Test Problems Based on Lamé Superspheres

Michael T.M. Emmerich and André H. Deutz

Leiden Institute of Advanced Computer Science, Leiden University,
2333 CA Leiden, The Netherlands
{deutz,emmerich}@liacs.nl
http://www.liacs.nl

Abstract. Pareto optimization methods are usually expected to find well-distributed approximations of Pareto fronts with basic geometry, such as smooth, convex and concave surfaces. In this contribution, test-problems are proposed for which the Pareto front is the intersection of a Lamé supersphere with the positive $\mathbb{R}^n$-orthant. Besides scalability in the number of objectives and decision variables, the proposed test problems are also scalable in a characteristic we introduce as *resolvability of conflict*, which is closely related to convexity/concavity, curvature and the position of knee-points of the Pareto fronts.

As a very basic bi-objective problem we propose a generalization of Schaffer's problem. We derive closed-form expressions for the efficient sets and the Pareto fronts, which are arcs of Lamé supercircles. Adopting the bottom-up approach of test problem construction, as used for the DTLZ test-problem suite, we derive test problems of higher dimension that result in Pareto fronts of superspherical geometry.

Geometrical properties of these test-problems, such as concavity and convexity and the position of knee-points are studied. Our focus is on geometrical properties that are useful for performance assessment, such as the dominated hypervolume measure of the Pareto fronts. The use of these test problems is exemplified with a case-study using the SMS-EMOA, for which we study the distribution of solution points on different 3-D Pareto fronts.

1 Introduction

Next to introducing a manageable mathematical foundation for meta-heuristic approaches in multiobjective optimization, constructing a repository of scalable and multimodal test problems is of vital importance [1, 3, 6, 8, 10]. In analyzing a test problem family with well-defined properties, we make a contribution to this ongoing effort of the multiobjective optimization community.

One reason for obtaining the complete Pareto front (PF) of a problem instead of a single non-dominated solution, is that the shape of the PF provides the decision maker with useful extra information about the nature of the conflict. A qualitative approach to this problem is to distinguish between concave, convex and linear (parts of) PFs, as it is well known that on convex PFs it is easier to

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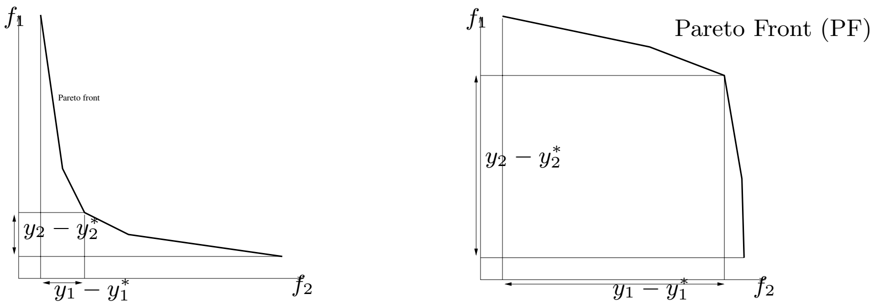


Fig. 1. Visualization of a measure for conflict resolvability. The left figure displays a scenario in which a good compromise exists, while in the scenario displayed on the right hand side they do not exists. The conflict resolvability is computed as the maximum of $y_1 - y_1^*$ and $y_2 - y_2^*$ at the position which minimizes this value, where y_1^* and y_2^* are the coordinates of the ideal point.

find good compromises than on concave ones, and also the behavior of algorithms is often different on both types of geometry.

Taking this into account, we aim for test problems that capture all three types of PFs (concave, convex, and linear). However, the test-case we propose allows also to scale quantitatively the resolvability of conflict. As a measure of the *resolvability of conflict* one may consider:

$$\mathrm{RoC}(PF) = 1 - \frac{\min_{\mathbf{y}\in PF} \max_{i\in\{1,\ldots,m\}} |y_i - y_i^*|}{\max_{\mathbf{y}\in PF} \max_{i\in\{1,\ldots,m\}} |y_i - y_i^*|} \tag{1}$$

Here, $\mathbf{y}^*$ denotes the ideal solution, m is the number of objective functions, and PF denotes the PF. Note, that in case of the denominator being zero, we define $\mathrm{RoC}(PF)$ to be one. Ideally, this value should be close to 1, meaning that all objectives are complementary.

In Figure 1 two situations are depicted. In the left figure a convex PF for a problem is depicted, for which good compromise solutions exists. In the right figure, a concave PF is depicted for which there exists no good *compromise*. We construct a highly symmetrical class of functions for which the geometry of the PF can be varied gradually from convex shapes with high resolvability of conflicts, to linear shapes, and concave shapes with low resolvability of conflicts (cf. figure 1). The problem family we propose is highly symmetrical and only introduces the difficulty of obtaining well-spread solutions on the different shapes of PFs. We consider these problems as interesting, as they can be used for analyzing metaheuristics in a controlled way, i. e. by isolating difficulties. However, we note that the complexity of the test problems can be gradually increased by adding difficulties in a managed way.

The problems can be considered as generalizations of models with spherical symmetries, that are frequently used as elementary test problems in single

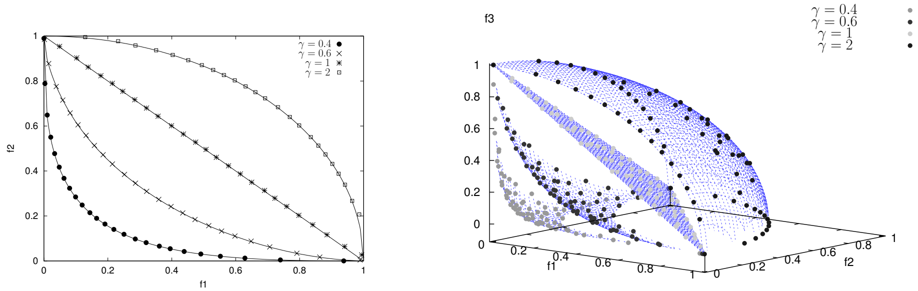


Fig. 2. PFs of different shapes as obtained with the test problem generator. The parameter γ controls the curvature and also the convexity, linearity and concavity of the PF. The points exemplify an approximation sets achieved with the SMS-EMOA.

objective optimization. Moreover we provide an analysis of the geometry of the PFs.

