wiener measure. Moreover, let V be a Gaussian element independent of $\mathcal{F}_\infty$, with zero mean and variance $w > 0$. Suppose the enlarging σ-algebra $\mathcal{A}$ is generated by the random variable

$$W_1 + V\,.$$

One can easily verify that three random variables X, Y and Z satisfy

$$I^p(X,(Y,Z)) \leqslant I^p(X,Z) + I^p(X,Y|Z) \quad (p \geqslant 1)\,.$$

Consequently, we obtain for the mutual information between $\mathrm{id}_{\mathcal{A}}$ and W

$$\begin{aligned} I^p(W,\mathrm{id}_{\mathcal{A}}) &= I^p(W_1+V,(W_1,(W_t)_{0\leqslant t<1})) \\ &\leqslant I^p(W_1+V,W_1) + I^p(W_1+V,(W_t)_{0\leqslant t<1}|W_1) \\ &= I^p(W_1,W_1+V) < \infty\,. \end{aligned}$$

Thus, for all $p \geqslant 1$ and $q, r \geqslant 0$ such that $\frac{1}{r} = \frac{1}{2p} + \frac{1}{q}$, the mapping $\mathcal{S}^q(\mathcal{F}_t) \to \mathcal{S}^r(\mathcal{G}_t), X \mapsto X$, is continuous.

References

[ADI06] S. Ankirchner, S. Dereich, and P. Imkeller. The Shannon information of filtrations and the additional logarithmic utility of insiders. *Ann. Probab.* 34(2): 743–778, 2006

[AIS98] J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stoch. Process. Appl.* 75(2): 263–286, 1998

[Ank05] S. Ankirchner. Information and semimartingales. Ph.D. Thesis, Humboldt-Universität, 2005

[DM82] C. Dellacherie and P.A. Meyer. *Probabilities and Potential.* North Holland, Amsterdam, 1982

[FI93] H. Föllmer and P. Imkeller. Anticipation cancelled by a Girsanov transformation: a paradox on Wiener space. *Ann. Inst. H. Poincaré Probab. Statist.* 29(4): 569–586, 1993

[GP98] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *Int. J. Theor. Appl. Finance*, 1: 331–347, 1998

[GV03] D. Gasbarra and E. Valkeila. Initial enlargement: a Bayesian approach. Preprint, 2003

[Imk96] P. Imkeller. Enlargement of the Wiener filtration by an absolutely continuous random variable via Malliavin's calculus. *Probab. Theor. Relat. Field.* 106(1): 105–135, 1996

[Imk03] P. Imkeller. Malliavin's calculus in insider models: additional utility and free lunches. *Math. Finance*, 13(1): 153–169, 2003. Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001)

[IPW01] P. Imkeller, M. Pontier, and F. Weisz. Free lunch and arbitrage possibilities in a financial market model with an insider. *Stoch. Process. Appl.* 92(1): 103–130, 2001

[Jac79] J. Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin Heidelberg New York 1979

[Jac85] J. Jacod. Grossissement initial, hypothese (H'), et théorème de Girsanov. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*, pages 15–35. Springer, Berlin Heidelberg New York 1985

[JY85] Th. Jeulin and M. Yor, editors. *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*. Springer, Berlin Heidelberg New York 1985. Papers from the seminar on stochastic calculus held at the Université de Paris VI, Paris, 1982–1983

[LV87] F. Liese and I. Vajda. *Convex statistical distances*, volume 95 of *Teubner-Texte zur Mathematik* (*Teubner Texts in Mathematics*). BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1987. With German, French and Russian summaries

[Pro04] Ph. Protter. *Stochastic Integration and Differential Equations*. Springer, Berlin Heidelberg New York, second edition, 2004

[SY78] C. Stricker and M. Yor. Calcul stochastique dépendant d'un paramètre. *Z. Wahrsch. Verw. Gebiete*, 45(2): 109–133, 1978

[Yoe85] C. Yoeurp. Grossissement de filtration et théorème de Girsanov généralisé. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*. Springer, Berlin Heidelberg New York 1985

[Yor85] M. Yor. Entropie d'une partition, et grossissement initial d'une filtration. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*, pages 45–58. Springer, Berlin Heidelberg New York 1985

On a Lemma by Ansel and Stricker

Marzia De Donno[1] **and Maurizio Pratelli**[1,2]

[1] Dipartimento di Matematica, Università di Pisa
Largo B. Pontecorvo 5, 56127 Pisa, Italy
[2] *e-mail: pratelli@dm.unipi.it*

Let S be a local martingale with values in $\mathbb{R}^d$, and let H be a d-dimensional predictable process, such that the stochastic integral $H \cdot S$ does exist: if the process $(H \cdot S)_t$ is uniformly bounded from below by a constant (or, more in general, by an integrable random variable), then $H \cdot S$ is a local martingale, hence a supermartingale.

This result, which is inspired from a proposition by Emery in [8] for the case $d = 1$, is due to Ansel and Stricker ([1], Corollary 3.5). Though obtained as a corollary to a more general proposition, it has become a fundamental result in mathematical finance. For instance, it was stated (as Theorem 2.9) and widely used by Delbaen and Schachermayer in their seminal paper on the fundamental theorem of asset pricing [5].

The purpose of this short note is to provide a different proof of the Ansel and Stricker's lemma, which also allows us to give a formulation of this result for the stochastic integral of measure-valued processes with respect to a family of semimartingales, indexed by a continuous parameter.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$ be a filtered probability space, which satisfies the usual conditions.

Theorem 1 *Let X be an adapted càdlàg process and (M^n) a sequence of martingales such that*

(i) $\sup_{t \leqslant T} |M^n_t - X_t|$ *tends to 0 in probability as $n \to \infty$;*

(ii) there exist an increasing sequence (η_k) of stopping times which converges stationarily to T and a sequence θ_k of integrable random variables, such that $X_{t \wedge \eta_k} \geqslant \theta_k$.

(iii) for every stopping time τ, we have that $(\Delta M^n_\tau)^- \leqslant (\Delta X_\tau)^-$ and $(\Delta M^n_\tau)^+ \leqslant (\Delta X_\tau)^+$ (where $\Delta X_t = X_t - X_{t-}$).

Then, X is a local martingale.

Proof. We can assume, for simplicity, that $M_0^n = X_0 = 0$. Define a sequence (τ_n) of stopping times as follows:

$$\tau_n = \inf\{t > 0 : X_t > n \text{ or } M_t^n > X_t + 1 \text{ or } M_t^n < X_t - 1\} \wedge T\,.$$

Because of (i), we have that $\lim_n \tau_n = T$, $\mathbb{P}$-a.s. Possibly up to a subsequence we can assume that $\sum_n \mathbb{P}(\tau_n < T) < \infty$. We then define the stopping times $\sigma_n = (\inf_{m \geqslant n} \tau_m) \wedge \eta_n$: the sequence σ_n is increasing and converges to T.

We will show that for all m, the stopped process $X_t^{\sigma_m} = X_{t \wedge \sigma_m}$ is a martingale. For every t, the sequence $M^n_{t \wedge \sigma_m}$ goes to $X_{t \wedge \sigma_m}$ in probability. Thanks to (ii) and the definition of σ_m, the jump ΔX_{σ_m} is such that $(\Delta X_{\sigma_m})^- \leqslant m - \theta_m$; condition (iii) implies that $(\Delta M^n_{\sigma_m})^- \leqslant m - \theta_m$ as well. Since $M_t^n \geqslant X_t - 1$ for $n \geqslant m$ and $t < \sigma_m$, we have that

$$M^n_{t \wedge \sigma_m} \geqslant \theta_m - 1 - (m - \theta_m) = 2\theta_m - m - 1\,.$$

We can then apply Fatou's lemma to find that

$$\mathbb{E}\left[X_{t \wedge \sigma_m}\right] \leqslant \liminf_{n \to \infty} \mathbb{E}\left[M^n_{t \wedge \sigma_m}\right] = 0\,.$$

This shows that $X_{t \wedge \sigma_m}$ is integrable: in particular, taking $t = T$, we obtain that X_{σ_m} is integrable and, as a consequence, ΔX_{σ_m} is integrable.

In an analogous way, we find that, for $n \geqslant m$,

$$M^n_{t \wedge \sigma_m} \leqslant m + 1 + (\Delta M^n_{\sigma_m})^+ \leqslant m + 1 + (\Delta X_{\sigma_m})^+\,.$$

So, we can apply Lebesgue theorem and obtain that for every fixed m and t, the sequence of random variables $M^n_{t \wedge \sigma_m}$ converges to $X_{t \wedge \sigma_m}$ in $L^1(\mathbb{P})$: this implies that $X_t^{\sigma_m}$ is a martingale. □

As a corollary, we deduce the lemma of Ansel and Stricker:

Corollary 2 *Let S be a d-dimensional local martingale and let H be a S-integrable predictable process. If there exists some constant $C > 0$ such that $(H \cdot S) \geqslant -C$ for all t, then $H \cdot S$ is a local martingale.*

Proof. We set $X = H \cdot S$, $H^n = H\mathbf{1}_{\{\|H\| \leqslant n\}}$ and $M^n = H^n \cdot S$. Every M^n is a local martingale, hence we can find an increasing sequence (τ_m) of stopping times such that $\lim_m \tau_m = \infty$ and $M^n_{\tau_m}$ is a martingale ([6], Theorem 3). So, up to a standard localization, we can assume that every M^n is a martingale.

The claim follows from Proposition 1 as soon as we check that conditions (i) and (iii) are fulfilled (condition (ii) is contained in the assumptions of the corollary). It is well-known that if H is integrable with respect to S, then $\sup_{t \leqslant T} |M_t^n - X_t|$, tends to 0 in probability, whence condition (i). Condition (iii) follows trivially once we have observed that $\Delta M_\tau^n = \Delta(H \cdot S)_\tau \mathbf{1}_{\{\|H\|_\tau \leqslant n\}}$. Hence the claim is proved. □

Now we briefly show how the previous arguments can be applied to the case of *measure-valued* integrands.

Let $\mathbf{M} = (M^x)_{x\in I}$ be a family of locally square integrable martingales, where I is a compact subset of $\mathbb{R}$. We denote by $\mathcal{P}$ the predictable σ-field and suppose that $\mathbf{M}$ satisfies the following:

Assumption 1. *There exist an increasing predictable process A_t and a function Q defined on $\Omega\times[0,T]\times I\times I$, measurable with respect to $\mathcal{P}\otimes\mathcal{B}(I)\otimes\mathcal{B}(I)$, such that, for almost all $(\omega,s)\in\Omega\times[0,T]$:*

(i) the function $(x,y)\mapsto Q_{\omega,s}(x,y)$ is symmetric, non-negative definite and continuous;
(ii) the function $(x,y)\mapsto\int_0^t Q_{\omega,s}(x,y)dA_s(\omega)$, is symmetric, non-negative definite and continuous;
(iii) for fixed $x,y\in I$ and for all $t\in[0,T]$, we have that:

$$\langle M^x, M^y\rangle_t(\omega) = \int_0^t Q_{\omega,s}(x,y)dA_s(\omega) \qquad \textit{for } \mathbb{P}\textit{-a.e. } \omega.$$

With this assumption, a stochastic integral with respect to $\mathbf{M}$ can be defined on an appropriate class of measure-valued processes, by making use of a theory on cylindrical integration developed by Mikulevicius and Rozovskii [9] (see also [3], Section 3). More in details, consider a stochastic process ϕ with values in the set of the Radon measures on I (the dual set of the space of continuous functions $\mathcal{C}=\mathcal{C}(I,\mathbb{R})$), such that, for every $f\in\mathcal{C}$, the process $\langle\phi_s,f\rangle_{\mathcal{M},\mathcal{C}}$ is predictable. We indicate by $\langle\phi_s,Q_s\psi_s\rangle = \langle\phi_s,Q_s\psi_s\rangle_{\mathcal{M},\mathcal{C}}$ the bilinear form $\int_I\phi_s(\mathrm{d}x)\int_I Q_s(x,y)\psi_s(\mathrm{d}y)$.
Suppose that

$$\mathbb{E}\left[\int_0^T\langle\phi_s,Q_s\phi_s\rangle\,\mathrm{d}A_s\right]<\infty:$$

then it is possible to define the stochastic integral $\phi\cdot\mathbf{M}$ which is a square-integrable martingale (see [9] for details). Moreover, if $\int_0^t\langle\phi_s,Q_s\phi_s\rangle\,\mathrm{d}A_s$ is locally integrable, the stochastic integral $\phi\cdot\mathbf{M}$ is defined and is a locally square-integrable martingale.

More general stochastic integrals can be defined, in a similar way to what happens for the finite dimensional case (see [2] page 130).

Let, for every n, $\phi^n = \phi\mathbf{1}_{\{\langle\phi,Q\phi\rangle\leqslant n\}}$: we say that ϕ is $\mathbf{M}$-*integrable* if the sequence of square-integrable martingales $\phi^n\cdot\mathbf{M}$ is convergent for the semimartingale topology (see [7] for the definition of this topology) and by definition $\phi\cdot\mathbf{M}=\lim_{n\to\infty}\phi^n\cdot\mathbf{M}$. Note that, if $X=\phi\cdot\mathbf{M}$, then $\phi^n\cdot\mathbf{M} = \mathbf{1}_{\{\langle\phi,Q\phi\rangle\leqslant n\}}\cdot X$.

Exactly as for the finite-dimensional case, the process $\phi\cdot\mathbf{M}$ might not be a local-martingale (see for instance, [8]), but the analogue of Corollary 2 holds (whit a proof similar to that of Corollary 2).

Proposition 3 *Let* ϕ *be a measure-valued integrable process. If there exists some constant* C *such that* $(\phi \cdot \mathbf{M})_t \geqslant -C$ *for all* t*, then* $\phi \cdot \mathbf{M}$ *is a local martingale.*

Remark: We point out that a stochastic integral $\mathbf{H}.\mathbf{M}$ has been defined in [4] for a wider class of integrands $\mathbf{H}$, and that in this more general framework the analogue of the Ansel–Stricker's lemma is false (see [4], Example 2.1).

References

1. Ansel, J.P., Stricker, C. (1994): Couverture des actifs contingents et prix maximum. *Ann. Inst. H. Poincaré Probab. Statist.* **30**(2): 303–315
2. Chou, C.S., Meyer, P.A., Stricker, C. (1980): Sur les Intégrales Stochastiques des processus prévisibles non bornés. *Séminaire de Probabilités XIV,* Lecture Notes in Math. **784**, 128–139, Springer, Berlin Heidelberg New York
3. De Donno, M., Pratelli, M. (2004): On the use of measure-valued strategies in bond markets. *Finance Stoch.* **8**, 87–109
4. De Donno, M., Pratelli, M. (2005): A theory of stochastic integration for bond markets. *Ann. Appl. Probab.* **15**(4), 2773–2791
5. Delbaen, F., Schachermayer, W. (1994): A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300**, 463–520
6. Dellacherie, C. (1978): Quelques applications du lemme de Borel-Cantelli à la théorie des semimartingales. *Séminaire de Probabilités XII*, Lecture Notes in Math. **649**, 742–745, Springer, Berlin Heidelberg New York
7. Emery, M. (1979): Une topologie sur l'espace des semi-martingales. *Séminaire de Probabilités XIII,* Lecture Notes in Math. **721**, 260–280, Springer, Berlin Heidelberg New York
8. Emery, M. (1980): Compensation de Processus V.F. non localement intégrables. *Séminaire de Probabilités XIV,* Lecture Notes in Math. **784**, 152–160, Springer, Berlin Heidelberg New York
9. Mikulevicius, R., Rozovskii, B.L. (1998): Normalized Stochastic Integrals in Topological Vector Spaces. *Séminaire de Probabilités XXXII*, Lecture notes in Math. **1686**, 137–165, Springer, Berlin Heidelberg New York

General Arbitrage Pricing Model: I – Probability Approach

Alexander Cherny

Department of Probability Theory, Faculty of Mechanics and Mathematics,
Moscow State University
119992 Moscow, Russia
e-mail: alexander.cherny@gmail.com
http://mech.math.msu.su/~cherny/

Summary. The purpose of this paper is to present a unified approach to pricing contingent claims through a new concept of *generalized arbitrage*.

First, we prove the fundamental theorem of asset pricing and establish the form of the fair price intervals within the framework of a *general arbitrage pricing model*.

Furthermore, these results are "projected" on several models, including:

- a dynamic model with an infinite number of assets;
- a model with European call options as basic assets;
- a mixed model.

This leads us, in particular, to revise the fundamental theorem of asset pricing for continuous-time models. Our variant of this theorem states that the absence of generalized arbitrage is equivalent to the existence of an equivalent measure, with respect to which the discounted price process is a true martingale. In a model with infinite time horizon, uniformly integrable martingales come into play.

The general approach mentioned above allows us to narrow the fair price intervals by taking into consideration the current prices of traded derivatives.

Key words: Change of numéraire, General arbitrage pricing model, Generalized arbitrage, Fair price, Fundamental theorem of asset pricing, Martingale measure, Martingale measure with given marginals, Risk-neutral measure, Set of attainable incomes

1 Introduction

1.1 Purpose of the paper

In the fundamental work [22], Harrison and Kreps introduced a general model of pricing by arbitrage. Their paper formed the basis of the martingale approach to arbitrage pricing. However, there are some technical problems inherent in their model. The main one stems from the assumption that the so-called marketed contingent claims should belong to L^2 (the model proposed later by Kreps [32] enables one to relax this assumption to the L^p-integrability with $p \geqslant 1$). This restriction is not very natural as shown by the example below.

Consider the following simple model for the (discounted) price evolution of an asset: $S_0 = 1$, $S_1 = \xi$, $S_2 = \xi\eta$, where ξ and η are independent random variables, each taking on values 1/2 and 3/2 with probability 1/2 (S_n represents the discounted price of some asset at time n). Let $(\mathcal{F}_n)_{n=0,1,2}$ be a filtration such that $\mathcal{F}_0$ is trivial, S is an $(\mathcal{F}_n)$-martingale, and $\mathcal{F}_1$ is rich enough, so that there exists an $\mathcal{F}_1$-measurable random variable H that is not integrable. Then $H(S_2 - S_1)$ is a natural candidate for a marketed contingent claim. However, it does not belong to L^1.

Further development of arbitrage pricing theory was mainly concentrated on dynamic models with a finite number of assets, which may be viewed as particular cases of the model proposed by Harrison and Kreps. Harrison and Pliska [23] introduced the admissibility condition on the trading strategies as a substitute for the integrability restriction described above. The fundamental theorem of asset pricing (FTAP) for a discrete-time model was established in the papers [11, 23] (alternative proofs were given in [26, 29, 30, 35, 37, 41]). The FTAP for a continuous-time model was established in the papers [12, 15] (another proof was given in [28]). In a series of papers [15, 18, 19, 31], the form of upper and lower prices of a contingent claim in a continuous-time model was established. However, there are some serious problems inherent in the mentioned approach to continuous-time models (these problems are described in Examples 4.3–4.5, and especially in Example 4.6).

In this paper, we propose a *general arbitrage pricing model* that has the same spirit as the model of Harrison and Kreps, but avoids the problems described above. This approach allows us to consider in a simple and unified manner various models of arbitrage pricing theory, some of which have so far been investigated separately and by different techniques. These include

- static as well as dynamic models; (see Sections 4, 5);
- Models with an infinite number of assets (in particular, this allows us to consider models with traded derivatives as basic assets, which makes it possible to narrow considerably fair price intervals – see Section 6);
- models with transaction costs (these will be considered in the paper [5], which is a continuation of this paper);
- combinations of various models (see Section 7).

In the paper [5], we extend our results to models with transaction costs. Our approach to these models turns out to be different from the existing ones. Furthermore, in the paper [6], we introduce the *possibility approach* to arbitrage pricing, which enables one to get rid of such a vague object as the original probability measure.

1.2 General arbitrage pricing model

A general arbitrage pricing model is a quadruple $(\Omega, \mathcal{F}, \mathsf{P}, A)$, where $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space and A (it is called the *set of attainable incomes*) is a collection of random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ representing the set of discounted incomes one can obtain by trading certain assets. For a model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we introduce a notion of *No Generalized Arbitrage* (NGA). The NGA condition might be viewed as a strengthening of the No Free Lunch condition known in financial mathematics (the necessity to strengthen the latter is illustrated by Example 6.4). Furthermore, we define an equivalent *risk-neutral* measure as a measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}} X^- \geqslant \mathsf{E}_{\mathsf{Q}} X^+$ for any $X \in A$ (X^- and X^+ denote the negative part and the positive part of X, respectively; the expectations $\mathsf{E}_{\mathsf{Q}} X^-$, $\mathsf{E}_{\mathsf{Q}} X^+$ here are allowed to take on the value $+\infty$). Although risk-neutral measures are a classical concept in financial mathematics, this particular definition seems to be new. It turns out to be very convenient as illustrated by considerations in Sections 4–7.

The first basic result of the paper is Theorem 3.6, which may be called the FTAP for the general arbitrage pricing model. It states (under some assumption that is automatically satisfied in the particular models considered below) that a model satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.

Next, we consider the problem of pricing contingent claims. We define a fair price of a contingent claim F (F is a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$) as a real number x such that the extended model $(\Omega, \mathcal{F}, \mathsf{P}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGA condition. The second basic result of the paper is Theorem 3.10. It states (under some natural assumptions) that the set of fair prices of F coincides with the interval $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\}$, where $\mathcal{R}$ denotes the set of equivalent risk-neutral measures.

1.3 Particular models

Various models of arbitrage pricing can be viewed as particular cases of the general model described above. In order to embed a particular model into this general framework, one should

1. specify the set A of attainable incomes;
2. find out the structure of the set of equivalent risk-neutral measures (typically, the risk-neutral measures in a particular model admit a simpler description than the general definition of a risk-neutral measure).

Once this is done, Theorem 3.6 gives the necessary and sufficient conditions for the absence of generalized arbitrage, while Theorem 3.10 yields the form of the set of fair prices of a contingent claim.

When "projected" to a discrete-time model with a finite number of assets, our results agree with the classical ones. Namely, the class of risk-neutral measures coincides with the class of martingale measures, while our intervals of fair prices coincide with the classical No Arbitrage intervals.

However, for continuous-time models (considered in Sections 4, 5), our results differ from the traditional ones. First of all, it should be mentioned that, unlike discrete-time models, continuous-time models do not possess a unique universally accepted approach to pricing by arbitrage. The two most well-known approaches are: the "L^2-approach" proposed by Harrison and Kreps [22] and the approach developed in a series of papers [12,15,18,19,28,31], and others. Our approach is different from the "L^2-approach" because we never impose any integrability restrictions on price processes or trading strategies.

Let us now describe the differences between our approach and the second one mentioned above. First, we consider the model with an arbitrary number of assets, while the traditional approach deals with a finite number of assets. Second, we consider only simple (i.e., piecewise constant) trading strategies with no admissibility condition imposed. Third, our FTAP states that a model with a finite time horizon satisfies the NGA condition if and only if there exists an equivalent measure, with respect to which the discounted price process is a true martingale; a model with infinite time horizon satisfies the NGA condition if and only if there exists an equivalent measure, with respect to which the discounted price process is a uniformly integrable martingale. This is different from the traditional FTAP provided by Delbaen and Schachermayer [12, 15] (another proof was given by Kabanov [28]), which states that a model satisfies the No Free Lunch with Vanishing Risk (NFLVR) condition (defined through the general predictable admissible strategies) if and only if there exists an equivalent measure, with respect to which the discounted price process is a sigma-martingale (this class of processes has been introduced by Chou [8]). Let us also point out in this connection that for the continuous-time model with a finite number of assets, Sin [40] and Yan [43] introduced some strengthening of the NFLVR condition and proved that these strengthening are equivalent to the existence of an equivalent measure, with respect to which the discounted price process is a true martingale. Thus, our FTAP agrees with these results although our NGA condition is different from the variants of No Arbitrage in these papers. Fourth, our definition of the interval of fair prices differs from the traditional one. We discuss in Section 4 the problems of the traditional theory of arbitrage pricing that arise when one considers admissible strategies, sigma-martingale measures, and traditional intervals of fair prices. These problems do not arise in our framework. Furthermore, it turns out that, unlike the NFLVR property, the NGA property is preserved under a change of numéraire (see Theorem 4.8).

The intervals of fair prices provided by arbitrage considerations are known to be unacceptably large in incomplete models. Several ways to overcome this problem have been proposed in financial mathematics. One of them is to consider traded derivatives as basic assets. Typically, this leads to models with an infinite number of assets, and this often creates serious theoretical problems. Our approach can easily be applied to models with an infinite number of assets, and the traded derivatives can be taken into consideration as follows. The set A depends on the amount of traded securities that we take into account; the set $\mathcal{R}$ depends on A; the interval of fair prices depends on $\mathcal{R}$. Diagrammatically,

$$\text{Assets} \longrightarrow A \longrightarrow \mathcal{R} \longrightarrow \text{Interval of fair prices.}$$

When the amount of assets taken into consideration is enlarged (i.e., more prices of traded contracts are taken into account), the set A is enlarged, the set $\mathcal{R}$ is reduced, and the sets of fair prices are reduced.

In Section 6, we consider a model, which takes into account traded European call options on a fixed asset with a fixed maturity T. It is shown that if options with all positive strike prices are traded (of course, this is an idealized assumption, but it is typical for the theory), then the risk-neutral measure is unique. As a corollary, the fair price of a contingent claim depending only on the price of the asset at time T (for example, a binary option) is uniquely determined.

It should be mentioned that this model was first proposed by Breeden and Litzenberger [2] and is very popular in mathematical finance (a literature review on this model is given in [25]). Our approach to this model is different from the existing ones. In particular, we establish the form of fair price intervals based on the NGA considerations, while traditionally the fair price of a contingent claim in this model is derived by representing the payoff as a combination of (a continuum of) European call options. This trick requires the payoff function to be smooth (for instance, binary options do not satisfy this condition), while in our approach no smoothness or continuity requirements are imposed.

The general approach introduced in Section 3 admits an easy procedure of combination of models. The aim of this procedure is to narrow the sets of fair prices by taking into consideration the current prices of a larger amount of traded contracts. Thus, the models of Sections 4–6 may be viewed as "building blocks" for constructing mixed models. An example is provided in Section 7, where we consider a mixed static-dynamic model. The "building blocks" are provided by the models of Sections 4 and 6. We show that, for the mixed model, the set $\mathcal{R}$ consists of the equivalent *martingale measures with given marginals*, i.e., the measures, with respect to which the discounted price process is a martingale with preassigned marginal distributions. Such measures have recently attracted attention in the literature (see [3], [4; Sect. 4.1], and [33]).

