Confluence by Decreasing Diagrams

Converted

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Abstract. The decreasing diagrams technique is a complete method to reduce confluence of a rewrite relation to local confluence. Whereas previous presentations have focussed on the *proof* the technique is correct, here we focus on *applicability*. We present a simple but powerful generalisation of the technique, requiring peaks to be closed only by conversions instead of valleys, which is demonstrated to further ease applicability.

1 Introduction

The decreasing diagrams technique [1,2] is a method to reduce the problem of showing confluence of a rewrite relation to showing its local confluence. In exchange for localisation, the confluence diagrams need to be *decreasing* with respect to some labelling. The method is complete in the sense that any (countable) confluent rewrite relation *can be* equipped with such a labelling. But by undecidability of confluence completeness also entails that *finding* such a labelling is hard. The goal of this paper is to ease the latter, thus enhancing applicability of the technique. We try to achieve this in two ways.

First, in Sect. 3, we relax the local confluence constraint. Instead of requiring that for every pair of diverging steps a *pair of reductions* exists such that the resulting diagram is decreasing, we show it suffices that a *conversion* exists such that the resulting diagram is decreasing, by analogy with the way in which Winkler & Buchberger's confluence criterion [3, Lemma 3.1] relaxes Newman's Lemma [4, Theorem 3].

Next, in Sect. 4, we provide heuristics for finding appropriate labellings, illustrated by many examples from the literature, ranging from abstract rewriting via term rewriting and λ-calculi to process algebra.

In the examples we use results from the literature. Other than that, we assume only basic rewriting knowledge, which is recapitulated in Sect. 2. That section serves also to recapitulate from [1,2] the core of the decreasing diagrams technique. Those not yet familiar with that technique are advised to consult one of its textbook accounts [5,6] first.

* This research was supported by a two-month grant from LIX, Paris, France.

A. Voronkov (Ed.): RTA 2008, LNCS 5117, pp. 306–320, 2008.

2 Preliminaries

A *rewrite* relation is a binary relation on a set of objects. To stress we are interested in the direction of rewrite relations we use arrow-like notations like $\rightarrow$, $\succ$, $\vartriangleright$, and $\blacktriangleright$ to denote them. For a rewrite relation $\rightarrow$, we inductively define an object a to be *terminating*, if for all objects b such that $a \rightarrow b$, b is terminating. The rewrite relation $\rightarrow$ is *terminating* if all its objects are. For a rewrite relation denoted by an arrow-like notation $\rightarrow$, its converse is denoted by the converse $\leftarrow$ of the notation. We denote the union of two rewrite relations by the union of their notations, e.g. $\vartriangleleft\!\blacktriangleright$ denotes $\vartriangleleft \cup \blacktriangleright$, and $\leftrightarrow$ denotes $\leftarrow \cup \rightarrow$, the symmetric closure of $\rightarrow$. We use $\rightarrow \cdot \vartriangleright$ to denote the composition of $\rightarrow$ and $\vartriangleright$, and $\rightarrow^{=}$ and $\rightarrow^{+}$ to denote respectively the reflexive and transitive closure of $\rightarrow$. To denote the reflexive–transitive closure of $\rightarrow$, i.e. its 'repetition', we employ the 'repetition' $\twoheadrightarrow$ of its notation. When both $\vartriangleright$,$\blacktriangleright$ are defined, we abbreviate $\vartriangleright \cup \blacktriangleright$ to $\rightarrow$. Further notions and notations will be introduced on a by-need basis. We now state the decreasing diagrams theorem, illustrating it by means of a running example.

Definition 1. *A pair* $(\vartriangleright, \blacktriangleright)$ *of rewrite relations* commutes *if* $\vartriangleleft\!\vartriangleleft \cdot \blacktriangleright\!\blacktriangleright \subseteq \blacktriangleright\!\blacktriangleright \cdot \vartriangleleft\!\vartriangleleft$*, and commutes* locally *if* $\vartriangleleft \cdot \blacktriangleright \subseteq \blacktriangleright\!\blacktriangleright \cdot \vartriangleleft\!\vartriangleleft$*. A rewrite relation* $\rightarrow$ *is* confluent *if* $(\rightarrow, \rightarrow)$ *commutes, and* locally *confluent if* $(\rightarrow, \rightarrow)$ *commutes locally.*

Example 1. A rewrite relation is confluent if locally confluent and terminating.

The decreasing diagrams technique generalises this example, that is, Newman's Lemma, by weakening the termination assumption to *decreasingness*.

Definition 2. *A pair* $((\vartriangleright_\ell)_{\ell \in L}, (\blacktriangleright_m)_{m \in M})$ *of families of rewrite relations is* decreasing *if the union* $L \cup M$ *of their sets* L,M *of labels comes equipped with a terminating and transitive rewrite relation* $\succ$*. A pair of rewrite relations* $(\vartriangleright, \blacktriangleright)$ *is* decreasing, *if* $\vartriangleright = \bigcup_{\ell \in L} \vartriangleright_\ell$, $\blacktriangleright = \bigcup_{m \in M} \blacktriangleright_m$ *for such a decreasing pair of families, such that for all* $\ell \in L$,$m \in M$, $\vartriangleleft_\ell \cdot \blacktriangleright_m \subseteq \blacktriangleright\!\blacktriangleright_{\curlyvee \ell} \cdot \blacktriangleright^{=}_{m} \cdot \blacktriangleright\!\blacktriangleright_{\curlyvee \{\ell,m\}} \cdot \vartriangleleft\!\vartriangleleft_{\curlyvee \{\ell,m\}} \cdot \vartriangleleft^{=}_{\ell} \cdot \vartriangleleft\!\vartriangleleft_{\curlyvee m}$ *(see Fig. 1), where* $\curlyvee N = \{n \in L \cup M \mid \exists k \in N\ k \succ n\}$*, and* $\curlyvee n$ *abbreviates* $\curlyvee \{n\}$*.*

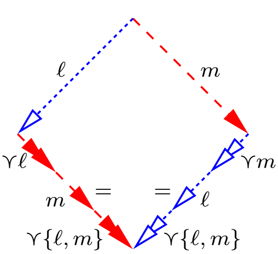

Fig. 1. Decreasingness

A family $(\rightarrow_\ell)_{\ell \in L}$ *of rewrite relations is* decreasing *if* $((\rightarrow_\ell)_{\ell \in L}, (\rightarrow_\ell)_{\ell \in L})$ *is. A rewrite relation* $\rightarrow$ *is* decreasing, *if* $(\rightarrow, \rightarrow)$ *is.*

Example 2. A rewrite relation $\to$ on A as in Example 1 is decreasing: The family $(\to_a)_{a\in A}$, with $\to_a$ defined as $\to$ with domain restricted to $\{a\}$, and with the set of labels A ordered by $\to^+$, is decreasing by the assumption that $\to$ is terminating. Clearly $\to = \bigcup_{a\in A} \to_a$, and as for any peak $b \leftarrow_a a \to_a c$, there is a valley between b and c by local confluence and since for any object, i.e. label, d in this valley it trivially holds $a \to^+ d$, we conclude to decreasingness of $\to$.

