Modularity of Confluence
Constructed

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Abstract. We present a novel proof of Toyama's famous modularity of confluence result for term rewriting systems. Apart from being short and intuitive, the proof is modular itself in that it factors through the decreasing diagrams technique for abstract rewriting systems, is constructive in that it gives a construction for the converging rewrite sequences given a pair of diverging rewrite sequences, and general in that it extends to opaque constructor-sharing term rewriting systems. We show that for term rewrite systems with extra variables, confluence is not preserved under *de*composition, and discuss whether for these systems confluence *is* preserved under composition.

1 Introduction

We present a novel proof of the classical result due to Toyama, that confluence is a modular property of term rewriting systems, that is, the disjoint union of two term rewriting systems is confluent if and only if both systems are confluent.

To illustrate both our proof method and its difference with existing proofs in the literature [1,2,3], we make use of the following example.

Example 1. Let $\mathcal{S}$ be the disjoint union $\mathcal{CL} \uplus \mathcal{E}$ of the term rewriting systems $\mathcal{CL} = (\{@, I, K, S\}, \{Ix \to x, Kxy \to x, Sxyz \to xz(yz)\})$ for combinatory logic (where @ is left implicit and associates to the left) and $\mathcal{E} = (\{*, a, b\}, \{x * x \to x, a \to b\})$ (with $*$ written infix).

Both these term rewriting systems are confluent, $\mathcal{CL}$ is since it is left-linear and non-overlapping (Rosen, see [4, Sect. 4.1]) and $\mathcal{E}$ is since it is terminating and all its critical pairs (none) are convergent (Huet, see [4, Lemma 2.7.15]).

Now consider the following peak in $\mathcal{S}$ for arbitrary $\mathcal{CL}$-terms t, s, u with $t \to_{\mathcal{CL}} u$:

$$(u * s)a \leftarrow_{\mathcal{CL}} (t * s)a \to_{\mathcal{E}} (t * s)b$$

Intuitively, it is easy to find a common reduct:

$$(u * s)a \to_{\mathcal{E}} (u * s)b \leftarrow_{\mathcal{CL}} (t * s)b$$

and indeed this valley is constructed by our proof. We will now informally explain our proof using classical modularity terminology from the papers mentioned above, which also can be found in the textbooks [5,4].

A. Armando, P. Baumgartner, and G. Dowek (Eds.): IJCAR 2008, LNCS 5195, pp. 348–363, 2008.

Our proof is based on a novel decomposition of a term into its *base vector* and its *base*. These notions will be defined formally in Section 3, but can be understood as the vector of aliens (also known as principal subterms) of *maximal* rank of the term, and its context, respectively. This decomposition of terms in turn induces a decomposition of reductions on base vectors and bases such that, roughly speaking, both are confluent and they commute.

Example 2. The source $(t * s)a$ of the initial peak has two aliens $t * s$ and a, only the first of which is of maximal rank 2. Accordingly, its base vector consists only of the former so is $(t * s)$, and its base is $(x_1)a$. Indeed the base vector and base compose, by substituting the former for x_1 in the latter, to the original term.

For the peak, the step on the left decomposes as a step on the base vector: $(u * s) \leftarrow (t * s)$ within base $(x_1)a$, whereas the step on the right decomposes as a step on the base: $(x_1)a \rightarrow (x_1)b$ with base vector $(t * s)$.

These steps commute, giving rise to the valley on the previous page, consisting of two steps. The step on the left is composed of the step on the base: $(x_1)a \rightarrow (x_1)b$ and base vector $(u * s)$, and the step on the right is composed of the step on the base vector: $(u * s) \leftarrow (t * s)$ and base $(x_1)b$.

Focussing attention on aliens of *maximal* rank is in a sense dual to the idea of the proofs in the literature [1,2,3], which are based on representing aliens by terms of *minimal* rank. To that end each of the latter three proofs relies on a test whether a layer may collapse or not (*root preservedness*, *inner preservedness*, and *cap-stability*, respectively). As the test whether a layer may collapse or not is an undecidable property, the proofs in the literature are non-constructive.

Example 3. To construct the common reduct for the peak above, each of these proofs depends on testing whether the term $t * s$ may collapse, i.e. rewrite to a $\mathcal{CL}$-term, or not. Since the rule for $*$ is non-left-linear, this requires testing whether t and s have a common reduct in $\mathcal{CL}$, which is an undecidable property.

In contrast, our proof yields modularity of *constructive* confluence: a pair of converging rewrite sequences can be constructed for every pair of diverging rewrite sequences, in the disjoint union of two term rewriting systems if and only if the same holds for each of the systems separately.

Example 4. $\mathcal{CL}$ is constructively confluent since a valley can be constructed for a given peak via the so-called orthogonal projections ([4, Chapter 8]), and $\mathcal{E}$ is constructively confluent since a valley can be constructed for a given peak by termination ([4, Chapter 6]). By our main result (Corollary 1) their disjoint union $\mathcal{S}$ is constructively confluent.

After recapitulating the relevant notions from rewriting in Section 2, our proof of modularity of confluence is presented in Section 3, and argued to be constructive in Section 4. In Section 5, we discuss (im)possible extensions of our technique.

2 Preliminaries

We recapitulate from the literature standard rewriting notions pertaining to this paper. The reader is referred to the original papers [1,2,3] for more background on modularity of confluence (Theorem 2) and to [6] for the confluence by decreasing diagrams technique (Theorem 1). As the textbooks [5,4] give comprehensive overviews of both topics, we have based our set-up on them, and we refer to them for the standard notions employed. The novel part of these preliminaries starts with Definition 1.

Rewriting preliminaries We employ arrow-like notations such as $\to$, $\rightsquigarrow$, $\triangleright$, $\blacktriangleright$ to range over *rewrite* relations which are binary relations on a set of *objects*. The *converse* $\to^{-1}$ of a rewrite relation $\to$ is also denoted by mirroring the symbol $\leftarrow$, its reflexive closure is denoted by $\to^{=}$ and its transitive closure by $\to^{+}$. Replicating a symbol is used to denote an idea of repetition; the replication $\twoheadrightarrow$ of $\to$ will be used simply as another notation for its reflexive–transitive closure $\to^{*}$; the meanings of $\triangleright\!\triangleright$ and $\blacktriangleright\!\blacktriangleright$ will be given below. A *peak* (*valley*) is a witnesses to $\twoheadleftarrow;\twoheadrightarrow$ ($\twoheadrightarrow;\twoheadleftarrow$), i.e. a pair of diverging (converging) rewrite sequences. A rewrite relation $\to$ is *confluent* $\mathrm{Con}(\to)$ if every peak can be completed by a valley, i.e. $\twoheadleftarrow ; \twoheadrightarrow \subseteq \twoheadrightarrow ; \twoheadleftarrow$, where ; and $\subseteq$ denote relation composition and set inclusion respectively, viewing relations as sets of ordered pairs.

