On Normalisation of Infinitary Combinatory Reduction Systems

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Abstract. For fully-extended, orthogonal infinitary Combinatory Reduction Systems, we prove that terms with perpetual reductions starting from them do not have (head) normal forms. Using this, we show that

1. needed reduction strategies are normalising for fully-extended, orthogonal infinitary Combinatory Reduction Systems, and that
2. weak and strong normalisation coincide for such systems as a whole and, in case reductions are non-erasing, also for terms.

1 Introduction

Infinitary higher-order rewrite systems extend infinitary TRSs (iTRSs) [1,2] with bound variables and nestings. Their introduction invalidates the Strip Lemma. Hence, new proof techniques are required to obtain confluence and normalisation results. The latter of these are the subject of this paper.

Failure of the Strip Lemma was first observed by Kennaway et al. [3] in infinitary λ-calculus. To prove confluence modulo certain subterms for this system, while avoiding the Strip Lemma, Kennaway et al. [3] and Kennaway and De Vries [2] use resp. a non-collapsing variant of the β-rule and standard reductions. To prove a similar confluence result for the infinite extension of Combinatory Reduction Systems (CRSs) [4], i.e. for infinitary CRSs (iCRSs) [5,6,7], Van Oostrom's technique of essential rewrite steps [8] was adapted.

Below we give an abstract formulation of Van Oostrom's technique under the name *projection pairs*. With the help of these pairs we show for fully-extended, orthogonal iCRSs that terms with perpetual reductions starting from them, i.e. reductions with an infinite number of root-steps, do not have (head) normal forms — the known proofs for iTRSs by Kennaway et al. [1] and by Klop and De Vrijer [9] do not carry over due to dependence on the Strip Lemma.

Using the above fact, we prove our main results: Needed reductions normalise for fully-extended, orthogonal iCRSs and weak and strong normalisation coincide for such systems as a whole and, in case of non-erasing reductions, also for terms.

Needed Reductions. Normalisation of needed reductions implies that any reduction strategy contracting only needed redexes, i.e. redexes a residual of which is contracted in every reduction to normal form, yields a normal form. Hence, these strategies are useful to obtain normal forms. We extend the classical result by Huet and Lévy [10] who show the same for orthogonal TRSs. This also extends

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identical results for orthogonal iTRSs by Kennaway et al. [1] and for orthogonal higher-order systems by Glauert and Khasidashvili [11].

Uniform Normalisation. Uniform normalisation [12], i.e. the coincidence of weak and strong normalisation, is special in the case of orthogonal iTRSs. As shown by Klop and De Vrijer [9], the property holds without any restrictions. As such, iTRSs behave different from TRSs, which need to be non-erasing [13].

However, the result for TRSs concerns terms and not systems. This result does not carry over to iTRSs, as noted by Kennaway et al. [14] and by Klop and De Vrijer [9]. Partial recovery is possible by considering non-erasing reductions instead of non-erasing rules, as indicated by Kennaway et al. [14]. We extend both this recovery and the result concerning systems to fully-extended, orthogonal iCRSs.

Overview. We give some preliminaries in Sect. 2. In Sect. 3, normalisation is introduced. Projection pairs are defined in Sect. 4 and used in Sect. 5 to obtain the result regarding perpetual reductions. In Sects. 6 and 7 we prove normalisation of needed reductions and uniform normalisation. We conclude in Sect. 8.

2 Preliminaries

We outline some basic facts concerning iCRSs; see [2,1,5,6,7] for more detailed accounts. Throughout, we denote the first infinite ordinal by ω, and arbitrary ordinals by α, β, γ, ... By $\mathbb{N}$ we denote the natural numbers including zero.

Terms and Substitutions. Let Σ be a signature with each element of finite arity. Moreover, assume a countably infinite set of variables and, for each finite arity, a countably infinite set of meta-variables — countably infinite sets suffice given 'Hilbert hotel'-style renaming.

Infinite terms are usually defined by metric completion [15,1,5]. Here, we give the shorter, but equivalent, definition from [6]:

Definition 2.1. *The set of* meta-terms *is defined by interpreting the following rules coinductively, where s and s_1, ..., s_n are again meta-terms:*

1. *each variable x is a meta-term,*
2. *if x is a variable, then $[x]s$ is a meta-term,*
3. *if Z is an n-ary meta-variable, then $Z(s_1, \ldots, s_n)$ is a meta-term, and*
4. *if $f \in \Sigma$ is n-ary, then $f(s_1, \ldots, s_n)$ is a meta-term.*

The set of finite *meta-terms, a subset of the set of meta-terms, is the set inductively defined by the above rules. A* term *is a meta-term without meta-variables and a* context *is a meta-term over $\Sigma \cup \{\Box\}$.*

We consider (meta-)terms modulo α-equivalence. A meta-term of the form $[x]s$ is called an *abstraction*; a variable x in s is called *bound* in $[x]s$. Meta-terms with meta-variables only occur in rewrite rules; rewriting itself is defined over terms. We have that $Z(Z(\ldots))$, and $Z([x]Z'([y]Z(\ldots)))$ are meta-terms. Moreover, $[x]f(Z(x))$ is a finite meta-term and $[x]x$ is a finite term.