Theorem 1 ([1]). *A pair of rewrite relations commutes if it is decreasing. A rewrite relation is confluent if it is decreasing.*

Proof. We recapitulate the core of the proof in [1] for easy adaptation later on. Instead of proving commutation one proves the stronger property:

($*$) Every peak $b \lhd\!\lhd_\sigma \cdot \blacktriangleright\!\blacktriangleright_\tau c$ can be completed by a valley $b \blacktriangleright\!\blacktriangleright_{\tau'} \cdot \lhd\!\lhd_{\sigma'} c$, into a so-called *decreasing* diagram, i.e. such that $|\sigma\tau'| \preceq_{mul} |\sigma| \uplus |\tau| \succeq_{mul} |\tau\sigma'|$.

where $|\sigma|$ is the *lexicographic maximum measure* of the string of labels σ, i.e. the multiset inductively defined by: $|\varepsilon| = \emptyset$ and $|\ell\sigma| = [\ell] \uplus |\sigma| - \curlyvee\ell$, and $\succ_{mul}$ is the (terminating) multiset extension of the (terminating) relation $\succ$ on the labels.

For such a peak, completability into a decreasing diagram is proved by $\succ_{mul}$-induction on its *measure* $|\sigma| \uplus |\tau|$. The proof being trivial in case either of the reductions in the assumption is empty, the interesting cases are seen to be of shape $\lhd\!\lhd_\sigma \cdot \lhd_\ell \cdot \blacktriangleright_m \cdot \blacktriangleright\!\blacktriangleright_\tau$, for which one concludes by the following three steps corresponding to the three components of Fig. 2:

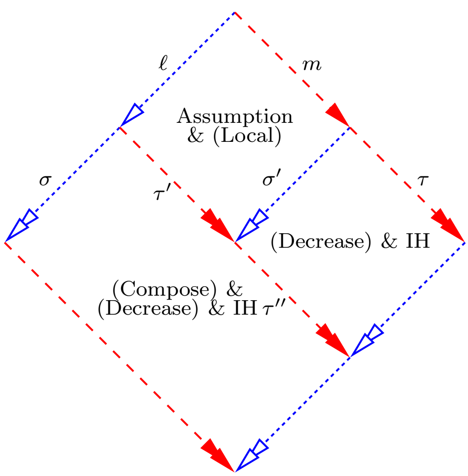

Fig. 2. Decreasing $\Rightarrow$ Commutes

1. by the decreasingness assumption on $(\rhd, \blacktriangleright)$ the local peak $\lhd_\ell \cdot \blacktriangleright_m$ can be completed by a valley $\blacktriangleright\!\blacktriangleright_{\tau'} \cdot \lhd\!\lhd_{\sigma'}$ yielding a decreasing diagram by (Local);
2. by the induction hypothesis, which applies by (Decrease) and (1), the peak $\lhd\!\lhd_{\sigma'} \cdot \blacktriangleright\!\blacktriangleright_\tau$ can be completed by a valley $\blacktriangleright\!\blacktriangleright_{\tau''} \cdot \lhd\!\lhd_{\sigma''}$ into a decreasing diagram;

3. by the induction hypothesis, which applies by (Decrease) and (Compose) applied to (1),(2), the peak $\lhd\!\!\lhd_{\sigma} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau'} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau''}$, can be completed into a decreasing diagram, which by another application of (Compose) proves the result;

where the following facts haven been used for diagrams D_i with $i \in \{1,2\}$, where a *diagram* D_i consists of a peak $\lhd\!\!\lhd_{\sigma_i} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau_i}$ completed by a valley $\blacktriangleright\!\!\blacktriangleright_{\tau'_i} \cdot \lhd\!\!\lhd_{\sigma'_i}$:

(Local) If D_1 is a local diagram, i.e. if its peak is of shape $\lhd_{\ell} \cdot \blacktriangleright_m$ for some labels ℓ,m, then D_1 is decreasing if and only if its valley is of shape $\blacktriangleright\!\!\blacktriangleright_{\curlyvee \ell} \cdot \blacktriangleright^{=}_{m} \cdot \blacktriangleright\!\!\blacktriangleright_{\curlyvee\{\ell,m\}} \cdot \lhd\!\!\lhd_{\curlyvee\{\ell,m\}} \cdot \lhd^{=}_{\ell} \cdot \lhd\!\!\lhd_{\curlyvee m}$ [1, Proposition 2.3.16].

(Decrease) If D_1 is a non-empty decreasing diagram, i.e. if its reduction $\blacktriangleright\!\!\blacktriangleright_{\tau_1}$ is not empty, then filling the peak $\lhd\!\!\lhd_{\sigma_1} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau_1} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau_2}$ with D_1 decreases the measure, i.e. $|\sigma_1| \uplus |\tau_1\tau_2| \succ_{mul} |\sigma'_1| \uplus |\tau_2|$ [1, Lemma 2.3.19].

(Compose) If diagrams D_1,D_2 are decreasing and can be composed, i.e. if their respective reductions $\lhd\!\!\lhd_{\sigma'_1}$ and $\lhd\!\!\lhd_{\sigma_2}$ coincide, then this composition, consisting of the peak $\lhd\!\!\lhd_{\sigma_1} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau_1} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau_2}$ completed by the valley $\blacktriangleright\!\!\blacktriangleright_{\tau'_1} \cdot \blacktriangleright\!\!\blacktriangleright_{\tau'_2} \cdot \lhd\!\!\lhd_{\sigma'_2}$, is decreasing again [1, Lemma 2.3.17]. ☐

Example 3. Theorem 1 applied to Example 2 yields a proof of Example 1.

Remark 1. Conversely, any *countable* confluent relation is decreasing [1, Corollary 2.3.30]. It is an open problem whether countability can be dropped, and also whether any pair of commuting relations, countable or not, is decreasing.