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2 Ordinary arbitrage

In this section, we briefly describe the classical arbitrage pricing theory in a static model with a finite number of assets. This material is well-known (for more details, one may consult, for instance, [20; Ch. 1]). The general arbitrage pricing model introduced in Section 3 may be regarded as the infinite-dimensional version of the model of this section (with the definitions of arbitrage and the definitions of fair prices appropriately reformulated).

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $S_0 \in \mathbb{R}^d$ and S_1 be an $\mathbb{R}^d$-valued random vector on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, S_n^i is the discounted price of the ith asset at time n. Consider the set

$$A = \left\{ \sum_{i=1}^{d} h^i (S_1^i - S_0^i) : h^i \in \mathbb{R} \right\} . \tag{1}$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading assets $1, \dots, d$ at times 0, 1.

Definition 2.1. *A model $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$ satisfies the No Arbitrage (NA) condition if $A \cap L^0_+ = \{0\}$ (L^0_+ denotes the set of $\mathbb{R}_+$-valued random variables on $(\Omega, \mathcal{F}, \mathsf{P})$).*

Definition 2.2. *An equivalent martingale measure is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q}|S_1| < \infty$ and $\mathsf{E}_\mathsf{Q} S_1 = S_0$. The set of equivalent martingale measures will be denoted by $\mathcal{M}$.*

Notation. Set $C = \overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_\mathsf{P} S_1$, where "$\overline{\operatorname{conv}}$" denotes the closed convex hull, "supp" denotes the support, and $\operatorname{Law}_\mathsf{P} S_1$ is the distribution of S_1 under P. Let C° denote the relative interior of C, i.e., the interior of C in the relative topology of the smallest affine subspace of $\mathbb{R}^d$ containing C.

Theorem 2.3 (FTAP). *For the model $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$, the following conditions are equivalent:*

(a) *NA*;
(b) $S_0 \in C^\circ$;
(c) $\mathcal{M} \neq \emptyset$.

Proof. Step 1. Let us prove the implication (a)⇒(b). If $S_0 \notin C^\circ$, then, by the separation theorem, there exists a vector $h \in \mathbb{R}^d$ such that $\langle h, (x - S_0)\rangle \geqslant 0$ for all $x \in C$ and $\langle h, (x - S_0)\rangle > 0$ for some $x \in C$. This means that $\langle h, (S_1 - S_0)\rangle \geqslant 0$ P-a.s. and $\mathsf{P}(\langle h, (S_1 - S_0)\rangle > 0) > 0$. But this contradicts the NA condition.

Step 2. Let us prove the implication (b)⇒(c). The set

$$E = \{\mathsf{E_Q} S_1 : \mathsf{Q} \sim \mathsf{P},\ \mathsf{E_Q}|S_1| < \infty\}$$

is convex, and the closure of E contains $\operatorname{supp} \operatorname{Law}_{\mathsf{P}} S_1$. Consequently, $E \supseteq C^\circ$.

Step 3. Let us prove the implication (c)⇒(a). Take $\mathsf{Q} \in \mathcal{M}$. Then $\mathsf{E_Q} X = 0$ for any $X \in A$. This implies the NA condition. □

Now, let F be a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. From the financial point of view, F is the discounted payoff of some contingent claim.

Definition 2.4. A real number x is a *fair price* of F if the model with $d+1$ assets $(\Omega, \mathcal{F}, \mathsf{P}, x, S_0^1, \dots, S_0^d, F, S_1^1, \dots, S_1^d)$ satisfies the NA condition. The set of fair prices of F will be denoted by $I(F)$.

Notation. Set $D = \overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_{\mathsf{P}}(F, S_1)$ and let D° denote the relative interior of D.

Theorem 2.5 (Pricing contingent claims). *Suppose that the model* $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$ *satisfies the NA condition. Then*

$$I(F) = \{x : (x, S_0) \in D^\circ\} = \{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}.$$

The expectation $\mathsf{E_Q} F$ *here is taken in the sense of finite expectations, i.e., we consider only those* Q*, for which* $\mathsf{E_Q}|F| < \infty$.

This is a direct consequence of Theorem 2.3. □

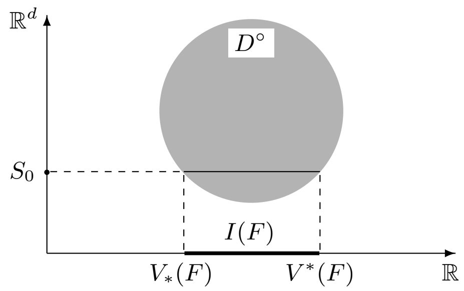

Fig. 1. The joint arrangement of $I(F)$, $V_*(F)$, $V^*(F)$, and D°

Remark. Another way to define the fair price interval (which is commonly used in financial mathematics) is as follows. We introduce the lower and upper prices by

$$V_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \leqslant F \ \mathsf{P}\text{-a.s.}\},$$
$$V^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \geqslant F \ \mathsf{P}\text{-a.s.}\},$$

and the fair price interval is defined as the interval with endpoints $V_*(F)$ and $V^*(F)$ (to be more precise, if $V_*(F) < V^*(F)$, we consider the interval $(V_*(F), V^*(F))$; if $V_*(F) = V^*(F)$, we consider the one-point interval $\{V_*(F)\}$). One can easily check that if the model $(\Omega, \mathcal{F}, \mathsf{P}, S_0, S_1)$ satisfies the NA condition, then the interval of fair prices defined this way coincides with the interval $I(F)$ introduced above (a proof can be found in [20; Th. 1.23]).

3 Generalized arbitrage

Definition 3.1. *A general arbitrage pricing model is a quadruple* $(\Omega, \mathcal{F}, \mathsf{P}, A)$*, where* $(\Omega, \mathcal{F}, \mathsf{P})$ *is a probability space and* A *is a convex cone in* L^0 *(*L^0 *is the space of real-valued random variables on* $(\Omega, \mathcal{F}, \mathsf{P})$ *considered up to indistinguishability). The set* A *will be called the set of attainable incomes.*

From the financial point of view, A is the set of discounted incomes that can be obtained by trading a certain amount of assets. An example is provided by (1). In frictionless models, A is a linear space. In models with transaction costs, A is a cone.

Notation. (i) Set

$$B = \Big\{ Z \in L^0 : \text{ there exist } (X_n)_{n \in \mathbb{N}} \in A \text{ and } a \in \mathbb{R} \\ \text{such that } X_n \geqslant a \ \mathsf{P}\text{-a.s. and } Z = \lim_{n \to \infty} X_n \ \mathsf{P}\text{-a.s.} \Big\}. \quad (2)$$

The elements of B might be regarded as generalized attainable incomes bounded below.

(ii) For $Z \in B$, put $\gamma(Z) = 1 - \operatorname{ess\,inf}_{\omega \in \Omega} Z(\omega)$ and set

$$A_1 = \{X - Y : X \in A,\ Y \in L^0_+\},$$
$$A_2(Z) = \left\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \right\},$$
$$A_3(Z) = A_2(Z) \cap L^\infty,$$
$$A_4(Z) = \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, L^1(\mathsf{P})). \quad (3)$$

Here L^0_+ is the set of $\mathbb{R}_+$-valued elements of L^0; L^∞ is the space of bounded elements of L^0; $\sigma(L^\infty, L^1(\mathsf{P}))$ denotes the weak topology on L^∞ induced by the space $L^1(\mathsf{P})$ of all P-integrable random variables on $(\Omega, \mathcal{F}, \mathsf{P})$.

Definition 3.2. *A model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the No Generalized Arbitrage (NGA) condition if for any $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.*

Remarks. (i) Note that $A_4(Z) \cap L^0_+ = \{0\}$ if and only if $A_5(Z) \cap L^0_+ = \{0\}$, where

$$A_5(Z) = \{(Z + \gamma(Z))X : X \in A_4(Z)\}\,. \tag{4}$$

The elements of $A_5(Z)$ may be regarded as generalized attainable incomes (i.e., one can approximate the elements of $A_5(Z)$ by the elements of A_1).

(ii) The existence of a generalized arbitrage opportunity means that there exist $Z \in B$, $W \in L^0_+ \setminus \{0\}$ and generalized sequences $(X_\lambda)_{\lambda \in \Lambda} \in A$, $(Y_\lambda)_{\lambda \in \Lambda} \in L^0_+$ and $(\alpha_\lambda)_{\lambda \in \Lambda} \in \mathbb{R}_+$ such that $|X_\lambda - Y_\lambda| \leqslant \alpha_\lambda(Z + \gamma(Z))$, $\lambda \in \Lambda$ and $(X_\lambda - Y_\lambda)$ converges to W in the sense that $\mathsf{E}_\mathsf{Q}(X_\lambda - Y_\lambda) \longrightarrow \mathsf{E}_\mathsf{Q} W$ for any probability measure $\mathsf{Q} \ll \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} Z < \infty$.

(iii) The NGA condition is similar to the *No Free Lunch* (NFL) condition introduced by Kreps [32] in a different framework. The NFL condition can be defined in our framework as: $A_4(0) \cap L^0_+ = \{0\}$. One can also define the *No Arbitrage* (NA) condition in our framework as: $A \cap L^0_+ = \{0\}$. The NGA condition is the strongest one: NGA$\Rightarrow$NFL, NGA$\Rightarrow$NA.

Definition 3.3. *An equivalent risk-neutral measure is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X^- \geqslant \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$ (we use the notation $X^- = (-X) \vee 0$, $X^+ = X \vee 0$). The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of equivalent risk-neutral measures will be denoted by $\mathcal{R}$.*

Notation. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of all probability measures $\mathsf{Q} \sim \mathsf{P}$ with the property: for any $X \in A$ such that $X \geqslant -\alpha Z - \beta$ P-a.s. with some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \leqslant 0$.

Lemma 3.4. *For any $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

Proof. Take $Z \in B$, $\mathsf{Q} \in \mathcal{R}$. It follows from the Fatou lemma that Z is Q-integrable. Thus, if $X \in A$ satisfies the inequality $X \geqslant -\alpha Z - \beta$ P-a.s with some $\alpha, \beta \in \mathbb{R}_+$, then $\mathsf{E}_\mathsf{Q} X^- < \infty$. By the definition of $\mathcal{R}$, $\mathsf{E}_\mathsf{Q} X^+ \leqslant \mathsf{E}_\mathsf{Q} X^-$. As a result, $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \leqslant 0$. □

The following basic assumption is satisfied in all particular models considered below.

Assumption 3.5. There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (in particular, both sets might be empty).

Theorem 3.6 (FTAP). *Suppose that Assumption 3.5 is satisfied. Then the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.*

The proof is based on a well-known result of Kreps [32] and Yan [42] (its proof can also be found in [37, 41], and other papers):

Lemma 3.7 (Kreps, Yan). *Let C be a $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone in L^∞ such that $C \supseteq L^\infty_-$ (L^∞_- is the set of negative elements of L^∞) and $C \cap L^0_+ = \{0\}$. Then there exists a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X \leqslant 0$ for all $X \in C$.*

Proof of Theorem 3.6. Step 1. Let us prove the "only if" implication. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 3.7 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A_4(Z_0)$ yields a probability measure $\mathsf{Q}_0 \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \leqslant 0$ for all $X \in A_4(Z_0)$. By the Fatou lemma, for any $X \in A$ such that $\frac{X}{Z_0+\gamma(Z_0)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0+\gamma(Z_0)} \leqslant 0$ (note that $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0+\gamma(Z_0)} \wedge c \leqslant 0$ for any $c > 0$). Consider the probability measure $\mathsf{Q} = \frac{c}{Z_0+\gamma(Z_0)} \mathsf{Q}_0$, where c is a normalizing constant (it exists since $Z_0 + \gamma(Z_0) \geqslant 1$). Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$.

Step 2. Let us prove the "if" implication. Take $\mathsf{Q} \in \mathcal{R}$ and $Z \in B$. It follows from the Fatou lemma that Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is a normalizing constant. For any $X \in A$ such that $\frac{X}{Z+\gamma(Z)}$ is bounded below by a constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_\mathsf{Q} X^- \leqslant \mathsf{E}_\mathsf{Q}(\alpha Z + \alpha\gamma(Z)) < \infty,$$

and consequently,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{X}{Z + \gamma(Z)} = c\mathsf{E}_\mathsf{Q} X \leqslant 0.$$

Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} X \leqslant 0$ for any $X \in A_4(Z)$. As a result, $A_4(Z) \cap L^0_+ = \{0\}$. □

It is seen from the above proof that the implication $\mathcal{R} \neq \emptyset \Rightarrow$ NGA is true without Assumption 3.5. The following example shows that this assumption is essential for the reverse implication.

Example 3.8. Let $(X_t)_{t\in[0,1]}$ be a collection of independent Gaussian random variables with mean 1 and variance 1 defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $\mathcal{F} = \sigma(X_t; t \in [0,1])$ and

$$A = \left\{ \sum_{n=1}^N h_n X_{t_n} : N \in \mathbb{N},\ t_n \in [0,1],\ h_n \in \mathbb{R} \right\}.$$

Clearly, the only element of A that is bounded below is 0. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition.

Suppose now that there exists an equivalent risk-neutral measure Q. Set $\rho = \frac{d\mathsf{Q}}{d\mathsf{P}}$. Note that $\mathcal{F} = \cup_C \sigma(X_t; t \in C)$, where the union is taken over all countable sets $C \subset [0,1]$. Hence, there exists a countable set $C_0 \subset [0,1]$ such that ρ is $\sigma(X_t; t \in C_0)$-measurable. For any $t \notin C_0$, we have

$$\mathsf{E}_\mathsf{Q} X_t = \mathsf{E}_\mathsf{P} \rho X_t = \mathsf{E}_\mathsf{P} \rho \cdot \mathsf{E}_\mathsf{P} X_t = \mathsf{E}_\mathsf{P} X_t = 1\,.$$

As a result, there exists no equivalent risk-neutral measure. □

Now, let F be a random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ representing the discounted payoff of a contingent claim.

Definition 3.9. *A real number x is a fair price of F if the extended model $(\Omega, \mathcal{F}, \mathsf{P}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGA condition. (From the financial point of view, $A + \{h(F-x) : h \in \mathbb{R}\}$ is the set of discounted incomes that can be obtained by trading the "original" assets as well as trading the contract F at the price x.) The set of fair prices of F will be denoted by $I(F)$.*

Theorem 3.10 (Pricing contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies Assumption 3.5 and the NGA condition, while F is bounded below. Then*

$$I(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\}.$$

The expectation $\mathsf{E}_\mathsf{Q} F$ here is taken in the sense of finite expectations, i.e., we consider only those Q, for which $\mathsf{E}_\mathsf{Q} F < \infty$.

Proof. Step 1. Let $x \in I(F)$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Set $Z_1 = Z_0 + (F - x)$. Then $Z_1 \in B'$, where B' is defined by (2) with A replaced by $A' := A + \{h(F-x) : h \in \mathbb{R}\}$. Lemma 3.7 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A'_4(Z_1)$ ($A'_4(Z_1)$ is defined by (3)) yields a probability measure $\mathsf{Q}_0 \sim \mathsf{P}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \leqslant 0$ for all $X \in A'_4(Z_1)$. By the Fatou lemma, for any $X \in A'$ such that $\frac{X}{Z_1 + \gamma(Z_1)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_1 + \gamma(Z_1)} \leqslant 0$. Consider the probability measure $\mathsf{Q} = \frac{c}{Z_1 + \gamma(Z_1)} \mathsf{Q}_0$, where c is a normalizing constant (it exists since $Z_1 + \gamma(Z_1) \geqslant 1$). Then $\mathsf{Q} \in \mathcal{R}(Z_1) \subseteq \mathcal{R}(Z_0) = \mathcal{R}$. Moreover, $\mathsf{E}_\mathsf{Q}(x - F) \leqslant 0$ and $\mathsf{E}_\mathsf{Q}(F - x) \leqslant 0$ since the random variables $\frac{x-F}{Z_1+\gamma(Z_1)}$ and $\frac{F-x}{Z_1+\gamma(Z_1)}$ are bounded below. Thus, $x = \mathsf{E}_\mathsf{Q} F$.

Step 2. Now, let $x = \mathsf{E}_\mathsf{Q} F$, where $\mathsf{Q} \in \mathcal{R}$. Take $Z \in B'$. Choose an arbitrary element $Y = X + h(F-x) \in A'$ (here $X \in A$) such that Y is bounded below. It follows from the condition $x = \mathsf{E}_\mathsf{Q} F$ that $\mathsf{E}_\mathsf{Q} X^- < \infty$. As $\mathsf{Q} \in \mathcal{R}$, we have $\mathsf{E}_\mathsf{Q} X \leqslant 0$. This implies that $\mathsf{E}_\mathsf{Q} Y \leqslant 0$. By the Fatou lemma, Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is a normalizing constant. For any $Y = X + h(F-x) \in A'$ such that $\frac{Y}{Z+\gamma(Z)}$ is bounded below by some constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_\mathsf{Q} Y^- \leqslant \mathsf{E}_\mathsf{Q}(\alpha Z + \alpha\gamma(Z)) < \infty\,.$$

Consequently, $\mathsf{E}_\mathsf{Q} X^- < \infty$, $\mathsf{E}_\mathsf{Q} X \leqslant 0$, and $\mathsf{E}_\mathsf{Q} Y \leqslant 0$. This means that $\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{Y}{Z+\gamma(Z)} \leqslant 0$. Hence, for any $Y \in A'_4(Z)$, we have $\mathsf{E}_{\widetilde{\mathsf{Q}}} Y \leqslant 0$. This implies that $A'_4(Z) \cap L^0_+ = \{0\}$. As a result, $x \in I(F)$. □

Remarks. (i) Theorem 3.10 remains valid if the condition "F is bounded below" is replaced by the condition "F is bounded above" (the proof remains the same).

(ii) Another way to define fair price intervals could be as follows. We introduce the lower and the upper prices by

$$V_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \leqslant F \text{ P-a.s.}\}\,, \qquad (5)$$
$$V^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \geqslant F \text{ P-a.s.}\}\,, \qquad (6)$$

and the fair price interval is defined as the interval with the endpoints $V_*(F)$ and $V^*(F)$. However, unlike the model of Section 2, in a general model the interval defined this way might be larger than $I(F)$ (see Example 6.5).

To conclude the section, we "project" our results on the model of Section 2.

Example 3.11. Consider the model of Section 2 and assume additionally that the components of S_1 are bounded below. Then, clearly, the class of risk-neutral measures coincides with the class of martingale measures. Consequently, the NGA turns out to be equivalent to the NA and the fair price interval based on the NGA coincides with the fair price interval based on the NA.

4 Dynamic model with finite time horizon

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t^i)_{t\in[0,T]}$, $i \in I$ be a family of real-valued $(\mathcal{F}_t)$-adapted càdlàg processes. Here, I is an arbitrary set (it might be finite or infinite). From the financial point of view, S_t^i is the discounted price of the ith asset at time t. Define the set of attainable incomes by

$$A = \Bigg\{\sum_{n=1}^{N}\sum_{i\in I} H_n^i(S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \leqslant \ldots \leqslant u_N \text{ are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H_n^i = 0 \text{ for all } i,\ \text{except for a finite set}\Bigg\}. \qquad (7)$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading assets from I on the interval $[0, T]$.

We will assume that each process S^i is bounded below (most financial assets automatically satisfy this condition). Moreover, we assume that there exists $Z_0 \in B$ (B is defined by (2)) with the property: for any $i \in I$, there exist $\alpha, \beta > 0$ such that $S_T^i \leqslant \alpha Z_0 + \beta$ a.s. This assumption is automatically satisfied in natural models.

Indeed, if I is finite, then the above assumption is satisfied with

$$Z_0 = \sum_{i\in I}(S_T^i - S_0^i)\,.$$

If I is countable, then the above assumption is satisfied with

$$Z_0 = \sum_{i \in I} \lambda^i (S_T^i - S_0^i)\,,$$

where constants $\lambda^i > 0$ are chosen in such a way that $\sum_{i \in I} \lambda^i S_T^i < \infty$ a.s. and $\sum_{i \in I} \lambda^i S_0^i < \infty$.

If S is the discounted price process of some asset and S^i is the discounted price process of a European call option on this asset with maturity T and strike price i, then $S_T^i = (S_T - i)^+$, and hence, the above assumption is satisfied with $Z_0 = S_T - S_0$ (we assume that the process S is included in the collection $(S^i)_{i \in I}$).

If S^i is the discounted price process of a zero-coupon bond with maturity i, then S^i takes on values in $[0,1]$, and the above assumption is satisfied with $Z_0 = 0$.

In order to get the FTAP and to obtain the form of the fair price intervals, it is sufficient to prove that Assumption 3.5 is satisfied and to find the structure of risk-neutral measures. We call the corresponding statement the *Key Lemma* of the section.

Notation. Set $\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for any } i \in I,\ S^i \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\}$.

Key Lemma 4.1. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(Z_0) = \mathcal{M}\,.$$

The proof employs the following statement (see [26] or [38; Ch. II, Section 1c]):

Lemma 4.2. *Let $(X_n)_{n=0,\dots,N}$ be an $(\mathcal{F}_n)$-local martingale such that $\mathsf{E}|X_0| < \infty$ and $\mathsf{E}X_N^- < \infty$. Then X is an $(\mathcal{F}_n)$-martingale.*

Proof of Key Lemma 4.1. Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 3.4.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. Fix $i \in I$. For any $u \in [0,T]$, the random variable $S_u^i - S_0^i$ is bounded below, and therefore, $\mathsf{E}_\mathsf{Q}(S_u^i - S_0^i) \leqslant 0$. In particular, S_u^i is Q-integrable. For any $u \leqslant v \in [0,T]$ and any $D \in \mathcal{F}_u$ such that S_u^i is bounded on D, the random variable $I_D(S_v^i - S_u^i)$ is bounded below, and hence, $\mathsf{E}_\mathsf{Q} I_D(S_v^i - S_u^i) \leqslant 0$. This proves that S^i is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale. It follows from the assumption $S_T^i \leqslant \alpha Z_0 + \beta$ and the definition of $\mathcal{R}(Z_0)$ that $\mathsf{E}_\mathsf{Q}(S_T^i - S_0^i) = 0$. This implies that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix

$$X = \sum_{n=1}^{N} \sum_{i \in I} H_n^i (S_{u_n}^i - S_{u_{n-1}}^i)\,.$$

The process

$$M_n = \sum_{k=1}^{n} \sum_{i \in I} H_k^i (S_{u_k}^i - S_{u_{k-1}}^i), \quad n = 0, \dots, N$$

is a Q-local martingale with respect to the filtration $(\mathcal{F}_{u_k})$. Now, it follows from Lemma 4.2 that $\mathsf{E_Q} X^- \geqslant \mathsf{E_Q} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

Remark. If the NGA condition is satisfied, then each S^i is an $(\mathcal{F}_t, \mathsf{P})$-semimartingale. This follows from the fact that the semimartingale property is preserved under an equivalent change of measure (see [27; Ch. III, Th. 3.13]).

For discrete-time models with a finite number of assets the approach proposed here agrees with the classical one: the NGA condition is equivalent to the existence of an equivalent martingale measure, which, in turn, is equivalent to the NA condition; the interval of fair prices of a contingent claim that is bounded below coincides with the classical one. However, for continuous-time models with a finite number of assets our approach turns out to be completely different from the traditional approach developed in [12, 15, 18, 19, 28, 31]. Let us briefly describe the latter one.

In the traditional approach, the discounted price process S is assumed to be an $\mathbb{R}^d$-valued $(\mathcal{F}_t, \mathsf{P})$-semimartingale. The "set of attainable incomes" (although this term is not used in the traditional approach) has the form

$$A = \Bigg\{ \int_0^T H_u \mathrm{d}S_u : H \text{ is an } \mathbb{R}^d\text{-valued } (\mathcal{F}_t)\text{-predictable } S\text{-integrable process satisfying the } \textit{admissibility} \text{ condition, i.e., there exists } a \in \mathbb{R} \text{ such that } \int_0^t H_u \mathrm{d}S_u \geqslant a \text{ for any } t \in [0, T] \Bigg\}. \tag{8}$$

(Here $\int_0^t H_u \mathrm{d}S_u$ is the vector stochastic integral; its definition can be found in [27; Ch. III, Section 6c] or [39]). Consider the sets

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L_+^0\}, \\ A_2 &= A_1 \cap L^\infty, \\ A_3 &= \text{closure of } A_2 \text{ in the norm topology of } L^\infty. \end{aligned}$$

The *No Free Lunch with Vanishing Risk* (NFLVR) condition is defined as:

$$A_3 \cap L_+^0 = \{0\}.$$

The traditional FTAP (see [15, 28]) states that a model satisfies the NFLVR condition if and only if there exists an equivalent *sigma-martingale measure*, i.e., a measure $\mathsf{Q} \sim \mathsf{P}$ such that S is an $(\mathcal{F}_t, \mathsf{Q})$-sigma-martingale. Recall that a process $(X_t)_{t \in [0,T]}$ is called a *sigma-martingale* if there exists a sequence of predictable sets $(D_n)_{n\in\mathbb{N}}$ such that $D_n \subseteq D_{n+1}$, $\bigcup_n D_n = \Omega \times [0,T]$, and, for any n, the stochastic integral $\int_0^{\cdot} I_{D_n}(s)\mathrm{d}X_s$ is a uniformly integrable martingale (this definition was proposed by Goll and Kallsen [21]; it is equivalent to the original definition of Chou [8]). The class of sigma-martingales contains the class of local martingales and is wider as shown by the Émery example (see [17]). However, an $\mathbb{R}^d_+$-valued sigma-martingale is necessarily a local martingale as shown by Ansel and Stricker [1].

The set of fair prices of a contingent claim F is defined as the interval with the endpoints $V_*(F)$ and $V^*(F)$, where

$$V_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \leqslant F \text{ P-a.s.}\},$$
$$V^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \geqslant F \text{ P-a.s.}\}$$

(here A is given by (8)). It follows from [15,18,19] that if the NFLVR condition is satisfied and F is bounded below, then

$$V^*(F) = \sup_{\mathsf{Q}\in\mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F, \tag{9}$$

where

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-sigma-martingale}\}. \tag{10}$$

Let us now give four examples and two remarks, which illustrate the problems that arise when one applies the traditional approach.