Theorem 1 (Decreasing Diagrams [6]). *Let* $\to = \bigcup_{\ell \in L} \to_\ell$ *where* L *is a set of* labels *with a terminating transitive relation* $\succ$. *If it holds that* $\leftarrow_\ell ; \to_k \subseteq \twoheadrightarrow_{\curlyvee \ell} ; \to_k^{=} ; \twoheadrightarrow_{\curlyvee\{\ell,k\}} ; \twoheadleftarrow_{\curlyvee\{k,\ell\}} ; \leftarrow_\ell^{=} ; \twoheadleftarrow_{\curlyvee k}$ *for all* ℓ,k, *then* $\to$ *is confluent. Here* $\curlyvee K = \{j \mid \exists i \in K, i \succ j\}$ *and* $\curlyvee j = \curlyvee\{j\}$.

Term rewriting preliminaries A *term* rewriting system (TRS) is a system (Σ, R) with Σ its signature and R its set of rules. A *signature* is a set of *(function) symbols* with each of which a natural number, its *arity*, is associated. A *rule* is a pair (l, r), denoted by $l \to r$, of terms over the signature extended with an implicit set of nullary *(term) variables*. For a term of shape $a(\boldsymbol{t})$, a is its *head*-symbol, which can be a variable or function symbol, and terms among $\boldsymbol{t}$ are its *direct subterms*. A *subterm* of a term is either the term itself, or a subterm of a direct subterm. Witnessing this inductive notion of subterm yields the notion of *occurrence* of a subterm *in* a term. As usual, we require that the head-symbol of the *left-hand side* l of a rule $l \to r$ not be a variable, and that the variables occurring in the *right-hand side* r occur in l. A rule is *left-linear* if variables occur at most once in its left-hand side, and *collapsing* if the head-symbol of its right-hand side is a variable. For a given TRS $\mathcal{T} = (\Sigma, R)$, its associated rewrite relation $\to_{\mathcal{T}}$ is the binary relation on terms over Σ defined by $t \to_{\mathcal{T}} s$ if $t = C[l^\tau]$ and $C[r^\tau] = s$ for some single-hole context C, rule $l \to r \in R$, and substitution τ. A *substitution* is a homomorphism on terms induced by the variables, and a *context* is a term over Σ extended with the *hole* $\Box$ (which will technically be treated as a fresh nameless variable). The application of a substitution τ to the term t is denoted by t^τ. The main operation on contexts is *hole filling*, i.e. replacing occurrences of the hole by terms. If C is a context and $\boldsymbol{t}$ a term vector with *length* $|\boldsymbol{t}|$ equal to the number of holes in C, then $C[\boldsymbol{t}]$ denotes filling the holes in C from left to right with $\boldsymbol{t}$. Properties of rewrite relations apply to TRSs $\mathcal{T}$ via $\to_{\mathcal{T}}$. Below, we fix TRSs $\mathcal{T}_i = (\Sigma_i, R_i)$ and their associated rewrite relations $\to_i$, for $i \in \{1, 2\}$.

Modularity preliminaries With $\mathcal{T}_1 \uplus \mathcal{T}_2 = (\Sigma, R)$ we denote the disjoint union of $\mathcal{T}_1$ and $\mathcal{T}_2$, where $\Sigma = \Sigma_1 \uplus \Sigma_2$ and $R = R_1 \uplus R_2$. Its associated rewrite relation is denoted by $\rightarrow$. A property P is *modular* if $P(\mathcal{T}_1 \uplus \mathcal{T}_2) \iff P(\mathcal{T}_1) \,\&\, P(\mathcal{T}_2)$.

Theorem 2 (Modularity of Confluence [1]). *$Con(\rightarrow) \iff Con(\rightarrow_1)$ & $Con(\rightarrow_2)$.*

A term t over Σ can be uniquely written as $C[\boldsymbol{t}]$ such that C is a non-empty context over one of the signatures Σ_1,Σ_2, and the head-symbols of the term vector $\boldsymbol{t}$ all *not* belong to that signature, where we let variables belong to both signatures; this situation is denoted by $C[\![\boldsymbol{t}]\!]$ and C and $\boldsymbol{t}$ are said to be the *top* and *alien* vector of t, respectively. The *rank* of t is 1 plus the maximum of the ranks of $\boldsymbol{t}$ and 0. A step $t \rightarrow s$ is *top*-collapsing if it is generated by a collapsing rule in the empty context and the head-symbol of s does not belong to the signature the head-symbol of t belongs to. Observe that the rank of t is then greater than the rank of s. For vectors of terms $\boldsymbol{t}$, $\boldsymbol{s}$ of the same length $|\boldsymbol{t}| = n = |\boldsymbol{s}|$, we write $\boldsymbol{t} \propto \boldsymbol{s}$ if $t_i = t_j$ entails $s_i = s_j$, for all $1 \leq i, j \leq n$. For referring to notions pertaining to the components of $\mathcal{T}_1 \uplus \mathcal{T}_2$, *colors* $c, b, w \in \{1, 2\}$ are employed. We refer to b and w as *black* and *white*, respectively, implicitly assuming they are distinct, i.e. $b = 3 - w$.

To illustrate our concepts and proof, we use the following running example.

Example 5. Let $\mathcal{T}$ be the disjoint union $\mathcal{T}_1 \uplus \mathcal{T}_2$ of the term rewriting systems $\mathcal{T}_1 = (\{a, f\}, \{f(x, x) \rightarrow x\})$ and $\mathcal{T}_2 = (\{I, J, K, G, H\}, \{G(x) \rightarrow I, I \rightarrow K, G(x) \rightarrow H(x), H(x) \rightarrow J, J \rightarrow K\})$. The goal will be to transform the peak

$$f(I, H(a)) \leftarrow f(I, G(a)) \leftarrow f(G(a), G(a)) \rightarrow G(a)$$

into a valley. For $b = 1$, the term $f(I, G(a))$ in this peak is top-black, has top

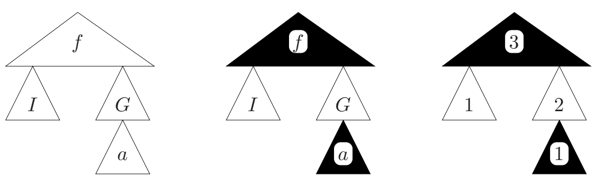

Fig. 1. Ranking a term

$f(\Box, \Box)$, alien vector $(I,G(a))$ and rank 3, since $G(a)$ is top-white, has top $G(\Box)$, alien vector (a) and rank 2, and I, a have no aliens so have rank 1 (see Figure 1).

Definition 1. *For a given rank r and color c, c-*rank-*r terms are terms either of rank r and top-c and then called* tall*, or of rank lower than r then called* short.

Every term of rank r is a c-rank-q term for *every* c and $q > r$, and a c-rank-r term for *some* color c. The aliens of a c-rank-$(r+1)$ term are $(3-c)$-rank-r-terms, and if tall its top has color c (the top of short terms can be of either color).

Example 6. The term $f(I, G(a))$ in the peak in Example 5, is short c-rank-r for *every* c and $r > 3$. Although the term is tall black-rank-3, it is *not* white-rank-3.

Rewriting a term does not increase its rank [5, p. 264][4, Proposition 5.7.6] by the constraints on the left- and right-hand sides of TRS rules, and it easily follows:

Proposition 1. *c-rank-r terms are closed under rewriting.*

3 Modularity of Confluence, by Decreasing Diagrams

We show modularity of confluence (Theorem 3) based on the decomposition of terms into bases and vectors of tall aliens, and of steps into tall and short ones, as outlined in the introduction. We show tall and short steps combine into decreasing diagrams, giving confluence by Theorem 1. To facilitate our presentation

we assume a rank r and a color b are given.