For the 2-D objective space, we generalize Schaffer's well known test problem $f_1(x) = x^2, f_2(x) = (1 - x)^2$ to higher dimensional search spaces. The generalized Schaffer problem [15], which has been used in some applied studies, reads: $f_1(x) = \frac{1}{n^\alpha}(\sum_{i=1}^n x_i^2)^\alpha \rightarrow \min$ and $f_2(x) = \frac{1}{n^\alpha}(\sum_{i=1}^n (1 - x_i)^2)^\alpha \rightarrow \min$ for $x_i \in \mathbb{R}_+$, where $i = 1, ..., n$. The parameter $\alpha \in \mathbb{R}_+$ controls the convexity/concavity and the resolvability of the PF, as it will be obtained in our analysis. For these generalized Schaffer functions explicit descriptions of the PF ($f_2 = (1 - f_1^\gamma)^{\frac{1}{\gamma}}$, $\gamma = \frac{1}{2\alpha}$) and efficient set are derived . More importantly the concavity changes for different choices of γ gradually from concave ($0 < \gamma < 1$), to linear ($\gamma = 1$) and convex ($1 < \gamma$). An interesting observation is that the shapes of the different Pareto curves are arcs of Lamé supercircles.

Adopting the bottom-up-approach of test-problem generation [3], the concept can be generalized to arbitrary numbers of objective functions. Again we obtain problems with super-spherical geometry of the PF for which the convexity/concavity and resolvability can be controlled by means of a single parameter. For higher dimensions the geometrical properties of such PFs are not obvious to see. Thus, we provide a detailed analysis of the geometry, focussing on properties like convexity/concavity and the size of dominated hypervolume. Based on the geometrical analysis, we provide explicit formulas for computing standard performance metrics that measure the quality of finite set approximations to the PFs, e.g., the average distance to the PF and percentage of dominated hypervolume. This will help the practitioner, who wants to assess the performance of meta-heuristics on these test problems.

The structure of this article is as follows: After the preliminaries (Section 2), Section 3 focusses on problems with two criteria. We derive an explicit formula for the PF of the generalized Schaffer problem. In Section 4 we study a general class of m-dimensional PFs for which the solution of the Schaffer problem is the

2-dimensional instance. In particular we will focus on the influence of the control parameter γ on the convexity/concavity of the PF. Adopting the bottom-up approach of multiobjective test problem construction, we provide in Section 5 a family of test problems with scalable geometrical properties. Section 6 deals with performance metrics. Finally, Section 7 illustrates the use of the test problems by means of a case study. Concluding discussions are in section 8.

2 Mathematical Preliminaries

Next, we are going to outline notions and definitions of Pareto optimality and non-dominance as they are used throughout this article. The notation is mainly borrowed from Ehrgott [4].

Definition 1. *Given two vectors* $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{y}' \in \mathbb{R}^m$ *we say* $\mathbf{y}$ *dominates* $\mathbf{y}'$ *(in symbols:* $\mathbf{y} \prec \mathbf{y}'$ *, iff* $\forall i = 1, \ldots, m : y_i \leq y'_i$ *and* $\exists i \in \{1, \ldots, m\} : y_i < y'_i$*. Moreover, we define* $\mathbf{y} \preceq \mathbf{y}' \Leftrightarrow \mathbf{y} \prec \mathbf{y}' \vee \mathbf{y} = \mathbf{y}'$*.*

Definition 2. *Given a set of points* $\mathcal{Y}$*, a point* $\mathbf{y}$ *is said to be non-dominated with respect to* $\mathcal{Y}$*, iff there does not exist* $\mathbf{y}' \in \mathcal{Y} : \mathbf{y}' \prec \mathbf{y}$*. Moreover, the subset of non-dominated points* $\mathbf{y}$ *in* $\mathcal{Y}$ *with respect to* $\mathcal{Y}$ *is called the non-dominated set of* $\mathcal{Y}$*. Also this set is referred to as the Pareto front (PF) of* $\mathcal{Y}$*.*

In the context of optimization problems $f_i(x) \rightarrow \min, i = 1, \ldots, m,\ x \in \mathcal{X}$ the concept of dominance is also defined on the search space.

Definition 3. *We say* $x \prec x' :\Leftrightarrow (f_1(x), \ldots, f_m(x)) \prec (f_1(x'), \ldots, f_m(x'))$*. Also, we define* $x \preceq x' :\Leftrightarrow (f_1(x), \ldots, f_m(x)) \preceq (f_1(x'), \ldots, f_m(x'))$*.*

Definition 4. *Given a set of points* $\mathcal{X}$ *the non-dominated subset of the set* $\{\mathbf{y} \mid y_1 = f_1(x), \ldots, y_m = f_m(x), x \in \mathcal{X}\ \}$ *is called the Pareto front (PF) with respect to the optimization problem. Moreover, the inverse image of this set in* $\mathcal{X}$ *is called the efficient set in* $\mathcal{X}$*, see[4]. The elements of this set are called efficient points.*

In the sequel $\mathbb{R}^m_+$ denotes the set $\{(y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_i \geqq 0, i = 1, \ldots, m\}$ and by $\mathbf{1}$ we denote $(1, \ldots, 1) \in \mathbb{R}^m$ where m is clear from the context.

3 Efficient Set and Pareto Front for the Generalized Schaffer Problem

In this section we derive a closed form expression for the solution of the problem

$$f_1(\mathbf{x}) = \frac{1}{n^\alpha}(\sum_{i=1}^{n} x_i^2)^\alpha \rightarrow \min,\ f_2(\mathbf{x}) = \frac{1}{n^\alpha}(\sum_{i=1}^{n}(x_i - 1)^2)^\alpha \rightarrow \min,\ \mathbf{x} \in \mathbb{R}^n_+ \quad (2)$$

Moreover, we will show that the efficient set of this problem will be the line segment $\mathcal{L}_n$:

$$\mathcal{L}_n = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \lambda \mathbf{1}, \lambda \in [0, 1]\} \quad (3)$$

Next, a number of lemmata for $\alpha = 1$ will be derived that provide building blocks for proving the interesting result for general $\alpha > 0$.