Table 1. The differences between the traditional approach to asset pricing in the continuous-time setting and the proposed approach

	Traditional approach	**Proposed approach**
The price process	$\mathbb{R}^d$-valued semimartingale	Infinite-dimensional process with adapted, càdlàg components bounded below
Trading strategies	Predictable strategies satisfying the integrability and the admissibility conditions	Simple strategies with no integrability and no admissibility conditions imposed
The variant of the NA condition	NFLVR	NGA
FTAP	NFLVR $\Longleftrightarrow$ existence of an equivalent sigma-martingale measure	NGA $\Longleftrightarrow$ existence of an equivalent martingale measure
Set of fair prices of a contingent claim	$(V_*(F), V^*(F))$	$I(F)$

The first two examples and the remark following them show that the admissibility condition leads to an inadmissible restriction of the class of strategies (by a strategy we mean a process H that appears in (8)).

Example 4.3. *Consider the Black–Scholes model, i.e.,* $S_t = \mathrm{e}^{\mu t+\sigma B_t}$, *where* B *is a Brownian motion. Let* $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. *Then the strategy* $H = -1$ *is not admissible. In other words, the admissibility condition prohibits in this model the strategy that consists in the short selling of the asset at time* 0 *and buying it back at time* T. □

Example 4.4. *Consider the exponential Lévy model, i.e.,* $S_t = \mathrm{e}^{X_t}$, *where* X *is a Lévy process. Let* $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F}{=}\mathcal{F}_T$. *Suppose that the jumps of* X *are not bounded from above (most exponential Lévy models used in modern financial mathematics satisfy this condition). One can check that if* H *is an admissible strategy, then* $H(\omega,t) \geqslant 0$ $\mathsf{P} \times \mu_L$*-a.e, where* μ_L *is the Lebesgue measure on* $[0,T]$. *In other words, the admissibility condition prohibits in this model all strategies employing short selling. Clearly, this is an unacceptable restriction: for example, when hedging a put option in practice, one employs strategies* H *with* $H < 0$ *(for more details, see [24; Ch. 14]).* □

Remark. Another drawback of the admissibility condition is as follows. Such a condition is not imposed in the discrete-time models, but it is imposed in the continuous-time models. This leads to an unpleasant unbalance. In particular, when one embeds a discrete-time model into a continuous-time model, then the set of attainable incomes defined for this continuous-time model by (8) does not coincide with the set of attainable incomes defined for the original discrete-time model.

The next example shows that in some models the traditional interval of fair prices is too wide.

Example 4.5. *Let* $S_t = I(t < T) + \xi I(t = T)$, *where* ξ *is an* $\mathbb{R}_+$*-valued random variable with the property: for any* $a > 0$, $\mathsf{P}(\xi < a) > 0$ *and* $\mathsf{P}(\xi > a) > 0$. *Let* $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. *Consider* $F = S_T$.

Let us find $V_*(F)$. *Let* H *be a predictable admissible strategy and* $x \in \mathbb{R}$ *be such that*

$$x - \int_0^T H_u \mathrm{d}S_u \leqslant F\,. \tag{11}$$

Note that

$$\int_0^T H_u \mathrm{d}S_u = H_T \Delta S_T = H_T(\xi - 1)\,.$$

Since H *is* $(\mathcal{F}_t)$*-predictable and* $\mathcal{F}_t = \{\emptyset, \Omega\}$ *for* $t < T$, H_T *is a real number. The admissibility condition, together with the property* $\mathsf{P}(\xi > a) > 0$ *for any* $a > 0$, *shows that* $H_T \geqslant 0$. *This, combined with* (11) *and with the property* $\mathsf{P}(\xi < a) > 0$ *for any* $a > 0$, *yields* $x \leqslant 0$. *Consequently,* $V_*(F) = 0$.

In a similar way one checks that $V^*(F) = 1$. *Thus, the interval of fair prices provided by the traditional approach is* $[0,1]$. *On the other hand, the* *interval of fair prices provided by common sense consists only of point* 1 *since F can be replicated by buying the asset (whose discounted price is S) at time* 0. □

Remark. In the model of the previous example, we have, due to the result of Ansel and Stricker [1],

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-local martingale}\}$$

($\mathcal{M}_\sigma$ is given by (10)). Furthermore, for any $(\mathcal{F}_t)$-stopping time τ, we have either $\tau = T$ P-a.s. or $\tau < T$ P-a.s. Consequently,

$$\mathcal{M}_\sigma = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\} = \{\mathsf{Q} \sim \mathsf{P} : \mathsf{E}_\mathsf{Q}\xi = 1\}.$$

Therefore, $\inf_{\mathsf{Q}\in\mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F = 1$. This shows that the equality $V_*(F) = \inf_{\mathsf{Q}\in\mathcal{M}_\sigma} \mathsf{E}_\mathsf{Q} F$, which is dual to (9), is not true for F bounded below.

One way to overcome this problem was proposed in [15]. Namely, the authors of that paper altered the definition of $V_*(F)$ and $V^*(F)$ by introducing the so-called w-admissibility condition as a substitute for the admissibility condition. However, a weak point of this definition is that it depends on the choice of a so-called weight function.

The fourth example is the most striking one. It shows that the use of the traditional approach may lead to mispricing contingent claims.

Example 4.6. *Let $S_t = |B_t|^{-1}$, where B is a 3-dimensional Brownian motion started at a point $B_0 \neq 0$. Let $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. Without loss of generality, $B_0^2 = B_0^3 = 0$. Note that*

$$\mathsf{E}S_T = \mathsf{E}((B_T^1)^2 + (B_T^2)^2 + (B_T^3)^2)^{-1/2} \leqslant \mathsf{E}((B_T^2)^2 + (B_T^3)^2)^{-1/2} = \frac{const}{\sqrt{T}}.$$

We take T large enough, so that $\mathsf{E}S_T < S_0$ (actually, $\mathsf{E}S_T < S_0$ for any $T > 0$). Consider $F = S_T$.

Let us find $V^*(F)$. Applying Itô's formula and P. Lévy's characterization theorem (see [36; Ch. IV, Th. 3.6]), we conclude that

$$S_t = S_0 + \int_0^t S_u^2 \mathrm{d}W_u, \quad t \in [0, T], \tag{12}$$

where W is an $(\mathcal{F}_t, \mathsf{P})$-Brownian motion. Furthermore, Itô's theorem (see [34; Th. 5.2.1]) guarantees that (S, W) is a strong solution of stochastic differential equation (12)*, i.e., $\mathcal{F}_t^S \subseteq \mathcal{F}_t^W$. It is clear from* (12) *that $\mathcal{F}_t^W \subseteq \mathcal{F}_t^S$, and hence, $\mathcal{F}_t^W = \mathcal{F}_t^S = \mathcal{F}_t$. Set $F_t = \mathsf{E}(F \,|\, \mathcal{F}_t)$. By the representation theorem for Brownian motion (see [36; Ch. V, Th. 3.5]), there exists an $(\mathcal{F}_t)$-predictable W-integrable process K such that*

$$F_t = \mathsf{E}F + \int_0^t K_u \mathrm{d}W_u, \quad t \in [0, T].$$

In view of (12),

$$F_t = \mathsf{E}F + \int_0^t \frac{K_u}{S_u^2}\,\mathrm{d}S_u = \mathsf{E}F + \int_0^t H_u \mathrm{d}S_u, \quad t \in [0,T]\,. \tag{13}$$

Since $F_t \geqslant 0$, the strategy H is admissible. Consequently, $V^*(F) \leqslant \mathsf{E}F$.

Similarly, by considering $F_t^n = \mathsf{E}(FI(F \leqslant n)\,|\,\mathcal{F}_t)$, we prove that $V_*(F) \geqslant \mathsf{E}F$. As a result, the fair price provided by the traditional approach is $\mathsf{E}F = \mathsf{E}S_T$. On the other hand, the fair price provided by common sense is S_0, which is not equal to $\mathsf{E}S_T$! □

The problems described above do not arise in the approach proposed in this paper.

Indeed, no admissibility restriction is imposed in this approach, which solves the problems described in Examples 4.3, 4.4, and the remark following Example 4.4.

In Example 4.5, we have, due to Theorem 3.10,

$$I(F) = \{\mathsf{E}_\mathsf{Q}F : \mathsf{Q} \in \mathcal{M}\} = \{\mathsf{E}_\mathsf{Q}F : \mathsf{Q} \sim \mathsf{P},\ \mathsf{E}_\mathsf{Q}\xi = 1\} = \{1\}\,,$$

which agrees with common sense.

By Theorem 3.10, the left endpoint of $I(F)$ coincides with $\inf_{\mathsf{Q}\in\mathcal{M}} \mathsf{E}_\mathsf{Q}F$ for all F bounded below, which solves the problem mentioned in the remark following Example 4.5.

Finally, in Example 4.6, P is the only local martingale measure for S. Indeed, if $\mathsf{Q} \sim \mathsf{P}$ is a local martingale measure for S, then S satisfies equation (12) with respect to Q. By Itô's theorem (see [34; Th. 5.2.1]), there are strong existence and pathwise uniqueness for this equation, and the Yamada–Watanabe theorem (see [36; Ch. IX, Th. 1.7]) implies uniqueness in law. Hence, $\mathsf{Q} = \mathsf{P}$. Since P is not a martingale measure, there exists no equivalent martingale measure. This means that the model considered in Example 4.6 does not satisfy the NGA condition, and the paradox is solved.

Remark. An "arbitrage opportunity" in the model of Example 4.6 can be constructed as follows. Consider the strategy $G = H - 1$, where H is given by (13). Then

$$\int_0^T G_u \mathrm{d}S_u = \int_0^T H_u \mathrm{d}S_u - S_T + S_0 = -\mathsf{E}_\mathsf{P}S_T + S_0 > 0\,.$$

The strategy G is not admissible, so it does not yield a free lunch with vanishing risk opportunity. It does not yield a generalized arbitrage opportunity either, but it can be used to construct a generalized arbitrage opportunity as follows. There exist simple strategies $(\widetilde{H}_n)_{n\in\mathbb{N}}$ such that

$$\sup_{t\in[0,T]} \left| \int_0^t \widetilde{H}_{nu} \mathrm{d}S_u - \int_0^t H_u \mathrm{d}S_u \right| \xrightarrow[n\to\inf_t y]{\mathsf{P}} 0\,.$$

Set

$$\tau_n = \inf\left\{t \in [0,T] : \int_0^t \widetilde{H}_{nu}\mathrm{d}S_u \leqslant -\mathsf{E}_\mathsf{P} S_T - 1\right\},$$

$$H_{nt} = \widetilde{H}_{nt} I(t \leqslant \tau_n), \quad t \in [0,T].$$

Since

$$\int_0^t H_u \mathrm{d}S_u \geqslant -\mathsf{E}_\mathsf{P} S_T, \quad t \in [0,T],$$

we get

$$\int_0^T H_{nu}\mathrm{d}S_u \xrightarrow[n\to\infty]{\mathsf{P}} \int_0^T H_u \mathrm{d}S_u = S_T - \mathsf{E}_\mathsf{P} S_T.$$

Set $G_n = H_n - 1$. Then, for $X_n = \int_0^T G_{nu}\mathrm{d}S_u$, we have $X_n \in A$, where A is given by (7). Furthermore, $X_n \geqslant -S_T + S_0 - \mathsf{E}_\mathsf{P} S_T - 1$ P-a.s. for any $n \in \mathbb{N}$ and

$$X_n \xrightarrow[n\to\infty]{\mathsf{P}} S_0 - \mathsf{E}_\mathsf{P} S_T > 0.$$

Note that $Z := S_T - S_0$ belongs to B, where B is given by (2). Take $Y_n \in L^0_+$, $n \in \mathbb{N}$ such that

$$\frac{X_n - Y_n}{Z + \gamma(Z)} = \frac{X_n}{Z + \gamma(Z)} \wedge (S_0 - \mathsf{E}_\mathsf{P} S_T),$$

and then

$$\frac{X_n - Y_n}{Z + \gamma(Z)} \xrightarrow[n\to\infty]{\sigma(L^\infty, L^1(\mathsf{P}))} \frac{S_0 - \mathsf{E}_\mathsf{P} S_T}{Z + \gamma(Z)}.$$

This yields a generalized arbitrage opportunity in the model of Example 4.6.

One of the problems associated with the model under consideration is related to the *change of numéraire*. It is as follows. Let $S = (S_t^1, \dots, S_t^d)_{t\in[0,T]}$ be the price process of d assets. We assume that each of its components is strictly positive. Fix $\alpha^1, \dots, \alpha^d \geqslant 0$ with $\sum_{i=1}^d \alpha^i > 0$ and define a numéraire as the combination $\sum_{i=1}^d \alpha^i S^i$. Now, define the discounted price process as $\overline{S} = S/\sum_{i=1}^d \alpha^i S^i$ and define the set of attainable $\overline{A}$ incomes by (7) or (8), depending on the choice of the approach. Now, choose another combination $\sum_{i=1}^d \beta^i S^i$ as a numéraire, define the new discounted process $\widetilde{S}$ as $\widetilde{S} = S/\sum_{i=1}^d \beta^i S^i$ and define the set of attainable incomes $\widetilde{A}$ through $\widetilde{S}$. The problem is whether the NFLVR/NGA property holds or not for both models $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ and $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ simultaneously.

In the traditional approach, the answer is negative as shown by the example below (it is borrowed from [13]). Let us mention in this connection the papers [14, 16] devoted to the study of conditions under which the NFLVR property is preserved under the change of numéraire.

Example 4.7. *Let $S^0 = 1$ and $S^1 = |B|^{-1}$, where B is a 3-dimensional Brownian motion started at a point $B_0 \neq 0$. Let $\mathcal{F}_t = \mathcal{F}_t^S$, $\mathcal{F} = \mathcal{F}_T$. If we take $\overline{S} = S/S^0$, then the model $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ ($\overline{A}$ is defined by (8)) satisfies* *the NFLVR condition since the process S^1 is a local martingale with respect to the original probability measure (see representation* (12)*). On the other hand, if we take $\widetilde{S} = S/S^1$, then $\widetilde{S}^0 = |B|$ (this is a 3-dimensional Bessel process). If* Q *is an equivalent sigma-martingale measure for $\widetilde{S}$, then, by the result of Ansel and Stricker [1], $\widetilde{S}^0$ is an $(\mathcal{F}_t, \mathsf{Q})$-local martingale. Using Itô's formula, one easily checks that the quadratic variation of $\widetilde{S}^0$ is given by $[\widetilde{S}^0]_t = t$. P. Lévy's characterization theorem (see [36; Ch. IV, Th. 3.8]) now implies that $\widetilde{S}^0$ is a* Q*-Brownian motion. But this contradicts the positivity of $\widetilde{S}^0$. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ does not satisfy the NFLVR condition.* □

In contrast, the change of numéraire preserves the NGA property as shown by the statement below.

Theorem 4.8 (Change of numéraire). *Let $\overline{A}$ (resp., $\widetilde{A}$) be defined through $\overline{S}$ (resp., $\widetilde{S}$) by* (7)*. Then the models $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ and $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ satisfy or do not satisfy the NGA condition simultaneously.*

The proof employs the following statement (see [27; Ch. III, Prop. 3.8]).

Lemma 4.9. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space and $\mathsf{Q} \ll \mathsf{P}$. Let $Z_t = \frac{d\mathsf{Q}|\mathcal{F}_t}{d\mathsf{P}|\mathcal{F}_t}$ be the density process of* Q *with respect to* P*. Then a process M is an $(\mathcal{F}_t, \mathsf{Q})$-martingale if and only if MZ is an $(\mathcal{F}_t, \mathsf{P})$-martingale.*

Proof of Theorem 4.8. Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, \overline{A})$ satisfies the NGA condition. Then there exists a probability measure $\overline{\mathsf{Q}} \sim \mathsf{P}$ such that $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale. Let $\overline{Z}$ denote the density process of $\overline{\mathsf{Q}}$ with respect to P. Consider the process

$$\widetilde{Z} = c\overline{Z}\,\frac{\sum_{i=1}^d \beta^i S^i}{\sum_{i=1}^d \alpha^i S^i}\,,$$

where the constant c is chosen in such a way that $\widetilde{Z}_0 = 1$. As $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale, then, by Lemma 4.9, $\widetilde{Z}$ is an $(\mathcal{F}_t, \mathsf{P})$-martingale. Hence, $\widetilde{Z}$ is the density process of the probability measure $\widetilde{\mathsf{Q}} = \widetilde{Z}_T\mathsf{P}$ with respect to P (note that $\widetilde{\mathsf{Q}} \sim \mathsf{P}$ since $\widetilde{Z}$ is strictly positive). As $\overline{S}$ is an $(\mathcal{F}_t, \overline{\mathsf{Q}})$-martingale, then, by Lemma 4.9, the process $\widetilde{S}\widetilde{Z} = c\overline{S}\,\overline{Z}$ is an $(\mathcal{F}_t, \mathsf{P})$-martingale, which (again by Lemma 4.9) implies that $\widetilde{S}$ is an $(\mathcal{F}_t, \widetilde{\mathsf{Q}})$-martingale. Hence, the model $(\Omega, \mathcal{F}, \mathsf{P}, \widetilde{A})$ satisfies the NGA condition. □

5 Dynamic model with infinite time horizon

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t^i)_{t\in\mathbb{R}_+}$, $i \in I$ be a family of real-valued $(\mathcal{F}_t)$-adapted càdlàg processes with components bounded below. From the financial point of view,

S_t^i is the discounted price of the ith asset at time t. Define the set of attainable incomes by

$$A = \Bigg\{ \sum_{n=1}^{N} \sum_{i \in I} H_n^i (S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \leqslant \cdots \leqslant u_N \text{ are } (\mathcal{F}_t)\text{-stopping times such that } \{u_N = \infty\} \subseteq \big\{ \text{for any } i,\ \exists \lim_{t \to \infty} S_t^i \big\}, \\ H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H_n^i = 0 \text{ for all } i, \text{ except for a finite set} \Bigg\}. \tag{14}$$

Notation. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for any } i \in I,\ S^i \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-uniformly integrable martingale}\}.$$

Key Lemma 5.1. *Suppose that I is countable and, for all i, the limit $S_\infty^i = \lim_{t \to \infty} S_t^i$ exists P-a.s. Then, for the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}\bigg(\sum_{i \in I} \lambda^i (S_\infty^i - S_0^i) \bigg) = \mathcal{M},$$

where the constants $\lambda^i > 0$ are chosen in such a way that $\sum_{i \in I} \lambda^i S_T^i < \infty$ a.s. and $\sum_{i \in I} \lambda^i S_0^i < \infty$.

Proof. Note that $(S_t^i)_{t \in \mathbb{R}_+}$ is an $(\mathcal{F}_t, \mathsf{Q})$-uniformly integrable martingale if and only if $(S_t^i)_{t \in [0,\infty]}$ is a $(\mathcal{G}_t, \mathsf{Q})$-martingale, where

$$\mathcal{G}_t = \begin{cases} \mathcal{F}_t & \text{if } t \in \mathbb{R}_+, \\ \mathcal{F} & \text{if } t = \infty \end{cases}$$

(this statement follows from [36; Ch. II, Th. 3.1]). The desired statement can now be proved in the same way as Key Lemma 4.1. □

Since Key Lemma 5.1 contains an additional assumption, Theorem 3.6 cannot be applied immediately, and the proof of the FTAP in this model requires a bit of additional work.

Corollary 5.2. *Suppose that I is countable. Then the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies the NGA condition if and only if there exists an equivalent uniformly integrable martingale measure (i.e., $\mathcal{M} \neq \emptyset$).*

Proof. Step 1. Let us prove the "only if" implication. Lemma 3.7 applied to the $\sigma(L^\infty, L^1(\mathsf{P}))$-closed convex cone $A_4(0)$ yields a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X \leqslant 0$ for all $X \in A$ that is bounded below. For any $i \in I$, any $u \leqslant v \in \mathbb{R}_+$, and any $D \in \mathcal{F}_u$ such that S_u^i is bounded on D, the random variable $I_D(S^i_v - S^i_u)$ is bounded below, and hence, $\mathsf{E}_\mathsf{Q} I_D(S^i_v - S^i_u) \leqslant 0$. This shows that S^i is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale. By Doob's supermartingale convergence theorem (see [36; Ch. II, Th. 2.10]), the limit $\lim_{t\to\infty} S^i_t$ exists Q-a.s., and hence, P-a.s. Now, Theorem 3.6, combined with Key Lemma 5.1, yields the desired statement.

Step 2. Let us prove the "if" implication. Take $\mathsf{Q} \in \mathcal{M}$. Then, by Doob's theorem, for any $i \in I$, $\lim_{t\to\infty} S^i_t$ exists Q-a.s., and hence, P-a.s. Now, Theorem 3.6, combined with Key Lemma 5.1, yields the desired statement. □

It has been shown in the proof of Corollary 5.2 that the NGA condition implies the existence of $\lim_{t\to\infty} S_t$ P-a.s. Hence, Theorem 3.10 can be applied with no additional assumptions.

It would be more natural to define the set of attainable incomes in this model as

$$A = \Bigg\{ \sum_{n=1}^{N} \sum_{i\in I} H^i_n (S^i_{u_n} - S^i_{u_{n-1}}) : N \in \mathbb{N},\ u_0 \leqslant \dots \leqslant u_N < \infty \text{ are } (\mathcal{F}_t)\text{-stopping times, } H^i_n \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H^i_n = 0 \text{ for all } i, \text{ except for a finite set} \Bigg\}.$$

However, for this choice of A we can only establish the equality $\mathcal{R} = \mathcal{M}$ (in the lemma below, I is arbitrary), but we cannot prove that Assumption 3.5 is satisfied.

Lemma 5.3. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have $\mathcal{R} = \mathcal{M}$.*

Proof. Step 1. The inclusion $\mathcal{M} \subseteq \mathcal{R}$ follows from the similar inclusion in Key Lemma 5.1.

Step 2. Let us prove the inclusion $\mathcal{R} \subseteq \mathcal{M}$. Choose $\mathsf{Q} \in \mathcal{R}$. Fix $i \in I$. For any $u \leqslant v \in \mathbb{R}_+$ and $D \in \mathcal{F}_u$, we have $\mathsf{E}_\mathsf{Q} I_D(S^i_v - S^i_u) = 0$ since S^i is bounded below. Hence, S^i is an $(\mathcal{F}_t, \mathsf{Q})$-martingale.

By Doob's supermartingale convergence theorem, there exists a limit $S^i_\infty = \text{(a.s.)} \lim_{t\to\infty} S^i_t$. By the Fatou lemma for conditional expectations,

$$\mathsf{E}_\mathsf{Q}(S^i_\infty \mid \mathcal{F}_t) \leqslant S^i_t, \quad t \geqslant 0. \tag{15}$$

In particular, $\mathsf{E}_\mathsf{Q} S^i_\infty \leqslant S^i_0$.

Suppose that $\mathsf{E}_\mathsf{Q} S^i_\infty < S^i_0$. The process $X_t = \mathsf{E}_\mathsf{Q}(S^i_\infty \mid \mathcal{F}_t)$, $t \geqslant 0$ has a càdlàg modification. Furthermore, $X_t \xrightarrow[t\to\infty]{\mathsf{Q}\text{-a.s.}} S^i_\infty$. Consequently, the stopping time

$$\tau = \inf \left\{ t \geqslant 0 : |S^i_t - X_t| \leqslant \frac{S^i_0 - \mathsf{E}_\mathsf{Q} S^i_\infty}{2} \right\}$$

is finite Q-a.s. From $\mathsf{Q} \in \mathcal{R}$ and the positivity of S^i it follows that $\mathsf{E}_\mathsf{Q} S^i_\tau = S^i_0$.

Thus,

$$\mathsf{E}_{\mathsf{Q}} X_\tau \geqslant S_0^i - \frac{S_0^i - \mathsf{E}_{\mathsf{Q}} S_\infty^i}{2} > \mathsf{E}_{\mathsf{Q}} S_\infty^i \,.$$

But this contradicts the equality $\mathsf{E}_{\mathsf{Q}} X_\tau = \mathsf{E}_{\mathsf{Q}} S_\infty^i$, which is a consequence of the optional stopping theorem for uniformly integrable martingales (see [36; Ch. II, Th. 3.2]). As a result, $\mathsf{E}_{\mathsf{Q}} S_\infty^i = S_0^i$. This, combined with (15), yields $\mathsf{E}_{\mathsf{Q}}(S_\infty^i \mid \mathcal{F}_t) = S_t^i$, $t \geqslant 0$. The proof is completed. □

The traditional approach to arbitrage pricing in dynamic models with infinite time horizon is the same as the one for continuous-time models with a finite time horizon. The only difference is that the set of attainable incomes given by (8) should be replaced by

$$A = \left\{ \int_0^\infty H_u \mathrm{d}S_u : H \text{ is } (\mathcal{F}_t)\text{-predictable, } S\text{-integrable,} \right.$$
$$\left. \text{admissible, and such that } \lim_{t\to\infty} \int_0^t H_u \mathrm{d}S_u \text{ exists } \mathsf{P}\text{-a.s.} \right\} .$$

Here $\int_0^\infty H_u \mathrm{d}S_u := \lim_{t\to\infty} \int_0^t H_u \mathrm{d}S_u$. (This might be called an improper stochastic integral. Alternatively, one can use the stochastic integral up to infinity; see [7]. The FTAP remains the same for these two types of integrals).

Many models with infinite time horizon that are arbitrage-free in the traditional approach (i.e., satisfy the NFLVR condition for predictable admissible strategies) are not arbitrage-free in the proposed approach (i.e., do not satisfy the NGA condition for simple strategies). This is illustrated by the following example.