Theorem 1 provides a method to prove properties stronger than commutation.

Theorem 2. *Let P be a property of diagrams which is closed under composition (defined as in the proof of Theorem 1). If $(\rhd, \blacktriangleright)$ is a decreasing pair of rewrite relations such that every local peak can be completed into a decreasing diagram having property P, then every peak can be so completed.*

Proof. Require the diagram in $(*)$ in the proof of Theorem 1 to satisfy P. ☐

Example 4. Consider the property P expressing that in a diagram D with peak $b \twoheadleftarrow a \twoheadrightarrow c$ and valley $b \twoheadrightarrow d \twoheadleftarrow c$, its 'left-reduction' $a \twoheadrightarrow b \twoheadrightarrow d$ is not longer than its 'right-reduction' $a \twoheadrightarrow b \twoheadrightarrow d$. As P is preserved under composition of diagrams, it suffices under the assumptions of Newman's Lemma to check that P holds for all local diagrams. If it does, then all maximal reductions from a given object end in the same normal form, *reached in the same number of steps* [7].

Example 5. The *strict* commutation property expressing that in a diagram with peak $b \lhd\!\!\lhd a \blacktriangleright\!\!\blacktriangleright c$ and valley $b \blacktriangleright\!\!\blacktriangleright d \lhd\!\!\lhd c$, if $a \blacktriangleright\!\!\blacktriangleright c$ is non-empty then so is $b \blacktriangleright\!\!\blacktriangleright d$, is easily seen to be preserved under composition. Hence, it suffices to verify that local decreasing diagrams are strict.

3 Conversion

We generalise the decreasing diagrams technique as presented in the previous section, by allowing local peaks to be completed by conversions instead of valleys.

Definition 3. *A pair of rewrite relations* $(\rhd, \blacktriangleright)$ *is* decreasing with respect to conversions*, if* $\rhd = \bigcup_{\ell \in L} \rhd_\ell$*,* $\blacktriangleright = \bigcup_{m \in M} \blacktriangleright_m$ *for a decreasing pair of families* $((\rhd_\ell)_{\ell \in L}, (\rhd_m)_{m \in M})$ *such that for all* $\ell \in L, m \in M$*,* $\lhd_\ell \cdot \blacktriangleright_m \subseteq \lhd\!\blacktriangleright^*_{\curlyvee \ell} \cdot \blacktriangleright^=_m \cdot \lhd\!\blacktriangleright^*_{\curlyvee \{\ell,m\}} \cdot \lhd^=_\ell \cdot \lhd\!\blacktriangleright^*_{\curlyvee m}$*. A rewrite relation* $\rightarrow$ *is* decreasing with respect to conversions*, if* $(\rightarrow, \rightarrow)$ *is.*

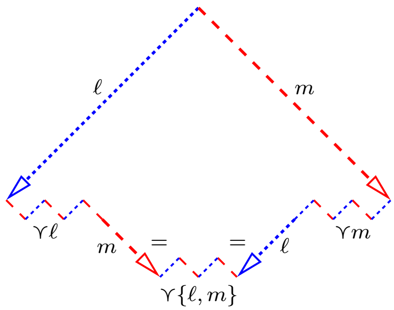

Fig. 3. Decreasingness with respect to conversions

Decreasingness is illustrated in Fig. 3, where, to avoid clutter, arrowheads in the conversions have been elided. We will henceforth refer to decreasingness in the sense of Definition 2 as *decreasing with respect to valleys*, abbreviated to $\diamondsuit$, and we abbreviate decreasingness in the present sense of Definition 3 to $\triangle$.

Example 6. Let $\rightarrow$ be a terminating rewrite relation such that for every local peak $b \leftarrow a \rightarrow c$ the objects b and c are *convertible below* a, i.e. $b = a_1 \leftrightarrow \ldots \leftrightarrow a_n = c$ with $a \rightarrow^+ a_i$ for all $1 \leq i \leq n$. From Example 2 we already know that labelling steps by their source and ordering the labels by $\rightarrow^+$ yields a decreasing labelling, and it is easy to see that the requirement that every local peak $b \leftarrow a \rightarrow c$ be convertible below a, entails that the rewrite relation $\rightarrow$ is decreasing with respect to conversions for this labelling.

Theorem 3. *A pair of rewrite relations commutes if it is decreasing with respect to conversions, and idem for confluence of a single rewrite relation.*

Proof. We adapt the proof of Theorem 1, keeping the same invariant and induction. Observe that the only difference arises in case the peak is of shape $b \blacktriangleleft\!\!\blacktriangleleft_\sigma b' \blacktriangleleft_\ell \cdot \rhd_m c' \rhd\!\!\rhd_\tau c$ for some $\ell \in L, m \in M$. By the assumption that $(\rhd, \blacktriangleright)$ is decreasing with respect to conversions, the local peak $b' \blacktriangleleft_\ell \cdot \rhd_m c'$ can be transformed into $b' \lhd\!\blacktriangleright^*_{\curlyvee \ell} \cdot \blacktriangleright^=_m \cdot \lhd\!\blacktriangleright^*_{\curlyvee \{\ell,m\}} \cdot \lhd^=_\ell \cdot \lhd\!\blacktriangleright^*_{\curlyvee m} c'$, see (1) in Fig. 4. We show decreasingness with respect to valleys, by transforming the conversion into a valley of shape $b' \blacktriangleright\!\!\blacktriangleright_{\curlyvee \ell} \cdot \blacktriangleright^=_m \cdot \blacktriangleright\!\!\blacktriangleright_{\curlyvee \{\ell,m\}} \cdot \lhd\!\!\lhd_{\curlyvee \{\ell,m\}} \cdot \lhd^=_\ell \cdot \lhd\!\!\lhd_{\curlyvee m} c'$, and conclude.