Lemma 1. *Decision vectors on the line segment $\lambda\mathbf{1}, \lambda \in [0,1]$ are mutually non dominated with respect to problem (2) and $\alpha = 1$.*

Proof. For any $(\lambda, \lambda') \in [0,1]^2$ with $\lambda' > \lambda$: $f_1(\lambda\mathbf{1}) = n\lambda^2 < n(\lambda')^2 = f_1(\lambda'\mathbf{1})$ and $f_2(\lambda\mathbf{1}) = n(1-\lambda)^2 > n(1-\lambda')^2 = f_2(\lambda'\mathbf{1})$ □

Lemma 2. *Let $\mathbf{x} \in \mathbb{R}^n_+ - \mathcal{L}_n$. Then $\lambda\mathbf{1} \prec \mathbf{x}$ for some $\lambda \in (0,1]$.*

Proof. Let $\mathbf{x} \in \mathbb{R}^n_+ - \mathcal{L}_n$. We consider the number $\lambda = \sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2}$. We first show that for this λ, $\lambda\mathbf{1} \prec \mathbf{x}$. Note that $\lambda > 0$ and possibly $\lambda > 1$. In case $\lambda \leq 1$ we are done. In case $\lambda > 1$ we easily see that $\mathbf{1} \prec \mathbf{x}$ (since $\mathbf{1} \prec \lambda\mathbf{1} \prec \mathbf{x}$). So in both cases the lemma obtains. In the remainder we will show that for the above chosen λ, $\lambda\mathbf{1} \prec \mathbf{x}$ holds:

It is clear that $f_1(\lambda\mathbf{1}) = f_1(\mathbf{x})$ holds, since $f_1(\lambda\mathbf{1}) = \frac{1}{n}\sum_{i=1}^n \lambda^2 = \lambda^2 = \frac{1}{n}\sum_{i=1}^n x_i^2 = f_1(\mathbf{x})$. Moreover, we can show that under the given assumptions $f_2(\lambda\mathbf{1}) < f_2(\mathbf{x})$ holds: $f_2(\lambda\mathbf{1}) < f_2(\mathbf{x}) \Leftrightarrow \frac{1}{n}\sum_{i=1}^n (1-\lambda)^2 < \frac{1}{n}\sum_{i=1}^n (1-x_i)^2 \Leftrightarrow 1 - 2\lambda + \lambda^2 < 1 - 2\frac{1}{n}\sum_{i=1}^n x_i + \frac{1}{n}\sum_{i=1}^n x_i^2 \Leftrightarrow \lambda^2 > \frac{1}{n^2}(\sum_{i=1}^n x_i)^2 \Leftrightarrow n\sum_{i=1}^n x_i^2 > (\sum_{i=1}^n x_i)^2$. The latter inequality holds in general for positive x_i, if for some pair $(i,j) \in \{1,\ldots,n\}^2$ the inequality $x_i \neq x_j$ holds. Since for at least one pair $(i,j) \in \{1,\ldots,n\}^2$ the inequality $x_i \neq x_j$ holds, we can show the relevant inequality as follows. To make the structure of the sum on the right hand side more visible let us format the expression in a matrix form:

$$(\sum_{i=1}^n x_i)^2 = \begin{array}{l} x_1x_1 + \cdots + x_1x_j + \cdots + x_1x_n + \\ \quad\vdots \qquad\qquad \vdots \\ x_ix_1 + \cdots + x_ix_j + \cdots + x_ix_n + \\ \quad\vdots \qquad\qquad \vdots \\ x_nx_1 + \cdots + x_1x_j + \cdots + x_nx_n \end{array} \tag{4}$$

Now, with $x_ix_j + x_jx_i \leq x_i^2 + x_j^2$ and in particular $x_ix_j + x_jx_i < x_i^2 + x_j^2$ for those pairs $(i,j) \in \{1,\ldots,n\}^2$ with $x_i \neq x_j$ the result can be simply obtained by overestimating all of the $(n-1)/n$ expressions $x_ix_j + x_jx_i$ by $x_i^2 + x_j^2$ for distinct i and j and then adding the diagonal x_i^2 which, in summary, results in an (strict) overestimator $n\sum_{i=1}^n x_i^2$. In other words we have shown that for the chosen λ, $\lambda\mathbf{1} \prec \mathbf{x}$ holds. □

Now, the lemmata can be assembled to prove the following central lemma:

Lemma 3. *The efficient set for problem (2) with $\alpha = 1$ is given by $\mathcal{L}_n$.*

Proof. Firstly we will show that $\mathcal{L}_n$ is a subset of the efficient set (ES) of problem (2). Secondly we will show that ES of problem (2) is a subset of $\mathcal{L}_n$. Let $\mathbf{x} \in \mathcal{L}_\mathbf{n}$. We want to show that $\mathbf{x}$ belongs to ES. Suppose the contrary. That is, $\exists \mathbf{x}' \in \mathbb{R}^\mathbf{n}_+$ such that $\mathbf{x}' \prec \mathbf{x}$. We distinguish two cases. Case I: $\mathbf{x}' \in \mathcal{L}_\mathbf{n}$. This leads to a contradiction because of Lemma 1. Next consider Case II: $\mathbf{x}' \in \mathbb{R}^{\mathbf{n}}_{+} - \mathcal{L}_{\mathbf{n}}$. This also leads to a contradiction because of Lemma 2 and Lemma 1.

Secondly we will show that ES is a subset of $\mathcal{L}_n$. Or equivalently: We will show that the assumption that there exists an $\mathbf{x}$ in ES such that $\mathbf{x} \notin \mathcal{L}_{\mathbf{n}}$ leads to a contradiction. Assume the existence of such an $\mathbf{x}$. Since $\mathbf{x} \notin \mathcal{L}_{\mathbf{n}}$ we know – according to Lemma 2 that there exists $\mathbf{x}' \in \mathcal{L}_{\mathbf{n}}$ such that $\mathbf{x}' \prec \mathbf{x}$. A contradiction. □

Now, by assuming general $\lambda > 0$ the proof can be extended to the central theorem of this section as follows:

Theorem 1. *The efficient set for* $f_1(\mathbf{x}) = (\frac{1}{n}\sum_{i=1}^{n} x_i^2)^\alpha \rightarrow \min$, $f_2(\mathbf{x}) = (\frac{1}{n}\sum_{i=1}^{n}(1-x_i)^2)^\alpha \rightarrow \min$ *and* $x \in \mathbb{R}^n_+$ *is given as* $\mathcal{L}_n = \{\lambda\mathbf{1} | \lambda \in [0,1]\}$. *Moreover the PF of this problem is* $y_2 = (1 - y_1^{1/2\alpha})^{2\alpha}$, $y_1 \in [0,1]$.