Example 5.4. *Let* $S_t = \mathrm{e}^{B_t - t/2}$*, where* B *is a Brownian motion. Let* $\mathcal{F}_t = \mathcal{F}_t^S$*,* $\mathcal{F} = \bigvee_{t\geqslant 0} \mathcal{F}_t$*. This model satisfies the NFLVR condition since the process* S *is a martingale (and hence, a sigma-martingale) with respect to the original probability measure. On the other hand, this model does not satisfy the NGA condition. Indeed, consider the stopping time* $v = \inf\{t \geqslant 0 : S_t = 1/2\}$*. The random variable* $-S_v + S_0 = 1/2$ *belongs to the set* A *given by* (14)*. Hence, the NGA condition is violated.*

From the financial point of view, the strategy providing generalized arbitrage in this model consists in short selling the asset at time 0 *and buying it back at time* v*. Note that this strategy is prohibited in the traditional approach by the admissibility condition.* □

Remark. A *"buy and hold" strategy* consists in buying an asset, waiting until its discounted price reaches some higher level, and selling it back at that time. The opposite (it may be called "sell and wait") strategy consists in the short selling of an asset, waiting until its discounted price reaches some lower level and buying it back at that time. In many models (like the one described above) such "sell and wait" strategies lead to arbitrage opportunities. In the traditional approach, these strategies are prohibited by the admissibility condition. In the approach proposed here, such strategies are allowed, but the models where they yield arbitrage opportunities are "prohibited," in the sense that they do not satisfy the NGA condition. Indeed, if the NGA condition is satisfied, then there exists an equivalent uniformly integrable martingale measure. But a uniformly integrable martingale with a strictly positive probability never reaches a preassigned level, so in models satisfying the NGA condition the "sell and wait" strategy does not yield an arbitrage opportunity.

To conclude the section, we show that no "stationary" model with infinite time horizon satisfies the NGA condition. We say that a real-valued process Z has *stationary increments* if $Z_{t+h} - Z_{s+h} \overset{\text{Law}}{=} Z_t - Z_s$ for all $s \leqslant t \in \mathbb{R}_+$, $h \in \mathbb{R}_+$.

Proposition 5.5. *Let $S_t = S_0 e^{Z_t}$ where Z has stationary increments and $\mathsf{P}(Z_t \neq Z_0) > 0$ for some $t \in \mathbb{R}_+$. Then the NGA condition is not satisfied.*

Proof. Suppose that the NGA condition is satisfied. Without loss of generality, we can assume that $\mathsf{P}(Z_t \neq Z_0) > 0$ for some $t \in \mathbb{R}_+$. The reasoning used in the proof of Corollary 5.2 shows that $\lim_{t\to\infty} S_t$ exists P-a.s. Hence, $\lim_{t\to\infty} Z_t =: Z_\infty$ exists P-a.s. (this limit takes on values in $[-\infty, \infty)$). Denote $\mathsf{P}(Z_\infty > -\infty)$ by p. Fix $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that $N > 1/\varepsilon$ and

$$\mathsf{P}(Z_\infty > -\infty \text{ and } |Z_n - Z_\infty| < \varepsilon \text{ for any } n \geqslant N) > p - \varepsilon\,.$$

Then

$$\mathsf{P}(Z_\infty > -\infty \text{ and } |Z_{2N} - Z_N| < 2\varepsilon) > p - \varepsilon\,.$$

Since $Z_{2N} - Z_N \overset{\text{Law}}{=} Z_N$, we get $\mathsf{P}(|Z_N| < 2\varepsilon) > p - \varepsilon$. As ε can be chosen arbitrarily small, we conclude that $\mathsf{P}(Z_\infty = 0) = p$. Hence, $Z_\infty = 0$ P-a.e. on the set $\{Z_\infty > -\infty\}$. This means that Z_∞ takes on only values $-\infty$ and 0.

Take $t \in \mathbb{R}_+$ such that $\mathsf{P}(Z_t \neq Z_0) > 0$. Choose $\alpha > 0$ such that $\mathsf{P}(|Z_t - Z_0| > \alpha) > 0$. For any $T \in \mathbb{R}_+$,

$$\mathsf{P}(|Z_{T+t} - Z_T| > \alpha) = \mathsf{P}(|Z_t - Z_0| > \alpha) > 0\,.$$

Consequently, $\mathsf{P}(Z_\infty = 0) < 1$. Thus, S_∞ takes on only values 0 and S_0, and $\mathsf{P}(S_\infty = 0) > 0$. Then $\mathcal{M} = \emptyset$, and, by Corollary 5.2, the NGA condition is not satisfied. □

Corollary 5.6. *Let $S_t = S_0 e^{Z_t}$ where Z is a Lévy process that is not identically equal to zero. Then the NGA condition is not satisfied.*

6 Model with european call options as basic assets

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $T \in [0, \infty]$. Let S_T be an $\mathbb{R}_+$-valued random variable. From the financial point of view, S_T is the price of some asset at time T. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices of European call options on this asset with maturity T (in practice $\mathbb{K}$ is finite, but in theory it is often assumed that $\mathbb{K} = \mathbb{R}_+$) and let $\varphi(K)$, $K \in \mathbb{K}$ be the price at time 0 of a European call option with the payoff $(S_T - K)^+$. Define the set of attainable incomes by

$$A = \left\{ \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) : N \in \mathbb{N},\ K_n \in \mathbb{K},\ h_n \in \mathbb{R} \right\}.$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading at times 0 and T European call options on "our" asset with maturity T. We assume that $0 \in \mathbb{K}$, which means the possibility to trade the underlying asset.

Notation. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \mathrm{Law}_{\mathsf{Q}}\, S_T \in \mathcal{D}\},$$

where

$$\mathcal{D} = \{\psi'' : \psi \text{ is convex on } \mathbb{R}_+,\ \psi'_+(0) \geqslant -1,\ \lim_{x \to \infty} \psi(x) = 0,$$
$$\text{and } \psi(K) = \varphi(K),\ K \in \mathbb{K}\}.$$

Here ψ'_+ denotes the right-hand derivative and ψ'' denotes the second derivative taken in the sense of distributions (i.e., $\psi''((a, b]) = \psi'_+(b) - \psi'_+(a)$) with the convention: $\psi''(\{0\}) = \psi'_+(0) + 1$ (thus, ψ'' is a probability measure provided that $\psi'_+(0) \geqslant -1$).

Key Lemma 6.1. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(S_T - \varphi(0)) = \mathcal{M}.$$

Proof. Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(S_T - \varphi(0))$ follows from Lemma 3.4.

Step 2. Let us prove the inclusion $\mathcal{R}(S_T - \varphi(0)) \subseteq \mathcal{M}$. Fix $\mathsf{Q} \in \mathcal{R}(S_T - \varphi(0))$. By considering the function $\psi(x) = \mathsf{E}_{\mathsf{Q}}(S_T - x)^+$, we conclude that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Fix $\mathsf{Q} \in \mathcal{M}$. Then

$$\mathsf{E}_{\mathsf{Q}}(S_T - K)^+ = \int_{\mathbb{R}_+} (x - K)^+ \psi''(\mathrm{d}x) = \psi(K) = \varphi(K), \quad K \in \mathbb{K}.$$

Consequently, $\mathsf{E}_{\mathsf{Q}} X = 0$ for all $X \in A$, which implies that $\mathsf{Q} \in \mathcal{R}$. □

Recalling Theorems 3.6 and 3.10, we get

Corollary 6.2. *Let $\mathbb{K} = \mathbb{R}_+$.*

(i) *The NGA is satisfied if and only if*

(a) *φ is convex;*

(b) $\varphi'_+(0) \geqslant -1$;
(c) $\lim_{x\to\infty} \varphi(x) = 0$;
(d) $\varphi'' \sim \mathrm{Law}_{\mathsf{P}}\, S_T$.

(ii) *Suppose that the NGA is satisfied. Let $F = f(S_T)$, where f is bounded below. Then*

$$I(F) = \begin{cases} \left\{ \int_{\mathbb{R}_+} f(x)\varphi''(\mathrm{d}x) \right\} & \text{if } \int_{\mathbb{R}_+} f(x)\varphi''(\mathrm{d}x) < \infty, \\ \emptyset & \text{otherwise.} \end{cases}$$

We conclude this section by three interesting examples. The first example shows that the ordinary NA condition (which means that $A \cap L^0_+ = \{0\}$) is too weak for the model under consideration.

Example 6.3. *Let $\mathbb{K} = \mathbb{R}_+$,*

$$\mathsf{P}(S_T \in A) = \frac{1}{2}\left(I(1 \in A) + \int_A \mathrm{e}^{-x}\mathrm{d}x \right), \quad A \in \mathcal{B}(\mathbb{R}_+),$$

and $\varphi(K) = \mathrm{e}^{-K}$. This model satisfies the NA condition. Indeed, suppose that there exists

$$X = \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

such that $X \geqslant 0$ P-a.s. and $\mathsf{P}(X > 0) > 0$. Note that X can be represented as $X = f(S_T)$ with a continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}_+} f(x)\mathrm{e}^{-x}\mathrm{d}x = \sum_{n=1}^{N} h_n \int_{\mathbb{R}_+} ((x - K_n)^+ - \mathrm{e}^{-K_n})\mathrm{e}^{-x}\mathrm{d}x = 0. \tag{16}$$

The above assumptions on X imply that $f \geqslant 0$ everywhere and f is not identically equal to zero. But this contradicts (16)*. Thus, the NA condition is satisfied.*

Consider now $F = I(S_T = 1)$. For every $\varepsilon > 0$, consider the function $f_\varepsilon(x) = (1 - \varepsilon^{-1}|x - 1|)^+$. It is seen from the representation

$$f_\varepsilon(S_T) = \frac{1}{\varepsilon}(S_T - 1 + \varepsilon)^+ - \frac{2}{\varepsilon}(S_T - 1)^+ + \frac{1}{\varepsilon}(S_T - 1 - \varepsilon)^+$$

that the random variables

$$X_\varepsilon = f_\varepsilon(S_T) - \int_{\mathbb{R}_+} f_\varepsilon(x)\mathrm{e}^{-x}\mathrm{d}x$$

belong to A *and*

$$X_\varepsilon + \int_{\mathbb{R}_+} f_\varepsilon(x)\mathrm{e}^{-x}\mathrm{d}x \geqslant F.$$

As $\int_{\mathbb{R}_+} f_\varepsilon(x)\mathrm{e}^{-x}\mathrm{d}x \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$, *it is reasonable to conclude that the fair price of* F *should not exceed 0 (thus, the fair price should equal 0 since* F *is positive). But on the other hand,* $\mathsf{P}(F = 1) = 1/2$, *so that we obtain a contradiction with common sense. The reason is that this model is not "fair" because one can construct "asymptotic arbitrage" taking* X_ε *with* $\varepsilon \downarrow 0$. □

The second example shows that the NFL condition (see Remark (iii) following Definition 3.2) is also too weak for the model under consideration.

Example 6.4. *Let* $\mathbb{K} = \mathbb{R}_+$, $\mathsf{P}(S_T \leqslant x) = 1 - \mathrm{e}^{-x}$, *and* $\varphi(K) = \mathrm{e}^{-K} + 1$. *This model satisfies the NFL condition. Indeed, let*

$$X = \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

be bounded below. Note that X *can be represented as* $X = f(S_T)$ *with a continuous function* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfying*

$$\lim_{x\to\infty} \frac{f(x)}{x} = \sum_{n=1}^{N} h_n\,.$$

The assumption on X *implies that* $\sum_{n=1}^{N} h_n \geqslant 0$. *Then we can write*

$$X \leqslant \sum_{n=1}^{N} h_n\big((S_T - K_n)^+ - \mathrm{e}^{-K_n}\big) = g(S_T) - \int_{\mathbb{R}_+} g(x)\mathrm{e}^{-x}\mathrm{d}x = g(S_T) - \mathsf{E}_\mathsf{P} g(S_T)$$

with $g(x) = \sum_{n=1}^{N} h_n(x - K_n)^+$. *This implies that, for any* $X \in A_4(0)$ *(*$A_4(0)$ *is defined by (3)), we have* $\mathsf{E}_\mathsf{P} X \leqslant 0$, *so that the NFL condition is satisfied.*

On the other hand, in this model the price of a European call option tend to 1 as the strike price tend to $+\infty$, *which contradicts common sense. Thus, this model is not "fair" since one can construct "asymptotic arbitrage" by selling European call options with strike price* $K \to +\infty$. □

The third example shows that $I(F)$ might not coincide with the interval whose endpoints are $V_*(F)$ and $V^*(F)$ defined by (5) and (6). Thus, in general the proposed approach to arbitrage pricing yields a finer interval of fair prices than the traditional approach based on sub- and super-replication.

Example 6.5. *Let* $\mathbb{K} = \mathbb{R}_+$, $\mathsf{P}(S_T \leqslant x) = 1 - \mathrm{e}^{-x}$, *and* $\varphi(K) = \mathrm{e}^{-K}$. *This model satisfies the NGA condition since* $\mathsf{P} \in \mathcal{M}$. *Choose* $D \in \mathcal{B}(\mathbb{R}_+)$ *such that, for any* $a < b \in \mathbb{R}_+$, *the sets* $D \cap [a, b]$ *and* $[a, b] \setminus D$ *have a strictly positive Lebesgue measure. Consider* $F = I(S_T \in D)$.

Let us find $V^*(F)$ *defined by* (6). *Let* $x \in \mathbb{R}$ *and*

$$X = \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) \in A$$

be such that $x + X \geqslant F$ P*-a.s. We can write*

$$X = g(S_T) - \sum_{n=1}^{N} h_n \mathrm{e}^{-K_n} = g(S_T) - \int_{\mathbb{R}_+} g(y)\mathrm{e}^{-y}\mathrm{d}y$$

with $g(y) = \sum_{n=1}^{N} h_n(y - K_n)^+$*. Thus,*

$$x + g(S_T) - \int_{\mathbb{R}_+} g(y)\mathrm{e}^{-y}\mathrm{d}y \geqslant I(S_T \in D) \quad \mathsf{P}\text{-}a.s.$$

Using the continuity of g *and the properties of* D*, we get*

$$x + g(z) - \int_{\mathbb{R}_+} g(y)\mathrm{e}^{-y}\mathrm{d}y \geqslant 1 \text{ for any } z \in \mathbb{R}_+ .$$

This implies that $x \geqslant 1$*. Consequently,* $V^*(F) = 1$*.*

In a similar way one checks that $V_*(F) = 0$*. On the other hand, by Corollary 6.2 (ii),* $I(F) = \left\{\int_D \mathrm{e}^{-y}\mathrm{d}y\right\}$*.*

7 Mixed model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial. Let $(S_t)_{t\in[0,T]}$ be an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted càdlàg process. From the financial point of view, S_t is the (discounted) price of some asset at time t. Let $\varphi_t(K)$ be the price of a European call option on this asset with maturity t and strike price K (we assume that such an option exists for any $t \in [0,T],\ K \in \mathbb{R}_+$). Define the set of attainable incomes by

$$A = \Bigg\{ \sum_{m=1}^{M} H_m(S_{u_m} - S_{u_{m-1}}) + \sum_{n=1}^{N} h_n((S_{v_n} - K_n)^+ - \varphi_{v_n}(K_n)) : \\ M, N \in \mathbb{N},\ u_0 \leqslant \dots \leqslant u_M \text{ are } (\mathcal{F}_t)\text{-stopping times}, \\ H_n \text{ is } \mathcal{F}_{u_n}\text{-measurable},\ h_n \in \mathbb{R},\ v_n \in [0,T],\ K_n \in \mathbb{R}_+ \Bigg\}.$$

From the financial point of view, A is the set of discounted incomes that can be obtained by trading "our" asset on the interval $[0,T]$ and trading European call options on this asset.

Notation. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale and } \operatorname{Law}_{\mathsf{Q}} S_t = \varphi_t'',\ t \in [0,T]\}$$

provided that, for all $t \in [0,T]$, the function φ_t is convex, $(\varphi_t)'_+(0) \geqslant -1$, and $\lim_{x\to\infty} \varphi_t(x) = 0$. Otherwise, we set $\mathcal{M} = \emptyset$.

Key Lemma 7.1. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}(S_T - S_0) = \mathcal{M}\,.$$

Proof. Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(S_T - S_0)$ follows from Lemma 3.4.

Step 2. Let us prove the inclusion $\mathcal{R}(S_T - S_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(S_T - S_0)$. The proof of Key Lemma 4.1 (Step 2) shows that S is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. For any $t \in [0,T]$, $K \in \mathbb{R}_+$, we have

$$\mathsf{E}_{\mathsf{Q}}(S_t - S_0 - (S_t - K)^+ + \varphi_t(K)) = 0$$

since the random variable under the expectation belongs to A and is bounded. By the martingale property of S, $\mathsf{E}_{\mathsf{Q}}(S_t - S_0) = 0$, which implies that $\mathsf{E}_{\mathsf{Q}}(S_t - K)^+ = \varphi_t(K)$. As a result, $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Fix

$$X = \sum_{m=1}^{M} H_m(S_{u_m} - S_{u_{m-1}}) + \sum_{n=1}^{N} h_n((S_{v_n} - K_n)^+ - \varphi_{v_n}(K_n)) = X_1 + X_2 \in A\,.$$

Clearly, X_2 is Q-integrable and $\mathsf{E}_{\mathsf{Q}} X_2 = 0$. The proof of Key Lemma 4.1 (Step 3) shows that $\mathsf{E}_{\mathsf{Q}} X_1^- \geqslant \mathsf{E}_{\mathsf{Q}} X_1^+$. This leads to the inequality $\mathsf{E}_{\mathsf{Q}} X^- \geqslant \mathsf{E}_{\mathsf{Q}} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

References

1. Ansel, J.P., Stricker, C (1994): Couverture des actifs contingents et prix maximum. Annales de l'Institut Henri Poincaré, **30**, No. 2, 303–315
2. Breeden, D.T., Litzenberger, R.H. (1978): Prices of state-contingent claims implicit in option prices. Journal of Business, **51**, No. 4, 621–651
3. Campi, L. (2004): Arbitrage and completeness in financial markets with given N-dimensional distributions. Decisions in Economics and Finance
4. Carr, P., Geman, H., Madan, D., Yor., M. (2003): Stochastic volatility for Lévy processes. Mathematical Finance, **13**, No. 3, 345–382
5. Cherny, A.S. (2007a): General arbitrage pricing model: II – Transaction costs (in this volume)
6. Cherny, A.S. (2007b): General arbitrage pricing model: III – Possibility approach (in this volume)
7. Cherny, A.S., Shiryaev, A.N. (2004): On stochastic integrals up to infinity and predictable criteria for integrability. Lecture Notes in Mathematics, **1857**, 165–185

8. Chou, C.S. (1979): Caractérisation d'une classe de semimartingales. Lecture Notes in Mathematics, **721**, 250–252
9. Cvitanić, J., Karatzas, I. (1996): Hedging and portfolio optimization under transaction costs: a martingale approach. Mathematical Finance, **6**, No. 2, 133–165
10. Cvitanić, J., Pham, H., Touzi, N. (1999): A closed-form solution to the problem of super-replication under transaction costs. Finance and Stochastics, **3**, No. 1, 35–54
11. Dalang, R.C., Morton, A., Willinger, W. (1990): Equivalent martingale measures and no-arbitrage in stochastic securities market models. Stochastics and Stochastic Reports, **29**, No. 2, 185–201
12. Delbaen, F., Schachermayer, W. (1994): A general version of the fundamental theorem of asset pricing. Mathematische Annalen, **300**, 463–520
13. Delbaen, F., Schachermayer, W. (1995): Arbitrage possibilities in Bessel processes and their relation to local martingales. Probability Theory and Related Fields, **102**, 357–366
14. Delbaen, F., Schachermayer, W. (1995): The no-arbitrage property under a change of numéraire. Stochastics and Stochastics Reports, **53**, 213–226
15. Delbaen, F., Schachermayer, W. (1998): The fundamental theorem of asset pricing for unbounded stochastic processes. Mathematische Annalen, **312**, 215–250
16. Delbaen, F., Shirakawa, H. (1996): A note on the no arbitrage condition for international financial markets. Financial Engineering and the Japanese Markets, **3**, 239–251
17. Émery, M. (1980): Compensation de processus v.f. non localement intégrables. Lecture Notes in Mathematics, **784**, 152–160
18. Föllmer, H., Kabanov, Yu.M. (1998): Optional decomposition and Lagrange multipliers. Finance and Stochastics, **2**, No. 1, 69–81
19. Föllmer, H., Kramkov, D.O. (1997): Optional decomposition under constraints. Probability Theory and Related Fields, **109**, No. 1, 1–25
20. Föllmer, H., Schied, A. (2004): Stochastic finance. An introduction in discrete time. 2nd Ed. Walter de Gruyter
21. Goll, T., Kallsen, J. (2003): A complete explicit solution to the log-optimal portfolio problem. Annals of Applied Probability, **13**, No. 2, 774–799
22. Harrison, J.M., Kreps, D.M. (1979): Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, **20**, 381–408
23. Harrison, J.M., Pliska, S.R. (1981): Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and Their Applications, **11**, No. 3, 215–260
24. Hull, J.C. (1997): Options, futures, and other derivatives. 3rd Ed. Prentice-Hall, Englewood Cliffs. NJ
25. Jackwerth, J. (1999): Option implied risk-neutral distributions and implied binomial trees: a literature review. Journal of Derivatives, **7**, No. 2, 66–82
26. Jacod, J., Shiryaev, A.N. (1998): Local martingales and the fundamental asset pricing theorems in the discrete-time case. Finance and Stochastics, **2**, No. 3, 259–273
27. Jacod, J., Shiryaev, A.N. (2003): Limit theorems for stochastic processes. 2nd Ed. Springer, Berlin Heidelberg New York
28. Kabanov, Yu.M. (1997): On the FTAP of Kreps–Delbaen–Schachermayer. Statistics and control of stochastic processes. The Liptser Festschrift. World Scientific, 191–203

29. Kabanov, Yu.M., Kramkov, D.O. (1994): No-arbitrage and equivalent martingale measures: a new proof of the Harrison–Pliska theorem. Theory of Probability and Its Applications, **39**, No. 3, 523–527
30. Kabanov, Yu.M., Stricker, C. (2001): A teacher's note on no-arbitrage criteria. Lecture Notes in Mathematics, **1755**, 149–152
31. Kramkov, D.O. (1996): Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. Probability Theory and Related Fields, **105**, No. 4, 459–479
32. Kreps, D. (1981): Arbitrage and equilibrium in economies with infinitely many commodities. Journal of Mathematical Economics, **8**, 15–35
33. Madan, D., Yor, M. (2002): Making Markov martingales meet marginals: with explicit constructions. Bernoulli, **8**, No. 4, 509–536
34. Øksendal, B. (2003): Stochastic differential equations: an introduction with applications. 6th Ed. Springer, Berlin Heidelberg New York
35. Rogers, L.C.G. (1994): Equivalent martingale measures and no-arbitrage. Stochastics and Stochastics Reports, **51**, No. 1, 41–49
36. Revuz, D., Yor, M. (1999): Continuous martingales and Brownian motion. 3rd Ed. Springer, Berlin Heidelberg New York
37. Schachermayer, W. (1992): A Hilbert space proof of the fundamental theorem of asset pricing in finite discrete time. Insurance: Mathematics and Economics, **11**, No. 4, 249–257
38. Shiryaev, A.N. (1999): Essentials of stochastic finance. World Scientific
39. Shiryaev, A.N., Cherny, A.S. (2002): Vector stochastic integrals and the fundamental theorems of asset pricing. Proceedings of the Steklov Mathematical Institute, **237**, 12–56
40. Sin, C.-A. (1996): Strictly local martingales and hedge ratios in stochastic volatility models. Ph.D. Thesis, Cornell University, Ithaka, NY
41. Stricker, C. (1990): Arbitrage et lois de martingale. Annales de l'Institut Henri Poincaré, Probab. et Statist., **26**, No. 3, 451–460
42. Yan, J.A. (1980): Caractérisation d'une classe d'ensembles convexes de L^1 ou H^1. Lecture Notes in Mathematics, **784**, 220–222
43. Yan, J.A. (1998): A new look at the fundamental theorem of asset pricing. Journal of Korean Mathematical Society, **35**, No. 3, 659–673

General Arbitrage Pricing Model: II – Transaction Costs

Alexander Cherny

Department of Probability Theory, Faculty of Mechanics and Mathematics,
Moscow State University
119992 Moscow, Russia
e-mail: alexander.cherny@gmail.com
http://mech.math.msu.su/~cherny/

Summary. In this paper we apply the general framework introduced in [2] to two models with transaction costs:

- a dynamic model with an infinite number of assets;
- a model with European call options as basic assets.

In particular, it is proved that a dynamic model with an infinite number of assets satisfies the No Generalized Arbitrage condition (this notion was introduced in [2]) if and only if there exist an equivalent measure and a martingale with respect to this measure that lies (componentwise) between the discounted ask and bid price processes. Furthermore, the set of fair prices of a contingent claim coincides with the set of expectations of the payoff with respect to these measures.

Our approach to arbitrage pricing in models with transaction costs differs from the existing ones.

Key words: Delta-martingale, Fair price, Fundamental theorem of asset pricing, General arbitrage pricing model, Generalized arbitrage, Risk-neutral measure, Set of attainable incomes, Transaction costs

1 Introduction

1.1 Purpose of the paper

Models with transaction costs have recently attracted much attention in the financial mathematics literature. Let us mention, in particular, the papers [4], [6], [12], [13], [14], [15], [16], [17], [19], [22] dealing with arbitrage pricing in such models. These papers differ in the level of generality, conditions imposed on price processes, definition of a strategy, definition of a price, and the form of representation of results.

In the paper [2], we introduced a unified approach to pricing contingent claims through a new concept of *generalized arbitrage*. (The necessary definitions and statements from [2] are collected in Section 2). In the framework of a general arbitrage pricing model, we proved in [2] the fundamental theorem of asset pricing and established the form of the fair price intervals. The general approach of [2] allows one to consider in a simple and unified manner various models of arbitrage pricing theory, some of which have so far been investigated separately and by different techniques. These include

- static as well as dynamic models;
- models with an infinite number of assets;
- models with transaction costs.

The purpose of this paper is to "project" the general results of [2] on two models with transaction costs.

1.2 Dynamic model with an infinite number of assets

This model is considered in Section 3. In order to apply the general results of [2], one only needs to establish the structure of the set of equivalent *risk-neutral measures* (see Definition 2.3). We prove that an equivalent measure Q is a risk-neutral measure if and only if there exists a Q-martingale that lies componentwise between the discounted ask and bid price processes. Then the general results of [2] show that the absence of generalized arbitrage (see Definition 2.2) is equivalent to the existence of such a measure, while the set of fair prices of a contingent claim coincides with the set of expectations of its payoff with respect to the class of these measures.