Under the assumption, we define *native* terms as black-rank-$(r+1)$ terms and use (non-indexed) t, s, u, v to range over them, and define *nonnative* terms as white-rank-r terms and use $\boldsymbol{t}$, $\boldsymbol{s}$, $\boldsymbol{u}$, $\boldsymbol{v}$ to range over vectors of nonnative terms, naming their individual elements by indexing and dropping the vector notation, e.g. t_1,s_n, etc.. Our choice of terminology should suggest that a term being nonnative generalises the traditional notion of the term being alien [5, p. 263], and indeed one easily checks that the aliens of a native term are nonnative, but unlike aliens, nonnative terms (and native terms as well) *are* closed under rewriting as follows from Proposition 1. This vindicates using $\rightarrow$ to denote rewriting them. By the assumption, tall native terms are top-black and tall nonnative terms top-white. Note an individual term can be native and nonnative at the same time, but the above conventions make a name unambiguously denote a term of either type: ordinary (non-indexed) names (t) denote natives, vector (boldface) names ($\boldsymbol{t}$) vectors of nonnatives, and indexed names (t_i) denote individual nonnatives.

Example 7. Assuming rank $r = 2$ and color $b = 1$, all terms $f(I, H(a))$, $f(I, G(a))$, $f(G(a), G(a))$, $G(a)$ in the peak in Example 5 are native and all but $G(a)$ are tall. The alien vector of $f(I, G(a))$ is $(I$,$G(a))$, consisting of the short nonnative I and the tall nonnative $G(a)$.

Base contexts, ranged over by C, D, E, are contexts obtained by replacing all tall aliens of some native term by the empty context $\Box$. Clearly, their rank does not exceed r.

Proposition 2. *If t is a native term, then there are a unique base context C and vector $\boldsymbol{t}$ of tall aliens,* the *base context and* base *vector of t, such that $t = C[\boldsymbol{t}]$.*

Proof. The base context is obtained by replacing all tall aliens of t by $\Box$. Uniqueness follows from uniqueness of the vector of tall aliens. $\Box$

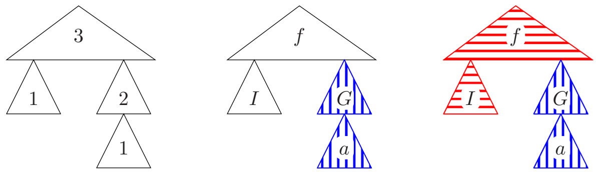

Fig. 2. From rank to base context–base vector decompositon

Example 8. For r,b as in Example 7, the base context of $f(I, G(a))$ is $f(I, \square)$, and its base vector is $(G(a))$ (see Figure 2, base vector vertically striped, base context horizontally). The base context of $G(a)$ is the term itself, and its base vector is empty.

In order to mediate between reductions in the base context and in the vector of aliens, it will turn out convenient to have a class of terms with 'named holes in which nonnative terms can be plugged in'. For that purpose we assume to have a set of *nonnative* variables, disjoint from the term variables, and we let $\boldsymbol{x}$ range over vectors of nonnative variables. *Nonnative* substitutions are substitutions of nonnative terms for nonnative variables, ranged over by τ, σ, υ, ϕ, η. In particular, we let η denote an arbitrary but fixed bijection from nonnative variables to tall nonnative terms. A *base* term, ranged over by c, d, e, f, is a term which is either obtained by applying η^{-1} to all tall aliens of some native term, or is a nonnative variable called an *empty* base.[1] Again, their rank does not exceed r.

Proposition 3. *If t is a native term, then there is a unique non-empty base term c,* the *base of t, such that $t = c^{\eta}$.*

Proof. As for Proposition 2, also using bijectivity of η for uniqueness. $\square$

The intuition that (non-empty) base terms are base contexts with 'named holes for nonnative terms' is easily seen to be correct: If C, $\boldsymbol{t}$, and c are the base context, base vector, and base of t, respectively, then $C = c^{x_i \mapsto \square}$ and $c = C[\eta^{-1}(\boldsymbol{t})]$.

Example 9. For r,b as above, let η map x_1 to the tall nonnative $G(a)$. The base of the tall native $f(I, G(a))$ is obtained by applying η^{-1} to its tall alien $G(a)$, yielding $f(I, x_1)$. The base of $G(a)$, now seen as short native, is the term itself.

Definition 2. *A step on the nonnative vector $\boldsymbol{t}$ is* tall *if the element t_i rewritten is. The* imbalance *$\#\boldsymbol{t}$ of a vector $\boldsymbol{t}$ of nonnative terms is the cardinality of its subset of tall ones, and the* imbalance *$\#t$ of a native term $t = C[\![\boldsymbol{t}]\!]$ is $\#\boldsymbol{t}$. We write $\boldsymbol{t} \rhd\!\!\rhd_{\iota} \boldsymbol{s}$ to denote that the vector $\boldsymbol{t}$ of nonnative terms rewrites in a positive number of tall steps to $\boldsymbol{s}$ having imbalance $\iota = \#\boldsymbol{s}$, and we write $t \rhd\!\!\rhd_{\iota} s$ for a native term $t = C[\![\boldsymbol{t}]\!]$, if $\boldsymbol{t} \rhd\!\!\rhd_{\iota} \boldsymbol{s}$ and $C[\boldsymbol{s}] = s$.*

The final line is justified by the fact that then $\#\boldsymbol{s} = \iota = \#s$.

[1] Directly defining base terms inductively is not difficult but a bit cumbersome.

Example 10. – $f(G(a), G(a)) \rhd\!\!\rhd_1 f(I, H(a))$ because the vector of tall aliens $(G(a),G(a))$ of the former rewrites by tall steps first to $(I,G(a))$ and then to $(I,H(a))$, which has imbalance 1 since it has only 1 tall element.
– $f(G(a), G(a)) \rhd\!\!\rhd_2 f(H(a), G(a))$ by rewriting the first tall alien $G(a)$ to $H(a)$. Imbalance was increased from 1 to 2 along this tall step.

Definition 3. *On nonnative vectors and native terms, a* short *step is defined to be a* $\to$*-step which is not of the shape of a single-step* $\rhd\!\!\rhd_\iota$ *rewrite sequence, and* $\blacktriangleright\!\!\blacktriangleright$ *denotes a positive number of short steps, with on native terms the condition that only the last step may be* tall-collapsing*, i.e. top-collapsing a tall term.*

Example 11. – $f(G(a), G(a)) \blacktriangleright\!\!\blacktriangleright G(a)$ because the former rewrites in one step to the latter, but not by a step in a tall alien. Although also $G(a) \blacktriangleright\!\!\blacktriangleright I$ (there are no tall aliens), we do *not* have $f(G(a), G(a)) \blacktriangleright\!\!\blacktriangleright I$, since the condition that only the last step may be tall-collapsing would then be violated.[2]
– $f(I, G(a)) \to f(I, I) \to f(I, K)$ induces $f(I, G(a)) \rhd\!\!\rhd_0 f(I, I) \blacktriangleright\!\!\blacktriangleright f(I, K)$ (see Figure 3). Note that the second, short, step does not change imbalance.