The set of *positions* [5] of a meta-term s, denoted $\mathcal{P}os(s)$, is a set of *finite* strings over $\mathbb{N}$, with each string denoting the 'location' of a subterm in s. If p is a position of s, then $s|_p$ is the *subterm of s at* position p. The length of p is denoted $|p|$. There exists a well-founded order $<$ on positions: $p < q$ iff p is a proper prefix of q. The concatenation of positions p and q is denoted $p \cdot q$.

A *valuation* [4], denoted $\bar{\sigma}$, substitutes terms for meta-variables in meta-terms and is defined by coinductively interpreting the rules of valuations for CRSs [5]. In CRSs, applying a valuation to a meta-term yields a unique term. This is not the case for iCRSs [5]. To alleviate this problem, the set of meta-terms satisfying the so-called 'finite chains property' is defined in [5]:

Definition 2.2. *Let s be a meta-term. A* chain *in s is a sequence of (context, position)-pairs $(C_i[\Box], p_i)_{i<\alpha}$, with $\alpha \leq \omega$, such that for each $(C_i[\Box], p_i)$ there exists a term t_i with $C_i[t_i] = s|_{p_i}$ and $p_{i+1} = p_i \cdot q$ where q is the position of the hole in $C_i[\Box]$. A* chain of meta-variables *in s is such that for each $i < \alpha$ it holds that $C_i[\Box] = Z(t_1, \ldots, t_n)$ with $t_j = \Box$ for exactly one $1 \leq j \leq n$.*

The meta-term s is said to satisfy the finite chains property *if no infinite chain of meta-variables occurs in s.*

Remark that $\Box$ only occurs in $C_i[\Box]$ if $i + 1 < \alpha$, otherwise $C_i[\Box] = s|_{p_i}$. The meta-term $[x_1]Z_1([x_2]Z_2(\ldots [x_n]Z_n(\ldots)))$ e.g. satisfies the finite chains property, while $Z(Z(\ldots Z(\ldots)))$ does not. Finite meta-terms always satisfy the finite chains property. The following is shown in [5]:

Proposition 2.3. *Let s be a meta-term satisfying the finite chains property and let $\bar{\sigma}$ be a valuation. There is a unique term that is the result of applying $\bar{\sigma}$ to s.*

Rewriting. To define rewriting, recall that a *pattern* is a finite meta-term each meta-variable of which has distinct bound variables as arguments and that a meta-term is *closed* if all variables occur bound [4].

Definition 2.4. *A* rewrite rule *is a pair of closed meta-terms (l, r), denoted $l \to r$, with l a finite pattern of the form $f(s_1, \ldots, s_n)$ and r satisfying the finite chains property such that all meta-variables occurring in r also occur in l.*

An infinitary Combinatory Reduction System (iCRS) *is a pair $\mathcal{C} = (\Sigma, R)$ with Σ a signature and R a set of rewrite rules.*

Left-linearity and orthogonality are defined as for CRSs [4], by virtue of left-hand sides of rewrite rules being finite. A rewrite rule is *collapsing* if the root of its right-hand side is a meta-variable. Moreover, a pattern is *fully-extended*, if, for each meta-variable Z and abstraction $[x]s$ with an occurrence of Z in its scope, x is an argument of that occurrence of Z; a rewrite rule is *fully-extended* if its left-hand side is and an iCRS is *fully-extended* if all its rewrite rules are.

Definition 2.5. *A* rewrite step *is a pair of terms (s, t) denoted $s \to t$ and adorned with a context $C[\Box]$, a rewrite rule $l \to r$, and a valuation $\bar{\sigma}$ such that $s = C[\bar{\sigma}(l)]$ and $t = C[\bar{\sigma}(r)]$. The term $\bar{\sigma}(l)$ is called an $l \to r$-*redex*. It* occurs *at position p and depth $|p|$ in s, where p is the position of the hole in $C[\Box]$.*

A position q of s occurs in the redex pattern *of the redex at position p if $q \geq p$ and if there does not exist a position q' with $q \geq p \cdot q'$ such that q' is the position of a meta-variable in l.*

Both $\bar{\sigma}(l)$ and $\bar{\sigma}(r)$ are well-defined, as left- and right-hand sides of rewrite rules satisfy the finite chains property (left-hand sides because they are finite).