Our approach to arbitrage pricing in dynamic models with transaction costs is different from the approaches of all papers mentioned above. First of all, our model is completely general in the sense that we consider an arbitrary Ω, the continuous-time case (so that the discrete-time case is covered as well), and arbitrary (not only proportional) transaction costs. There are no assumptions on the probabilistic structure of price evolution (like the assumption that the price is a geometric Brownian motion). We consider a model with arbitrarily many assets, while all papers mentioned above consider only finitely many assets. An important conceptual difference between our model and the majority of models mentioned above is as follows. In most of them a contingent claim is modeled as a multidimensional vector (its ith component means the amount of assets of type i obtained by a holder of the claim). In contrast, here we use the monetary representation, i.e., we consider a contingent claim as a one dimensional random variable. Another important distinctive feature of our approach is that the price of a contingent claim is defined not through sub- and super-replication, but directly through the No Generalized Arbitrage condition (see Definition 2.7).

In most aspects mentioned above, our model is similar to the model of Jouini and Kallal [12], but there is a number of essential differences between the two models. The approach of Jouini and Kallal might be considered a "transaction cost extension" of the approach of Harrison and Kreps [8], while our model is the "transaction cost projection" of the general arbitrage pricing model introduced in [2]. The most important difference between our approach and the approaches of [8, 12] is that these papers employ the L^2-setting (in particular, the price processes and the capital processes are assumed to be square-integrable and the densities $d\mathsf{Q}/d\mathsf{P}$ or risk-neutral measures should also be square-integrable), while we employ the L^0-setting.

We also study in our framework the convergence of the fair price intervals of a European call option $(S_T - K)^+$ in the Black–Scholes model with proportional transaction costs when the coefficient of transaction costs tend to zero. It is shown that the fair price interval tend to the trivial one, i.e., to $((S_0 - K)^+, S_0)$. Although our framework differs from the existing ones, this result agrees with the results of [5], [20], and [24], where the same problem was considered. The financial interpretation is as follows: in the model under consideration, the fair price interval obtained by dynamic hedging coincides with the fair price interval obtained by static hedging.

1.3 Model with european call options as basic assets

This model is considered in Section 4 (it is again a particular case of the general arbitrage pricing model). We provide a simple geometric representation of the class of risk-neutral measures. The frictionless variant of this model is very popular in financial mathematics (see, in particular, [1], [10]) and was analyzed in [2; Sect. 6] within our general framework. The main idea of considering such models is that taking into account the market prices of traded derivatives enables one to narrow considerably fair price intervals.

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2 Generalized arbitrage

Here we recall some basic definitions and facts from [2].

Definition 2.1. *A general arbitrage pricing model is a quadruple* $(\Omega, \mathbb{F}, \mathsf{P}, A)$*, where* $(\Omega, \mathbb{F}, \mathsf{P})$ *is a probability space and* A *is a convex cone in the space of all random variables.*

From the financial point of view, A is the set of all discounted incomes that can be obtained by trading a certain amount of assets. In the frictionless models, A is a linear space. In the models with transaction costs, A is a cone.

Notation. (i) Set

$$B = \Big\{ Z \in L^0 : \text{ there exist } (X_n)_{n\in\mathbb{N}} \in A \text{ and } a \in \mathbb{R} \\ \text{such that } X_n \geqslant a \ \mathsf{P}\text{-a.s. and } Z = \lim_{n\to\infty} X_n \ \mathsf{P}\text{-a.s.} \Big\}. \tag{1}$$

(ii) For $Z \in B$, denote $\gamma(Z) = 1 - \operatorname{ess\,inf}_{\omega\in\Omega} Z(\omega)$ and set

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L^0_+\}, \\ A_2(Z) &= \left\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \right\}, \\ A_3(Z) &= A_2(Z) \cap L^\infty, \\ A_4(Z) &= \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, L^1(\mathsf{P})). \end{aligned} \tag{2}$$

Here L^0_+ is the set of $\mathbb{R}_+$-valued elements of L^0; L^∞ is the space of bounded elements of L^0; $\sigma(L^\infty, L^1(\mathsf{P}))$ denotes the weak topology on L^∞ induced by the space $L^1(\mathsf{P})$ of the P-integrable random variables on $(\Omega, \mathbb{F}, \mathsf{P})$.

Definition 2.2. *A model $(\Omega, \mathbb{F}, \mathsf{P}, A)$ satisfies the No Generalized Arbitrage (NGA) condition if for all $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.*

Definition 2.3. *An equivalent risk-neutral measure is a probability measure $\mathsf{Q} \sim \mathsf{P}$ such that $\mathsf{E}_\mathsf{Q} X^- \geqslant \mathsf{E}_\mathsf{Q} X^+$ for all $X \in A$ (we use the notation $X^- = (-X) \vee 0$, $X^+ = X \vee 0$). The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of equivalent risk-neutral measures will be denoted by $\mathcal{R}$.*

Notation. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of all probability measures $\mathsf{Q} \sim \mathsf{P}$ with the property: for any $X \in A$ such that $X \geqslant -\alpha Z - \beta$ P-a.s. for some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \leqslant 0$.

Lemma 2.4. *For any $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

Assumption 2.5. There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (both these sets may be empty).

Theorem 2.6 (Fundamental theorem of asset pricing). *Suppose that Assumption 2.5 is satisfied. Then the model $(\Omega, \mathbb{F}, \mathsf{P}, A)$ satisfies the NGA condition if and only if there exists an equivalent risk-neutral measure.*

Now, let F be a random variable on $(\Omega, \mathbb{F}, \mathsf{P})$ meaning the discounted payoff of a contingent claim.

Definition 2.7. *A real number x is a fair price of F if the extended model $(\Omega, \mathbb{F}, \mathsf{P}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGA condition. The set of fair prices of F will be denoted by $I(F)$.*

Theorem 2.8 (Pricing contingent claims). *Suppose that the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$ satisfies Assumption 2.5 and the NGA condition, while F is bounded below. Then*

$$I(F) = \{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{R}\}.$$

The expectation $\mathsf{E}_{\mathsf{Q}} F$ here is taken in the sense of finite expectations, i.e., we consider only those Q, for which $\mathsf{E}_{\mathsf{Q}} F < \infty$.

Let us illustrate the setup introduced above by a static model with a finite number of assets.

Example 2.9. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $S_0^a, S_0^b \in \mathbb{R}^d$ and S_1^a, S_1^b be $\mathbb{R}^d$-valued random vectors. From the financial point of view, S_n^{ai} (resp., S_n^{bi}) is the discounted ask (resp., bid) price of the ith asset at time n (so that $S_n^a \geqslant S_n^b$ componentwise). Define the set of attainable incomes by

$$A = \left\{ \sum_{i=1}^{d} \left[g^i(S_1^{bi} - S_0^{ai}) + h^i(-S_1^{ai} + S_0^{bi}) \right] : g^i, h^i \in \mathbb{R}_+ \right\}.$$

Then the NGA condition is equivalent to the traditional No Arbitrage (NA) condition defined as: $A \cap L_+^0 = \{0\}$. (Consequently, the set of fair prices would remain unchanged if we replaced the NGA condition in the definition of a fair price by the NA condition.) Indeed, the implication NGA$\Rightarrow$NA is obvious, while the implication NA$\Rightarrow$NGA is proved as follows. Assume the NA condition and consider the measure $\mathsf{P}' = c(\|S_1^a\| \vee \|S_1^b\| \vee 1)^{-1}\mathsf{P}$, where c is the normalizing constant. By the Kreps–Yan theorem (see [18] or [25]), there exists a probability measure $\mathsf{Q} \sim \mathsf{P}'$ such that the density $d\mathsf{Q}/d\mathsf{P}'$ is bounded and $\mathsf{E}_{\mathsf{Q}} X \leqslant 0$ for all $X \in A$. Then $\mathsf{Q} \in \mathcal{R}$ and, by Theorem 2.6, the NGA is satisfied (note that the proof of the implication $\mathcal{R} \neq \emptyset \;\Rightarrow\;$ NGA in this theorem does not employ Assumption 2.5). □

3 Dynamic model with infinite number of assets

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathsf{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is P-trivial and $(\mathcal{F}_t)$ is right-continuous. Let $(S_t^{ai})_{t \in [0,T]}$ and $(S_t^{bi})_{t \in [0,T]}$, $i \in I$ be a family of real-valued $(\mathcal{F}_t)$-adapted càdlàg processes. From the financial point of view, S_t^{ai} (resp., S_t^{bi}) is the discounted ask (resp., bid) price of the ith asset at time t (so that $S_t^a \geqslant S_t^b$ componentwise). Define the set of attainable incomes by

$$\begin{aligned} A = \Bigg\{ & \sum_{n=0}^{N} \sum_{i \in I} \Big[-H_n^i I(H_n^i > 0) S_{u_n}^{ai} - H_n^i I(H_n^i < 0) S_{u_n}^{bi} \Big] : \\ & N \in \mathbb{N},\ u_0 \leqslant \dots \leqslant u_N \text{ are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_n}\text{-measurable,} \\ & H_n^i = 0 \text{ for all but finitely many } i, \text{ and } \sum_{n=0}^{N} H_n^i = 0 \text{ for all } i \Bigg\}. \end{aligned}$$

Here H_n^i represents the amount of ith asset bought at time u_n (so $\sum_{k=0}^n H_k^i$ is the total amount of ith asset held at time u_n).

Remark. Consider a model with no transaction costs (i.e., $S^a = S^b = S$). Then for any i and any H_n^i such that $\sum_{n=0}^N H_n^i = 0$, we can write

$$\sum_{n=0}^N \left[-H_n^i S_{u_n}^i\right] = \sum_{n=1}^N \left(\sum_{k=0}^{n-1} H_k^i\right) \left(S_{u_n}^i - S_{u_{n-1}}^i\right).$$

Thus, in this model the set A admits a simpler description:

$$A = \Bigg\{\sum_{n=1}^N \sum_{i\in I} H_n^i (S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \leqslant \dots \leqslant u_N \text{ are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable, and } H_n^i = 0 \text{ for all but finitely many } i\Bigg\}.$$

We will assume that each process S^{bi} is positive. We will also suppose that, for all $i \in I$, there exists a constant $\gamma^i > 0$ such that $S^{ai} \leqslant \gamma^i S^{bi}$. Finally, we assume that, for each $t \in [0, T]$, there exists $Y_t \in B$ (B is defined by (1)) with the property: for all $i \in I$, there exist $\alpha, \beta > 0$ such that $S_t^{bi} \leqslant \alpha Y_t + \beta$ a.s. This assumption is automatically satisfied in natural models.

Indeed, if I is finite, then the above assumption is satisfied with

$$Y_t = \sum_{i\in I} \left(S_t^{bi} - S_0^{ai}\right).$$

If I is countable, then the above assumption is satisfied with

$$Y_t = \sum_{i\in I} \lambda^i (S_t^{bi} - S_0^{ai}),$$

with constants $\lambda^i > 0$ chosen in such a way that $\sum_{i\in I} \lambda^i S_t^{bi} < \infty$ a.s. and $\sum_{i\in I} \lambda^i S_0^{ai} < \infty$.

If S^{bi} is the discounted bid price process of a zero-coupon bond with maturity i, then S^{bi} takes on values in $[0, 1]$, and the above assumption is satisfied with $Y_t = 0$.

In order to get the fundamental theorem of asset pricing and to obtain the form of the fair price intervals, it is sufficient to prove that Assumption 2.5 is satisfied and to find the structure of risk-neutral measures. We call the corresponding statement the *Key Lemma* of the section.

Notation. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \text{for each } i \in I, \text{there exists an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale } M^i \text{ such that, for all } t \in [0, T],\ S_t^{bi} \leqslant M_t^i \leqslant S_t^{ai}\ \ \mathsf{Q}\text{-a.s.}\}.$$

Here M^i need not be càdlàg.

Key Lemma 3.1. *For the model* $(\Omega, \mathcal{F}, \mathsf{P}, A)$*, we have*

$$\mathcal{R} = \mathcal{R}\Bigg(\sum_{t\in\mathbb{Q}\cap[0,T]} \lambda_t Y_t\Bigg) = \mathcal{M},$$

with constants $\lambda_t > 0$ *chosen in such a way that* $\sum_{t\in\mathbb{Q}\cap[0,T]} \lambda_t |Y_t| < \infty$ *a.s. and* $\sum_{t\in\mathbb{Q}\cap[0,T]} \lambda_t \operatorname{ess\,inf}_{\omega\in\Omega} Y_t(\omega) < \infty$.

The proof employs two auxiliary statements. The first of them was proved by Jouini and Kallal [12; Lem. 3] (see Choulli and Stricker [3] for a related result). Actually, Jouini and Kallal use the additional assumption that X and Y are càdlàg, but a slight modification of their proof allows one to get rid of this assumption.

Lemma 3.2. *Let* X *be a supermartingale and* Y *be a submartingale on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ *with a right-continuous filtration* (X *and* Y *are not necessarily càdlàg*)*. Suppose that, for all* $t \in [0,T]$*,* $X_t \leqslant Y_t$ *a.s. Then there exists an* $(\mathcal{F}_t)$*-martingale* M *such that, for all* $t \in [0,T]$*,* $X_t \leqslant M_t \leqslant Y_t$ *a.s.*

We will also need the following statement (see [11] or [23; Ch. II, Sect. 1c]):

Lemma 3.3. *Let* $(X_n)_{n=0,\dots,N}$ *be an* $(\mathcal{F}_n)$*-local martingale such that* $\mathsf{E}|X_0| < \infty$ *and* $\mathsf{E}X_N^- < \infty$*. Then* X *is an* $(\mathcal{F}_n)$*-martingale.*

Proof of Key Lemma 3.1. Denote $\sum_{t\in\mathbb{Q}\cap[0,T]} \lambda_t Y_t$ by Z_0.

Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.4.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Take $\mathsf{Q} \in \mathcal{R}(Z_0)$. Fix $i \in I$ and two $(\mathcal{F}_t)$-stopping times $u \leqslant v$. We shall prove that

$$\mathsf{E}_{\mathsf{Q}}(S_v^{bi} \mid \mathcal{F}_u) \leqslant S_u^{ai}. \tag{1}$$

For $n \in \mathbb{N}$, set

$$u_n = \sum_{k=1}^{n} \frac{kT}{n}\, I\bigg(\frac{(k-1)T}{n} < u \leqslant \frac{kT}{n}\bigg),$$

$$v_n = \sum_{k=1}^{n} \frac{kT}{n}\, I\bigg(\frac{(k-1)T}{n} < v \leqslant \frac{kT}{n}\bigg).$$

Then, for all $n \leqslant m$ and all $D \in \mathcal{F}_{u_m}$ such that $S_{u_m}^{ai}$ is bounded on D, we have $u_m \leqslant v_n$ and

$$\mathsf{E}_{\mathsf{Q}} I_D (S_{v_n}^{bi} - S_{u_m}^{ai}) \leqslant 0,$$

which implies that

$$\mathsf{E}_{\mathsf{Q}}\left(S_{v_n}^{bi} \mid \mathcal{F}_{u_m}\right) \leqslant S_{u_m}^{ai}. \tag{2}$$

As u_m decreases to u pointwise, we have $\mathcal{F}_{u_m} \subseteq \mathcal{F}_{u_{m-1}}$ and $\bigcap_{m=1}^{\infty} \mathcal{F}_{u_m} = \mathcal{F}_u$ (see [21; Ch. I, Ex. 4.17]). Therefore,

$$\mathsf{E}_{\mathsf{Q}}\left(S^{bi}_{v_n} \mid \mathcal{F}_{u_m}\right) \xrightarrow[m\to\infty]{\mathsf{Q}\text{-a.s.}} \mathsf{E}_{\mathsf{Q}}\left(S^{bi}_{v_n} \mid \mathcal{F}_u\right)$$

(see [21; Ch. II, Cor. 2.4]) and (2) yields

$$\mathsf{E}_{\mathsf{Q}}\left(S^{bi}_{v_n} \mid \mathcal{F}_u\right) \leqslant S^{ai}_u.$$

Applying the Fatou lemma for conditional expectations, we get (1).

We shall now prove that

$$\mathsf{E}_{\mathsf{Q}}(S^{ai}_v \mid \mathcal{F}_u) \geqslant S^{bi}_u. \tag{3}$$

For u_m, v_n defined above and all $D \in \mathcal{F}_{u_m}$, we have

$$\mathsf{E}_{\mathsf{Q}} I_D(-S^{ai}_{v_n} + S^{bi}_{u_m}) \leqslant 0$$

(recall that $S^{ai} \leqslant \gamma^i S^{bi}$). Thus,

$$\mathsf{E}_{\mathsf{Q}}(S^{ai}_{v_n} \mid \mathcal{F}_{u_m}) \geqslant S^{bi}_{u_m}.$$

The same argument as above gives

$$\mathsf{E}_{\mathsf{Q}}(S^{ai}_{v_n} \mid \mathcal{F}_u) \geqslant S^{bi}_u. \tag{4}$$

It follows that, for all $(\mathcal{F}_t)$-stopping time v,

$$S^{ai}_v \leqslant \gamma^i S^{bi}_v \leqslant \gamma^i \mathsf{E}_{\mathsf{Q}}(S^{ai}_T \mid \mathcal{F}_v) \leqslant (\gamma^i)^2 \mathsf{E}_{\mathsf{Q}}(S^{bi}_T \mid \mathcal{F}_v).$$

Using the fact that $\mathsf{Q} \in \mathcal{R}(Z_0)$, it is easy to check that S^{bi}_T is Q-integrable, and hence, the collection $(S^{ai}_{v_n})_{n=1}^{\infty}$ is Q-uniformly integrable. Now, (3) follows from (4).

Consider the Snell envelopes

$$X_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathsf{E}_{\mathsf{Q}}(S^{bi}_\tau \mid \mathcal{F}_t), \quad t \in [0,T],$$
$$Y_t = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathsf{E}_{\mathsf{Q}}(S^{ai}_\tau \mid \mathcal{F}_t), \quad t \in [0,T],$$

where $\mathcal{T}_t$ denotes the set of all $(\mathcal{F}_t)$-stopping times such that $\tau \geqslant t$. (Recall that $\operatorname{ess\,sup}_\alpha \xi_\alpha$ is the random variable ξ such that, for all α, $\xi \geqslant \xi_\alpha$ a.s. and for any other random variable ξ' with this property, we have $\xi \leqslant \xi'$ a.s.) Then X is an $(\mathcal{F}_t, \mathsf{Q})$-supermartingale, while Y is an $(\mathcal{F}_t, \mathsf{Q})$-submartingale (see [7; Th. 2.12.1]).

Let us prove that, for all $t \in [0,T]$, $X_t \leqslant Y_t$ Q-a.s. Assume that there exists t such that $\mathsf{P}(X_t > Y_t) > 0$. Then there exist $\tau, \sigma \in \mathcal{T}_t$ such that

$$\mathsf{Q}\big(\mathsf{E}_{\mathsf{Q}}(S^{bi}_\tau \mid \mathcal{F}_t) > \mathsf{E}_{\mathsf{Q}}(S^{ai}_\sigma \mid \mathcal{F}_t)\big) > 0.$$

This implies that $\mathsf{Q}(\xi > \eta) > 0$, where $\xi = \mathsf{E}_{\mathsf{Q}}(S^{bi}_\tau \mid \mathcal{F}_{\tau\wedge\sigma})$ and $\eta = \mathsf{E}_{\mathsf{Q}}(S^{ai}_\sigma \mid \mathcal{F}_{\tau\wedge\sigma})$. Assume first that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \leqslant \sigma\}) > 0$. On the set $\{\tau \leqslant \sigma\}$, we have

$$\xi = S^{bi}_\tau = S^{bi}_{\tau\wedge\sigma},$$
$$\eta = \mathsf{E}_{\mathsf{Q}}(S^{ai}_\sigma \mid \mathcal{F}_{\tau\wedge\sigma}) = \mathsf{E}_{\mathsf{Q}}(S^{ai}_{\tau\vee\sigma} \mid \mathcal{F}_{\tau\wedge\sigma}),$$

and we obtain a contradiction with (3). Similarly, if we assume that $\mathsf{Q}(\{\xi > \eta\} \cap \{\tau \geqslant \sigma\}) > 0$, then we arrive at a contradiction with (1). As a result, $X_t \leqslant Y_t$ Q-a.s. Now, an application of Lemma 3.2 shows that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Take $\mathsf{Q} \in \mathcal{M}$. Then, for each i, there exists an $(\mathcal{F}_t, \mathsf{Q})$-martingale M^i such that, for all $t \in [0, T]$, $S^{bi}_t \leqslant M^i_t \leqslant S^{ai}_t$ Q-a.s. Fix

$$X = \sum_{n=0}^{N} \sum_{i\in I} \bigl[-H^i_n I(H^i_n > 0) S^{ai}_{u_n} - H^i_n I(H^i_n < 0) S^{bi}_{u_n}\bigr] \in A.$$

Let $(\widetilde{\mathcal{F}}_t)$ denote the Q-completion of $(\mathcal{F}_t)$. The process M^i admits a càdlàg $(\widetilde{\mathcal{F}}_t)$-modification $\widetilde{M}^i$ (see [21; Ch. II, Th. 2.9]). We have

$$\begin{aligned} X &\leqslant \sum_{n=0}^{N} \sum_{i\in I} \Bigl[-H^i_n I(H^i_n > 0) \widetilde{M}^i_{u_n} - H^i_n I(H^i_n < 0) \widetilde{M}^i_{u_n}\Bigr] \\ &= \sum_{n=1}^{N} \sum_{i\in I} \left[\left(\sum_{k=0}^{n-1} H^i_k\right) \left(\widetilde{M}^i_{u_n} - \widetilde{M}^i_{u_{n-1}}\right)\right]. \end{aligned}$$

The process

$$M_l = \sum_{n=1}^{l} \sum_{i\in I} \left[\left(\sum_{k=0}^{n-1} H^i_k\right) \left(\widetilde{M}^i_{u_n} - \widetilde{M}^i_{u_{n-1}}\right)\right], \quad l = 0, \dots, N$$

is a Q-local martingale with respect to the filtration $(\mathcal{F}_{u_l})$. Now, it follows from Lemma 3.3 that $\mathsf{E}_{\mathsf{Q}} M^-_N \geqslant \mathsf{E}_{\mathsf{Q}} M^+_N$. Consequently, $\mathsf{E}_{\mathsf{Q}} X^- \geqslant \mathsf{E}_{\mathsf{Q}} X^+$. As a result, $\mathsf{Q} \in \mathcal{R}$. □

Let us now consider a model with proportional transaction costs, i.e., a model with $S^{bi} = (1-\lambda^i) S^{ai}$, where $\lambda^i \in [0, 1]$ is the coefficient of proportional transaction costs for the ith asset. We introduce the following definition.

Definition 3.4. An $\mathbb{R}_+$-valued process X is called an $(\mathcal{F}_t, \mathsf{P})$-*delta-martingale of order* a, where $a \in [0, 1]$ if

(a) X is $(\mathcal{F}_t)$-adapted and càdlàg;
(b) $\mathsf{E} X_t < \infty$, $t \in [0, T]$;
(c) for all $(\mathcal{F}_t)$-stopping times $u \leqslant v$, we have $a X_u \leqslant \mathsf{E}(X_v \mid \mathcal{F}_u) \leqslant a^{-1} X_u$.

It is seen from the proof of Key Lemma 3.1 that X is a delta-martingale of order a if and only if there exists a martingale M such that, for all $t \in [0,T]$, $aX_t \leqslant M_t \leqslant X_t$ a.s. Consequently, in models with proportional transaction costs

$$\begin{aligned}\mathcal{R} = \{\mathsf{Q} \sim \mathsf{P} : \text{for all } i \in I,\ S^{ai} \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-}\\ \text{delta-martingale of order } 1 - \lambda^i\}.\end{aligned} \tag{5}$$

Let us now study the following problem. Consider a one dimensional model having proportional transaction costs with coefficient λ and denote by $I_\lambda(F)$ the fair price interval in this model. *Is it true that* $I_\lambda(F) \xrightarrow[\lambda\downarrow 0]{} I(F)$, *where* $I(F)$ *is the fair price interval in the model with no transaction costs* (*i.e., with* $\lambda = 0$)? This problem was considered for the Black–Scholes model in the papers [5], [20], [24], and it was proved that the upper price of a European call option $(S_T - K)^+$ tend to S_0 as $\lambda \downarrow 0$. Our approach to arbitrage pricing in models with transaction costs is different from the one in the papers mentioned, but the same result turns out to be true in our approach as well. Thus, the answer to the question posed above is negative for natural continuous-time models.

Proposition 3.1. *Let* $S_t = S_0 \mathrm{e}^{\mu t + \sigma B_t}$, *where* $\mu \in \mathbb{R}$, $\sigma > 0$, *and* B *is a Brownian motion. Let* $\mathcal{F}_t = \mathcal{F}_t^B$, $S^a = S$, $S^b = (1-\lambda)S$, $F = (S_T - K)^+$. *Then*

$$I_\lambda(F) \xrightarrow[\lambda\downarrow 0]{} ((S_0 - K)^+,\ S_0)$$

in the sense that the left (*resp., right*) *endpoints of* $I_\lambda(F)$ *tend to* $(S_0 - K)^+$ (*resp.,* S_0) *as* $\lambda \downarrow 0$.

Proof. It is clear that $I_\lambda(F)$ decreases as $\lambda \downarrow 0$. Furthermore, using static considerations (i.e., considering trades at dates 0 and T only), one can easily see that, for any $\varepsilon > 0$, there exists $\lambda > 0$ such that $I_\lambda(F) \subseteq ((S_0 - K)^+ - \varepsilon, S_0 + \varepsilon)$. Thus, it will suffice to prove that, for any $\lambda > 0$,

$$I_\lambda(F) \supseteq ((S_0 - K)^+,\ S_0). \tag{6}$$

Clearly, we can assume from the outset that $\Omega = C([0,T])$, $S = X$, where X denotes the coordinate process (i.e., $X_t(\omega) = \omega(t)$), and $\mathcal{F}_t = \mathcal{F}_t^X$.