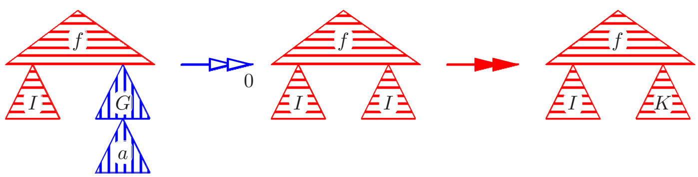

Fig. 3. Tall and short step

– Tall and short steps need not be disjoint: setting for this example rank $r = 1$ and color $b = 1$, the step $f(I, J) \to f(J, J)$ in the disjoint union of $\{f(x, y) \to f(y, y)\}$ and $\{I \to J\}$ can be both of the shape of a single-step $\rhd\!\!\rhd_\iota$ reduction and not, depending on whether I or the whole term is contracted. Hence, both $f(I, J) \rhd\!\!\rhd_1 f(J, J)$ and $f(I, J) \blacktriangleright\!\!\blacktriangleright f(J, J)$!

Proposition 4. *Confluence of* $\to$ *and* $\blacktriangleright\!\!\blacktriangleright \cup \bigcup_\iota \rhd\!\!\rhd_\iota$ *are equivalent, both on nonnative vectors and on native terms.*

Proof. In either case, it suffices to show that the reflexive–transitive closures of $\to$ and $\blacktriangleright\!\!\blacktriangleright \cup \bigcup_\iota \rhd\!\!\rhd_\iota$ coincide, which follows by monotonicity of taking reflexive–transitive closures from $\to \subseteq \blacktriangleright\!\!\blacktriangleright \cup \bigcup_\iota \rhd\!\!\rhd_\iota \subseteq \to^+$, which holds by definition. □

Lemma 1. *If* $t = c^\eta \blacktriangleright\!\!\blacktriangleright s$*, then* $s = d^\eta$ *for some* $c \twoheadrightarrow d$*, where* c *is the base of* t*.*

[2] Transitivity of $\blacktriangleright\!\!\blacktriangleright$ on native terms could be regained by dropping the condition that only the last step may be tall-collapsing. However, this would necessitate reformulating Lemma 1 below, and would make the proof of Theorem 3 a bit more complex.

Proof. We claim that if $t = c^\eta \blacktriangleright\!\!\blacktriangleright s$ by a single short step with c the base of t, then $s = d^\eta$ for some $c \twoheadrightarrow d$ such that d is either the base of s, or it is an empty base and the step was tall-collapsing. To see this, note the redex-pattern contracted does not occur in a tall alien as the step was assumed short, and neither does it overlap a tall alien as left-hand sides of rules are *monochromatic*, i.e. either black or white. Thus, the redex-pattern is entirely contained in c, and the claim easily follows. We conclude by induction on the length of the reduction using that only its last step may be tall-collapsing. □

Example 12. For $f(G(a), G(a)) \blacktriangleright\!\!\blacktriangleright G(a)$ as in the first item of Example 10, $c = f(x_1, x_1)$ and $d = x_1$. The final part of the same item shows that without the condition that only the last step may be tall-collapsing, the result would fail: I cannot be written as d^η with $f(x_1, x_1) \twoheadrightarrow d$; as I is not tall, taking x_1 fails.

The following lemma is key for dealing with non-left-linear rules. Assuming confluence, it allows to construct for diverging reductions on a vector of nonnative terms, a common reduct *as balanced as* the initial vector. Intuitively, if the initial vector 'instantiates' a non-linear left-hand side, the common reduct will do so too because it is as balanced. Moreover, the imbalance of each of the converging reductions does not exceed that of the corresponding diverging one, which will be (our) key for applying the decreasing diagrams technique (Theorem 1).

Lemma 2. *If $\to$ is confluent on nonnative vectors and $\boldsymbol{t} \rhd\!\!\rhd_{\iota_k} \boldsymbol{s_k}$ for $1 \leq k \leq n$, then there exist $\boldsymbol{v}$ such that $\boldsymbol{t} \propto \boldsymbol{v}$ and, for $1 \leq k \leq n$, there exists $\boldsymbol{u_k}$ such that $\boldsymbol{s_k} \rhd\!\!\rhd^{=}_{\#\boldsymbol{v}} \boldsymbol{u_k} \blacktriangleright\!\!\blacktriangleright^{=} \boldsymbol{v}$ and $\iota_k \geq \#\boldsymbol{v}$; if moreover $n = 1$ and $\boldsymbol{s_1} \neq \boldsymbol{u_1}$, then $\iota_1 > \#\boldsymbol{v}$.*

Proof. We proceed in *stages*, which are defined to be vectors of length $n|\boldsymbol{t}|$, written using underlining in order to distinguish them from the boldfaced vectors of length $|\boldsymbol{t}|$. Starting at $\underline{s} = \boldsymbol{s_1} \dots \boldsymbol{s_n}$, each stage $\underline{v}$ will be such that $\underline{s} \twoheadrightarrow \underline{v}$ and $\underline{s} \propto \underline{v}$, until finally $\boldsymbol{t}^n \propto \underline{v}$ holds, where $\boldsymbol{t}^n$ denotes the n-fold repetition of $\boldsymbol{t}$.

By the latter condition, if the procedure stops, then $\underline{v}$ is the n-fold repetition of some vector, say $\boldsymbol{v}$. By the invariant both $\boldsymbol{s_k} \twoheadrightarrow \boldsymbol{v}$ and $\boldsymbol{s_k} \propto \boldsymbol{v}$. By the latter, we may assume that reductions of the former taking place on identical elements of $\boldsymbol{s_k}$, are in fact identical. Using that the rank of terms does not increase along rewriting, $\boldsymbol{s_k} \twoheadrightarrow \boldsymbol{v}$ has a *tall–short* factorisation[3] of shape $\boldsymbol{s_k} \rhd\!\!\rhd^{=}_{\#\boldsymbol{v}} \boldsymbol{u_k} \blacktriangleright\!\!\blacktriangleright^{=} \boldsymbol{v}$ with $\iota_k = \#\boldsymbol{s_k} \geq \#\boldsymbol{u_k}$ (since identical elements are reduced identically), and $\#\boldsymbol{u_k} = \#\boldsymbol{v}$ (since base reductions do not affect imbalance). This gives partial correctness of the construction. To get total correctness it must also terminate.

If stage $\underline{v}$ is not yet final, there are $\underline{t}_i = \underline{t}_j$ such that $\underline{v}_i \neq \underline{v}_j$. Per construction $\underline{t} \twoheadrightarrow \underline{s} \twoheadrightarrow \underline{v}$ hence $\underline{v}_i \twoheadleftarrow \underline{t}_i = \underline{t}_j \twoheadrightarrow \underline{v}_j$ giving some $\underline{v}_i \twoheadrightarrow w \twoheadleftarrow \underline{v}_j$ by the confluence assumption. The next stage is obtained by applying the reductions $\underline{v}_i \twoheadrightarrow w$ and $\underline{v}_j \twoheadrightarrow w$ at all indices of $\underline{v}$ to which either is applicable, i.e. to all elements identical to either $\underline{v}_i$ or $\underline{v}_j$. As the cardinality (as set) of the resulting stage is smaller than that of $\underline{v}$ (elements which were identical still are and the elements at indices i, j have become so), the procedure terminates in less than $n|\boldsymbol{t}|$ steps.