Proof. The generalization to $\alpha > 0$ follows from the fact, that f_1 and f_2 are transformed by the same strictly monotonous function $y \mapsto y^\theta$, $\theta > 0$, such that for any two points $\mathbf{x}$ and $\mathbf{x}'$: $f_1(\mathbf{x}) > f_1(\mathbf{x}') \Leftrightarrow f_1(\mathbf{x})^\theta > f_1(\mathbf{x}')^\theta$, $f_1(\mathbf{x}) \geq f_1(\mathbf{x}') \Leftrightarrow f_1(\mathbf{x})^\theta \geq f_1(\mathbf{x}')^\theta$ and the same for f_2. Hence, also the pre-order defined on the decision space remains equal for the problem with $\alpha = 1$ and any other $\alpha > 0$. The expression for the PF can be derived as follows: Let $\mathbf{x}$ denote an arbitrary vector $(\lambda, \ldots, \lambda) \in \mathcal{L}_n$, $\lambda \in [0,1]$. Then $f_1(\mathbf{x}) = (\lambda^2)^\alpha$ and $f_2(\mathbf{x}) = ((1-\lambda)^2)^\alpha$. From the first equation we get $\lambda = f_1^{1/2\alpha}$, which is then to be substituted in f_2, resulting in $f_2 = (1 - f_1^{1/2\alpha})^{2\alpha}$ □

For $\alpha = 1$ an alternative geometric proof of Theorem 1 can be given.

One thing that is apparent is that the parameter α plays an important role for the shape of the PF. It is easily seen that for $\alpha = 0.5$ the PF is linear, for $\alpha > 0.5$ it gets convex and for $\alpha < 0.5$ it gets concave (see also Theorem 2 and Fig. 2). Moreover the PF is symmetric w.r.t the main bisector line between the f_1 and f_2 coordinate axes and takes its extremal values (extremal solutions) in the points $\mathbf{y_1^*} = (0,1)^T$ and $\mathbf{y}_2^* = (1,0)^T$. The Nadir point is $\mathbf{y}^N = (1,1)^T$. Note, that for the more general problem $1/n^\alpha(\sum_{i=1}^{n}|x_i|^q)^\alpha \rightarrow \min$ and $1/n^\alpha(\sum_{i=1}^{n}|1-x_i|^q)^\alpha \rightarrow \min$ similar expressions for the non-dominated front can be found. However, for $0 \leq q \leq 1$ the efficient set is no longer a line segment. Emmerich [6] derived that the efficient set for $q = 1$ is the hypercube of dimension m. The problem was used in [5]. An open question for future research would be how to extend further Schaffer's problem for more than two objective functions, where each objective function is a distance function to a fixed point in $\mathbb{R}^n$. For linearly independent points, we conjecture that the convex hull of the points is the efficient set.

A crucial observation is, that the PFs of this class of problems are Lamé supercircles [12], i. e. zero sets of $|y_1|^\gamma + |y_2|^\gamma$, intersected with $\mathbb{R}^2_+$. The connection between γ and α is established as $\gamma = \frac{1}{2\alpha}$. These curves have interesting geometric properties that can be exploited to compute performance metrics (cf. section 6). A generalization of Lamé curves are the m-dimensional superspheres discussed in the next section.

4 N-Dimensional Pareto Fronts with Superspherical Geometry

In this section we will define and look at superspheres of arbitrary dimension. More specifically we will consider parts of superspheres which consist of mutually non-dominated points, of which the solutions of the aforementioned generalized Schaffer problems are special cases. It is easy to see that they arise as zero sets of strictly concave or strictly convex functions. Alternatively they can be viewed as graphs of strictly convex or strictly concave functions. Since the sets we are considering consist of mutually non dominated points, we can view them as PFs arising from multiobjective optimization problems.

4.1 Convexity and Concavity of Superspheres

In the sequel we will use notions on concavity and convexity such as convexity of sets, convexity/concavity of functions, strict convexity/concavity of functions etcetera without defining them. Instead we refer the reader to the standard literature (for instance, Convex Analysis by R. Tyrrel Rockafeller [14]).

The following definition introduces the main building blocks of the test problems we study: postive parts of superspheres and hyperspheres.

Definition 5. *Consider the set*

$$\{(y_1, \ldots, y_m) \in \mathbb{R}^m \mid |y_1|^\gamma + \cdots + |y_m|^\gamma - 1 = 0\}, \tag{5}$$

*where $\gamma \in \mathbb{R}_+$ is arbitrary and fixed. We will call such zero-sets γ–*superspheres *or more precisely the $m-1$-*dimensional γ–supersphere *(notation: S_γ^{m-1}). The supersphere which arises for $\gamma = 2$ is usually called the $m-1$-*dimensional hypersphere *(notation: S^{m-1}).*

In the sequel we will only consider the "positive" parts of the γ-superspheres, i.e., we consider sets of the form:

$$\{(y_1, \ldots, y_m) \in \mathbb{R}_+^m \mid y_1^\gamma + \cdots + y_m^\gamma = 1\}, \tag{6}$$

where $\gamma \in \mathbb{R}_+$ is arbitrary but fixed. We denote these "positive" parts of hyperspheres ($\gamma = 2$) by $S^{m-1,+}$ and those of superspheres by $S_\gamma^{m-1,+}$

Theorem 2 shows that we can view the (positive parts) of the γ-superspheres as graphs of concave ($\gamma > 1$) or convex ($0 < \gamma < 1$) functions.

Theorem 2. *Let $\gamma \in \mathbb{R}_+$ and let $\mathcal{X}_\gamma = \{(y_1, \ldots, y_{m-1}) \in \mathbb{R}_+^{m-1} \mid y_1^\gamma + \cdots + y_{m-1}^\gamma \leq 1\}$. For each positive γ, define a function $h_\gamma : \mathcal{X}_\gamma \longrightarrow \mathbb{R}$ by $h(\mathbf{y}) = (1 - (y_1^\gamma + \cdots + y_{m-1}^\gamma))^{\frac{1}{\gamma}}$, where $\mathbf{y} \in \mathbb{R}^{m-1}$. Then h_γ is strictly concave for $\gamma > 1$ and strictly convex for $0 < \gamma < 1$. For $\gamma = 1$, h_γ is convex and concave (but neither is strict).* □

Alternatively, the next Theorem shows that the positive parts of γ-superspheres can also be viewed as zero-sets of strictly concave or strictly convex functions – we again omit the proof of this theorem:

Theorem 3. *Let* $\gamma \in \mathbb{R}_+$ *and let* $f_\gamma : \mathbb{R}_+^m \longrightarrow \mathbb{R}$ *be the function defined by* $f_\gamma(\mathbf{y}) := y_1^\gamma + \cdots + y_m^\gamma - 1$.