In addition to collapsing rewrite rules, a redex and a rewrite step are *collapsing* if the employed rewrite rule is. Using rewrite steps, we define reductions:

Definition 2.6. *A* transfinite reduction *with domain $\alpha > 0$ is a sequence of terms $(s_\beta)_{\beta<\alpha}$ such that $s_\beta \to s_{\beta+1}$ for all $\beta + 1 < \alpha$. In case $\alpha = \alpha' + 1$, the reduction is* closed *and of length α'. In case α is a limit ordinal, the reduction is* open *and of length α.*

The reduction is weakly *or* Cauchy continuous *if for every limit ordinal $\gamma < \alpha$ it holds that s_β converges to s_γ as β approaches γ from below. The reduction is* weakly *or* Cauchy convergent *if it is weakly continuous and closed.*

For each rewrite step $s_\beta \to s_{\beta+1}$, let d_β denote the depth of the contracted redex. The reduction is strongly continuous *if it is weakly continuous and if, for every limit ordinal $\gamma < \alpha$, the depth d_β tends to infinity as β approaches γ from below. The reduction is* strongly convergent *if strongly continuous and closed.*

Consider the rules $a \to a$ and $f(Z) \to g(f(Z))$ and the term $f(a)$. The following reduction of length ω is both weakly and strongly continuous:

$$f(a) \to f(a) \to \cdots \to f(a) \to \cdots .$$

Extending the reduction with $f(a)$ yields a weakly convergent reduction but not a strongly convergent one. The reduction

$$f(a) \to g(f(a)) \to \cdots \to g^n(f(a)) \to \cdots g^\omega ,$$

also of length ω and where g^ω denotes $g(g(\ldots g(\ldots)))$, is strongly convergent.

Reductions are ranged over by D, S, and T. We mostly consider *strongly convergent* reductions: By $s \twoheadrightarrow^\alpha t$, resp. $s \twoheadrightarrow^{\leq\alpha} t$, we denote a strongly convergent reduction of length α, resp. of length at most α. By $s \twoheadrightarrow t$, resp. $s \to^* t$, we denote a strongly convergent reduction of arbitrary length, resp. of finite length.

Across strongly convergent reductions we assume that a position that occurs in the redex pattern of a contracted redex does not have any descendants; likewise for residuals [5]. We write $P/(s \twoheadrightarrow t)$ for the descendants of a set of positions $P \subseteq \mathcal{P}os(s)$ across a strongly convergent reduction $s \twoheadrightarrow t$ and $\mathcal{U}/(s \twoheadrightarrow t)$ for the residuals of a set $\mathcal{U}$ of subterms of s across $s \twoheadrightarrow t$.

In the remainder we appeal to a number of properties of iCRSs. The first is immediate by the proof of the compression property in [5].

Theorem 2.7 (Compression). *For every fully-extended, left-linear iCRS, if $s \twoheadrightarrow^\alpha t$, then $s \twoheadrightarrow^{\leq\omega} t$. Moreover, if $s \twoheadrightarrow^\alpha t$ has a root-step, then does $s \twoheadrightarrow^{\leq\omega} t$.*

Assuming orthogonality, let $\mathcal{U}$ be a set of redexes of a term s. A *development* of $\mathcal{U}$ is a reduction $s \twoheadrightarrow t$ each step of which contracts a residual of a redex in $\mathcal{U}$. A development $s \twoheadrightarrow t$ is *complete* if $\mathcal{U}/(s \twoheadrightarrow t) = \emptyset$; in this case we also write $s \Rightarrow t$, where the arrow is adorned with $\mathcal{U}$ as needed. We have the following:

Proposition 2.8 (See [6]). *Let s be a term in an orthogonal iCRS. If $\mathcal{U}$ is a set of redexes of s with a complete development $s \Rightarrow t$ and if v is a redex of s, then the following diagram exists:*

$$\begin{array}{ccc} s & \xrightarrow{\;v\;} & t' \\ \Downarrow \mathcal{U} & & \Downarrow \mathcal{U}/(s \to^{v} t') \\ t & \underset{v/(s \Rightarrow t)}{\Longrightarrow} & s' \end{array}$$

A term s is *hypercollapsing* if for all $s \twoheadrightarrow t$ there exists a $t \twoheadrightarrow t'$ such that t' is a collapsing redex. We write $s \sim_{hc} t$ if t can be obtained from s by replacing hypercollapsing subterms in s by other hypercollapsing subterms. We have:

Theorem 2.9. *Fully-extended, orthogonal iCRSs are confluent modulo $\sim_{hc}$, i.e. if $s \twoheadrightarrow s'$ and $t \twoheadrightarrow t'$ with $s \sim_{hc} t$, then $s' \twoheadrightarrow s''$ and $t' \twoheadrightarrow t''$ with $s'' \sim_{hc} t''$.*

The above is shown in [6] under assumption that rewrite rules have finite right-hand sides; in [7] the result is extended to allow for infinite right-hand sides.

3 Weak and Strong Normalisation

We define (head) normal forms together with weak and strong normalisation. Ample motivation for the definitions is given by Klop and De Vrijer [9].