Step 1. Let $a > 0$ and Z be a solution of the stochastic differential equation

$$\mathrm{d}Z_t = -a\,\mathrm{sgn}(Z_t - S_0) I(t \leqslant \tau)\mathrm{d}t + \sigma Z_t \mathrm{d}W_t, \quad Z_0 = S_0,$$

where $\tau = \inf\{t \geqslant 0 : |Z_t - S_0| \geqslant \Delta\}$, $\Delta = \lambda S_0/10$, and W is a Brownian motion. The process Z is defined on a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathsf{P}})$. Fix $(\widetilde{\mathcal{F}}_t)$-stopping times $u \leqslant v$. On the set $\{\tau \leqslant u\}$, we have $\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) = Z_u$. On the set $\{\tau > u\}$, we have $\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) = \mathsf{E}_{\widetilde{\mathsf{P}}}(Z_{v\wedge\tau} \mid \widetilde{\mathcal{F}}_u)$, $|Z_u - S_0| \leqslant \Delta$, and $|Z_{v\wedge\tau} - S_0| \leqslant \Delta$. Thus,

$$\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) \geqslant (1-\lambda)Z_u, \qquad \mathsf{E}_{\widetilde{\mathsf{P}}}((1-\lambda)Z_v \mid \widetilde{\mathcal{F}}_u) \leqslant Z_u.$$

Now, set $\mathsf{Q}(a) = \mathrm{Law}(Z_t; t \leqslant T)$. Then, for all $(\mathcal{F}_t)$-stopping times $u \leqslant v$, we have

$$\mathsf{E}_{\mathsf{Q}(a)}(X_v \mid \mathcal{F}_u) \geqslant (1-\lambda)X_u, \qquad \mathsf{E}_{\mathsf{Q}(a)}((1-\lambda)X_v \mid \mathcal{F}_u) \leqslant X_u.$$

Furthermore, Girsanov's theorem guarantees that $\mathsf{Q}(a) \sim \mathsf{P}$. In view of (5), $\mathsf{Q}(a)$ is a risk-neutral measure.

Let us prove that

$$\lim_{a\to\infty} \mathsf{E}_{\mathsf{Q}(a)} F = (S_0 - K)^+. \tag{7}$$

For any $b > 2S_0$, we have, by the Itô-Tanaka-Meyer formula,

$$(Z_t - b)^+ = \int_0^t I(Z_s > b)\sigma Z_s \mathrm{d}W_s + \frac{1}{2} L_t^b(Z), \quad t \geqslant 0,$$

where $L_t^b(Z)$ denotes the local time spent by the process Z at point b up to time t. It follows from this representation that $\mathsf{E}(Z_T - b)^+ \leqslant \mathsf{E}(Z_\sigma - b)^+$, where $\sigma = TI(\tau \geqslant T) + (T+\tau)I(\tau < T)$. Using this inequality and the property that $(Z_T - b)^+ = 0$ on $\{\tau \leqslant T\}$, we can write

$$\begin{aligned}
\mathsf{E}(Z_T - b)^+ &\leqslant \mathsf{E}(Z_{T+\tau} - b)^+ I(\tau < T, Z_\tau = S_0 - \Delta) \\
&\quad + \mathsf{E}(Z_{T+\tau} - b)^+ I(\tau < T, Z_\tau = S_0 + \Delta) \\
&= \mathsf{E}\big(Y_T^{S_0-\Delta} - b\big)^+ \mathsf{P}(\tau < T, Z_\tau = S_0 - \Delta) \\
&\quad + \mathsf{E}\big(Y_T^{S_0+\Delta} - b\big)^+ \mathsf{P}(\tau < T, Z_\tau = S_0 + \Delta),
\end{aligned}$$

where Y^x is a solution of the stochastic differential equation

$$\mathrm{d}Y_t^x = \sigma Y_t^x \mathrm{d}W_t, \quad Y_0^x = x.$$

It is seen from the inequality proved above that $\mathsf{E}(Z_T - b)^+$ converges to 0 uniformly in $a > 0$ as $b \to \infty$. Furthermore, it is clear that Z_T converge weakly to S_0 as $a \to \infty$. This yields (7).

Step 2. Let $a, b, c > 0$ and $(Z, \widetilde{Z})$ be a solution of the system

$$\begin{aligned}
&\mathrm{d}Z_t = -a\,\mathrm{sgn}(Z_t - \widetilde{Z}_t)I(t \leqslant \tau)\mathrm{d}t + \sigma Z_t \mathrm{d}W_t, \quad Z_0 = S_0, \\
&\mathrm{d}\widetilde{Z}_t = b\widetilde{Z}_t \mathrm{d}\widetilde{W}_t, \quad \widetilde{Z}_0 = S_0,
\end{aligned}$$

where $\tau = \inf\{t \geqslant 0 : |Z_t - \widetilde{Z}_t| \geqslant \Delta \text{ or } Z_t \leqslant c\}$, $\Delta = \lambda S_0/10$, and W, $\widetilde{W}$ are independent Brownian motions. The process $(Z, \widetilde{Z})$ is defined on a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathsf{P}})$. Arguing in the same way as above, we check that, for all $(\widetilde{\mathcal{F}}_t)$-stopping times $u \leqslant v$,

$$\mathsf{E}_{\widetilde{\mathsf{P}}}(Z_v \mid \widetilde{\mathcal{F}}_u) \geqslant (1-\lambda)Z_u, \qquad \mathsf{E}_{\widetilde{\mathsf{P}}}((1-\lambda)Z_v \mid \widetilde{\mathcal{F}}_u) \leqslant Z_u$$

provided that $c < S_0$. Hence, the measure $\mathsf{Q}(a,b,c) = \mathrm{Law}(Z_t;\, t \leqslant T)$ is a risk-neutral measure. Clearly,

$$\lim_{c\downarrow 0} \lim_{a\to\infty} \mathrm{Law}(Z_t;\, t \leqslant T) = \mathrm{Law}(\widetilde{Z}_t;\, t \leqslant T),$$

and therefore,

$$\liminf_{b\to\infty} \liminf_{c\downarrow 0} \liminf_{a\to\infty} \mathsf{E}_{\mathsf{Q}(a,b,c)} F \geqslant \liminf_{b\to\infty} \mathsf{E}(\widetilde{Z}_T - K)^+ = \lim_{b\to\infty} \mathsf{E}(\widetilde{Z}_T - K)^+ = S_0. \tag{8}$$

Relations (7) and (8) taken together yield (6). □

4 Model with european call options as basic assets

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and $T \in [0,\infty)$. Let S_T be an $\mathbb{R}_+$-valued random variable. From the financial point of view, S_T is the ask price of some asset at time T. For simplicity, we consider only proportional transaction costs on the underlying assets, i.e., the bid price of the ith asset at time T is $(1-\lambda)S_T$, where $\lambda \in [0,1]$. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices K of traded European call options on this asset with maturity T. Let $\varphi^a(K)$ and $\varphi^b(K)$, $K \in \mathbb{K}$ be the ask and bid prices at time 0 of such an option. Define the set of attainable incomes by

$$A = \Bigg\{ \sum_{n=1}^{N} [g_n(((1-\lambda)S_T - K_n)^+ - \varphi^a(K_n)) + h_n(-(S_T - K_n)^+ + \varphi^b(K_n))] : \\ N \in \mathbb{N},\ K_n \in \mathbb{K},\ g_n, h_n \in \mathbb{R}_+ \Bigg\}.$$

We assume that $0 \in \mathbb{K}$, which means the possibility to trade the underlying asset.

Notation. Set

$$\mathcal{M} = \{\mathsf{Q} \sim \mathsf{P} : \mathrm{Law}_{\mathsf{Q}}\, S_T \in \mathcal{D}\},$$

where

$$\mathcal{D} = \{\varphi'' : \varphi \text{ is convex on } \mathbb{R}_+,\ \varphi'_+(0) \geqslant -1,\ \lim_{x\to\infty} \varphi(x) = 0, \\ \varphi((1-\lambda)^{-1}K) \leqslant (1-\lambda)^{-1}\varphi^a(K) \text{ and } \varphi(K) \geqslant \varphi^b(K),\ K \in \mathbb{K}\}.$$

Here φ'_+ denotes the right-hand derivative and φ'' denotes the second derivative taken in the sense of distributions (i.e., $\varphi''([a,b]) = \varphi'_+(b) - \varphi'_+(a)$) with the convention: $\varphi''(\{0\}) = \varphi'_+(0) + 1$ (thus, φ'' is a probability measure provided that $\varphi'_+(0) \geqslant -1$).

Key Lemma 4.1. *For the model $(\Omega, \mathcal{F}, \mathsf{P}, A)$, we have*

$$\mathcal{R} = \mathcal{R}((1-\lambda)S_T - \varphi^a(0)) = \mathcal{M}.$$

Proof. Denote $(1-\lambda)S_T - \varphi^a(0)$ by Z_0.

Step 1. The inclusion $\mathcal{R} \subseteq \mathcal{R}(Z_0)$ follows from Lemma 2.4.

Step 2. Let us prove the inclusion $\mathcal{R}(Z_0) \subseteq \mathcal{M}$. Fix $\mathsf{Q} \in \mathcal{R}(Z_0)$. By considering the function $\varphi(x) = \mathsf{E}_\mathsf{Q}(S_T - x)^+$, we conclude that $\mathsf{Q} \in \mathcal{M}$.

Step 3. Let us prove the inclusion $\mathcal{M} \subseteq \mathcal{R}$. Fix $\mathsf{Q} \in \mathcal{M}$ with $\mathrm{Law}_\mathsf{Q}\, S_T = \varphi''$. Then

$$\mathsf{E}_\mathsf{Q}(S_T - K)^+ = \int_{\mathbb{R}_+} (x-K)^+ \varphi''(\mathrm{d}x) = \varphi(K), \quad K \in \mathbb{R}_+.$$

Consequently, $\mathsf{E}_\mathsf{Q} X \leqslant 0$ for all $X \in A$, which means that $\mathsf{Q} \in \mathcal{R}$. □

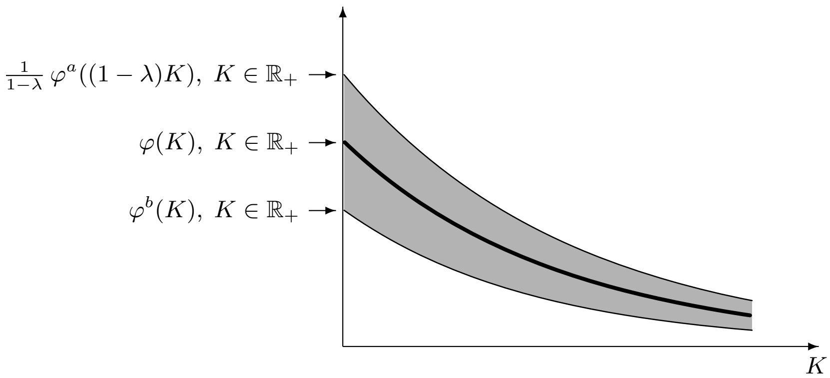

Fig. 1a. The structure of $\mathcal{D}$ when $\mathbb{K} = \mathbb{R}_+$. The set $\mathcal{D}$ consists of the second derivatives φ'', where φ is convex on $\mathbb{R}_+$, $\varphi'_+(0) \geqslant -1$, $\lim\limits_{x\to\infty} \varphi(x) = 0$, and φ lies in the shaded region

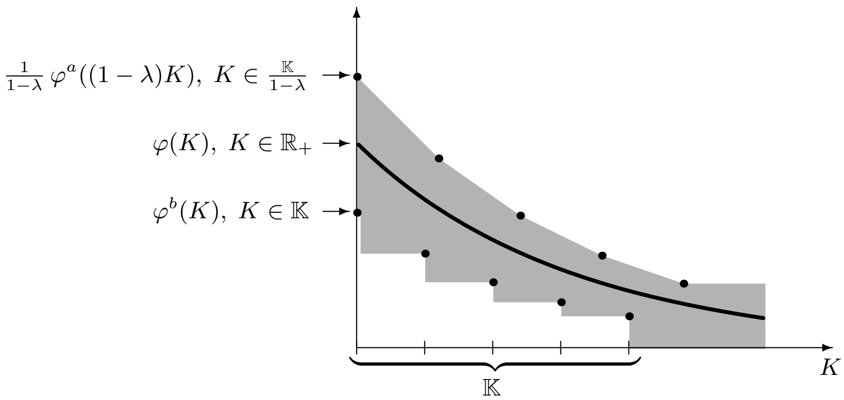

Fig. 1b. The structure of $\mathcal{D}$ when $\mathbb{K}$ is finite. The set $\mathcal{D}$ consists of the second derivatives φ'', where φ is convex on $\mathbb{R}_+$, $\varphi'_+(0) \geqslant -1$, $\lim\limits_{x\to\infty} \varphi(x) = 0$, and φ lies in the shaded region

References

1. Breeden, D.T., Litzenberger, R.H. (1978): Prices of state-contingent claims implicit in option prices. Journal of Business, **51**, No. 4, 621–651
2. Cherny, A.S. (2007): General arbitrage pricing model: I – Probability approach (in this volume)
3. Choulli, T., Stricker, C. (1998): Séparation d'une sur- et d'une sousmartingale par une martingale. Lecture Notes in Mathematics, **1686**, 67–72
4. Cvitanić, J., Karatzas, I. (1996): Hedging and portfolio optimization under transaction costs: A martingale approach. Mathematical Finance, **6**, No. 2, 133–165
5. Cvitanić, J., Pham, H., Touzi, N. (1999): A closed-form solution to the problem of super-replication under transaction costs. Finance and Stochastics, **3**, No. 1, 35–54
6. Delbaen, F., Kabanov, Yu.M., Valkeila, E. (2002): Hedging under transaction costs in currency markets: A discrete-time model. Mathematical Finance, **12**, No. 1, 45–61
7. El Karoui, N. (1981): Les aspects probabilistes du contrôle stochastique. Lecture Notes in Mathematics, **876**, 73–238
8. Harrison, J.M., Kreps, D.M. (1979): Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, **20**, 381–408
9. Ikeda, N., Watanabe, S. (1989): Stochastic differential equations and diffusion processes, 2nd Edition. North-Holland, Amsterdam
10. Jackwerth, J. (1999): Option implied risk-neutral distributions and implied binomial trees: A literature review. Journal of Derivatives, **7**, No. 2, 66–82
11. Jacod, J., Shiryaev, A.N. (1998): Local martingales and the fundamental asset pricing theorems in the discrete-time case. Finance and Stochastics, **2**, No. 3, 259–273
12. Jouini, E., Kallal, H. (1995): Martingales and arbitrage in securities markets with transaction costs. Journal of Economic Theory, **66**, No. 1, 178–197
13. Kabanov, Yu.M. (1999): Hedging and liquidation under transaction costs in currency markets. Finance and Stochastics, **3**, No. 2, 237–248
14. Kabanov, Yu.M., Last, G. (2002): Hedging under transaction costs in currency markets: a continuous-time model. Mathematical Finance, **12**, No. 1, 63–70
15. Kabanov, Yu.M., Rásonyi, M., Stricker, C. (2002): No-arbitrage criteria for financial markets with efficient friction. Finance and Stochastics, **6**, No. 3, 371–382
16. Kabanov, Yu.M., Stricker, C. (2001): The Harrison-Pliska arbitrage pricing theorem under transaction costs. Journal of Mathematical Economics, **35**, 185–196
17. Koehl, P.F., Pham, H., Touzi, N. (2000): On super-replication in discrete time under transaction costs. Theory of Probability and its Applications, **45**, 783–788
18. Kreps, D. (1981): Arbitrage and equilibrium in economies with infinitely many commodities. Journal of Mathematical Economics, **8**, 15–35
19. Kusuoka, S. (1995): Limit theorems on option replication with transaction costs. Annals of Applied Probability, **5**, 198–221
20. Leventhal, S., Skorokhod, A.V. (1997): On the possibility of hedging options in the presence of transaction costs. Annals of Applied Probability, **7**, 410–443

21. Revuz, D., Yor, M. (1999): Continuous martingales and Brownian motion, 3rd Edition. Springer, Berlin Heidelberg New York
22. Schachermayer, W. (2004): The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time. Mathematical Finance, **14**, No. 1, 19–48
23. Shiryaev, A.N. (1999): Essentials of stochastic finance. World Scientific, Singapore
24. Soner, H.M., Shreve, S.E., Cvitanić, J. There is no nontrivial hedging portfolio for option pricing with transaction costs. Annals of Applied Probability, **5**, 327–355
25. Yan, J.A. (1980): Caractérisation d'une classe d'ensembles convexes de L^1 ou H^1. Lecture Notes in Mathematics, **784**, 220–222

General Arbitrage Pricing Model: III – Possibility Approach

Alexander Cherny

Department of Probability Theory, Faculty of Mechanics and Mathematics,
Moscow State University
119992 Moscow, Russia
e-mail: alexander.cherny@gmail.com
http://mech.math.msu.su/~cherny/

Summary. We introduce the *possibility approach* to pricing by arbitrage. The characteristic feature of this approach is that it does not employ the historic probability measure.

The study is performed on two levels of generality:

- for a static model with a finite number of assets;
- for a general arbitrage pricing model introduced in [3].

The main results obtained for each of these models are: the fundamental theorem of asset pricing and the representation of the fair price intervals.

Key words: Fair price, Fundamental theorem of asset pricing, General arbitrage pricing model, Generalized arbitrage, Possibility space, Risk-neutral measure, Set of attainable incomes, Set of possible elementary events, Transaction costs

1 Introduction

1.1 Purpose of the paper

When a coin is tossed, everyone agrees that there exists a probability measure on the set of elementary outcomes, and this measure assigns the mass 1/2 to each of the two outcomes. When shooting at a target is performed, everyone agrees that there exists a probability measure on the set of elementary outcomes. The exact form of this measure cannot be found by pure thought, but can be estimated by repeated trials. In both examples, the legitimacy of a probability measure is based on the existence of a fixed set of conditions that admits an unlimited number of repetitions. The importance of such a set of conditions was stressed by Kolmogorov [7; Ch. I, Sect. 2].

In the problems that finance deals with, such a fixed set of conditions does not seem to exist at all. Therefore, it is questionable whether there exists a historic measure P, which serves as an input to the overwhelming majority of arbitrage pricing models. It is unquestionable that even if such a measure exists, then no one knows exactly what it is.

But let us now recall that the origin of arbitrage pricing lies in decomposing a complicated contract into simpler contracts, and this does not require any probability considerations. Another example: when calculating the exchange rate through the triangular arbitrage, the probability measure is not needed. Yet another example: the trivial interval $((S_0 - K)^+, S_0)$ of fair prices of a European call option is obtained with no probability at all. However, in more complicated models the structure of P is essential. The basic example in this line is the Black–Scholes model, in which it is the particular structure of P that yields the completeness. Of course, if P is eliminated in such a model, then we get unacceptably wide intervals of fair prices. But nevertheless, in some cases the following effect takes place: if the market prices of a sufficient number of traded derivatives are taken into account, then one can obtain a reasonably small fair price interval of a new contract without relying on the original probability measure. This method was successfully employed (for various models) in the papers [1], [2], [6].

The above observation justifies the general possibility approach to arbitrage pricing. It requires as first input the set of all possible outcomes and does not require that probabilities be assigned to these outcomes. To be more precise, the possibility approach is based on a *possibility space* $(\Omega, \mathcal{F})$ instead of a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. We call Ω the *set of possible elementary events*. Usually it can be defined by pure thought (i.e., without using the real data) in an unambiguous way. For example, a natural set of possible prices of an equity is $\mathbb{R}_{++}(= (0, \infty))$; a natural set of possible prices of d equities is $\mathbb{R}^d_{++}$. Typically, the set of possible elementary events admits a natural topology, and $\mathcal{F}$ is taken as its Borel σ-field.

The possibility approach is introduced on two levels of generality: first, we consider a static model with a finite number of assets, and then we consider the general arbitrage pricing model introduced in [3].

1.2 Static model with a finite number of assets

This is a classical model of financial mathematics (a review of arbitrage pricing in this model can be found, for example, in [5; Ch. 1] or [3; Sect. 2]). In Section 2, we consider the possibility version of this model.

We introduce the possibility variant of the No Arbitrage (NA) condition and prove that this condition is satisfied if and only if for any nonempty set $D \in \mathcal{F}$, there exists a martingale measure Q such that $\mathsf{Q}(D) > 0$. A geometric criterion is presented as well.

Furthermore, using the possibility version of the NA condition, we define the set of fair prices of a contingent claim F and prove that it coincides up to endpoints with the interval $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{M}\}$, where $\mathcal{M}$ denotes the set of martingale measures. A geometric representation of this set is given as well.

1.3 General arbitrage pricing model

In [3], we presented a unified approach to pricing contingent claims through a new concept of *generalized arbitrage*. The No Generalized Arbitrage (NGA) condition is a strengthening of the classical NA condition. This was done within the framework of a *general arbitrage pricing model*. Various models of arbitrage pricing theory, including

- static as well as dynamic models;
- models with an infinite number of assets;
- models with transaction costs (see [4]),

can be viewed as particular cases of this general model.

In Section 3, we consider the possibility version of a general arbitrage pricing model. It is defined as a triple $(\Omega, \mathcal{F}, A)$, where $(\Omega, \mathcal{F})$ is a possibility space and A is a convex cone in the space of all $\mathcal{F}$-measurable real-valued functions. From the financial point of view, A is the set of discounted incomes that can be obtained in the model under consideration.

For a model $(\Omega, \mathcal{F}, A)$, we introduce the possibility variant of the NGA condition. Similarly to [3], we define a *risk-neutral measure* as a measure Q on $\mathcal{F}$ such that $\mathsf{E}_\mathsf{Q} X^- \geqslant \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$ (X^- and X^+ denote the negative part and the positive part of X, respectively; the expectations $\mathsf{E}_\mathsf{Q} X^-$, $\mathsf{E}_\mathsf{Q} X^+$ here are allowed to take on the value $+\infty$).

Theorem 3.6 states (under a natural assumption) that the NGA condition is satisfied if and only if for any nonempty set $D \in \mathcal{F}$, there exists a risk-neutral measure Q such that $\mathsf{Q}(D) > 0$. Thus, a risk-neutral measure appears to be a more fundamental object than a historic probability measure. A nice illustration is provided by bookmaking, where the "true" distribution on the set of outcomes is completely unclear, while the "market-estimated" distribution is easily recovered from the bets.

Next we consider the problem of pricing contingent claims. We define a fair price of a contingent claim F (F is a measurable function on $(\Omega, \mathcal{F})$) as a real number x such that the extended model $(\Omega, \mathcal{F}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGA condition. Theorem 3.9 states (under some natural assumptions) that the set of fair prices of F coincides up to endpoints with the interval $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\}$, where $\mathcal{R}$ denotes the set of risk-neutral measures.

1.4 Particular models

In order to apply the general results of Section 3 to a particular model, one should

1. specify the set A of attainable incomes (this is typically done in a straightforward way);
2. find out the structure of the set of risk-neutral measures (typically, the risk-neutral measures in a particular model admit a simpler description than the general definition of a risk-neutral measure).

Once this is done, Theorem 3.6 gives the necessary and sufficient conditions for the absence of generalized arbitrage, while Theorem 3.9 yields the form of the set of fair prices of a contingent claim. Both procedures 1 and 2 were implemented in [3], [4] for a number of particular models.

However, the possibility framework gives rise to an interesting question: Is the NGA condition (in its possibility version) satisfied in a particular model? The answer depends on the "geometry" of the price structure. In Sections 4–5, we study this problem for a number of particular models, namely

- a discrete-time model with a finite number of assets (Section 4);
- a continuous-time model with a finite number of assets (Section 5);
- a model with European call options as basic assets (Section 6).

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2 Static model with finite number of assets

The reader is invited to compare this section with [3; Sect. 2].

Definition 2.1. *A possibility space is a pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a σ-field on Ω. We call Ω the set of possible elementary events.*

Let $(\Omega, \mathcal{F})$ be a possibility space. Let $S_0 \in \mathbb{R}^d$ and S_1 be an $\mathbb{R}^d$-valued $\mathcal{F}$-measurable function. From the financial point of view, S_n^i is the discounted price of the ith asset at time n. Define the set of attainable incomes by

$$A = \left\{ \sum_{i=1}^{d} h^i (S_1^i - S_0^i) : h^i \in \mathbb{R} \right\}.$$

Definition 2.2. *A model $(\Omega, \mathcal{F}, S_0, S_1)$ satisfies the No Arbitrage (NA) condition if $A \cap L_+^0 = \{0\}$ (L_+^0 denotes the set of $\mathbb{R}_+$-valued $\mathcal{F}$-measurable functions).*

Definition 2.3. *A martingale measure is a probability measure Q on $\mathcal{F}$ such that $\mathsf{E}_\mathsf{Q}|S_1| < \infty$ and $\mathsf{E}_\mathsf{Q} S_1 = S_0$. The set of martingale measures will be denoted by $\mathcal{M}$.*

Notation. Set $C = \overline{\operatorname{conv}}\{S_1(\omega) : \omega \in \Omega\}$ and let C° denote the relative interior of C.

Theorem 2.4 (Fundamental theorem of asset pricing). *For the model $(\Omega, \mathcal{F}, S_0, S_1)$, the following conditions are equivalent:*

(a) *NA*;
(b) $S_0 \in C^\circ$;
(c) *for any $D \in \mathcal{F} \setminus \{\emptyset\}$, there exists $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(D) > 0$.*

Proof. Step 1. Let us prove the implication (a)⇒(b). If $S_0 \notin C^\circ$, then, by the separation theorem, there exists $h \in \mathbb{R}^d$ such that $\langle h, (S_1 - S_0)\rangle \geqslant 0$ pointwise and $\langle h, (S_1(\omega) - S_0(\omega))\rangle > 0$ for some $\omega \in \Omega$. This contradicts the NA condition.