1. *for* $\gamma > 1$, *the function* f_γ *is strictly convex and the below set* $B = \{\mathbf{y} \in \mathbb{R}_+^m \mid f(\mathbf{y}) \le 0\}$ *is convex and* $S_\gamma^{m-1,+} = \{\mathbf{y} \in \mathbb{R}_+^m \mid f(\mathbf{y}) = 0\}$ *is a subset of the boundary of* B. *(The remaining boundary points lie in the coordinate hyperplanes and the set of these remaining points is described by* $\bigcup_{i=1}^m \{(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_m) \in \mathbb{R}_+^m \mid y_1^\gamma + \cdots + x_{i-1}^\gamma + 0 + x_{i+1}^\gamma + \cdots y_m^\gamma - 1 \le 0\}$. *Or equivalently by* $\bigcup_{i=1}^n B \cap \{(y_1, \ldots, y_m) \in \mathbb{R}_+^m \mid y_i = 0\}$.*)*
2. *for* $0 < \gamma < 1$, *the function* f_γ *is strictly concave and the above set* $A = \{\mathbf{y} \in \mathbb{R}_+^m \mid f(\mathbf{y}) \ge 0\}$ *is convex and* $S_\gamma^{m-1,+} = \{\mathbf{y} \in \mathbb{R}_+^m \mid f(\mathbf{y}) = 0\}$ *is a subset of the boundary of* A. *(The remaining boundary points of* A *lie in the coordinate hyperplanes and the set of these remaining boundary points is described by* $\bigcup_{i=1}^m \{(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_m) \in \mathbb{R}_+^m \mid y_1^\gamma + \cdots + x_{i-1}^\gamma + 0 + x_{i+1}^\gamma + \cdots y_m^\gamma - 1 \ge 0\}$. *Or equivalently by* $\bigcup_{i=1}^m A \cap \{(y_1, \ldots, y_m) \in \mathbb{R}_+^m \mid y_i = 0\}$.*)*

Moreover the part of the boundary which is equal to $S_\gamma^{m-1,+}$ *in each of the above cases is equal to the graph of the function* h_γ *defined in Theorem 2.* □

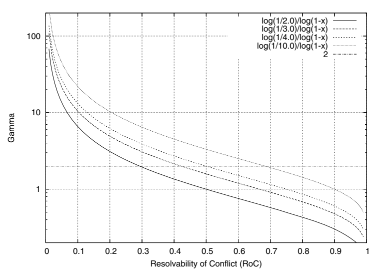
For a proof of these theorems the reader is referred to [7].

4.2 Resolvability/Intractability of Conflict Versus γ

Next, we discuss how the measure of resolvability of conflict is related to γ. It would be desirable for the user, to provide only the desired value for the RoC and then compute a corresponding value for γ. In order to come up with an expression for γ, we exploit that the Minkowski distance to the ideal point gets minimized in points with $y_1 = \cdots = y_m$. Moreover, we use $y_1^\gamma + \ldots y_m^\gamma = 1$. Combining these expressions yields the desired equation:

$$\gamma = \frac{\log(1/m)}{\log(1 - \mathrm{RoC}_m)} \tag{7}$$

For a visualization and some explicitly computed values, see Figure 3.



m	γ	$\mathrm{RoC}_m(\gamma)$
2	$\frac{1}{2}$	$\frac{3}{4}$
2	1	$\frac{1}{2}$
2	2	$1 - \frac{\sqrt{2}}{2}$
3	$\frac{1}{2}$	$\frac{8}{9}$
3	1	$\frac{2}{3}$
3	2	$1 - \frac{\sqrt{3}}{3}$

n	γ	$\mathrm{RoC}_m(\gamma)$
4	$\frac{1}{2}$	$\frac{15}{16}$
4	1	$\frac{3}{4}$
4	2	$\frac{1}{2}$
10	$\frac{1}{2}$	$\frac{99}{100}$
10	1	$\frac{9}{10}$
10	2	$1 - \frac{\sqrt{10}}{10}$

Fig. 3. RoC versus γ (logscale)

5 Construction of Test Problems

Now we are ready to use the superspheres as PFs of test problems. First we introduce parametrizations for superspheres. After that we can apply the methods introduced by Deb et al. [3] to generate test problems of which the superspheres are the PFs. We are afforded scalability in the number of objectives and also in the number of decision variables and a complete control over the extent of convexity/concavity and therefore resolvability of conflict.

5.1 Parametrizations of Hyperspheres and Superspheres

Consider the following representation of an $m-1$-dimensional γ-super-sphere (notation: S_γ^{m-1}) as a subset of $\mathbb{R}^m$

$$S_\gamma^{m-1} = \{(y_1, y_2, ..., y_m) \in \mathbb{R}^m \mid |y_1|^\gamma + |y_2|^\gamma + \dots |y_m|^\gamma = 1\} \quad (8)$$

where $\gamma \in \mathbb{R}_+$ is fixed. Since super-spheres with $\gamma = 2$ – they are usually called hyperspheres – admit parametrizations, we easily get parametrizations for any γ-super-sphere. For consider an (m-1)-dimensional hypersphere S^{m-1}:

$$S^{m-1} = \{(y_1, y_2, ..., y_m) \in \mathbb{R}^m \mid y_1^2 + y_2^2 + \dots y_m^2 = 1\} \quad (9)$$

Such a hypersphere admits parametrizations (for example:

$$\begin{cases} y_1 &= \cos(\theta_1) \\ y_2 &= \sin(\theta_1)\cos(\theta_2) \\ y_3 &= \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\ \dots &= \dots \\ y_{m-1} &= \sin(\theta_1)\sin(\theta_2)\dots\sin(\theta_{m-2})\cos(\theta_{m-1}) \\ y_m &= \sin(\theta_1)\sin(\theta_2)\dots\sin(\theta_{m-2})\sin(\theta_{m-1}), \end{cases} \quad (10)$$

where $\theta_1 \dots \theta_{m-1}$ are in $\mathbb{R}$ (or if need be in bounded intervals of $\mathbb{R}$)). For each parametrization of the hypersphere S^{m-1}, we obtain a parametrization of an $(m-1)-\gamma$-super-sphere, for a fixed $\gamma \in \mathbb{R}_+$. For let

$$y_i = p_i(\theta_1, \dots, \theta_{m-1}), \quad (11)$$

where $i = 1, \dots, m$ be a parametrization of S^{m-1}, then

$$y_i = \pm(p_i(\theta_1, \dots, \theta_{m-1}))^{\frac{2}{\gamma}}, \quad (12)$$

where $i = 1, \dots, m$ is a parametrization of S_γ^{m-1} (if we are willing to tread carefully with raising to the power $\frac{2}{\gamma}$ in (12), i.e., first raise to the power 2 and subsequently to the power $\frac{1}{\gamma}$).