First observe that if the peak of a decreasing diagram consists only of labels in $\curlyvee M$ then so does its valley. Thus by repeatedly applying the induction hypothesis, the peaks in the conversions $\lhd\!\blacktriangleright^*_{\curlyvee \ell}, \lhd\!\blacktriangleright^*_{\curlyvee m}$ can be transformed into valleys smaller than ℓ,m, yielding $b' \blacktriangleright\!\!\blacktriangleright_{\curlyvee \ell} \cdot \lhd\!\!\lhd_{\curlyvee \ell} \cdot \blacktriangleright^=_m \cdot \lhd\!\blacktriangleright^*_{\curlyvee \{\ell,m\}} \cdot \lhd^=_\ell \cdot \blacktriangleright\!\!\blacktriangleright_{\curlyvee m} \cdot \lhd\!\!\lhd_{\curlyvee m} c'$,

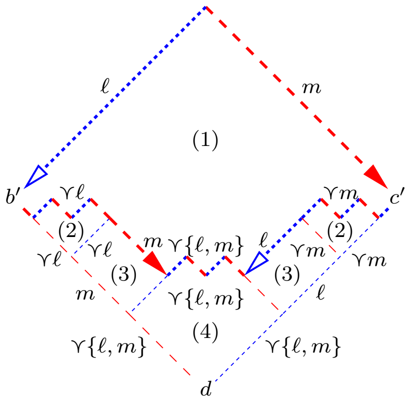

Fig. 4. Decreasing with respect to conversions ⇒ Commutes

see (2) in Fig. 4. Applying the induction hypothesis to the peaks $\lhd\!\lhd_{\curlyvee\ell} \cdot \blacktriangleright^{=}_{m}$ and $\lhd^{=}_{\ell} \cdot \blacktriangleright\!\!\blacktriangleright_{\curlyvee m}$ gives by analogous reasoning, valleys of shapes $\blacktriangleright\!\!\blacktriangleright_{\curlyvee\ell} \cdot \blacktriangleright^{=}_{m} \cdot \blacktriangleright\!\!\blacktriangleright_{\curlyvee\{\ell,m\}} \cdot \lhd\!\lhd_{\curlyvee\{\ell,m\}}$ and $\blacktriangleright\!\!\blacktriangleright_{\curlyvee\{\ell,m\}} \cdot \lhd\!\lhd_{\curlyvee\{\ell,m\}} \cdot \lhd^{=}_{\ell} \cdot \lhd\!\lhd_{\curlyvee m}$, see (3) in Fig. 4, giving $b' \blacktriangleright\!\!\blacktriangleright_{\curlyvee\ell} \cdot \blacktriangleright^{=}_{m} \cdot \lhd\!\blacktriangleright^{*}_{\curlyvee\{\ell,m\}} \cdot \lhd^{=}_{\ell} \cdot \lhd\!\lhd_{\curlyvee m}\ c'$. Finally, repeatedly applying the induction hypothesis to the peaks in the conversion $\lhd\!\blacktriangleright^{*}_{\curlyvee\{\ell,m\}}$ transforms it into a valley $\blacktriangleright\!\!\blacktriangleright_{\curlyvee\{\ell,m\}}\ d \lhd\!\lhd_{\curlyvee\{\ell,m\}}$, see (4) in Fig. 4, resulting in a decreasing diagram with respect to valleys. □

Example 7. Theorem 3 applied to Example 6 yields Winkler & Buchberger's result [3, Lemma 3.1] stating that any terminating rewrite relation such that the targets of any local peak are convertible below its source, is confluent.

Remark 2. It would be interesting to see whether the proofs of confluence by decreasing diagrams as presented in [8,9] can be adapted in a similar way.

Observe that from the proof of Theorem 3 it follows that, for any *given* labelling, decreasingness with respect to valleys is equivalent to decreasingness with respect to conversions, Although equivalent, the latter is in principle easier to check: one 'just' has to find an appropriate conversion instead of an appropriate valley for each local peak $\leftarrow_{\ell}\cdot\rightarrow_{m}$. Of course, the 'search space' for conversions is in general much larger than for valleys. To keep searching feasible nonetheless, observe first that one only needs to search (forward or backward) rewrite steps having labels smaller than or equal to ℓ or m, and second that one may opt to linearly bound the amount of time spent on searching decreasing conversions by that spent on valleys, where the latter restriction is complete by the observation above. That searching for conversions instead of valleys can be advantageous is witnessed by the following example, where searching conversions for all local peaks takes time linear in n whereas searching for valleys takes quadratic time.

Example 8. Consider for every natural number n, the confluent rewrite relation given by $b_i \leftarrow a_i \rightarrow c_i$, $b_i \rightarrow b_{i+1}$, and $c_i \rightarrow c_{i+1}$, for $1 \leq i \leq n$, with $b_{n+1} = c_{n+1}$. Completing a local peak $b_i \leftarrow a_i \rightarrow c_i$ by a valley takes time $n - i$, whereas completing it by a conversion takes constant time, say 4. This proves our claim since $\sum_{i=1}^{n} n - i$ is quadratic in n and $\sum_{i=1}^{n} 4$ is linear.

Remark 3. In his dissertation Geser argues [10, p. 38] that checking for valleys is less complex than checking for conversions. The above shows on the contrary that both can be combined fruitfully without changing the worst case O-behaviour. Whether such combinations are useful, i.e. less complex on average, in practice or in theory, for all or some labellings, remains to be investigated.

Like Theorem 1, also Theorem 3 provides a method to prove properties stronger than commutation. However, compared to Theorem 2, the property now has to apply to all *conversion* diagrams, i.e. diagrams consisting of a peak completed by a conversion instead of a valley, and be *closed* not only under composition, but also under filling a peak of the conversion by another conversion diagram.

Theorem 4. *Let P be a property of conversion diagrams which is closed. If $(\triangleright, \blacktriangleright)$ is a decreasing pair of rewrite relations such that every local peak can be completed into a decreasing diagram with respect to conversions, having property P, then every peak can be completed into a decreasing diagram with respect to valleys, having property P.*

Proof. Load the induction hypothesis in the proof of Theorem 3 with P. □

We generalise Examples 4 and 5 to conversion diagrams.

Example 9. Consider the property P expressing that the *distance* $d(D)$ of the diagram D with peak $b \twoheadleftarrow a \twoheadrightarrow c$ and conversion $b \leftrightarrow^{*} c$, is not positive, where $d(D)$ is the integer defined as the number of forward steps $(\rightarrow)$ minus the number of backward steps $(\leftarrow)$ on the cycle $a \twoheadrightarrow b \leftrightarrow^{*} c \twoheadleftarrow a$. The property P is closed, since the distance of a conversion diagram obtained by 'glueing' two such diagrams together is the sum of their distances (note that shared steps contribute oppositely). Hence under the assumptions of Winkler & Buchberger's Lemma it suffices to verify that P holds for local conversion diagrams. If it does, then all maximal reductions from a given object end in the same normal form, reached in the same number of steps, generalizing Example 4.