Step 2. Let us prove the implication (b)⇒(c). Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. The set

$$E = \left\{ \sum_{k=0}^{m} \alpha_k S_1(\omega_k) : m \in \mathbb{N},\ \omega_1, \dots, \omega_m \in \Omega,\ \alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}, \sum_{k=0}^{m} \alpha_k = 1 \right\}$$

is convex, and the closure of E contains $\{S_1(\omega) : \omega \in \Omega\}$. Consequently, $E \supseteq C^\circ$. Thus, there exist $\omega_1, \dots, \omega_m \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $\sum_{k=0}^m \alpha_k = 1$ and $\sum_{k=0}^m \alpha_k S_1(\omega_k) = S_0$. Then the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$ belongs to $\mathcal{M}$ and $\mathsf{Q}(D) > 0$ (δ_ω denotes the point mass concentrated on $\{\omega\}$, i.e., $\delta_\omega(A) = I(\omega \in A)$).

Step 3. Let us prove the implication (c)⇒(a). Suppose that the NA condition is not satisfied, i.e., there exists $X \in A \cap (L^0_+ \setminus \{0\})$. Consider $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(X > 0) > 0$. Then $\mathsf{E_Q} X > 0$. On the other hand, as $\mathsf{Q} \in \mathcal{M}$, we should have $\mathsf{E_Q} X = 0$. The obtained contradiction shows that the NA condition is satisfied. □

Now, let F be a real-valued $\mathcal{F}$-measurable function. From the financial point of view, F is the discounted payoff of some contingent claim.

Definition 2.5. *A real number x is a fair price of F if the model with $d+1$ assets $(\Omega, \mathcal{F}, x, S_0^1, \dots, S_0^d, F, S_1^1, \dots, S_1^d)$ satisfies the NA condition. The set of fair prices of F will be denoted by $I(F)$.*

Notation. Set $D = \overline{\text{conv}}\{(F(\omega), S_1(\omega)) : \omega \in \Omega\}$ and let D° denote the relative interior of D.

For two subsets I, J of the real line, by the notation $I \approx J$ we will mean that the interiors of I and J coincide and the closures of I and J coincide. In particular, if $I \approx J$ and J is an interval (that may be closed, open, or semi-open), then I is also an interval, and I coincides with J up to the endpoints.

Theorem 2.6. *Suppose that the model $(\Omega, \mathcal{F}, S_0, S_1)$ satisfies the NA condition. Then*

$$I(F) = \{x : (x, S_0) \in D^\circ\} \approx \{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}. \tag{1}$$

The expectation $\mathsf{E_Q} F$ here is taken in the sense of finite expectations, i.e. we consider only those Q, for which $\mathsf{E_Q}|F| < \infty$.

Proof. Theorem 2.4 implies that

$$I(F) = \{x : (x, S_0) \in D^\circ\} \subseteq \{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}. \tag{2}$$

Let $x \in \{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{M}\}$. Take $\mathsf{Q}_0 \in \mathcal{M}$ such that $x = \mathsf{E}_{\mathsf{Q}_0} F$. One can find $\mathsf{Q}_1 \in \mathcal{M}$ such that $\mathsf{E}_{\mathsf{Q}_1}|F| < \infty$ and $\overline{\text{conv}}\, \text{supp}\, \text{Law}_{\mathsf{Q}_1}(F, S_1) = D$

(Q_1 can be found in the form $\sum_{n=1}^{\infty} \alpha_n \delta_{\omega_n}$). For any $\varepsilon \in (0,1)$, the measure $\mathsf{Q}(\varepsilon) = (1-\varepsilon)\mathsf{Q}_0 + \varepsilon \mathsf{Q}_1$ belongs to $\mathcal{M}$ and $\overline{\operatorname{conv}} \operatorname{supp} \operatorname{Law}_{\mathsf{Q}(\varepsilon)}(F, S_1) = D$. Therefore, $\mathsf{E}_{\mathsf{Q}(\varepsilon)}(F, S_1) \in D^\circ$, which means that

$$\mathsf{E}_{\mathsf{Q}(\varepsilon)} F \in \{x : (x, S_0) \in D^\circ\}.$$

Furthermore, $\mathsf{E}_{\mathsf{Q}(\varepsilon)} F \xrightarrow[\varepsilon \downarrow 0]{} x$. This, together with (2), proves the approximate equality in (1). □

Remarks. (i) Let $V_*(F)$ (resp., $V^*(F)$) denote the left (resp., right) endpoint of $I(F)$. Let F be such that $V_*(F) < V^*(F)$. It follows from the equality $I(F) = \{x : (x, S_0) \in D^\circ\}$ that $I(F) = (V_*(F), V^*(F))$. As for the interval $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\}$, it has the endpoints $V_*(F)$ and $V^*(F)$, but may contain them. For instance, this interval contains $V^*(F)$ if and only if

$$(V^*(F), S_0) \in \rightarrow \{(F(\omega), S_1(\omega)) : \omega \in \Omega\}.$$

(ii) Another way to define the fair price interval could be as follows. We introduce the lower and the upper prices as

$$V_*(F) = \sup\{x : \text{there exists } X \in A \text{ such that } x - X \leqslant F \text{ pointwise}\},$$
$$V^*(F) = \inf\{x : \text{there exists } X \in A \text{ such that } x + X \geqslant F \text{ pointwise}\},$$

and the fair price interval is defined as the interval with the endpoints $V_*(F)$ and $V^*(F)$. Using the equality $I(F) = \{x : (x, S_0) \in D^\circ\}$ and elementary geometric considerations, one can check that if the model $(\Omega, \mathcal{F}, S_0, S_1)$ satisfies the NA condition, then the values $V_*(F)$ and $V^*(F)$ defined this way coincide with the values defined in the previous remark.

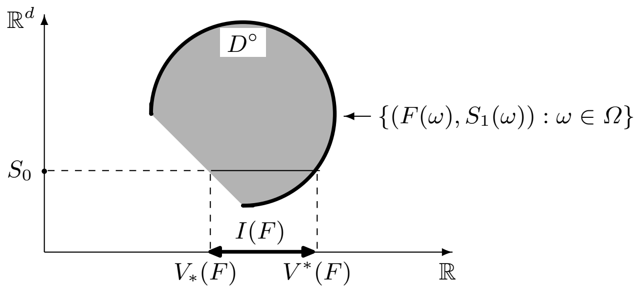

Fig. 1. The joint arrangement of $I(F)$, $V_*(F)$, $V^*(F)$, $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\}$, and D°. In the example shown here, $I(F) = (V_*(F), V^*(F))$, while $\{\mathsf{E}_{\mathsf{Q}} F : \mathsf{Q} \in \mathcal{M}\} = (V_*(F), V^*(F)]$.

3 General arbitrage pricing model

The reader is invited to compare this section with [3; Sect. 3].

Definition 3.1. A *general arbitrage pricing model* is a triple $(\Omega, \mathcal{F}, A)$, where $(\Omega, \mathcal{F})$ is a possibility space and A is a convex cone in L^0 (L^0 is the space of real-valued $\mathcal{F}$-measurable functions). The set A will be called the *set of attainable incomes.*

Notation. (i) Set

$$B = \Big\{ Z \in L^0 : \text{ there exist } (X_n)_{n\in\mathbb{N}} \in A \text{ and } a \in \mathbb{R} \text{ such that } X_n \geqslant a \text{ pointwise and } Z = \lim_{n\to\infty} X_n \text{ pointwise} \Big\}. \tag{1}$$

(ii) For $Z \in B$, denote $\gamma(Z) = 1 - \inf_{\omega\in\Omega} Z(\omega)$ and set

$$\begin{aligned} A_1 &= \{X - Y : X \in A,\ Y \in L^0_+\}, \\ A_2(Z) &= \left\{ \frac{X}{Z + \gamma(Z)} : X \in A_1 \right\}, \\ A_3(Z) &= A_2(Z) \cap L^\infty, \\ A_4(Z) &= \text{closure of } A_3(Z) \text{ in } \sigma(L^\infty, M_F). \end{aligned} \tag{2}$$

Here L^0_+ is the set of $\mathbb{R}_+$-valued elements of L^0; L^∞ is the space of bounded elements of L^0; $\sigma(L^\infty, M_F)$ denotes the weak topology on L^∞ induced by the space M_F of finite σ-additive measures on $\mathcal{F}$ (i.e. signed measures with finite variation).

Definition 3.2. The model $(\Omega, \mathcal{F}, A)$ satisfies the *No Generalized Arbitrage* (NGA) condition if for all $Z \in B$, we have $A_4(Z) \cap L^0_+ = \{0\}$.

Definition 3.3. A *risk-neutral measure* is a probability measure Q on $\mathcal{F}$ such that $\mathsf{E}_\mathsf{Q} X^- \geqslant \mathsf{E}_\mathsf{Q} X^+$ for any $X \in A$. The expectations $\mathsf{E}_\mathsf{Q} X^-$ and $\mathsf{E}_\mathsf{Q} X^+$ here may take on the value $+\infty$. The set of risk-neutral measures will be denoted by $\mathcal{R}$.

Notation. For $Z \in B$, we will denote by $\mathcal{R}(Z)$ the set of probability measures Q on $\mathcal{F}$ with the property: for all $X \in A$ such that $X \geqslant -\alpha Z - \beta$ pointwise for some $\alpha, \beta \in \mathbb{R}_+$, we have $\mathsf{E}_\mathsf{Q}|X| < \infty$ and $\mathsf{E}_\mathsf{Q} X \leqslant 0$.

The following lemma is almost the same as [3; Lem. 3.4].

Lemma 3.4. *For all $Z \in B$, we have $\mathcal{R} \subseteq \mathcal{R}(Z)$.*

The following basic assumption is satisfied in all particular models considered below.

Assumption 3.5. There exists $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$ (both these sets may be empty).

Theorem 3.6 (Fundamental theorem of asset pricing). *Suppose that Assumption 3.5 is satisfied. Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition if and only if for any* $D \in \mathcal{F} \setminus \{\emptyset\}$*, there exists a risk-neutral measure* Q *such that* $\mathsf{Q}(D) > 0$.

The proof of Theorem 3.6 follows the same lines as the proof of its probability analogue in [3; Th. 3.6]. It is based on the following possibility analogue of the Kreps–Yan theorem:

Lemma 3.7. *Let* C *be a* $\sigma(L^\infty, M_F)$*-closed convex cone in* L^∞ *such that* $C \supseteq L^\infty_-$ *(*L^∞_- *is the set of negative elements of* L^∞*). Let* $W \in L^\infty \setminus C$*. Then there exists a probability measure* Q *on* $\mathcal{F}$ *such that* $\mathsf{E}_\mathsf{Q} X \leqslant 0$ *for all* $X \in C$ *and* $\mathsf{E}_\mathsf{Q} W > 0$.

Proof. By the Hahn-Banach separation theorem (see [9; Ch. II, Th. 9.2]), there exists a measure $\mathsf{Q}_0 \in M_F$ such that $\mathsf{E}_{\mathsf{Q}_0} W \notin \{\mathsf{E}_{\mathsf{Q}_0} X : X \in C\}$. Without loss of generality, $\mathsf{E}_{\mathsf{Q}_0} W > 0$. As C is a cone, $\mathsf{E}_{\mathsf{Q}_0} X \leqslant 0$ for any $X \in C$. Since $C \supseteq L^\infty_-$, Q_0 is positive. Then the measure $\mathsf{Q} = c\mathsf{Q}_0$, where c is a normalizing constant, satisfies the desired properties. □

Proof of Theorem 3.6. Step 1. Let us prove the "only if" implication. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Set $W = I_D$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 3.6 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4(Z_0)$ and to the point W yields a probability measure Q_0 on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \leqslant 0$ for any $X \in A_4(Z_0)$ and $\mathsf{E}_{\mathsf{Q}_0} W > 0$. By the Fatou lemma, for any $X \in A$ such that $\frac{X}{Z_0 + \gamma(Z_0)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_0 + \gamma(Z_0)} \leqslant 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_0 + \gamma(Z_0)} \mathsf{Q}_0$, where c is a normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$ and

$$\mathsf{Q}(D) = \mathsf{E}_{\mathsf{Q}_0} \frac{cW}{Z_0 + \gamma(Z_0)} > 0.$$

Step 2. Let us prove the "if" implication. Suppose that the NGA condition is not satisfied. Then there exist $Z \in B$ and $W \in A_4(Z) \cap (L^0_+ \setminus \{0\})$. Take $\mathsf{Q} \in \mathcal{R}$ such that $\mathsf{Q}(W > 0) > 0$. It follows from the Fatou lemma that Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z + \gamma(Z))\mathsf{Q}$, where c is a normalizing constant. For any $X \in A$ such that $\frac{X}{Z + \gamma(Z)}$ is bounded below by a constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_\mathsf{Q} X^- \leqslant \mathsf{E}_\mathsf{Q}(\alpha Z + \alpha\gamma(Z)) < \infty,$$

and consequently,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{X}{Z + \gamma(Z)} = c\mathsf{E}_\mathsf{Q} X \leqslant 0.$$

Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} X \leqslant 0$ for any $X \in A_4(Z)$. On the other hand,

$$\mathsf{E}_{\widetilde{\mathsf{Q}}} W = c\mathsf{E}_\mathsf{Q}(Z + \gamma(Z))W > 0.$$

The obtained contradiction shows that the NGA condition is satisfied. □

Remark. It is seen from the above proof that the necessity part in Theorem 3.6 is true without Assumption 3.5. It can be shown that this assumption is essential for the sufficiency part.

Now, let F be an $\mathcal{F}$-measurable function meaning the discounted payoff of some contingent claim.

Definition 3.8. A real number x is a *fair price* of F if the extended model $(\Omega, \mathcal{F}, A + \{h(F-x) : h \in \mathbb{R}\})$ satisfies the NGA condition. The set of fair prices of F will be denoted by $I(F)$.

Theorem 3.9 (Pricing contingent claims). *Suppose that the model* $(\Omega, \mathcal{F}, A)$ *satisfies Assumption 3.5 and the NGA condition, while* F *is bounded below and* $\mathsf{E}_\mathsf{Q} F < \infty$ *for any* $\mathsf{Q} \in \mathcal{R}$. *Then*

$$I(F) \approx \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\}.$$

Proof. Step 1. Let us prove the inclusion

$$I(F) \subseteq \Big[\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F, \ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\Big].$$

Let $x \in I(F)$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Set $Z_1 = Z_0 + (F - x)$. Then $Z_1 \in B'$, where B' is defined by (1) with A replaced by

$$A' = \{X + h(F-x) : X \in A, \ h \in \mathbb{R}\}. \tag{3}$$

Set $W \equiv 1$. Lemma 3.7 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A'_4(Z_1)$ ($A'_4(Z_1)$ is defined by (2) with A replaced by A') and to the point W yields a probability measure Q_0 on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \leqslant 0$ for any $X \in A'_4(Z_1)$. By the Fatou lemma, for any $X \in A'$ such that $\frac{X}{Z_1+\gamma(Z_1)}$ is bounded below, we have $\mathsf{E}_{\mathsf{Q}_0} \frac{X}{Z_1+\gamma(Z_1)} \leqslant 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_1+\gamma(Z_1)} \mathsf{Q}_0$, where c is a normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_1) \subseteq \mathcal{R}(Z_0) = \mathcal{R}$. Moreover, $\mathsf{E}_\mathsf{Q}(x - F) \leqslant 0$ and $\mathsf{E}_\mathsf{Q}(F - x) \leqslant 0$ since the functions $\frac{x-F}{Z_1+\gamma(Z_1)}$ and $\frac{F-x}{Z_1+\gamma(Z_1)}$ are bounded below. Thus, $\mathsf{E}_\mathsf{Q} F = x$.

Step 2. Suppose that $\mathsf{E}_\mathsf{Q} F = \mathsf{E}_{\mathsf{Q}'} F$ for any $\mathsf{Q}, \mathsf{Q}' \in \mathcal{R}$. Let us prove the inclusion $\mathsf{E}_\mathsf{Q} F \in I(F)$. Denote $\mathsf{E}_\mathsf{Q} F$ by x. Suppose that $x \notin I(F)$, i.e. the model $(\Omega, \mathcal{F}, A')$, where A' is given by (3), does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A'_4(Z) \cap (L^0_+ \setminus \{0\})$. Take $Z_0 \in B$ such that $\mathcal{R} = \mathcal{R}(Z_0)$. Lemma 3.7 applied to the $\sigma(L^\infty, M_F)$-closed convex cone $A_4(Z_0)$ and to the point W yields a probability measure Q_0 on $\mathcal{F}$ such that $\mathsf{E}_{\mathsf{Q}_0} X \leqslant 0$ for any $X \in A_4(Z_0)$ and $\mathsf{E}_{\mathsf{Q}_0} W > 0$. Consider the measure $\mathsf{Q} = \frac{c}{Z_0+\gamma(Z_0)} \mathsf{Q}_0$, where c is a normalizing constant. Then $\mathsf{Q} \in \mathcal{R}(Z_0) = \mathcal{R}$ and $\mathsf{E}_\mathsf{Q} W > 0$. Moreover, $\mathsf{E}_\mathsf{Q} F = x$.

Choose an arbitrary $Y = X + h(F - x) \in A'$ (here $X \in A$) such that Y is bounded below. It follows from the condition $\mathsf{E}_\mathsf{Q} F = x$ that $\mathsf{E}_\mathsf{Q} X^- < \infty$. As $\mathsf{Q} \in \mathcal{R}$, we have $\mathsf{E}_\mathsf{Q} X \leqslant 0$. This, combined with the condition $\mathsf{E}_\mathsf{Q} F = x$, implies that $\mathsf{E}_\mathsf{Q} Y \leqslant 0$. By the Fatou lemma, Z is Q-integrable. Consider the measure $\widetilde{\mathsf{Q}} = c(Z+\gamma(Z))\mathsf{Q}$, where c is a normalizing constant. For any $Y = X + h(F-x) \in A'$ (here $X \in A$) such that $\frac{Y}{Z+\gamma(Z)}$ is bounded below by some constant $-\alpha$ ($\alpha \in \mathbb{R}_+$), we have

$$\mathsf{E}_\mathsf{Q} Y^- \leqslant \mathsf{E}_\mathsf{Q}(\alpha Z + \alpha\gamma(Z)) < \infty.$$

Consequently, $\mathsf{E}_\mathsf{Q} X^- < \infty$, $\mathsf{E}_\mathsf{Q} X \leqslant 0$, and $\mathsf{E}_\mathsf{Q} Y \leqslant 0$. This means that $\mathsf{E}_{\widetilde{\mathsf{Q}}} \frac{Y}{Z+\gamma(Z)} \leqslant 0$. Hence, $\mathsf{E}_{\widetilde{\mathsf{Q}}} W \leqslant 0$. But this is a contradiction since $\widetilde{\mathsf{Q}} \sim \mathsf{Q}$ and $\mathsf{E}_\mathsf{Q} W > 0$. As a result, $x \in I(F)$.

Step 3. Let us prove the inclusion

$$\Bigl(\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F,\ \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F\Bigr) \subseteq I(F).$$

Let x belong to the left-hand side of this inclusion, i.e.

$$\inf_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F < x < \sup_{\mathsf{Q}\in\mathcal{R}} \mathsf{E}_\mathsf{Q} F.$$

Suppose that $x \notin I(F)$, i.e. the model $(\Omega, \mathcal{F}, A')$, where A' is defined by (3), does not satisfy the NGA condition. Then there exist $Z \in B'$ and $W \in A'_4(Z) \cap (L^0_+ \setminus \{0\})$. Applying the same reasoning as in the previous step, we find a measure $\mathsf{Q}_1 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_1} W > 0$. By the conditions of the theorem, $\mathsf{E}_{\mathsf{Q}_1}|F| < \infty$. Find measures $\mathsf{Q}_2, \mathsf{Q}_3 \in \mathcal{R}$ such that $\mathsf{E}_{\mathsf{Q}_2} F < x$, $\mathsf{E}_{\mathsf{Q}_3} F > x$. Clearly, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_{++}$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\mathsf{E}_\mathsf{Q} F = x$, where $\mathsf{Q} = \alpha_1\mathsf{Q}_1 + \alpha_2\mathsf{Q}_2 + \alpha_3\mathsf{Q}_3$. Note that $\mathsf{Q} \in \mathcal{R}$ due to the convexity of $\mathcal{R}$ and $\mathsf{E}_\mathsf{Q} W > 0$. The proof is now completed in the same way as in the previous step. □

The following example shows that the equality $I(F) = \{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\}$ (which is true in the probability setting; see [3; Th. 3.10]) can be violated.

Example 3.10. Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $A = \{0\}$. Consider $F(\omega) = \omega$. Then $I(F) = (0,1)$, while $\{\mathsf{E}_\mathsf{Q} F : \mathsf{Q} \in \mathcal{R}\} = [0,1]$. □

The next example shows that the assumption "$\mathsf{E}_\mathsf{Q} F < \infty$ for any $\mathsf{Q} \in \mathcal{R}$" in Theorem 3.9 is essential.

Example 3.11. Let $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, and

$$A = \Bigl\{\sum_{n=1}^N h_n X_{a_n b_n} : N \in \mathbb{N},\ a_n < b_n \in \mathbb{R}_+,\ h_n \in \mathbb{R}\Bigr\},$$

where

$$X_{ab}(\omega) = I(a < \omega \leqslant b) - I(\omega > 0)\int_a^b \mathrm{e}^{-x}\mathrm{d}x, \quad \omega \in \Omega.$$

Consider $F(\omega) = \mathrm{e}^\omega$.

If $\mathsf{Q} \in \mathcal{R}$, then, for any $a > 0$, we have $\mathsf{E_Q} X_{0a} = 0$ (note that X_{0a} is bounded), which means that

$$\mathsf{Q}((0,a]) = \mathsf{Q}(\mathbb{R}_{++}) \int_0^a \mathrm{e}^{-x}\mathrm{d}x, \quad a \in \mathbb{R}_{++}.$$

Hence, Q has the form $\alpha_1\mathsf{Q}_1 + \alpha_2\mathsf{Q}_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_1 + \alpha_2 = 1$, $\mathsf{Q}_1 = \delta_0$, and Q_2 is the exponential distribution on $\mathbb{R}_+$ with parameter 1. Clearly, any measure of this form belongs to $\mathcal{R}$. We have $\mathsf{E_Q} F = 1$ if $\mathsf{Q} = \mathsf{Q}_1$ and $\mathsf{E_Q} F = \infty$ otherwise. Consequently, $\{\mathsf{E_Q} F : \mathsf{Q} \in \mathcal{R}\} = \{1\}$.

Take now $x \in I(F)$. For $n \in \mathbb{N}$, set

$$F_n(\omega) = \begin{cases} 0 & \text{if } \omega = 0, \\ \mathrm{e}^m & \text{if } \omega \in (m, m+1],\ m = 0, \dots, n-1, \\ 0 & \text{if } \omega > n, \end{cases}$$

$$x_n = \int_0^\infty F_n(x)\mathrm{e}^{-x}\mathrm{d}x.$$

Then $F_n - x_n \in A$. Since $x_n \to \infty$, there exists n_0 such that $x_{n_0} > x$. Then

$$(F(\omega) - x) - (F_{n_0}(\omega) - x_{n_0}) \geqslant x_{n_0} - x > 0, \quad \omega \in \Omega.$$

But

$$(F - x) - (F_n - x_n) \in A' = \{X + h(F - x) : X \in A,\ h \in \mathbb{R}\}.$$

This contradicts the choice of x. As a result, $I(F) = \emptyset$. □

4 Discrete-time model with finite number of assets

We will consider a model with no transaction costs. Thus, we are given a possibility space $(\Omega, \mathcal{F})$ endowed with a filtration $(\mathcal{F}_n)_{n=0,\dots,N}$ and an $\mathbb{R}^d$-valued $(\mathcal{F}_n)$-adapted sequence $(S_n)_{n=0,\dots,N}$. From the financial point of view, S_n^i is the discounted price of the ith asset at time n. The set of attainable incomes is defined as

$$A = \biggl\{ \sum_{n=1}^N \sum_{i=1}^d H_n^i (S_n^i - S_{n-1}^i) : H_n \text{ is } \mathcal{F}_{n-1}\text{-measurable} \biggr\}.$$

We will assume that, for any $n = 0, \dots, N-1$, $\omega \in \Omega$, there exists an atom $\mathfrak{a}_n(\omega)$ of $\mathcal{F}_n$ that contains ω. (Recall that an *atom* of a σ-field $\mathcal{F}$ is a set $\mathfrak{a} \in \mathcal{F}$ such that $\mathfrak{a} \neq \emptyset$ and, for any $D \in \mathcal{F}$, we have either $D \supseteq \mathfrak{a}$ or $D \cap \mathfrak{a} = \emptyset$.)

Notation. Set $C_n(\omega) = \overline{\mathrm{conv}}\{S_{n+1}(\omega') : \omega' \in \mathfrak{a}_n(\omega)\}$ and let $C_n^\circ(\omega)$ denote the relative interior of $C_n(\omega)$.

Let $\mathcal{M}$ denote the set of probability measures on $\mathcal{F}$, with respect to which S is an $(\mathcal{F}_n)$-martingale.

Theorem 4.1 (Fundamental theorem of asset pricing). *For the model* $(\Omega, \mathcal{F}, A)$*, the following conditions are equivalent:*

(a) *NGA*;
(b) *NA* (*i.e.* $A \cap L^0_+ = \{0\}$);
(c) $S_n(\omega) \in C^\circ_n(\omega)$*,* $n = 0, \dots, N-1$*,* $\omega \in \Omega$*;*
(d) *for every* $D \in \mathcal{F} \setminus \{\emptyset\}$*, there exists* $\mathsf{Q} \in \mathcal{M}$ *such that* $\mathsf{Q}(D) > 0$*.*

Lemma 4.2. *Suppose that condition* (c) *of Theorem 4.1 is satisfied. Let* $\omega_0 \in \Omega$*. Then there exist* $\omega_1, \dots, \omega_m \in \Omega$ *and* $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ *such that* $\sum_{k=0}^m \alpha_k = 1$ *and* $\sum_{k=0}^m \alpha_k X(\omega_k) = 0$ *for any* $X \in A$*.*

Proof. We will prove this statement by induction on N.

Base of induction. For $N = 1$, the statement is verified by the same arguments as those used in the proof of Theorem 2.4 (Step 2).