Definition 3.1. *A term s is a* normal form *if no redexes occur in s and a* head normal form *if it is not reducible to a redex by a strongly convergent reduction. In addition, s is* weakly normalising *if a strongly convergent reduction exists from s to a normal form and s is* strongly normalising *if for all open strongly continuous reductions starting in s there exists a term that extends the reduction such that it becomes strongly convergent.*

An iCRS is weakly normalising, *resp.* strongly normalising, *if all terms are.*

Consider again the rules $a \to a$ and $f(Z) \to g(f(Z))$, introduced below Definition 2.6. The term $f(a)$ is weakly normalising by the second reduction below the definition; g^{ω} is a normal form. The term is not strongly normalising, as the first reduction below the definition cannot be extended such that it becomes strongly convergent. On the other hand, $f(x)$ is strongly normalising, as the only open strongly continuous reduction starting from it is

$$f(x) \to g(f(x)) \to \cdots \to g^{n}(f(x)) \to \cdots ,$$

which extends to a strongly convergent reduction by adding g^{ω}.

The definition of weak normalisation is taken from finitary rewriting. To understand strong normalisation, consider the following proposition, which is immediate by the fact that strongly convergent reductions have a finite number of reduction steps at each depth [1,5]:

Proposition 3.2. *An open strongly continuous reduction extends to a strongly convergent one iff the number of reduction steps is finite at every depth.*

Hence, the definition of strong normalisation from finitary rewriting is relaxed: A finite number of steps in total implies a finite number of steps at each depth. Given that a term does not need to have a maximum depth in the current setting, this seems a reasonable way to relax the definition.

As in the finite case, strong normalisation implies weak normalisation: To start, remark that any strongly normalising term reduces to a head normal form, otherwise it has an open strongly continuous reduction starting from it with an infinite number of root-steps. Next, as the same holds for each subterm of the head normal form, again by strong normalisation, iteration gives a term each subterm of which is a head normal form, i.e. it gives a normal form.

4 Projection Pairs

We give an abstract formulation of Van Oostrom's technique of essential rewrite steps [8] and its adaptation to iCRSs [6,7]. This requires an auxiliary definition:

Definition 4.1. *Let s and t be terms and $P \subseteq \mathcal{P}os(s)$. The set P is a* prefix set *of s if P is* finite *and if all prefixes of positions in P are also in P. Moreover, t* mirrors *s in P, if for all $p \in P$ it holds that $p \in \mathcal{P}os(t)$ and $root(t|_p) = root(s|_p)$.*

Van Oostrom's technique is a termination argument focusing on prefix sets P and finite sequences of complete developments D, i.e. reductions D consisting of a finite number of such developments. Given a prefix set P of the final term of D, the defined measure assigns to D a tuple of natural numbers of the same length as the sequence. In addition, a map is defined which, intuitively, given P yields a prefix set of its initial term such that the function symbols that occur at the positions in obtained prefix set are those 'responsible', across D, for what occurs at the positions of P in the final term of D.

At the core of the technique lies a projection. Given a reduction step from the initial term of D, which is called *essential* in case it occurs in the obtained prefix of the initial term of D and *inessential* otherwise, the projection yields a finite sequence of complete developments D', starting in the term created by the reduction step, such that the final term of D' mirrors the final one of D in P. The projection is such that the measure decreases in case of an essential reduction step and stays equal otherwise, facilitating the termination argument.

Moving away from tuples, the measure and the map on prefixes can abstractly be defined as follows:

Definition 4.2. *Given a well-founded order $\prec$, a* projection pair *is a pair (μ, ε) of maps over finite sequences of complete developments D and prefix sets P of the final term of the chosen D such that:*

- *$\mu_P(D)$ maps to an element of the well-founded order $\prec$, and*
- *$\varepsilon_P(D)$ maps to a prefix of the initial term of D,*

and such that if D' is a sequence of complete developments strictly shorter than D with P' a prefix set of the final term of D', then $\mu_{P'}(D') \prec \mu_P(D)$.

The map μ is the measure and ε is the map for prefix sets. The measure requires a sequence that is strictly shorter than D to map to a smaller element in the well-founded order. Although of a technical nature, this property is easily obtained in case tuples are used to define the well-founded order, as described above, and the tuples are first compared length-wise and next lexicographically.

The existence of the projection mentioned above can now be formulated as the soundness of a projection pair:

Definition 4.3. *Let $\prec$ be a well-founded order. A projection pair (μ, ε) is* sound *iff for each finite sequence of complete development D, prefix set P of the final term of D, and $s \twoheadrightarrow t$, with s the initial term of D, it holds that:*

- *if $s \twoheadrightarrow t$ consists of a* single *step contracting a redex u at a position in $\varepsilon_P(D)$, with no residual from u/D occurring at a position in P, then there exists a D' such that $\mu_P(D') \prec \mu_P(D)$, and*
- *if $s \twoheadrightarrow t$* only *contracts redexes at positions* outside *$\varepsilon_P(D)$, then there exists a D' such that $\mu_P(D') = \mu_P(D)$ and $\varepsilon_P(D') = \varepsilon_P(D)$,*

where in both cases D' is a finite sequence of complete developments with initial term t such that the final term of D' mirrors the final one of D in P.