Example 10. Strictness as in Example 5 can easily be extended to a closed property of conversion diagrams, by requiring in a diagram with peak $b \lhd\!\lhd\, a \blacktriangleright\!\!\blacktriangleright c$ and conversion $b \lhd\!\blacktriangleright^{*} c$, if $a \blacktriangleright\!\!\blacktriangleright c$ is non-empty then $b \lhd\!\blacktriangleright^{*} c$ contains some $\blacktriangleright$-step. Again, it suffices to verify that local decreasing conversion diagrams are strict.

4 Application

We apply the results of the previous section, providing heuristics for finding decreasing labellings along the way. First, we present the 'self-labelling' heuristic and show that it can be used to deal with several known commutation and confluence results for abstract rewrite relations. More generally, we cover and systematise *all* such results in [6, Chapter 1] and [11]. Finally, we present the 'rule-labelling' and 'self-duplication' heuristics and show they can be used to deal with commutation and confluence problems in term rewriting.

4.1 Abstract Rewriting

The labelling employed in case of Newman's Lemma and Winkler & Buchberger's Lemma (Examples 2 and 6) may seem like sorcery, but is in fact an instance of a general idea: self-labelling. Let us try to explain this by showing what *fails* if one would try to devise a decreasing labelling in case of Kleene's standard counterexample to the implication 'local-confluence $\Rightarrow$ confluence'.

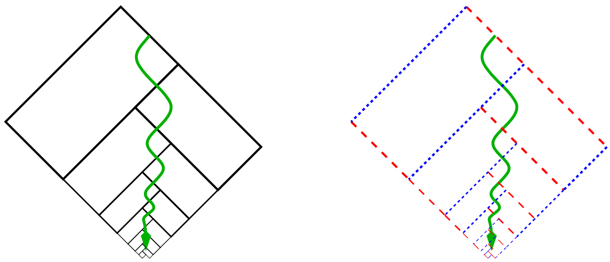

Fig. 5. Failure of local confuence (commutation) $\Rightarrow$ confluence (commutation)

Example 11. Consider the rewrite relation $\to$ given by $b \leftarrow a \leftrightarrows a' \to c$. Since every local peak can be completed by means of a valley, e.g. $b \leftarrow a \to a'$ can be completed by means of $b \leftarrow a \leftarrow a'$, the rewrite relation is locally confluent. However it is not confluent, as e.g. the peak $b \leftarrow a \to a' \to c$ cannot be completed. Fig. 5 (left) illustrates what goes wrong when trying a proof of confluence for this peak by means of tiling: the tiling process never terminates so does not lead to a completed confluence diagram. The (green) curved downward arrow, intersecting steps all of shape $a \leftrightarrow a'$, shows how such an infinite regress must *transfer* to an infinitely decreasing sequence of labels, preventing the construction of a decreasing labelling. Vice versa, in case of Newman's Lemma such an infinite decreasing sequence of labels cannot occur since it would, by concatenating the $a \leftrightarrow a'$-steps intersected by the arrow, immediately *transfer* into an infinite $\to$-reduction, contradicting the assumed termination of $\to$.

Remark 4. In [12] it is noted that the commuting version of Kleene's counterexample, Fig. 5 (right), plays a similar obstructive rôle in process algebra in proving similarity: Taking $\triangleright$ as reduction, $\mathbf{0} \triangleleft a \overset{\triangleleft}{\blacktriangleright} \tau.a \blacktriangleright \mathbf{0}$ witnesses that despite $\blacktriangleright$ being a weak simulation modulo transitivity, it is not contained in weak similarity.

The example suggests one may *transfer* termination of a (rewrite) relation on the objects, to a decreasing labelling by means of the following heuristic:

(H_1: **Self-labelling**) Given a terminating relation on the objects, try using steps (or objects) *themselves* as labels, ordered by the transitive closure of the relation, and label a step $a \to b$ by itself (or its source a or target b).

One may think that this heuristic is so much geared towards Newman's Lemma that it doesn't apply to any other interesting cases. But in fact it was inspired by self-labelling as used in proving termination by means of monotone algebras, and below we will see several other important instances of this heuristic.

We proceed by systematically treating *all* the abstract confluence and commutation results by analysis of local peaks as found in [6, Chapter 1] and [11] and some more. On the one hand, the systematisation is based on relating results based on decreasing *valleys* and *conversions* (the rows in Fig. 6) for a *given* labelling. On the other hand, it is based on the different *trace patterns* which arise when labelling the diagrams (the columns in Fig. 6). On the gripping hand, we relate *commutation* to *confluence* results (the tables below).

Unqualified references in tables are to [6, Chapter 1].

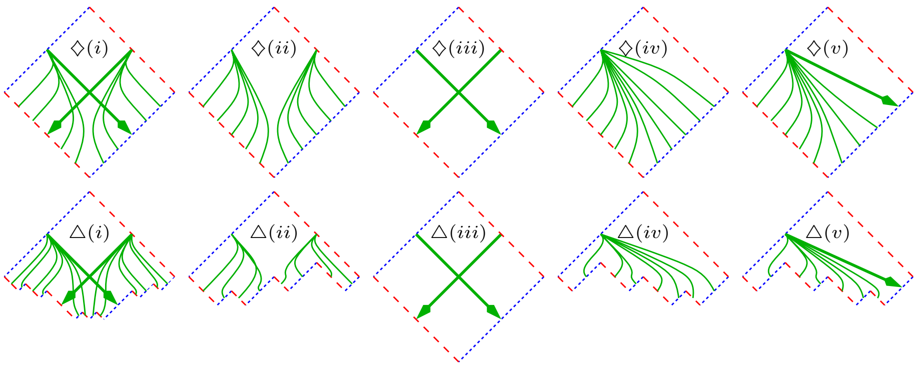

Fig. 6. Valleys (top) vs. conversions (bottom) for various trace patterns

Trace patterns [8,9,6] as displayed in Fig. 6 are the *patterns* obtained by *tracing* each label in the conclusion of a decreasing (valley or conversion) diagram back to a label in its hypothesis (the peak), which is either the same (the thick arrows in the figure) or greater (the thin lines) in concordance with the requirements imposed by decreasingness. E.g. Fig. 6 $\triangle(i)$ displays the general trace pattern for decreasing conversions of Theorem 3, and the patterns to its right are special cases of interest to us.