5.2 Test Problems

Next we proceed to use parametrizations for the *positive part* of γ-superspheres in the construction of test problems. We will single out one family of parametrizations for the γ-superspheres which arise from the parametrization of the hypersphere as in (10). In more detail we define the positive parts of γ-superspheres as follows. Let $\gamma > 0$ be fixed. We let $\tilde{p}_1(\theta_1, \ldots, \theta_{m-1}) = (\cos(\theta_1))^{\frac{2}{\gamma}}$, $\tilde{p}_i(\theta_1, \ldots, \theta_{m-1}) = (\sin(\theta_1) \cdot \sin(\theta_2) \cdot \ldots \cdot \sin(\theta_{i-1}) \cdot \cos(\theta_i))^{\frac{2}{\gamma}}$. where $1 < i < m$, and finally $\tilde{p}_m(\theta_1, \ldots, \theta_{m-1}) = (\sin(\theta_1) \cdot \sin(\theta_2) \cdot \ldots \cdot \sin(\theta_{m-1}))^{\frac{2}{\gamma}}$ with $0 \leq \theta_j \leq \frac{\Pi}{2}$ and $j = 1, \ldots, m-1$ be a parametrization of $S_\gamma^{m-1,+}$.

Adopting the approach of Deb et al. [3] for spheres and linear surfaces, the parametrization for the γ-superspheres can be used to construct test problems as follows. We let

$$f_i(\theta, r) = (1 + g(r))\tilde{p}_i(\theta) \tag{13}$$

for $i = 1, \ldots, m$ and where $0 \leq \theta_j \leq \frac{\Pi}{2}$ for $j = 1, \ldots, m-1$ and $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and the test problem consists of the minimization of the f_i for $i = 1, \ldots, m$ under the constraints $0 \leq \theta_j \leq \frac{\pi}{2}$, for $j = 1, \ldots, m-1$.

The variables $\theta_\mathbf{i}$ with $i = 1, \ldots, m-1$ are viewed as meta-variables and one can, for instance, map the decision variables to θ_i as follows: $\theta_i = \frac{\pi}{2} x_i$ for $i = 1, \ldots, m-1$ with the restriction $0 \leq x_i \leq 1$. But one can choose other mappings as well. Also given the function g we can view the body we obtain as the image of $(f_i)_{i=1}^m$ as a layered 'onion' where each layer corresponds to a function value of g. Each of the layers can be described as follows: $\{(x_1, \ldots, x_m) \in \mathbb{R}_+^m \mid x_1^\gamma + \cdots + x_m^\gamma = (1 + g(r))^\gamma\}$ if we fix r or if we fix a function value $g(r)$. Also r can be considered as a meta-variable. The PF occurs for the minimum of the function g.

From the above it is clear that it is straightforward to apply the methods developed in Deb et al. [3] to the case of superspheres.

5.3 Uni- and Multimodal Test Problems and Their Mirror Problems

As an *unimodal test problem* with $n \geq m$ decision variables we propose the problem ED1: $g(r) = r$, $r = \sqrt{(x_m^2 + \cdots + x_n^2)}$.

For a given γ with convex (concave) PF, we can obtain a problem (we will term it the *mirror problem*) with *congruent* concave (convex) PF by setting: $f_i(\mathbf{x}) = 1/(g(\mathbf{x}) + 1)\tilde{p}_i(\theta_1, \ldots, \theta_{m-1}), i = 1, \ldots, m$ For $\gamma = 2$ the PF of the mirror problem is given in Figure 4 (left).

In order to make the problem multimodal, we propose to choose $g(r) = F_{\text{natmin}}(r)$, for a function F_{natmin} that we define next (cf. Figure 4 (right)):

$$F_{\text{natmin}}(r) = b + (r - a) + 0.5 + 0.5\cos(2\pi(r - a) + \pi), a \approx 0.051373, b \approx 0.0253235 \tag{14}$$

The function F_{natmin} has the 'nice' property that it takes its minima at the natural numbers. The optimal function values are the natural numbers. Therefore, it can be checked which of the local PFs has been reached. The constants

a and b have been computed numerically by setting the derivative of F_{natmin} to zero and at the same time setting the function F_{natmin} to zero and solving (numerically) for a and b.

The resulting optimization problem, we will name it ED2, is a multimodal test-problem with equidistant local Pareto-fronts with respect to the radii $||\mathbf{x}||_\gamma$. Many-to-one mappings can be introduced by replacing this function by the function $\sin^2(x/\pi)$ in a similar manner as discussed for the F_{natmin} function.

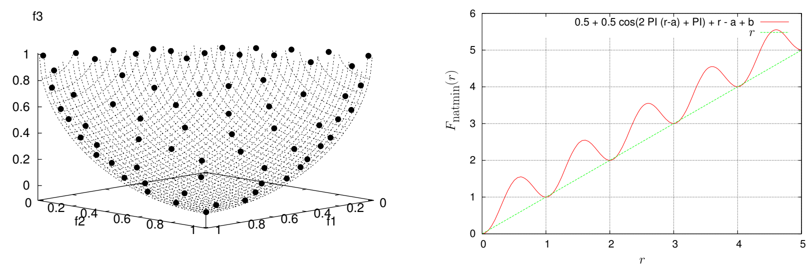


Fig. 4. (l) Convex PF of the mirror problem for $\gamma = 2$. (r) The function $F_{\text{natmin}} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that takes its minima at the natural numbers is depicted.

In conclusion, we note that in the bicriteria case and $\gamma = 2$ the new test problems result in the same PF as Schaffer's 1D problem [15]. For $m > 2$ and $\gamma = 1$ and $\gamma = 2$ the PFs we propose are equivalent to those of DTLZ problems [3]. However, we extended this benchmark by introducing scalability w.r.t. the RoC. Huband et al. [8] compiled a list of recommendations and desirable features. Regarding this list we note that we focus on the difficulties of scalability in dimensions, number of objective functions, and curvature (e.g. concavity/convexity, RoC). Moreover, we discussed multimodal versions of the test problems. Many extra difficulties, such as non-separability, plateaus etc., can be introduced in a managed way by designing adequate functions for g and using coordinate transformations.

6 Implementation of Performance Metrics

The measure of dominated hypervolume and the average distance of points to the PF are two measures that are frequently used to measure the quality of a pareto set approximation [2, 5].