In the first clause the redex is essential and in the second clause all are inessential. Intuitively, the restriction in the first clause stating that no residual from u/D occurs in P ensures that the projection preserves P. Together the clauses formalise the intuition behind ε, i.e. that P only depends on positions in $\varepsilon_P(D)$. The map is constant for reductions contracting only redexes outside $\varepsilon_P(D)$ and, obviously, any term in such a reduction mirrors all the other terms in $\varepsilon_P(D)$.

Remark 4.4. The first clause of Definition 4.3 deals neither with reductions where residuals from u/D occur in P nor with infinite reductions. In the next section, we deal with the first through the restriction on strictly shorter sequences of complete developments and with the second through strong convergence.

This leaves to show that sound projection pairs actually exist. For *fully-extended*, *orthogonal* iCRSs this is done in [6] in case all rewrite rules have finite right-hand sides. In [7] the result is extended to iCRSs that allow for infinite right-hand sides. The lengthy definitions from [6] and [7] are omitted here; the abstract definitions suffice.

5 Perpetual Reductions

To show our main results, we prove that terms with perpetual reductions starting from them do not have (head) normal forms. Except for the final lemma of this section, the proofs in this section differ from the proofs for head normal forms [1] and normal forms [9] of iTRSs, which depend on the Strip Lemma.

We assume *fully-extended*, *orthogonal* iCRSs. Perpetual reductions, not to be mistaken for perpetual reduction strategies [16], are defined as in [1]:

Definition 5.1. *A* perpetual reduction *is an open strongly continuous reduction with an infinite number of root-steps.*

Any perpetual reduction can be 'compressed' to one of length ω:

Lemma 5.2. *Let s be a term. If there is a perpetual reduction starting from s, then there also is a perpetual reduction of length ω starting from it.*

Proof. By definition, we may write a perpetual reduction starting from s as:

$$s = s_0 \twoheadrightarrow s'_0 \to s_1 \twoheadrightarrow s'_1 \to s_2 \twoheadrightarrow \cdots ,$$

with $s'_i \to s_{i+1}$ a root-step and no root-steps occurring in $s_i \twoheadrightarrow s'_i$ for each $i \in \mathbb{N}$.

We inductively define a perpetual reduction of length ω:

$$s = t_0 \to^* t'_0 \to t_1 \to^* t'_1 \to t_2 \to^* \cdots ,$$

where for all $i \in \mathbb{N}$ we have that $t'_i \to t_{i+1}$ is a root-step and $t_i \to^* t'_i$ is finite and without root-steps. First, define $t_0 = s_0 = s$. Next, assume we have defined a term t_i with $t_i \twoheadrightarrow s_i$. Compression of $t_i \twoheadrightarrow s_i \twoheadrightarrow s'_i \to s_{i+1}$ yields a reduction $t_i \to^* t'_i \to t_{i+1} \twoheadrightarrow^{\leq \omega} s_{i+1}$ with $t'_i \to t_{i+1}$ a root-step and $t_i \to^* t'_i$ finite and without root-steps. We thus obtain a perpetual reduction with the required properties. □

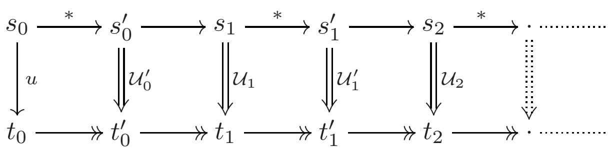
The following lemma, which projects perpetual reductions over single steps, is the iCRS analogue of Proposition 17 in [9]. Its proof is the only in the current paper explicitly dealing with nestings; in all other cases these are either 'hidden' by the current result or the use of projection pairs.

Lemma 5.3. *Let s and t be terms with $s \to t$. If there is a perpetual reduction starting from s, then there is a perpetual reduction starting from t.*

Proof. Define $s_0 = s$, $t_0 = t$, and suppose u is the redex contracted in $s \to t$. By Lemma 5.2, we may write the perpetual reduction starting from s_0 as:

$$s_0 \to^* s'_0 \to s_1 \to^* s'_1 \to s_2 \to^* \cdots ,$$

where for all $i \in \mathbb{N}$, we have that $s'_i \to s_{i+1}$ is a root-step and $s_i \to^* s'_i$ is finite and without root-steps. By repeated application of Proposition 2.8, we obtain:



Write S_i for $s_i \to^* s'_i \to s_{i+1} \to^* \cdots$ and T_i for $t_i \twoheadrightarrow t'_i \twoheadrightarrow t_{i+1} \twoheadrightarrow \cdots$. If we can show for each $i \in \mathbb{N}$ that a root-step occurs in T_i, then an infinite number of root-steps occur in T_0, implying that the reduction is perpetual.