Example 12. Consider the labelling src which labels each step $a \to b$ by its *source* as $a \to_a b$. Then, if $\to$ is terminating, we may order the labels via the transitive closure $\to^+$, giving rise to a decreasing labelling. The trace pattern corresponding to the commutation version of Newman's Lemma (Example 2) is Fig. 6 $\Diamond(ii)$, and the one corresponding to the commutation version of Winkler & Buchberger's Lemma (Examples 6) is Fig. 6 $\triangle(ii)$. This, together with corresponding references to [6], is summarised in Table 1. Note that $\to^+$ could be replaced by any transitive and terminating relation $\succ$ such that $\to \subseteq \succ$, as in the usual presentation of Winkler & Buchberger's result.

Table 1. Source self-labelling with termination of $\to$

src	confluence	commutation
$\Diamond$	(Theorem 1.2.1) Newman	(Exercise 1.3.2) folklore
$\triangle$	(Exercise 1.3.12) Winkler & Buchberger	new?

Table 2. The empty order $\emptyset$

$\emptyset$	confluence	commutation
$\Diamond = \triangle$	(Theorem 1.2.2(iii)) Hindley & Rosen	folklore

Example 13. Any family can be made decreasing simply by equipping it with the empty order. However, a local peak $\triangleleft_\ell \cdot \blacktriangleright_m$ can then only be turned into a decreasing diagrams by means of $\blacktriangleright_m^= \cdot \triangleleft_\ell^=$, so decreasing valleys (Fig. 6 $\Diamond(iii)$) and conversions (Fig. 6 $\triangle(iii)$) coincide, yielding Table 2. For a singleton set of labels the result (subcommutativity implies confluence) goes back to [4].

Based on the previous two examples one could be led to believe the decreasing diagrams technique is just the 'sum of Newman's Lemma and the Lemma of Hindley–Rosen'. The following examples show it is much more powerful than a simple 'sum'; it is also our second instance of the self-labelling heuristic.

Example 14. Consider the labelling tgt which labels each step $a \to b$ by its *target* as $a \to_b b$. If the relation $\blacktriangleright$ relative *to* $\triangleleft$, defined by $\blacktriangleright/\triangleleft = \twoheadleftarrow \cdot \blacktriangleright \cdot \twoheadleftarrow$, is terminating, then ordering the labels via the transitive closure $(\blacktriangleright/\triangleleft)^+$ gives a decreasing labelling. If we also require that only *smaller* labels be used, then to have decreasing valleys coincides with strict commutation (see Example 5), to have decreasing conversions with *quasi*-commutation, i.e. $\triangleleft \cdot \blacktriangleright \subseteq \blacktriangleright \cdot \triangleleft\blacktriangleright^*$, and trace patterns are as in Fig. 6(iv), since labelling $b \triangleleft a \blacktriangleright c$ yields $b \triangleleft_b a \blacktriangleright_c c$. Since $b \blacktriangleright/\triangleleft c$, if c is greater than some label, b is so as well. Although the requirement that labels be *smaller* allows to reduce ([6, Exercise 1.3.19] Bachmair & Dershowitz) termination of $\blacktriangleright/\triangleleft$ to that of $\blacktriangleright$ as in the usual presentation of these results, cf. Table 3, it is not a necessary requirement. To wit, $b \triangleleft a \blacktriangleright c \triangleright b$ *is* decreasing although $c \triangleright_b b$ is equal to not smaller than $a \triangleright_b b$, with trace pattern as in Fig. 6 $\Diamond(v)$. For confluence this result is not interesting as $\to/\leftarrow$ is never terminating, for non-empty $\to$.

Example 15. Extending Example 13, a pair $(\triangleright, \blacktriangleright)$ can be made decreasing by letting $\triangleright$ be *stronger* than $\blacktriangleright$, i.e. by ordering $\triangleright$ above $\blacktriangleright$. A local peak $\triangleleft \cdot \blacktriangleright$ can

Table 3. Target self-labelling with termination of $\blacktriangleright/\triangleleft$

tgt	commutation
$\Diamond$	(Exercise 1.3.15) Geser
$\triangle$	[10, Sect. 3.3] Geser

Table 4. The stronger-than order $\triangleright \succ \blacktriangleright$

$\triangleright \succ \blacktriangleright$	confluence	commutation
$\Diamond = \triangle$	(Exercise 1.3.11) Huet	(Exercise 1.3.6) Hindley

be completed into a decreasing valley or conversion (only) by means of $\blacktriangleright\!\!\blacktriangleright \cdot \triangleleft^{=}$, its trace pattern being a special case of Fig. 6 $\Diamond(v)$. Interestingly, of the ensuing results in Table 4, Huet's result that *strong* confluence, i.e. $\leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \leftarrow^{=}$, implies confluence, is more recent than Hindley's that *strong* commutation, i.e. $\triangleleft \cdot \blacktriangleright \subseteq \blacktriangleright\!\!\blacktriangleright \cdot \triangleleft^{=}$, implies commutation, despite being an instance. In fact, also Staples' later result [6, Exercise 1.3.7] that $\triangleleft\!\!\triangleleft \cdot \blacktriangleright \subseteq \blacktriangleright\!\!\blacktriangleright \cdot \triangleleft\!\!\triangleleft$ implies commutation is seen to be an instance of Hindley's result, noting that $\triangleright\!\!\triangleright^{*} = \triangleright\!\!\triangleright^{=} = \triangleright\!\!\triangleright$.

Whereas *stronger-than* as in the previous example orders one family above another, *requests* as in the next example orders within families.

Example 16. If both $\triangleright$,$\blacktriangleright$ are $\{1,2\}$-labelled families and all diagrams are decreasing with respect to $1 \succ 2$, then 1 *requests* 2. The reason for this terminology becomes clear when considering the most general shapes concrete decreasing diagrams, say for decreasing valleys, may have: $\triangleleft_1 \cdot \blacktriangleright_1 \subseteq \blacktriangleright\!\!\blacktriangleright_2 \cdot \blacktriangleright_1^{=} \cdot \blacktriangleright\!\!\blacktriangleright_2 \cdot \triangleleft\!\!\triangleleft_2 \cdot \triangleleft_1^{=} \cdot \triangleleft\!\!\triangleleft_2$ with trace pattern $\Diamond(i)$, or $\triangleleft_1 \cdot \blacktriangleright_2 \subseteq \blacktriangleright\!\!\blacktriangleright_2 \cdot \triangleleft\!\!\triangleleft_2 \cdot \triangleleft_1^{=}$ or its symmetric version $\triangleleft_2 \cdot \blacktriangleright_1 \subseteq \blacktriangleright_1^{=} \cdot \blacktriangleright\!\!\blacktriangleright_2 \cdot \triangleleft\!\!\triangleleft_2$ both with trace pattern $\Diamond(v)$, or $\triangleleft_2 \cdot \blacktriangleright_2 \subseteq \blacktriangleright_2 \cdot \triangleleft_2$ with trace pattern $\Diamond(iii)$; for commutation a 1-step may *request* 2-steps to find the common reduct, but not the other way around. This generalises the classical notion of request as employed in the results of Table 5, strengthening these.