[3] W.l.o.g. we assume the factorisation to depend functionally on the given reduction.

If moreover $n = 1$, note that tall nonnative terms are only reduced when they are joined with some other term. Then note that joining, both with another tall term and with another short term (to a short term!), decreases imbalance. □

Example 13. For the nonnative vector $\boldsymbol{t} = (G(a),G(a))$ with imbalance 1, consider both $\boldsymbol{t} \rhd\!\!\rhd_2 (H(a),G(a)) = \boldsymbol{s_1}$ and $\boldsymbol{t} \rhd\!\!\rhd_1 (G(a),I) = \boldsymbol{s_2}$. Then in the proof of the lemma the following vectors of nonnatives may be constructed:

$$\begin{aligned} \boldsymbol{t}^2 = \underline{t} &= (\ G(a),G(a),G(a),G(a)\ ) \\ \boldsymbol{s_1 s_2} = \underline{s} &= (\ H(a),G(a),G(a),\ \ I\ \ \ ) \\ \underline{u} &= (\ H(a),H(a),H(a),\ \ I\ \ \ ) \\ \boldsymbol{v}^2 = \underline{v} &= (\ \ K\ \ ,\ \ K\ \ ,\ \ K\ \ ,\ \ K\ \ ) \end{aligned}$$

In stage $\underline{s}$, $\underline{s}_1 = H(a) \neq G(a) = \underline{s}_2$, but $\underline{t}_1 = \underline{t}_2$. By confluence, the latter are joinable, e.g., by $H(a) \leftarrow G(a)$, and applying this to all terms in $\underline{s}$ yields $\underline{u}$.

In stage $\underline{u}$, $\underline{s}_1 = H(a) \neq I = \underline{s}_4$, but $\underline{t}_1 = \underline{t}_4$. By confluence, the latter are joinable, e.g., by $H(a) \twoheadrightarrow K \leftarrow I$, and applying these to all terms in $\underline{u}$ yields $\underline{v}$.

Therefore, for $1 \leq k \leq 2$, $\boldsymbol{s_k} \twoheadrightarrow (K,K) = \boldsymbol{v}$, with tall–short factorisations $\boldsymbol{s_1} \rhd\!\!\rhd_0 (J,J) \blacktriangleright\!\!\blacktriangleright \boldsymbol{v}$ and $\boldsymbol{s_2} \rhd\!\!\rhd_0 (J,I) \blacktriangleright\!\!\blacktriangleright \boldsymbol{v}$, both with imbalance (0) not exceeding that of $\boldsymbol{t}$ (1).

Lemma 3. *1. If $\boldsymbol{t} \rhd\!\!\rhd_\iota \boldsymbol{s} \blacktriangleright\!\!\blacktriangleright \boldsymbol{u}$, then $C[\boldsymbol{t}] \rhd\!\!\rhd_\iota C[\boldsymbol{s}] \blacktriangleright\!\!\blacktriangleright C[\boldsymbol{u}]$, for base contexts C.*
2. If $c \twoheadrightarrow d$, then for any nonnative substitution τ, $c^\tau \blacktriangleright\!\!\blacktriangleright^= d^\tau$ and $\#c^\tau \geq \#d^\tau$.

Proof. The items follow from closure of rewriting under contexts and substitutions; $\#c^\tau \geq \#d^\tau$ since reducing c can only replicate the tall nonnatives in τ. □

Example 14.
1. Filling the base context $f(\Box,\Box)$ with the tall–short factorisations of $\boldsymbol{s_k} \twoheadrightarrow \boldsymbol{v}$ in Example 13, yields $f(H(a),G(a)) \rhd\!\!\rhd_0 f(J,J) \blacktriangleright\!\!\blacktriangleright f(K,K)$ and $f(G(a),I) \rhd\!\!\rhd_0 f(J,I) \blacktriangleright\!\!\blacktriangleright f(K,K)$.
2. Since $f(x_1,x_1) \rightarrow x_1$, we have $f(G(a),G(a)) \blacktriangleright\!\!\blacktriangleright G(a)$ by mapping x_1 to $G(a)$, but also $f(a,a) \blacktriangleright\!\!\blacktriangleright a$ by mapping x_1 to a.

The proof of our main theorem is by induction on the rank of a peak. The idea is to decompose a peak into tall and short peaks, which being of lower rank can both be completed into valleys by the induction hypothesis, which then can be combined to yield decreasing diagrams by measuring tall reductions by their imbalance, giving confluence by the decreasing diagrams theorem.

Theorem 3. *$\rightarrow$ is confluent if and only if both $\rightarrow_1$ and $\rightarrow_2$ are.*

Proof. The direction from left to right is simple: For any color c, a $\rightarrow_c$-peak 'is' a $\rightarrow$-peak of c-rank-1 terms. By Con($\rightarrow$) and Proposition 1, the latter peak can be completed by a $\rightarrow$-valley of c-rank-1 terms, which 'is' a $\rightarrow_c$-valley.

The direction from right to left is complex. Assuming Con($\rightarrow_1$) and Con($\rightarrow_2$), we prove by induction on the rank, that any $\rightarrow$-peak of terms of at most that rank can be completed by a $\rightarrow$-valley of such terms again.

The base case, a peak of terms of rank 1, is simple and symmetrical to the above: such a peak 'is' a $\to_c$-peak for some color c. By $\mathrm{Con}(\to_c)$ it can be completed by a $\to_c$-valley, which 'is' a $\to$-valley of c-rank-1 terms.

Using the terminology introduced above, the step case corresponds to proving confluence of reduction on native (black-rank-$(r+1)$) terms, having confluence of reduction on nonnative (white-rank-r) terms as induction hypothesis. By Proposition 4 it suffices to show confluence of $\blacktriangleright\!\!\blacktriangleright \cup \bigcup_\iota \rhd\!\!\rhd_\iota$. To that end, we show the preconditions of Theorem 1 are satisfied taking the $\rhd\!\!\rhd_\iota$ and $\blacktriangleright\!\!\blacktriangleright$ as *steps*, ordering $\rhd\!\!\rhd_\iota$ above $\rhd\!\!\rhd_\kappa$ if $\iota > \kappa$, and ordering all these above $\blacktriangleright\!\!\blacktriangleright$.

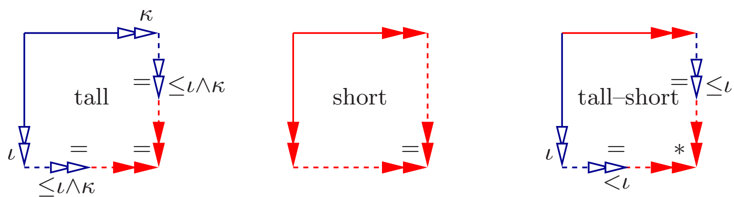

Fig. 4. Decreasing diagrams case analysis

We distinguish cases (see Figure 4) on the types, short or tall, of the steps in a peak having the native term t as source, and we let C, $\boldsymbol{t}$, and c be the base context, base vector, and base of t as given by Propositions 2 and 3, respectively.