6.1 Dominated Hypervolume of Pareto Fronts

Next, we provide expression for the measure of the dominated hypervolume [16] for PFs of the problem. Let $\mathcal{S}(P)$ denote the measure of the dominated hypervolume of a PF P. For different γ and m it suffices to provide formulas for a reference point of $\mathbf{r} = \mathbf{1} \in \mathbb{R}^m$ as it just dominates all points of the PF. Note, that in case of reference points that are dominated by $\mathbf{r}$ the computation of the remaining part reduces to computing a sum of hyperbox volumes. The Lebesgue measure of the dominated hypervolume for $\mathbf{r} = \mathbf{1}$ and different γ and m will be denoted with $\mathcal{S}_{\gamma,m}$.

For the objective space with dimension $m = 2$ (and $m = 3$) the equations are obtained as complements of quarters of areas of super-circles (super-spheres) [12, 11]:

$$1 - \mathcal{S}_{\gamma,2} = \frac{4^{1-\frac{1}{\gamma}}\sqrt{\pi}\Gamma(1+\frac{1}{\gamma})}{\Gamma(\frac{1}{2}+\frac{1}{\gamma})}/4, \quad 1 - \mathcal{S}_{\gamma,3} = \frac{8[\Gamma(1+1/\gamma)]^3}{\Gamma(1+3/\gamma)}/8 \tag{15}$$

Note, that we considered only the first quadrant and a constant radius of $r = 1$. For higher dimension, we know no general formula for the hypervolume. However, for $\gamma = 2$ the volume of the m-dimensional unit hypersphere S^m can be used to compute the $\mathcal{S}_{2,m}$.

$$1 - \mathcal{S}_{2,m} = \frac{\pi^{m/2}}{\Gamma(0.5m)}/(2^m) \tag{16}$$

Expressions for integer m allow for a simplified formulas (cf. [13]). Finally, for the linear case we get the simple expression $\mathcal{S}_{1,m} = 1 - 1/m!$. Some special cases are reported in the table below[1]:

$\mathcal{S}_{\gamma,m}$	$\gamma = 2$	$\gamma = 1$	$\gamma = 0.5$	$\gamma > 0$
$m = 2$	$1 - \pi/4$	$\frac{1}{2}$	$1/6$	$\mathcal{S}_{\gamma,2}$
$m = 3$	$1 - \frac{\pi}{6}$	1- $\frac{1}{6}$	$1 - 8/6!$	$\mathcal{S}_{\gamma,3}$
$m > 3$	$\frac{\pi^{m/2}}{\Gamma(\frac{m}{2}+1)}$	$1 - \frac{1}{m!}$	?	?

6.2 Distance to the Pareto Front

The average distance of points to the PF is another measure that can be used to measure the quality of PF approximations. Usually, it is combined with other measures that take the coverage of the PF into account. This measure also turns out to be useful as an indicator for local convergence to a local PF, as it may easily occur for the multimodal test function proposed in Section 5.3.

A straightforward approach to compute the distance of a point to the Pareto-front that also fits with the detection of local convergence is to compute directly the supersphere radius r_γ of a point with respect to γ, i.e.: $d(\mathbf{x}) = -1 + r_\gamma(\mathbf{x})$. This measure becomes zero, if and only if the point $\mathbf{y}$ lies on the PF. Otherwise it is a value that decreases with the Minkowski distance of that point to the PF. Moreover, the value can be used to detect the local PF to which a run has converged.

[1] Question marks indicate results unknown to us.

6.3 A Note on Knee-Points

It has been often stated that, in case of convex parts of the PF it is desirable to obtain knee-points in Pareto optimization, as they are considered to be good compromise solutions. In case of *our* test problems knee-points for convex PFs are exactly given by $(1 - \mathrm{RoC}_m(\gamma))\mathbf{1} \in \mathbb{R}^m$. Hence, the distance of a solution to a knee-point can be easily checked.

7 Case Study

In a case study, illustrating the use of the test problem, we looked at the results of the SMS-EMOA [5, 9]. This steady state EMO algorithm aims at finding well-spread solution sets by maximizing their dominated hypervolume. It is an interesting question how exactly this algorithm distributes points on PFs of different shape. Test runs for the 2-D case are reported in Figure 2 (left). A population size of 15 was used and 15000 objective function evaluations. The generalized Schaffer problem with $\mathbf{x} \in [-2, 2]^5$ has been solved for different values of γ $(= \frac{1}{2\alpha})$. The results indicate well-distributed point sets and the almost all solutions are on the PF. For high RoC the density of points grows near the knee point, while for the linear case points are distributed uniformly. This corresponds to the optimal distribution w.r.t. S-metric maximization [7].

In addition we computed PFs for the following 3-D problem: Setting $g = (\sum_{i=m}^{n} x_i^2)^{0.5}$ and

$$f_1 = ((\cos(x_1))^2)^{\frac{1}{\gamma}}(1 + g(\mathbf{x})) \rightarrow \min \tag{17}$$

$$f_2 = ((\sin(x_1)\cos(x_2))^2)^{\frac{1}{\gamma}}(1 + g(\mathbf{x})) \rightarrow \min \tag{18}$$

$$f_3 = ((\sin(x_1)\sin(x_1))^2)^{\frac{1}{\gamma}}(1 + g(\mathbf{x})) \rightarrow \min \tag{19}$$

$$\mathbf{x} \in [0, \pi/2]^2 \times [0, 1]^{n-2}, n = 7 \tag{20}$$

The population size was changed to 70, while the other settings remained the same. for different γ are displayed in Figure 5 and, for the convex mirror problem of $\gamma = 2$, in 4 (right). From another viewpoint the same results are shown in Figure 2 (right). Again, in the linear case points are distributed evenly across the PF. In case of convex and concave fronts, compromise regions as well as regions at the boundary are sampled with higher density. The results for $\gamma = 1$ and $\gamma = 2$ conform to the SMS-EMOA results on the DTLZ test-suite reported in [9]. A more detailed study of the SMS-EMOA (and similar algorithms) on the new test problems, though interesting, is beyond the scope of this paper .

8 Conclusions

We proposed and studied test problems with PFs being parts of Lamé superspheres, i.e. zero sets of $y_1^\gamma + \cdots + y_m^\gamma$, for $\gamma \in \mathbb{R}^+$. They can be scaled from concave problems with low conflict resolvability, to linear problems, and finally convex problems with high conflict resolvability.