Table 5. Requests for ordering $1 \succ 2$ within a family $\{1,2\}$

$1 \succ 2$	confluence	commutation
$\Diamond$	(Exercise 1.3.8) Rosen & Staples	new?
$\triangle$	(Exercise 1.3.10) van Oostrom	new?

A third instance of the self-labelling heuristic is obtained by noting that in the commuting version of Example 12, it is in fact not necessary to have termination of $\triangleright \cup \blacktriangleright$, but termination of $\triangleright^{+} \cdot \blacktriangleright^{+}$ suffices; translated to Fig. 5 (right) this reads: It suffices that there's no infinite *zigzag*, alternating $\triangleright$ and $\blacktriangleright$ reductions.

Example 17. If $(\triangleright, \blacktriangleright)$ commutes locally and $\triangleright^{+} \cdot \blacktriangleright^{+}$ is terminating, then $(\triangleright, \blacktriangleright)$ commutes [12, Corollary 4.6]. To see this, consider the stp-labelling which labels a step of either type, say, $a \triangleright b$, by *itself* as $a \triangleright_{a \triangleright b} b$, which is ordered above a step of the *other* type $c \blacktriangleright_{c \blacktriangleright d} d$ if $b \twoheadrightarrow c$. Note that the stp-labelling is decreasing since an infinite decreasing sequence of labels would entail an infinitely zigzagging reduction, contradicting termination of $\triangleright^{+} \cdot \blacktriangleright^{+}$. As a local peak $b \triangleleft_{a \triangleright b} a \blacktriangleright_{a \blacktriangleright c} c$ can by assumption be completed by a commuting valley the steps of which are either reachable by a zig from b or a zag from c, we conclude.

Variations on this example like e.g. [12, Lemma 4.5] are easily accomodated by our techniques as well, and even in a modular way by choosing an appropriate property to load the induction in Theorem 2.

The above examples cover the results in [6, Chapter 1], which in turn subsume those in [11], except for [11, Exercise 2.0.8(11)] which we deal with now.

Example 18. Let $\rightarrow\, = \,\triangleright \cup \blacktriangleright$ and consider the labelling pred which may label any step $a \rightarrow b$ as $a \rightarrow_{c,i} b$, where c is any *predecessor* of a, i.e. $c \twoheadrightarrow a$, and i is set to 1 for a backward step and to 2 for a forward step. If $\triangleright/\blacktriangleright$ is terminating, then the lexicographic product of $(\triangleright/\blacktriangleright)^+$ and $1 \succ 2$ gives a decreasing labelling. We verify by case analysis that then the combination of $\triangleright$ being locally confluent and $\blacktriangleright$ being *non-splitting*, i.e. $\leftarrow \cdot \blacktriangleright \subseteq \twoheadrightarrow \cdot \leftarrow^=$, results in decreasing diagrams:

for a peak $\triangleleft_{a,1} \cdot \triangleright_{b,2}$, local confluence of $\triangleright$ yields a valley $\triangleright\!\!\triangleright \cdot \triangleleft\!\!\triangleleft$ the steps of which can be labelled simply by their sources to result in a decreasing diagram;

for a peak $\triangleleft_{a,1} \cdot \blacktriangleright_{b,2}$, non-splittingness of $\blacktriangleright$ yields a valley $\twoheadrightarrow \cdot \leftarrow^=$, which after labelling the steps in $\twoheadrightarrow$ by their source and the $\leftarrow$-step as $\leftarrow_{a,1}$, results in a decreasing diagram;

a peak $\blacktriangleleft_{a,1} \cdot \triangleright_{b,2}$ is dealt with symmetric to the previous case;

for a peak $\blacktriangleleft_{a,1} \cdot \blacktriangleright_{b,2}$ non-splittingness of $\blacktriangleright$ yields a valley $\twoheadrightarrow \cdot \leftarrow^=$, which after labelling as in the second item except for labelling the steps of $\twoheadrightarrow$ up to and including the first $\triangleright$-step (if any) as $\rightarrow_{a,2}$, results in a decreasing diagram.

Although a bit more involved this labelling and case analysis directly cover the Full Localisation Lemma, the most complex (p.71–73) confluence result in [10], stating $\rightarrow\, = \,\triangleright \cup \blacktriangleright$ is confluent, if $\triangleright/\blacktriangleright$ is terminating and $\rightarrow$ is locally confluent, with the condition that in case a local peak $b \leftarrow a \blacktriangleright c$ needs to be completed by a valley of shape $b \twoheadrightarrow d \leftarrow^+ a' \blacktriangleleft c$ then $a\ (\triangleright/\blacktriangleright)^+\ a'$.

Remark 5. The final two examples both are covered by the original decreasing diagrams technique. It would be interesting to consider their conversion versions.

4.2 Term Rewriting

We show the usefulness of the rule-labelling heuristic in first- or higher-order term rewriting systems.

(H_2: **Rule-labelling**) Try labelling steps by the rule applied.

Example 19. Consider the TRS with rules [13, Example 2]: (1) $\mathsf{nats} \rightarrow 0 : \mathsf{inc}(\mathsf{nats})$, (2) $\mathsf{inc}(x:y) \rightarrow \mathsf{s}(x) : \mathsf{inc}(y)$, (3) $\mathsf{hd}(x:y) \rightarrow x$, (4) $\mathsf{tl}(x:y) \rightarrow y$, (5) $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \rightarrow \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$. The rule-labelling heuristic which labels every step by the rule applied, yields, by left- and right-linearity of the rules, that $\leftarrow_i \cdot \rightarrow_j \subseteq \rightarrow_j^= \cdot \leftarrow_i^=$ for all $i, j \in \{1, \ldots, 5\}$ and non-overlapping steps. The only critical pair arises from the local peak $\mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \leftarrow_5 \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \rightarrow_1 \mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats})))$ which can be completed by $\mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \rightarrow_1 \mathsf{tl}(\mathsf{inc}(0 : \mathsf{inc}(\mathsf{nats}))) \rightarrow_2 \mathsf{tl}(\mathsf{s}(0) : \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \rightarrow_4 \mathsf{inc}(\mathsf{inc}(\mathsf{nats})) \leftarrow_4 \mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats})))$. As the latter diagram can be easily made decreasing, e.g. by ordering $5 \succ 1, 2, 4$, we conclude confluence.