To show that a root-step occurs in T_i we distinguish two cases: (1) a root-step occurs in S_i not contracting a residual of u, and (2) all root-steps in S_i contract a residual of u. We deal with each of these cases in turn:

1. In this case there exists a root-step $s'_j \to s_{j+1}$ with $j \geq i$ such that the contracted redex, say v, is not a residual of u. Since $\mathcal{U}'_j$ contracts only residuals of u, we have by orthogonality that a residual of v occurs at the root of t'_j and that no other residuals of v occur in t'_j. By construction, $t'_j \twoheadrightarrow t_{j+1}$ contracts precisely all residuals of v. Hence, $t'_j \twoheadrightarrow t_{j+1}$ is a root-step.
2. In this case, the infinite number of root-steps of S_i each contract a residual of u. Hence, all terms in S_i have a chain of residuals of u at the root and u is collapsing. All the chains are finite, as only a finitely many steps occur

before each term and as right-hand sides of rewrite rules only allow for finite chains of meta-variables.

Residuals of u cannot create further nestings of other residuals of u: This requires a residual of u to occur on the path between the redex pattern and a bound variable of another residual of u. Such a situation cannot occur by definition of rewrite rules and valuations. Thus, for each step following s_i, we have that each residual in the chain at the root of s_i has at most one residual. Eventually, no residuals are left, as an infinite number of root-steps occur in S_i. Since the residuals always occur in a chain starting at the root, the last residual is contracted by means of a root-step, say $s'_j \to s_{j+1}$.

Suppose now that no redex contracted in $s_i \to^* s_{j+1}$ has a residual occurring at the root of one of the terms in $t_i \twoheadrightarrow t_{j+1}$. As u is collapsing and as each development of $\mathcal{U}_k$ and $\mathcal{U}'_k$ contracts only residuals of u, which occur in finite chains, it follows that a fixed function symbol occurs at the root of each the terms in $t_i \twoheadrightarrow t_{j+1}$. Moreover, as residuals of u cannot create further nestings of other residuals of u, the fixed function symbol also occurs at the root of s_{j+1}, i.e. no residual of u occurs at the root of s_{j+1}, contradiction. Hence, a root-step occurs in $t_i \twoheadrightarrow t_{j+1}$.

As required, we have that a root-step occurs in each T_i. Hence, T_0 is a perpetual reduction starting from $t_0 = t$. □

We next show that reduction to a redex is preserved if no root-steps occur. In the proof we assume the existence of a sound projection pair (μ, ε), which is possible in case of fully-extended, orthogonal iCRSs, as remarked in Sect. 4.

Lemma 5.4. *If no root-steps occur in $s \twoheadrightarrow t$ and s reduces to a redex, then t reduces to a redex.*

Proof. Using ordinal induction, we show that every term s_α in $s \twoheadrightarrow t$ reduces to a redex by a finite sequence of complete developments D_α. Denote by P_α the set of positions of the redex pattern at the root of the final term of D_α; to facilitate the induction we also show for $\beta \leq \alpha$ that either $\mu_{P_\alpha}(D_\alpha) \prec \mu_{P_\beta}(D_\beta)$ or $\mu_{P_\alpha}(D_\alpha) = \mu_{P_\beta}(D_\beta)$ and $\varepsilon_{P_\alpha}(D_\alpha) = \varepsilon_{P_\beta}(D_\beta)$.

For $s_0 = s$, it follows by assumption that s_0 reduces to a redex. In fact, by strong convergence and compression, s_0 reduces to a redex by a finite reduction D_0. As any finite reduction is a finite sequence of complete developments, where each set of redexes is a singleton set, the result follows.

For $s_{\alpha+1}$ there are two cases, depending on the occurrence of a residual of u, the redex contracted in $s_\alpha \to s_{\alpha+1}$, at the root of the final term of D_α:

- If no residual of u occurs at the root of the final term of D_α, the result is immediate by soundness of the pair (μ, ε) and the induction hypothesis.
- If a residual of u does occur at the root of the final term of D_α, a root-step not contracting a residual of u occurs in D_α. Otherwise, no residual of u occurs at the root of the final term of D_α, because $s_\alpha \to s_{\alpha+1}$ is not a root-step. Hence, there is a finite sequence D'_α of complete developments, strictly shorter than D_α, that has a redex at the root of its final term which is not a

residual of u. By definition of projection pairs, $\mu_{P'_\alpha}(D'_\alpha) \prec \mu_{P_\alpha}(D_\alpha)$, where P'_α is the set of positions of the redex pattern at the root of the final term of D'_α. The case in which no residual of u occurs at the root of the final term of the complete development now applies and the result follows.