(tall–tall) Suppose $s \lhd\!\!\lhd_\iota t \rhd\!\!\rhd_\kappa u$. By definition $s = C[\boldsymbol{s}] \lhd\!\!\lhd_\iota C[\boldsymbol{t}] \rhd\!\!\rhd_\kappa C[\boldsymbol{u}] = u$ for some $\boldsymbol{s} \lhd\!\!\lhd_\iota \boldsymbol{t} \rhd\!\!\rhd_\kappa \boldsymbol{u}$. By the induction hypothesis we may apply Lemma 2 giving $\boldsymbol{s} \rhd\!\!\rhd^{=}_{\#\boldsymbol{v}} \boldsymbol{s'} \blacktriangleright\!\!\blacktriangleright^{=} \boldsymbol{v} \blacktriangleleft\!\!\blacktriangleleft \boldsymbol{u'} \lhd\!\!\lhd_{\#\boldsymbol{v}} \boldsymbol{u}$ for some $\boldsymbol{s'}, \boldsymbol{v}, \boldsymbol{u'}$ with $\iota \geq \#\boldsymbol{v} \leq \kappa$. We conclude to $s = C[\boldsymbol{s}] \rhd\!\!\rhd^{=}_{\#\boldsymbol{v}} C[\boldsymbol{s'}] \blacktriangleright\!\!\blacktriangleright^{=} C[\boldsymbol{v}] \blacktriangleleft\!\!\blacktriangleleft^{=} C[\boldsymbol{u'}] \lhd\!\!\lhd^{=}_{\#\boldsymbol{v}} C[\boldsymbol{u}] = u$ from Lemma 3(1), giving a decreasing diagram (Figure 4 left).

(short–short) Suppose $s \blacktriangleleft\!\!\blacktriangleleft t \blacktriangleright\!\!\blacktriangleright u$. Lemma 1 entails $s = d^\eta \blacktriangleleft\!\!\blacktriangleleft c^\eta \blacktriangleright\!\!\blacktriangleright e^\eta = u$ for some nonnative peak $d \twoheadleftarrow c \twoheadrightarrow e$. By the induction hypothesis for this peak, $d \twoheadrightarrow f \twoheadleftarrow e$ for some f. We conclude to $s = d^\eta \blacktriangleright\!\!\blacktriangleright^{=} f^\eta \blacktriangleleft\!\!\blacktriangleleft^{=} e^\eta = u$ by Lemma 3(2), giving a decreasing diagram (Figure 4 middle).

(tall–short) Suppose $s \lhd\!\!\lhd_\iota t \blacktriangleright\!\!\blacktriangleright u$. By definition and Lemma 1 we have $s = C[\boldsymbol{s}] \lhd\!\!\lhd_\iota C[\boldsymbol{t}] = t = c^\eta \blacktriangleright\!\!\blacktriangleright e^\eta = u$ for some $\boldsymbol{s} \lhd\!\!\lhd_\iota \boldsymbol{t}$ and $c \twoheadrightarrow e$.
By the induction hypothesis and Lemma 2, $\boldsymbol{s} \lhd\!\!\lhd_\iota \boldsymbol{t}$ entails $\boldsymbol{s} \rhd\!\!\rhd^{=}_{\#\boldsymbol{v}} \boldsymbol{s'} \blacktriangleright\!\!\blacktriangleright^{=} \boldsymbol{v}$ for some $\boldsymbol{s'}, \boldsymbol{v}$ with $\boldsymbol{t} \propto \boldsymbol{v}$ and $\iota \geq \#\boldsymbol{v}$, and if $\boldsymbol{s} \neq \boldsymbol{s'}$ then $\iota > \#\boldsymbol{v}$. Lemma 3(1) yields $C[\boldsymbol{s}] \rhd\!\!\rhd^{=}_{\#\boldsymbol{v}} C[\boldsymbol{s'}] \blacktriangleright\!\!\blacktriangleright^{=} C[\boldsymbol{v}]$ and $\boldsymbol{t} \propto \boldsymbol{v}$ gives $C[\boldsymbol{v}] = c^\sigma$ for some $\sigma \twoheadleftarrow \eta$. By Lemma 3(2), $c \twoheadrightarrow e$ entails $c^\sigma \blacktriangleright\!\!\blacktriangleright^{=} e^\sigma$ with $\#c^\sigma \geq \#e^\sigma$. To conclude (as in Figure 4 right) we distinguish cases on whether e is empty or not.

(empty) If e is empty, say $e = x_i$ then $\sigma \twoheadleftarrow \eta$ entails $e^\sigma = \sigma(x_i) \blacktriangleleft\!\!\blacktriangleleft^{=} \eta(x_i) = e^\eta$, thus $s = C[\boldsymbol{s}] \rhd\!\!\rhd^{=}_{\#\boldsymbol{v}} C[\boldsymbol{s'}] \blacktriangleright\!\!\blacktriangleright^{=} C[\boldsymbol{v}] = c^\sigma \blacktriangleright\!\!\blacktriangleright^{=} e^\sigma \blacktriangleleft\!\!\blacktriangleleft^{=} e^\eta = u$, giving a decreasing diagram since $\iota > \#\boldsymbol{v}$ if the $\rhd\!\!\rhd^{=}_{\#\boldsymbol{v}}$-step is non-empty.

(non-empty) If e is not empty, let E and $\boldsymbol{x}$ be the (unique) base context and vector of nonnative variables such that $E[\boldsymbol{x}] = e$. Then $\sigma \twoheadleftarrow \eta$

entails $\boldsymbol{x}^\sigma \twoheadleftarrow \boldsymbol{x}^\eta$, any tall–short factorisation (see Lemma 2) of which yields $e^\sigma = E[\boldsymbol{x}^\sigma] \twoheadleftarrow^= E[\boldsymbol{x}^\phi] \lhd\!\!\lhd_\kappa^= E[\boldsymbol{x}^\eta] = e^\eta$ for some ϕ by Lemma 3(1), thus $s = C[\boldsymbol{s}] \rhd\!\!\rhd_{\#\boldsymbol{v}}^= C[\boldsymbol{s'}] \twoheadrightarrow^= C[\boldsymbol{v}] = c^\sigma \twoheadrightarrow^= e^\sigma \twoheadleftarrow^= E[\boldsymbol{x}^\phi] \lhd\!\!\lhd_\kappa^= e^\eta = u$, giving a decreasing diagram since $\iota > \#\boldsymbol{v}$ if the $\rhd\!\!\rhd_{\#\boldsymbol{v}}^=$-step is non-empty as before, and $\iota \geq \#\boldsymbol{v} = \#C[\boldsymbol{v}] = \#c^\sigma \geq \#e^\sigma = \#E[\boldsymbol{x}^\phi] = \kappa$. ☐

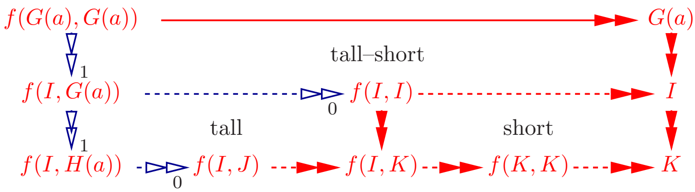

Fig. 5. The three cases in Theorem 3

Example 15. Successively completing the peaks of Example 5, written as:

$$f(I, H(a)) \lhd\!\!\lhd_1 f(I, G(a)) \lhd\!\!\lhd_1 f(G(a), G(a)) \twoheadrightarrow G(a)$$

into valleys, gives rise to Figure 5, illustrating the three cases in Theorem 3.