More generally, for any finite left- and right-linear term rewriting system, it is decidable whether the rule-labelling entails decreasingness, simply by trying all possible orderings of the rules,[1] refuting the claim of [13, Footnote 1] that this requires 'careful and smart design choices'. Of course, this does not allow to deal with non-right-linear rules:

Example 20. Consider the TRS with rules [13, Example 1]: (1) $g(a) \to f(g(a))$, (2) $g(b) \to c$, (3) $a \to b$, (4) $f(x) \to h(x,x)$, (5) $h(x,y) \to c$. Since the TRS is not right-linear, the above observation does not apply. In particular, the rule-labelling cannot work as rule 4 can *self-duplicate*, consider e.g. the term $f(f(c))$! Still, it is easy to find a decreasing labelling noting the duplicated variable has on the right-hand side less f-symbols above it than on its left-hand side: labelling steps by first the number of f-symbols above it and then the rule, and ordering these lexicographically by first $>$ and then $\succ$ given by $3 \succ 2, 4, 5$ does the job.

The trick in the example fails if variables occur 'deeper' in the right-hand side than in the left-hand side of a rule. Even in such cases the heuristic might be applicable, by solving the problem of self-duplication by brute force:

(H_3: **Self-duplication**) First try to separate out self-duplicating rules, and then switch for these to 'multi' steps in which an arbitrary number of redexes for that rule may be contracted.

This technique may be applied to prove confluence of orthogonal term rewriting systems, but also of some term rewriting systems *with* critical pairs, as is nicely illustrated in the following example.

Extending Bloo and Rose's $\lambda\mathtt{x}$-calculus with explicit substitutions, with a rule encoding the substitution lemma of the λ-calculus, yields the $\lambda\mathtt{xc}$-calculus.

Theorem 5. *The following CRS [6, Chapter 11]) for $\lambda\mathtt{xc}$-calculus is confluent.*

$$\begin{aligned}
(\lambda y.X(y))Y &\to X(y)[y{:=}Y] \\
X[y{:=}Y] &\to X \\
y[y{:=}Y] &\to Y \\
(X_1(y)X_2(y))[y{:=}Y] &\to (X_1(y)[y{:=}Y])(X_2(y)[y{:=}Y]) \\
(\lambda x.X(y))[y{:=}Y] &\to \lambda x.X(y)[y{:=}Y] \\
X(y,z)[y{:=}Y(z)][z{:=}Z] &\to X(y,z)[z{:=}Z][y{:=}Y(z)[z{:=}Z]]
\end{aligned}$$

Proof. It will turn out handy to split the set of rules as follows. The first rule is the $\mathtt{Beta}$-rule, the next four are the $\mathtt{x}$-rules, and the final rule is the $\mathtt{c}$-rule.

Let $\multimap_{\mathtt{Beta}}$ denote the contraction of any number of $\mathtt{Beta}$-redexes, let $\to_{\mathtt{x}}$ denote contracting an arbitrary (possibly garbage collecting) $\mathtt{x}$-redex, and let $\multimap_{\mathtt{c}}$ denote a $\mathtt{c}$-reduction which is a prefix of contracting all c-redexes in the term (see [14]). To show confluence of $\lambda\mathtt{xc}$, it then suffices to prove confluence of these relations, since $\to_{\lambda\mathtt{xc}} \subseteq \multimap_{\mathtt{Beta}} \cup \to_{\mathtt{x}} \cup \multimap_{\mathtt{c}} \subseteq \twoheadrightarrow_{\lambda\mathtt{xc}}$. In the rest of the proof, the three types of rules are referred to simply as $\mathtt{Beta}$, $\mathtt{x}$, and $\mathtt{c}$. It suffices to show that they are decreasing with respect to the order $\succ$ given by $\mathtt{Beta} \succ \mathtt{c} \succ \mathtt{x}$,

[1] This is analogous to the way one may proceed when checking whether a TRS is terminating via recursive path orders.

using the src-labelling with respect to (terminating) x-reduction to order x-steps among each other. Distinguish cases on the types of the rules in a local peak.

- If both are Beta, then the result follows from Beta being *linear orthogonal*, in the sense that a common reduct is found in either zero or one Beta-steps on both sides.
- If $s \leftarrow_{\mathtt{x}} t \mathrel{\circ\!\!\!\to}_{\mathtt{Beta}} r$, then $s \to_{\mathtt{x}} \cdot \mathrel{\circ\!\!\!\to}_{\mathtt{Beta}} q$ and $r \mathrel{\circ\!\!\!\to}_{\mathtt{c}} q$ in case the x-step overlaps one of the Beta-redexes, and $s \mathrel{\circ\!\!\!\to}_{\mathtt{Beta}} q \twoheadleftarrow_{\mathtt{x}} r$ otherwise.
- Beta is orthogonal to c, and they commute in a single step on either side.
- If both are x, a common reduct is found in at most two further x-steps having smaller sources.
- If $s \leftarrow_{\mathtt{x}} t \mathrel{\circ\!\!\!\to}_{\mathtt{c}} r$, then $s \twoheadrightarrow_{\mathtt{x}} \cdot \mathrel{\circ\!\!\!\to}_{\mathtt{c}} q \twoheadleftarrow_{\mathtt{x}} r$ in case the x-step overlaps one of the c-redexes (needing a number of garbage collection steps). Otherwise the steps simply commute (c may duplicate x, but not *vice versa*).
- If both rules are c, a common reduct is found in at most one further c-step, which holds since the c-rule is an instance of self-distributivity [14]. □

Note that a common reduct is found in an amount of work which is linear in the diverging steps, measuring each step by the number of steps performed by it. This is not that good, but still better than always reducing to x-normal form as is done in proofs relying on the so-called interpretation method.

5 Conclusion

We have improved upon our earlier decreasing diagrams technique. It was shown that in many cases it is not difficult to find a labelling showing decreasingness. The heuristics presented could be a stepping stone for constructing an automatic confluence prover.

We conclude by noting that the generalization does straightforwardly extend to Ohlebusch's confluence by decreasing diagrams *modulo an equivalence relation* results [5, Chapter 2].

Acknowledgments. We thank the anonymous referees, Alfons Geser, and Jean-Pierre Jouannaud for useful discussions, and the latter and Delia Kesner for their hospitality during my research stay in Paris, where this paper was conceived.

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