For s_α, with α a limit ordinal, it follows by the induction hypothesis, strong convergence, and the well-foundedness of $\prec$ that there exists a $\beta < \alpha$ such that all steps in $s_\beta \twoheadrightarrow s_\alpha$ occur at positions outside $\varepsilon_{P_\beta}(D_\beta)$. Hence, the result follows by the second clause of Definition 4.3. □

Using the above we can prove the result we are after, which generalises Proposition 8.9 in [1] for head normal forms and Corollary 20 in [9] for normal forms:

Lemma 5.5. *Let s be a term. If s has a perpetual reduction starting from it, then s does not have a (head) normal form.*

Proof. Assume a perpetual reduction starting from s and let $s \twoheadrightarrow t$ be arbitrary. By compression and strong convergence, we may write $s \to^* t' \twoheadrightarrow^{\leq \omega} t$, where all root-steps occur in $s \to^* t'$. By repeated application of Lemma 5.3, there exists a perpetual reduction starting from t'. Thus, t' reduces to a redex. Since $t' \twoheadrightarrow t$ contains no root-steps, we have by Lemma 5.4 that t also reduces to a redex. As $s \twoheadrightarrow t$ is arbitrary, it follows that s does not have a (head) normal form. □

The reverse of the above lemma only holds for head normal forms: Suppose the term s does not have a head normal form. Hence, each reduct of s reduces to a redex. Repeatedly contracting the redexes obtained yields a perpetual reduction.

In case of normal forms consider the rule $a \to a$. The term $f(a)$ does not have a normal form, as the term reduces to itself, but no perpetual reduction starts from the term either, as $f(a)$ is a head normal form.

6 Needed Reductions

Assuming again *fully-extended*, *orthogonal* iCRSs, we show that needed reductions are normalising. We define needed redexes and reductions as in [1]:

Definition 6.1. *A redex u in a term s is* needed *if in every strongly convergent reduction from s to normal form some residual of u is contracted. A* needed reduction *is a weakly continuous reduction contracting only needed redexes.*

Non-neededness is due to the erasure of residuals. As in the finite case, this can be the result of the absence of residuals after a certain rewrite step, while residuals did occur earlier. In addition, a redex can also be 'pushed out' of a term by an infinite reduction. To see this, consider the rules $a \to a$ and $f(Z) \to g(f(Z))$ from Sect. 2. The $a \to a$-redex in the term $f(a)$ can be 'pushed out' of the term in the reduction to the normal form g^ω without contracting it.

We next proceed in two steps: First, we show that a term with a normal form has a needed redex. Thereafter, we prove the actual result.

Existence of Needed Redexes. To prove that a term with a normal form has a needed redex, we adapt a proof by Middeldorp [17], who shows for TRSs that a non-root-stable term has a root-needed redex. The proof deviates from the one by Huet and Lévy [10] and its analogue for iTRSs by Kennaway et al. [1]; it does not require the introduction of external redexes, although the redex eventually identified in Lemma 6.4 has the property of being external.

We start by proving the iCRS analogues of Lemmas 3.3 and 4.2 in [17], where we write $s \check{\twoheadrightarrow} t$ in case all contracted redexes in $s \twoheadrightarrow t$ occur below the root.

Lemma 6.2. *Let $s \check{\twoheadrightarrow} s'$ and $t \check{\twoheadrightarrow} t'$. If $s \sim_{hc} t$, where it suffices to replace hypercollapsing subterms below the root, then $s' \check{\twoheadrightarrow} s''$ and $t' \check{\twoheadrightarrow} t''$ with $s'' \sim_{hc} t''$, where it also suffices to replace hypercollapsing subterms below the root.*

Proof. Let $s \sim_{hc} t$, where it suffices to replace hypercollapsing subterms below the root. By assumption, $s = f(s_1, \ldots, s_n) \check{\twoheadrightarrow} f(s'_1, \ldots, s'_n) = s'$ and $t = f(t_1, \ldots, t_n) \check{\twoheadrightarrow} f(t'_1, \ldots, t'_n) = t'$. Moreover, $s_i \sim_{hc} t_i$ for all $1 \leq i \leq n$. Hence, by Theorem 2.9 it holds for all $1 \leq i \leq n$ that $s'_i \twoheadrightarrow s''_i$ and $t'_i \twoheadrightarrow t''_i$ with $s''_i \sim_{hc} t''_i$. The result follows by defining $s'' = f(s''_1, \ldots, s''_n)$ and $t'' = f(t''_1, \ldots, t''_n)$. □

Lemma 6.3. *Let s be a term. If s reduces to a redex, then the rule used to contract the first such redex is independent of the reduction.*

Proof. Suppose s reduces to a redex. We may assume that all rewrite steps occur below the root, otherwise s reduces to a redex by a shorter reduction. Let $s \check{\twoheadrightarrow} \bar{\sigma}_1(l_1)$ and $s \check{\twoheadrightarrow} \bar{\sigma}_2(l_2)$, where l_1 and l_2 are left-hand sides of rewrite rules. By Lemma 6.2 and since $s \sim_{hc} s$, there exist $\bar{\sigma}_1(l_1) \check{\twoheadrightarrow} t_1$ and $\bar{\sigma}_2(l_2) \check{\twoheadrightarrow} t_2$ with $t_1 \sim_{hc} t_2$, where it is suffices to replace hypercollapsing subterms below the root. Since a redex at the root cannot be destroyed by either replacing hypercollapsing subterms below the root or contracting of redexes below the root, by orthogonality and fully-extendedness, we have $l_1 = l_2$. □

We now show the presence of needed redexes in terms with normal forms. The proof is based on the one of Theorem 4.3 in [17], although the induction employed there no longer applies as terms may be infinite:

Lemma 6.4. *Let s be a term which is not a normal form. If s has a normal form, then s has a needed redex.*

Proof. Suppose s has a normal form. As s is not a normal form, there exists a minimal position p in s such that $s|_p$ is not a head normal form. There are two possibilities: either $s|_p$ is a redex or not.