The main difference between our proof of modularity of confluence and those in the literature [1,2,3] is the way in which they deal with the *identity-check* of the terms occurring at the non-linear argument places of the left-hand side of a rule. Reducing these terms may cause that the identity-check ceases to succeed, upon which so-called *balancing* reductions must be performed on these terms, in order to make the identity-check succeed again.

Our proof relies on the *local* reduction history, i.e. on the reductions performed since the moment the identity-check *did* succeed, to construct the balancing reductions using Lemma 2. In the above example, in order to apply the rule $f(x, x) \to x$ to the term $f(I, G(a))$, the reduction from $f(G(a), G(a))$ for which the identity-check *did* succeed, is used to construct the balancing reduction to $f(I, I)$ for which the identity-check succeeds again.

In contrast, the proofs in the literature rely on the *global* reduction history by mapping convertible terms (so certainly those terms for which the identity-check once did succeed) to some witness. In order to guarantee that all the convertible terms in fact *reduce* to the witness, i.e. in order to obtain balancing *reductions*, the witness is chosen to be of minimal rank, cf. Example 3, which has the side-effect of making the proofs non-constructive.

4 Modularity of Confluence, Constructed

A proof is *constructive* if it demonstrates the existence of a mathematical object by providing a procedure for creating it. A rewriting system with a constructive proof of confluence is *constructively* confluent. More precisely, we require to have a procedure which for any given peak, say given as proof terms in rewriting logic [4, Chapter 8], constructs the corresponding valley. In other words, we require the existence of effective confluence functions f, g as in Figure 6.

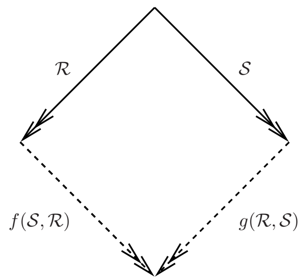

Fig. 6. Effective confluence functions

Example 16. The term rewriting system $\mathcal{CL}$ in the introduction is constructively confluent, which can be seen by setting both f and g to the residual/projection function, which is effective [4, Definition 8.7.54] (giving rise to least upper bounds [4, Figure 8.7.52]), because $\mathcal{CL}$ is an orthogonal TRS.

The term rewriting system $\mathcal{E}$ is constructively confluent, which can be seen by setting both f and g to the binary function computing the normal form of (the target of) its second argument. This procedure is effective for any constructively terminating TRS, such as $\mathcal{E}$, the critical pairs of which are convergent (the upper bounds computed are greatest with respect to the induced partial order $\twoheadrightarrow$).

Corollary 1. *Constructive confluence is modular.*

Proof. All constructions in Section 3 are effective. In fact, the proof by induction of Theorem 3 gives rise to a program which in order to compute a common reduct for a peak of a given rank, relies on the ability to decompose the peak into peaks of lower rank, on making recursive calls to itself giving valleys for those lower ranks, and the ability to compose these valleys again.

That the program eventually terminates, i.e. that it does not produce an infinitely regressing sequence of peaks for which common reducts need to be found, relies on Theorem 1 the proof of which is seen to be constructive.[4] □

Example 17. The disjoint union of $\mathcal{CL} \uplus \mathcal{E}$ is constructively confluent, computing least (greatest) upper bounds on $\mathcal{CL}$ ($\mathcal{E}$) components.

Note that since the confluence proofs in the literature are not constructive, they yield no method better than a blind search to find a common reduct, the

[4] Although, as far as we know, Theorem 1 itself has not been formalized in some proof checker, its 'point version' (where objects instead of steps are labelled) has [7].

termination of which will not be constructive. Even if one is not concerned with that, such a blind search is of course extremely inefficient.

Remark 1. The complexity of our construction should be studied. It should be interesting to formalize our proof in, say Coq, and extract an effective confluence function which computes for any pair of diverging reductions in a disjoint union, the corresponding converging reductions, by making calls to the confluence functions for the component term rewriting systems.

5 (Im)possible Extensions

5.1 Constructor-Sharing TRSs

Two (possibly non-disjoint) TRSs $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be *constructor-sharing* if the symbols shared by their alphabets (possibly none) are *constructors*, i.e. do not appear as head-symbols of left-hand sides of rules. The following well-known example [5, Examples 8.2.1,8.5.24] shows that the union of confluent constructor-sharing TRSs need not be confluent, in general.

Example 18. Let $\mathcal{T}_1$ be $\{\infty \to S(\infty)\}$ and $\mathcal{T}_2$ be $\{E(x,x) \to \mathit{true}, E(x,S(x)) \to \mathit{false}\}$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are confluent, and share the constructor S, but their union $\mathcal{T}_1 \cup \mathcal{T}_2$[5] is not confluent: $\mathit{true} \leftarrow E(\infty,\infty) \to E(\infty,S(\infty)) \to \mathit{false}$. A similar counterexample arises when setting $\mathcal{T}_1$ to $\{\infty \to I(S(\infty)), I(x) \to x\}$.

The counterexamples can be barred by forbidding the TRSs to have rules with a shared constructor or variable as head-symbol of the right-hand side, forbidding $\infty \to S(\infty)$ and $I(x) \to x$ in the example. Modularity of (constructive) confluence of such *opaque* TRSs [5, Corollary 8.5.23] reduces to ordinary modularity:

Definition 4. *For $\mathcal{T} = (\Sigma, R)$ an opaque TRS, $\underline{\mathcal{T}}$ is $(\underline{\Sigma}, \underline{R})$ with:*

- *$\underline{\Sigma}$ a new signature having opaque Σ-contexts as function symbols, where a Σ-context is* opaque *if only its head-symbol is not a constructor; its arity as function symbol is the number of holes in it as context.*
- *For a Σ-term t not headed by a shared constructor, let $\underline{t}$ denote the (unique) corresponding $\underline{\Sigma}$-term. Then $\underline{R}$ is a new set of rules consisting of $\underline{l^\gamma} \to \underline{r^\gamma}$ for all $l \to r \in R$ and all substitutions γ of shared constructor terms.*

Example 19. For the opaque TRS $\mathcal{T}_2$ of Example 18, $\underline{\mathcal{T}}_2$ consists of *false*, *true*, $E(\Box,\Box)$, $E(S(\Box),\Box)$, $E(\Box,S(\Box))$, $E(S(\Box),S(\Box))$, $E(S(S(\Box)),\Box)$,...

Lemma 4. *For any opaque TRS $\mathcal{T}$, $Con(\mathcal{T}) \iff Con(\underline{\mathcal{T}})$.*

Proof (Sketch). Since shared constructor symbols are inert, confluence of $\mathcal{T}$ reduces to confluence of non-shared-constructor-headed terms. We conclude as by opaqueness $\underline{\cdot}$ is a bisimulation [4, Section 8.4.1] between $\to_{\mathcal{T}}$ and $\to_{\underline{\mathcal{T}}}$ on such terms.