Step of induction. Assume that the statement is true for $N-1$. Let us prove it for N. By the induction hypothesis, there exist $\widetilde{\omega}_1, \dots, \widetilde{\omega}_l \in \Omega$ and $\widetilde{\alpha}_0, \dots, \widetilde{\alpha}_l \in \mathbb{R}_{++}$ such that $\widetilde{\omega}_0 = \omega_0$, $\sum_{i=0}^l \widetilde{\alpha}_i = 1$, and $\sum_{i=0}^l \widetilde{\alpha}_i X(\widetilde{\omega}_i) = 0$ for any $X \in A'$, where

$$A' = \left\{ \sum_{n=1}^{N-1} \sum_{i=1}^{d} H^i_n (S^i_n - S^i_{n-1}) : H^i_n \text{ is } \mathcal{F}_{n-1}\text{-measurable} \right\}.$$

For all $i = 0, \dots, l$, there exist $\widetilde{\omega}_{i0}, \dots, \widetilde{\omega}_{il(i)} \in \mathfrak{a}_{N-1}(\widetilde{\omega}_i)$ and $\widetilde{\alpha}_{i0}, \dots, \widetilde{\alpha}_{il(i)} \in \mathbb{R}_{++}$ such that $\widetilde{\omega}_{i0} = \widetilde{\omega}_i$, $\sum_{j=0}^{l(i)} \widetilde{\alpha}_{ij} = 1$, and

$$\sum_{j=0}^{l(i)} \widetilde{\alpha}_{ij} (S_N(\widetilde{\omega}_{ij}) - S_{N-1}(\widetilde{\omega}_{ij})) = 0.$$

Let $(i(0), j(0)), \dots, (i(m), j(m))$ be a numbering of the set $\{(i,j) : i = 0, \dots, l,\ j = 0, \dots, l(i)\}$. We arrange this numbering in such a way that $i(0) = j(0) = 0$. Set $\omega_k = \widetilde{\omega}_{i(k)j(k)}$, $\alpha_k = \widetilde{\alpha}_{i(k)} \widetilde{\alpha}_{i(k)j(k)}$, $k = 0, \dots, m$. Then, for any

$$X = \sum_{n=1}^{N} \langle H_n, (S_n - S_{n-1}) \rangle \in A\,,$$

we have

$$\sum_{k=0}^{m}\alpha_k X(\omega_k)=\sum_{n=1}^{N-1}\sum_{i=0}^{l}\sum_{j=0}^{l(i)}\widetilde{\alpha}_i\widetilde{\alpha}_{ij}\langle H_n(\widetilde{\omega}_{ij}),(S_n(\widetilde{\omega}_{ij})-S_{n-1}(\widetilde{\omega}_{ij}))\rangle$$
$$+\sum_{i=0}^{l}\sum_{j=0}^{l(i)}\widetilde{\alpha}_i\widetilde{\alpha}_{ij}\langle H_N(\widetilde{\omega}_{ij}),(S_N(\widetilde{\omega}_{ij})-S_{N-1}(\widetilde{\omega}_{ij}))\rangle$$

$$=\sum_{n=1}^{N-1}\sum_{i=0}^{l}\widetilde{\alpha}_i\langle H_n(\widetilde{\omega}_i),(S_n(\widetilde{\omega}_i)-S_{n-1}(\widetilde{\omega}_i))\rangle$$
$$+\sum_{i=0}^{l}\widetilde{\alpha}_i\Big\langle H_N(\widetilde{\omega}_i),\sum_{j=0}^{l(j)}\widetilde{\alpha}_{ij}(S_N(\widetilde{\omega}_{ij})-S_{N-1}(\widetilde{\omega}_{ij}))\Big\rangle=0\,.$$

In the second equality, we used the fact that H_n, $n=0,\dots,N$ and S_n, $n=0,\dots,N-1$ are constant on the atoms of $\mathcal{F}_{N-1}$. Thus, $\omega_1,\dots,\omega_m$ and $\alpha_0,\dots,\alpha_m$ satisfy the desired conditions. □

Proof of Theorem 4.1. Step 1. The implication (a)⇒(b) is obvious.

Step 2. Let us prove the implication (b)⇒(c). Suppose that there exist $m\in\{0,\dots,N-1\}$ and $\omega_0\in\Omega$ such that $S_m(\omega_0)\notin C_m^{\circ}(\omega_0)$. By the separation theorem, there exists $h\in\mathbb{R}^d$ such that $\langle h,(S_{m+1}(\omega)-S_m(\omega))\rangle\geqslant 0$ for every $\omega\in\mathfrak{a}_m(\omega_0)$ and $\langle h,(S_{m+1}(\omega)-S_m(\omega))\rangle>0$ for some $\omega\in\mathfrak{a}_m(\omega_0)$. Set

$$H_n(\omega)=\begin{cases}hI(\omega\in\mathfrak{a}_m(\omega_0)) & \text{if } n=m+1,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\sum_{n=1}^{N}\sum_{i=1}^{d}H_n^i(S_n^i-S_{n-1}^i)\in A\cap(L_+^0\setminus\{0\})\,,$$

which contradicts the NA condition.

Step 3. Let us prove the implication (c)⇒(d). Fix $D\in\mathcal{F}\setminus\{\emptyset\}$. Choose $\omega_0\in D$. Take $\omega_1,\dots,\omega_m\in\Omega$ and $\alpha_0,\dots,\alpha_m\in\mathbb{R}_{++}$ provided by Lemma 4.2. Then the measure $\mathsf{Q}=\sum_{k=0}^{m}\alpha_k\delta_{\omega_k}$ belongs to $\mathcal{M}$ and $\mathsf{Q}(D)>0$.

Step 4. Let us prove the implication (d)⇒(a). It has been shown in [3; Lem. 4.1] that $\mathcal{M}\subseteq\mathcal{R}$. Now it follows from Theorem 3.6 that the NGA is satisfied (note that the proof of the "if" part of that theorem does not employ Assumption 3.5). □

Corollary 4.3. *Suppose that* $S_0\in\mathbb{R}_{++}^d$*,*

$$\{(S_1(\omega),\dots,S_N(\omega)):\omega\in\Omega\}=(\mathbb{R}_{++}^d)^N,$$

$\mathcal{F}_n=\mathcal{F}_n^S$*, and* $\mathcal{F}=\mathcal{F}_N$*. Then the model* $(\Omega,\mathcal{F},A)$ *satisfies the NGA condition.*

Proof. It is sufficient to note that, for every $\omega\in\Omega$ and $n=0,\dots,N-1$, we have $C_n^{\circ}(\omega)=\mathbb{R}_{++}^d$, so that condition (c) of Theorem 4.1 is satisfied. □

5 Continuous-time model with finite number of assets

We will first consider the frictionless model (its probability version was discussed in [3; Sect. 4]). Thus, we are given a possibility space $(\Omega, \mathcal{F})$ endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and a family $(S_t)_{t\in[0,T]}$ of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable functions such that, for each ω, the map $t \mapsto S_t(\omega)$ is càdlàg. We will assume that each component of S is strictly positive (this condition is naturally satisfied if, for example, each S^i is the price process of an equity or an option). The set of attainable incomes is defined by

$$A = \biggl\{\sum_{n=1}^{N}\sum_{i=1}^{d} H_n^i(S_{u_n}^i - S_{u_{n-1}}^i) : N \in \mathbb{N},\ u_0 \leqslant \dots \leqslant u_N \\ \text{are } (\mathcal{F}_t)\text{-stopping times, } H_n^i \text{ is } \mathcal{F}_{u_{n-1}}\text{-measurable}\biggr\}.$$

It follows from the results of [3; Sect. 4] that if each component of S is bounded below, then

$$\mathcal{R} = \mathcal{R}\biggl(\sum_{i=1}^{d}(S_T^i - S_0^i)\biggr) = \{\mathsf{Q} : S \text{ is an } (\mathcal{F}_t, \mathsf{Q})\text{-martingale}\}.$$

We present two sufficient conditions for the absence of generalized arbitrage.

Proposition 5.1. *Suppose that $S_0 \in \mathbb{R}_{++}^d$,*

$$\{S_{\cdot}(\omega) : \omega \in \Omega\} = \{f : f \text{ is a càdlàg piecewise constant function } [0,T] \to \mathbb{R}_{++}^d \\ \text{with a finite number of jumps, } f(0) = S_0\},$$

$\mathcal{F}_t = \mathcal{F}_t^S$, and $\mathcal{F} = \mathcal{F}_T$. Then the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition.

Proof. Fix $D \in \mathcal{F}\backslash\{\emptyset\}$. Take $\omega_0 \in D$. Let $0 < t_1 < \dots < t_N \leqslant T$ be the jump times of $S_{\cdot}(\omega_0)$. We set $t_0 = 0$, $t_{N+1} = T$. Consider the sequence $\widetilde{S}_n = S_{t_n}$. For this sequence, the set $C_n^{\circ}(\omega)$ defined in the previous section equals $\mathbb{R}_{++}^d$ for all $\omega \in \Omega$ and $n = 0, \dots, N-1$. Thus, we can apply Lemma 4.2, which yields the existence of $\omega_1, \dots, \omega_m \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $S_{\cdot}(\omega_k)$ is constant on (t_l, t_{l+1}), $k = 0, \dots, m$, $l = 0, \dots, N$, $\sum_{k=0}^m \alpha_k = 1$, and the sequence $(S_{t_0}, \dots, S_{t_{N+1}})$ is an $(\mathcal{F}_{t_0}, \dots, \mathcal{F}_{t_{N+1}})$-martingale with respect to the measure $\mathsf{Q} = \sum_{k=0}^m \alpha_k \delta_{\omega_k}$. As S is Q-a.s. constant on (t_l, t_{l+1}), $l = 0, \dots, N$, the process $(S_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. This means that $\mathsf{Q} \in \mathcal{R}$. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 3.6 completes the proof. □

Proposition 5.2. *Suppose that* $S_0 \in \mathbb{R}^d_{++}$,

$$\{S_\cdot(\omega) : \omega \in \Omega\} = \big\{f : f \text{ is a càdlàg function } [0,T] \to \mathbb{R}^d_{++} \text{ with finite variation such that, for each } i, \ \inf_{t\in[0,T]} f^i(t) > 0, \text{and } f(0) = S_0\big\}, \tag{1}$$

$\mathcal{F}_t = \mathcal{F}^S_t$*, and* $\mathcal{F} = \mathcal{F}_T$*. Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

Proof. Fix $D \in \mathcal{F}\backslash\{\emptyset\}$. Take $\omega_0 \in D$. Set $\varphi(t) = S_t(\omega_0)$, $\psi^i(t) = \ln \varphi^i(t)$, $i = 1, \dots, d$, $t \in [0,T]$. For each i, the function ψ^i can be represented as $\psi^i = \psi^i_+ - \psi^i_-$, where ψ^i_+ and ψ^i_- are càdlàg and increasing. Set

$$\lambda^i_+(t) = \frac{\psi^i_-(t)}{\mathrm{e}-1}, \quad \lambda^i_-(t) = \frac{\psi^i_+(t)}{1-\mathrm{e}^{-1}}, \quad i = 1, \dots, d, \ t \in [0,T].$$

Let N^i_+, N^i_-, $i = 1, \dots, d$ be independent Poisson processes with intensity 1. For each $i = 1, \dots, d$, the process

$$Z^i_t = \exp\big\{(N^i_+)_{\lambda^i_+(t)} - (N^i_-)_{\lambda^i_-(t)} - (\mathrm{e}-1)\lambda^i_+(t) + (1-\mathrm{e}^{-1})\lambda^i_-(t)\big\}, \quad t \in [0,T]$$

is a martingale with respect to its natural filtration. Let us denote the space of functions standing in (1) by $\mathcal{V}$. It is equipped with the σ-field $\mathcal{G} = \sigma(X_t; t \in [0,T])$, where $X_t(f) = f(t)$. Set $\mathsf{Q}_0 = \mathrm{Law}(Z_t; t \in [0,T])$. Then X is an $(\mathcal{F}^X_t, \mathsf{Q}_0)$-martingale. In view of the representation

$$Z^i_t = \varphi^i(t) \exp\big\{(N^i_+)_{\lambda^i_+(t)} - (N^i_-)_{\lambda^i_-(t)}\big\}, \quad i = 1, \dots, d, \ t \in [0,T],$$

we have $\mathsf{Q}_0(\{\varphi\}) > 0$. Define the measure Q on $\big\{S_\cdot^{-1}(C) : C \in \mathcal{G}\big\}$ by $\mathsf{Q}(S_\cdot^{-1}(C)) := \mathsf{Q}_0(C)$. Note that $\{S_\cdot^{-1}(C) : C \in \mathcal{G}\} = \mathcal{F}$ and Q is correctly defined. Then S is an $(\mathcal{F}_t, \mathsf{Q})$-martingale. This means that $\mathsf{Q} \in \mathcal{R}$. Moreover, the set $S_\cdot^{-1}(\{\varphi\})$ contains ω_0 and is an atom of $\mathcal{F}$. Hence, $S_\cdot^{-1}(\{\varphi\}) \subseteq D$, and therefore, $\mathsf{Q}(D) > 0$. An application of Theorem 3.6 completes the proof. □

The following statement is rather surprising.

Proposition 5.3. *Suppose that* $S_0 \in \mathbb{R}^d_{++}$,

$$\{S_\cdot(\omega) : \omega \in \Omega\} = \{f : f \text{ is a continuous function } [0,T] \to \mathbb{R}^d_{++}, \ f(0) = S_0\},$$

$\mathcal{F}_t = \mathcal{F}^S_t$*, and* $\mathcal{F} = \mathcal{F}_T$*. Then the model* $(\Omega, \mathcal{F}, A)$ *does not satisfy the NGA condition.*

Proof. Suppose that the NGA condition is satisfied. By Theorem 3.6, there exists a measure $\mathsf{Q} \in \mathcal{M}$ such that $\mathsf{Q}(D) > 0$, where $D = \{S^1_t = 1 + t, t \in [0,T]\}$. Then S should be an $(\mathcal{F}_t, \mathsf{Q})$-martingale (see [3; Sect. 4]). Moreover, S is continuous. On the set D, the quadratic variation of S^1 is 0. This implies that $S^1_T = S^1_0$ Q-a.e. on D (see [8; Ch. IV, Prop. 1.13]). The obtained contradiction shows that the NGA condition is not satisfied. □

Let us now consider the model with proportional transaction costs (its probability version was discussed in [4; Sect. 3]). For this model, the set of discounted incomes is defined as

$$A = \Bigg\{ \sum_{n=0}^{N} \sum_{i=1}^{d} \big[-H_n^i I(H_n^i > 0) S_{u_n}^i - H_n^i I(H_n^i < 0)(1-\lambda^i) S_{u_n}^i \big] : \\ N \in \mathbb{N},\ u_0 \leqslant \dots \leqslant u_N \text{ are } (\mathcal{F}_t)\text{-stopping times}, \\ H_n^i \text{ is } \mathcal{F}_{u_n}\text{-measurable, and } \sum_{n=0}^{N} H_n = 0 \Bigg\} .$$

Here $\lambda^i \in [0,1]$ means the coefficient of proportional transaction costs for the ith asset. For this model, we are able to prove the absence of generalized arbitrage under more natural assumptions than those used for the frictionless model.

Proposition 5.4. *Suppose that* $S_0 \in \mathbb{R}_{++}^d$,

$$\{S_\cdot(\omega) : \omega \in \Omega\} = \{f : f \text{ is a continuous function } [0,T] \to \mathbb{R}_{++}^d,\ f(0) = S_0\},$$

$\mathcal{F}_t = \mathcal{F}_t^S$, *and* $\mathcal{F} = \mathcal{F}_T$. *Suppose moreover that* $\lambda^i > 0$ *for all* i. *Then the model* $(\Omega, \mathcal{F}, A)$ *satisfies the NGA condition.*

Proof. Fix $D \in \mathcal{F} \setminus \{\emptyset\}$. Take $\omega_0 \in D$. Consider the function $\varphi(t) = S_t(\omega_0)$. Fix $i \in \{1, \dots, d\}$ and set $\Delta^i = \inf_{t \in [0,T]} S_t^i(\omega_0)$. We can find points $0 = t_0 < \dots < t_M = T$ such that

$$|S_t^i(\omega_0) - S_{t_m}^i(\omega_0)| < \lambda^i \Delta^i / 3 \quad \text{for } m = 0, \dots, M-1,\ t \in [t_m, t_{m+1}) .$$

Then the function ψ^i defined as $(1 - \lambda^i/2)\varphi^i(t_m)$ for $t \in [t_m, t_{m+1})$ is piecewise constant, $\psi^i(0) = S_0(\omega_0)$, and

$$(1 - \lambda^i)\varphi^i(t) \leqslant \psi^i(t) \leqslant \varphi^i(t), \quad i = 1, \dots, d,\ t \in [0,T] . \tag{2}$$

Set $\psi(t) = (\psi^1(t), \dots, \psi^d(t))$ and take $\omega_0' \in \Omega$ such that $S_t(\omega_0') = \psi(t)$. The reasoning used in the proof of Proposition 5.1 shows that there exist $\omega_1', \dots, \omega_m' \in \Omega$ and $\alpha_0, \dots, \alpha_m \in \mathbb{R}_{++}$ such that $\sum_{k=0}^m \alpha_k = 1$ and S is a martingale with respect to the measure $\mathsf{Q}_0 = \sum_{k=0}^m \lambda_k \delta_{\omega_k'}$. Set $\omega_k = \omega_k'$, $k = 1, \dots, m$. Consider an arbitrary element

$$X = \sum_{n=0}^{N} \sum_{i=1}^{d} \big[-H_n^i I(H_n^i > 0) S_{u_n}^i - H_n^i I(H_n^i < 0)(1-\lambda^i) S_{u_n}^i \big] \in A.$$

Set

$$Y = \sum_{n=0}^{N} \sum_{i=1}^{d} \big[-H_n^i I(H_n^i > 0) S_{u_n}^i - H_n^i I(H_n^i < 0) S_{u_n}^i \big].$$

In view of (2), $X(\omega_0) \leqslant Y(\omega_0')$, and hence,

$$\sum_{k=0}^{m} \alpha_k X(\omega_k) \leqslant \alpha_0 Y(\omega_0') + \sum_{k=1}^{m} \alpha_k X(\omega_k) \leqslant \sum_{k=0}^{m} \alpha_k Y(\omega_k') = \mathsf{E}_{\mathsf{Q}_0} Y.$$

Using the fact that S is an $(\mathcal{F}_t, \mathsf{Q}_0)$-martingale and employing the representation

$$Y = \sum_{n=1}^{N} \sum_{i=1}^{d} \left(\sum_{k=1}^{n-1} H_k^i \right) (S_{u_n}^i - S_{u_{n-1}}^i),$$

we conclude that $\mathsf{E}_{\mathsf{Q}_0} Y = 0$ (note that Q_0 is concentrated on a finite set of points). Thus, the measure $\mathsf{Q} = \sum_{k=0}^{m} \alpha_k \delta_{\omega_k}$ belongs to $\mathcal{R}$. Moreover, $\mathsf{Q}(D) > 0$. An application of Theorem 3.6 completes the proof (Assumption 3.5 is satisfied in this model; see [3; Lem. 3.1]). □

6 Model with european call options as basic assets

We will consider a model with no transaction costs (its probability version was discussed in [3; Sect. 6]). Thus, we are given a possibility space $(\Omega, \mathcal{F})$ and $T \in [0, \infty]$. Let S_T be an $\mathbb{R}_+$-valued $\mathcal{F}$-measurable function. From the financial point of view, S_T is the price of some asset at time T. Let $\mathbb{K} \subseteq \mathbb{R}_+$ be the set of strike prices of European call options on this asset with maturity T and let $\varphi(K)$, $K \in \mathbb{K}$ be the price at time 0 of a European call option with the payoff $(S_T - K)^+$. The set of attainable incomes is defined as

$$A = \left\{ \sum_{n=1}^{N} h_n((S_T - K_n)^+ - \varphi(K_n)) : N \in \mathbb{N},\ K_n \in \mathbb{K},\ h_n \in \mathbb{R} \right\}.$$

We assume that $0 \in \mathbb{K}$, which means the possibility to trade the underlying asset. Consider $Z_0 = S_T - \varphi(0)$. Then, for all $\mathsf{Q} \in \mathcal{R}(Z_0)$ and all $X \in A$, we have $\mathsf{E}_{\mathsf{Q}}|X| < \infty$ and $\mathsf{E}_{\mathsf{Q}} X = 0$. Thus, Assumption 3.5 is satisfied.

This model will be studied in two (most important) cases:

1. the case, where $\mathbb{K} = \mathbb{R}_+$;
2. the case, where $\mathbb{K}$ is finite.

Propositions 6.1 and 6.2 show that in case 1 the NGA condition is not satisfied in most natural situations, while in case 2 the NGA condition is satisfied in most natural situations.

Below φ'_+ denotes the right-hand derivative and φ'' denotes the second derivative of a convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ taken in the sense of distributions (i.e., $\varphi''([a, b]) = \varphi'_+(b) - \varphi'_+(a)$) with the convention: $\varphi''(\{0\}) = \varphi'_+(0) + 1$ (thus, φ'' is a probability measure provided that $\varphi'_+(0) \geqslant -1$).

Proposition 6.1. *Let $\mathbb{K} = \mathbb{R}_+$. Then the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition if and only if*

(a) *φ is convex;*
(b) $\varphi'_+(0) \geqslant -1$;
(c) $\lim_{x\to\infty} \varphi(x) = 0$;
(d) *the set $C := \{S_T(\omega) : \omega \in \Omega\}$ is countable;*
(e) *φ'' is concentrated on C;*
(f) *$\varphi''(\{x\}) > 0$ for any $x \in C$.*

Proof. Step 1. Let us prove the "only if" implication. If the NGA is satisfied, then (by Theorem 3.6), for any $a \in C$, there exists a risk-neutral measure Q such that $\mathsf{Q}(S_T = a) > 0$. We have

$$\mathsf{E}_{\mathsf{Q}}(S_T - K)^+ = \varphi(K), \quad K \in \mathbb{R}_+ ,$$

which immediately implies (a)–(c). Furthermore, it follows that $\operatorname{Law}_{\mathsf{Q}} S_T = \varphi''$. In particular, $\varphi''(\{a\}) = \mathsf{Q}(S_T = a) > 0$, which yields (f), and (f) leads to (d). Employing once more the property $\operatorname{Law}_{\mathsf{Q}} S_T = \varphi''$, we get (e).

Step 2. Let us prove the "if" part. Let $a_1, a_2, \dots$ be a numbering of C. Find $\omega_1, \omega_2, \dots$ such that $S_T(\omega_i) = a_i$ and consider the measure $\mathsf{Q} = \sum_i \varphi''(\{a_i\})\delta_{\omega_i}$. Then

$$\operatorname{Law}_{\mathsf{Q}} S_T = \sum_i \varphi''(\{a_i\})\delta_{a_i} = \varphi'' .$$

Hence,

$$\mathsf{E}_{\mathsf{Q}}(S_T - K)^+ = \int_{\mathbb{R}_+} (x - K)^+ \varphi''(\mathrm{d}x) = \varphi(K), \quad K \in \mathbb{R}_+ ,$$

which means that Q is a risk-neutral measure. Furthermore, $\mathsf{Q}(S_T = a) = \varphi''(\{a\}) > 0$ for any $a \in C$. By Theorem 3.6, the NGA is satisfied. □

Proposition 6.2. *Suppose that $\mathbb{K}$ is finite, $0 \in \mathbb{K}$, and $\{S_T(\omega) : \omega \in \Omega\} = \mathbb{R}_{++}$. Then the model $(\Omega, \mathcal{F}, A)$ satisfies the NGA condition if and only if*

(a) *φ is strictly positive on $\mathbb{K}$;*
(b) *φ is strictly convex on $\mathbb{K}$;*
(c) *φ is strictly decreasing on $\mathbb{K}$;*
(d) *$\varphi(x) > \varphi(0) - x$, $x \in \mathbb{K} \setminus \{0\}$.*

Proof. Step 1. Let us prove the "only if" implication. If the NGA is satisfied, then, for any $a \in \mathbb{R}_{++}$, there exists a risk-neutral measure Q such that $\mathsf{Q}(S_T = a) > 0$. The function $\psi(x) := \mathsf{E}_{\mathsf{Q}}(S_T - x)^+$, $x \in \mathbb{R}_+$ is positive, convex, decreasing, $\psi(x) \geqslant \psi(0) - x$ for any $x \in \mathbb{R}_+$, ψ' has a jump at the point a, and ψ coincides with φ on K. This yields (a)–(d).

Step 2. Let us prove the "if" part. Fix $a \in \mathbb{R}_{++}$. We can find a piecewise linear function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is convex, $\psi'_+(0) = -1$,

$\lim_{x\to\infty}\psi(x) = 0$, $\psi''(\{a\}) > 0$, and ψ coincides with φ on K. The measure ψ'' is concentrated on a countable set $\{a_1, a_2, \dots, \}$. Find ω_i such that $S_T(\omega_i) = a_i$ and consider the measure $\mathsf{Q} = \sum_i \varphi''(\{a_i\})\delta_{\omega_i}$. Then

$$\mathsf{E}_{\mathsf{Q}}(S_T - K)^+ = \psi(K) = \varphi(K), \quad K \in \mathbb{K},$$

which means that Q is a risk-neutral measure. Furthermore, $\mathsf{Q}(S_T = a) = \psi''(\{a\}) > 0$. By Theorem 3.6, the NGA condition is satisfied. □

References

1. Breeden, D.T., Litzenberger, R.H. (1978): Prices of state-contingent claims implicit in option prices. Journal of Business, **51**, No. 4, 621–651
2. Brown, H., Hobson, D., Rogers, L.C.G. (2001): Robust hedging of barrier options. Mathematical Finance, **11**, No. 3, 285–314
3. Cherny, A.S. (2007): General arbitrage pricing model: I – Probability approach (in this volume)
4. Cherny, A.S. (2007): General arbitrage pricing model: II – Transaction costs. (in this volume)
5. Föllmer, H., Schied, A. (2004): Stochastic finance. An introduction in discrete time, 2nd Edition. Walter de Gruyter, New York
6. Hobson, D.G. (1998): Robust hedging of the lookback option. Finance and Stochastics, **2**, No. 4, 329–347
7. Kolmogorov, A.N. (1933): Grundbegriffe der Wahrscheinlichkeitsrechnung. Springer, Berlin Heidelberg New York
8. Revuz, D., Yor, M. (1999): Continuous martingales and Brownian motion, 3rd Edition. Springer, Berlin Heidelberg New York
9. Schaefer, H.H. (1971): Topological vector spaces. Springer, Berlin Heidelberg New York

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