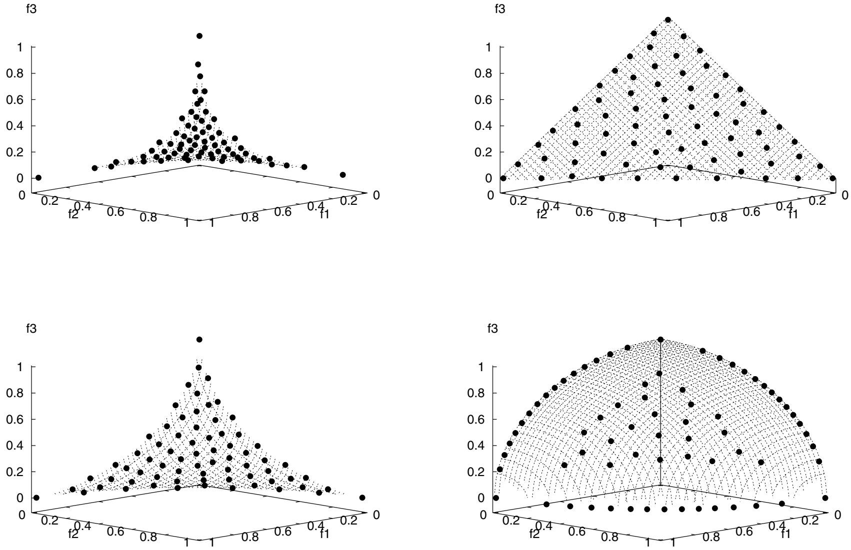


Fig. 5. Results of the SMS-EMOA for different settings of γ on the 3-D test-problems

As a first class of such functions we introduced generalized versions of Schaffer's test-problem $f_1(\mathbf{x}) = \frac{1}{m^\alpha}(\sum_{i=1}^{m} |x_i|^q)^\alpha \rightarrow \min$, $f_2(\mathbf{x}) = \frac{1}{m^\alpha}(\sum_{i=1}^{m} |1 - x_i|^q)^\alpha \rightarrow \min$ and provided closed form expressions for their efficient set and PFs, which turned out to be arcs of Lamé supercircles with $\gamma = \frac{1}{2\alpha}$. Then, adopting the construction methods proposed by Deb et al. [3], we constructed test problems that result in PFs that are parts of m-dimensional super-spheres. For all test problems we provided means to compute performance measures, such as the dominated hypervolume measure, the (average) distance to the PF and distance of solutions to the knee point. A case study with the SMS-EMOA illustrated the use of these test problems.

Our primary goal was to introduce and study a set of test problems with basic geometry, rather than benchmark problems that include all kinds of difficulties in an intertwined way. This way, properties of the algorithm and its solution sets can be studied in a isolated manner on well-understood geometries. However, these elementary problems can be used as building blocks for more complex test problems in benchmark suites.

In future, it would be interesting to extend the test problems, in order to capture problems, for which subsets of the objectives lead to highly resolvable problems and for which other subsets do not. A promising approach is to use individual γ values for different dimensions, i.e. look at zero sets of the form $y_1^{\gamma_1} + \cdots + y_m^{\gamma_m} = 1$. However, such families of PFs deserve a thorough study that would extend the scope of this paper.

Supporting material (`C++`-implementation of test problems, related technical reports) is provided under `www.liacs.nl/~emmerich/superspheres.html`.

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References

[1] K. Deb: Multi-objective genetic algorithms: Problem difficulties and construction of test problems. ECJ 7 (1999) pp. 205-230

[2] K. Deb, S. Agrawal, A. Pratab, T. Meyarivan: *A Fast Elitist Non-Dominated Sorting Genetic Algorithm for Multi-Objective Optimization: NSGA-II*, PPSN VI Conference, Springer. Lecture Notes in Computer Science No. 1917, Paris, France, Marc Schoenauer et al. (Edt.), pp. 849–858, 2000

[3] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler: *Scalable Test Problems for Evolutionary Multiobjective Optimization*, TIK-Technical Report 112, Institut für Technische Informatik und Kommunikationsnetze, ETH Zürich, Switzerland, 2001

[4] M. Ehrgott: *Multicriteria Optimization*, Springer, Berlin, 2005

[5] M. Emmerich, N. Beume, and B. Naujoks: *An EMO Algorithm Using the Hypervolume Measure as Selection Criterion*, C. A. Coello Coello, A. Hernández Aguirre and E. Zitzler, Proc. Evolutionary Multi-Criterion Optimization: Third Int'l Conference (EMO 2005), Springer, Berlin, 2005, pp. 62–76, Lecture Notes in Computer Science, 3410, 2005

[6] M. Emmerich: *A Rigorous Analysis of Two Bi-Criteria Problem Families with Scalable Curvature of the Pareto Fronts*, LIACS-TR 2005-05, Leiden Univ., 2005

[7] M. Emmerich and André Deutz: Multiobjective Test Problems with Superspherical Pareto Fronts, LIACS TR 2006-05, Leiden Univ., 2006

[8] S. Huband, Ph. Hingston, L. Barone, L. Whyle: *A Review of Multiobjective Test Problems and a Scalable Test Problem Kit*, IEEE Transactions on Evolutionary Computation, Vol. 10, No5, pp 477-506, 2006

[9] B. Naujoks, N. Beume, and M. Emmerich: Multi-objective optimisation using S-metric selection: Application to three-dimensional solution spaces: Proc. Conf. IEEE-CEC'05, IEEE-Press, Piscataway, NJ, 2005, pp. 1282-1289

[10] V.A. Van Veldhuizen and G.B. Lamont: Multiobjective Evolutionary Algorithm Test Suites, 1999 ACM Symp. on Applied Computing, ACM (1999), pp. 351-357

[11] E. W. Weissstein: *Superellipsoid*. From MathWorld - A Wolfram Web Resource. `http://mathworld.wolfram.com/Superellipsoid.html,August2004`

[12] E. W. Weissstein: *Superellipse*. From MathWorld - A Wolfram Web Resource. `http://mathworld.wolfram.com/Superellipse.html,September2005`

[13] E. W. Weissstein: *Hypersphere*. From MathWorld - A Wolfram Web Resource. `http://mathworld.wolfram.com/Hypersphere.html`, April 2006

[14] R. Tyrrel Rockafeller: *Convex Analysis*, Princeton University Press, 1996

[15] J. D. Schaffer: *Multiple objective optimization with vector evaluated genetic algorithms.* In: Proceedings of the first International Conference on Genetic Algorithms. Lawrence Erlbaum Ass. (1985),

[16] E. Zitzler. Evolutionary Algorithms for Multiobjective Optimization: Methods and Applications. PhD thesis, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland, November 1999.