[5] Huet's counterexample to confluence of non-overlapping TRS [4, Example 4.1.4(i)].

Theorem 4. *(Constructive) confluence is modular for opaque TRSs.*

Proof. Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are confluent opaque TRSs. By the lemma and Theorem 3: $\mathrm{Con}(\mathcal{T}_1 \cup \mathcal{T}_2) \iff \mathrm{Con}(\underline{\mathcal{T}_1 \cup \mathcal{T}_2}) \iff \mathrm{Con}(\underline{\mathcal{T}}_1 \cup \underline{\mathcal{T}}_2) \iff \mathrm{Con}(\underline{\mathcal{T}}_1 \uplus \underline{\mathcal{T}}_2) \iff \mathrm{Con}(\underline{\mathcal{T}}_1) \,\&\, \mathrm{Con}(\underline{\mathcal{T}}_2) \iff \mathrm{Con}(\mathcal{T}_1) \,\&\, \mathrm{Con}(\mathcal{T}_2)$. □

5.2 Extra-Variable TRSs

Definition 5. *An* extra-variable *TRS is a term rewriting system where the condition that variables of the right-hand side must be contained in those of the left-hand side, is dropped.*

Modularity of confluence for extra-variable TRSs was studied in [8] in an abstract categorical setting. However, the following surprising example shows that confluence for extra-variable TRSs is *not* preserved under *de*composition, hence that the conditions on the categorical setting in [8] must exclude a rather large class of extra-variable TRSs, in order not to contradict the statement in [9, Section 4.3] that in that setting confluence *is* preserved under decomposition.

Example 20. Consider the following extra-variable TRSs, where the variable z in the right-hand side of the first rule of $\mathcal{T}_1$ is not contained in the variables of its left-hand side:

$$\mathcal{T}_1 = \{f(x,y) \to f(z,z), f(b,c) \to a, b \to d, c \to d\}$$

$$\mathcal{T}_2 = \{M(y,x,x) \to y, M(x,x,y) \to y\}$$

Clearly $\mathcal{T}_1$ is not confluent since $a \leftarrow f(b,c) \to f(t,t) \not\twoheadrightarrow a$ as no term t reduces to both b and c. However the disjoint union of $\mathcal{T}_1$ and $\mathcal{T}_2$ *is* confluent, as then

$$f(x,y) \to f(M(b,d,c), M(b,d,c)) \to^2 f(M(b,d,d), M(d,d,c)) \to^2 f(b,c) \to a$$

Formally: this justifies adjoining the rule $\varrho : f(x,y) \to a$, after which the first two rules of $\mathcal{T}_1$ are derivable: $f(x,y) \to_\varrho a \leftarrow_\varrho f(z,z)$ and $f(b,c) \to_\varrho a$. Removing these two rules we obtain an ordinary TRS $\mathcal{T}$ confluence of which entails confluence of the disjoint union as per construction $\leftrightarrow^*_{\mathcal{T}_1 \uplus \mathcal{T}_2} = \leftrightarrow^*_{\mathcal{T}}$ and $\twoheadrightarrow_{\mathcal{T}} \subseteq \twoheadrightarrow_{\mathcal{T}_1 \uplus \mathcal{T}_2}$. Confluence of $\mathcal{T}$ holds by Huet's Critical Pair Lemma [4, Thm. 2.7.15], as $\mathcal{T}$ is terminating and its critical pair, arising from $x \leftarrow M(x,x,x) \to x$, is trivial.

Remark 2. Note that in the above counterexample, the disjoint union $\mathcal{T}_1 \uplus \mathcal{T}_2$ is neither orthogonal nor left- or right-linear. We conjecture that requiring either of these three properties suffices for establishing that the components of a confluent extra-variable TRS are confluent again.

At present, we do not know whether our method does extend to show the other direction, preservation of confluence under disjoint unions, for extra-variable TRSs. We conjecture it does, but the following two examples exhibit some proof-invariant-breaking phenomena.

Example 21. It is not always possible to avoid creating aliens to find a common reduct. For instance, the peak $H(a) \leftarrow f(H(a)) \rightarrow g(H(a), a)$ in the union of

$$\mathcal{T}_1 = \{f(x) \rightarrow x, f(x) \rightarrow g(x, a), g(x, y) \rightarrow g(x, z), g(x, x) \rightarrow x\}$$

with the empty TRS $\mathcal{T}_2$ over $\{H\}$, can only be completed into a valley of shape $H(a) \leftarrow g(H(a), H(a)) \twoheadleftarrow g(H(a), a)$, i.e. by creating an alien $H(a)$.

Example 22. It is not always possible to erase an alien once created. For instance, consider extending the rules expressing that $*$ is an associative–commutative symbol with the creation *ex nihilo* rule $x * y \rightarrow x * y * z$. This gives a confluent TRS, where finding a common reduct can be thought of as constructing a multiset union. Obviously, reduction in the disjoint union with an arbitrary other (non-erasing) TRS may create terms of arbitrary rank which can never be erased.

Remark 3. According to [10], the proof of [3] that confluence is preserved does extend to extra-variable TRSs.

5.3 Pseudo-TRSs

For *pseudo*-TRSs [4, p. 36], i.e. extra-variable TRSs where also the condition that the head-symbol of the left-hand side must be a function symbol, is dropped, i.e. TRSs where both the left- and right-hand side are arbitrary terms, modularity trivially fails in both directions:

- The disjoint union of the confluent pseudo-TRSs $\mathcal{T}_1 = \{x \rightarrow f(x)\}$ and $\mathcal{T}_2 = \{G(A) \rightarrow A\}$ is not confluent. To wit $A \leftarrow G(A) \rightarrow G(f(A))$, and A, $G(f(A))$ do not have a common reduct as one easily verifies.
- Vice versa, the disjoint union of the pseudo-TRSs $\mathcal{T}_1 = \{a \rightarrow b, a \rightarrow c\}$ and $\mathcal{T}_2 = \{x \rightarrow D\}$ is confluent since any term reduces in one step to D, but clearly $\mathcal{T}_1$ is not confluent.

6 Conclusion

We have presented a novel proof of modularity of (constructive) confluence for term rewriting systems.

- Our proof is relatively short, in any case when omitting the illustrative examples from Section 3. Still a better measure for that would be obtained by formalising it, which we intend to pursue.
- Our proof is itself modular in that it is based on the decreasing diagrams technique. Following [11], it would be interesting to pursue this further and try to factorize other existing proof methods in this way as well. For instance, the commutation of outer and inner reductions for preserved terms ([4, Lemma 5.8.7(ii)],[5, Lemma 8.5.15]) can be proven by labelling inner steps by the imbalance of their target, since that gives rise to the *same* decreasing diagrams as in Figure 4, after replacing short (tall) by inner (outer).[6]

[6] Still, that wouldn't make these proofs constructive, as the projection to preserved terms they rely on is not constructive.

– It would be interesting to see whether our novel way of decomposing terms and steps into short and tall parts, could be meaningfully employed to establish other results (constructively), e.g. persistence of confluence [12], modularity of uniqueness of normal forms [13], or modularity of confluence for conditional TRSs [14].

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