If $s|_p$ is a redex, it is needed: By minimality of p, $s|_q$ is a head normal form for each $q < p$. Hence, by orthogonality and fully-extendedness, residuals of $s|_p$ cannot be erased or occur at increasingly greater depths in the reducts of s.

If $s|_p$ is not a redex, it reduces to one, otherwise $s|_p$ is a head normal form. By Lemma 6.3, the rule used in the first redex to which $s|_p$ reduces is independent of the reduction. Assume l is the left-hand side of this rule. Since $s|_p$ is not a redex, there exists a non-root position q in the intersection of $\mathcal{P}os(s|_p)$ and the set of positions in the redex pattern of l such that $root(s|_{p \cdot q}) \neq root(l|_q)$ — if q would be the root position, then $s|_p$ reduces to a redex by a shorter reduction. Consider $s|_{p \cdot q}$. If $s|_{p \cdot q}$ is a redex, then it is needed, otherwise we can reduce s to a normal form without reducing $s|_p$ to a redex, which is impossible by minimality of p. If

$s|_{p \cdot q}$ is not a redex, then the argument for $s|_p$ can be repeated with p replaced by $p \cdot q$. Repeating the argument, a needed redex must eventually be encountered. If not, then $s|_p$ is a head normal form, contradicting assumptions. □

Normalisation. To prove that needed reductions are normalising, we need to show that these reductions are strongly convergent for terms with normal forms. To this end, we first prove the iCRS analogues of Theorem 8.10 and Corollary 8.11 in [1]: Reductions outside subterms without a head normal form are strongly convergent and redexes in that do occur in such subterms are never needed.

Lemma 6.5. *Reductions in which all contracted redexes occur outside subterms without a head normal form are strongly convergent.*

Proof (Sketch). Identical to the proof of Theorem 8.10 in [1]: A non-strongly convergent reduction yields a subterm with a perpetual reduction starting from it. By Lemma 5.5 this implies the subterm is without a head normal form. □

Lemma 6.6. *Let s be a term with a normal form. A redex in s which occurs in a subterm without a head normal form is never needed.*

Proof (Sketch). Identical to the proof of Corollary 8.11 in [1], employing the previous lemma instead of Theorem 8.10 in [1]. □

Our intermediate result is now easily obtained and is the iCRS analogue of Corollary 8.12 in [1]:

Lemma 6.7. *Let s be a term with a normal form. Every needed reduction starting from s is strongly convergent.*

Proof. By Lemma 6.6 no needed redexes occur in subterms without a head normal form. Hence, the result follows by Lemma 6.5. □

By the previous lemma and Lemma 6.4, we now immediately obtain:

Theorem 6.8. *In fully-extended, orthogonal iCRSs, needed reductions of terms with normal forms are strongly convergent and normalising.*

Although needed reductions are countable, by definition of strong convergence, no bound exists on the maximum length of such reductions. To see this, consider the rule $f(Z) \to g(Z)$ and the term f^ω, i.e. $f(f(\ldots f(\ldots)))$. Obviously, all redexes in f^ω are needed with respect to the unique normal form g^ω. Assume that δ is a bijection between any countable, infinite ordinal α and $\mathbb{N}$ and note that for each depth there is precisely one position in f^ω. Define $(s_\beta)_{\beta<\alpha+1}$ with $s_0 = f^\omega$ and $s_\alpha = g^\omega$ such that $s_\beta \to s_{\beta+1}$ contracts the redex at depth $\delta(\beta)$. As δ is a bijection, all rewrite steps in the reduction $(s_\beta)_{\beta<\alpha+1}$ of length α exist and by Proposition 3.2 the reduction is strongly convergent.

Remark 6.9. Needed reductions are not hypernormalising, i.e. if a *finite* number of arbitrary steps occur between each step contracting a needed redex, then the obtained reduction need not to be strongly convergent. This is contrary to the finite higher-order case [8].

To see this, consider the rules $a \to f(a)$, $b \to b$, and $g(Z, Z') \to Z$. Moreover, consider for each $n \in \mathbb{N}$ the term $g(f^n(a), b)$, where the root-redex and the

$a \to f(a)$-redex are needed, but where the $b \to b$-redex is not. We have a reduction contracting a needed redex in every other step:

$$g(a, \underline{b}) \to g(\underline{a}, b) \to g(f(a), \underline{b}) \to g(f(\underline{a}), b) \to \cdots \\ \to g(f^n(a), \underline{b}) \to g(f^n(\underline{a}), b) \to g(f^{n+1}(a), \underline{b}) \to \cdots ,$$

where the contracted redexes are underlined. The reduction is not strongly convergent, as an infinite number of b redexes are contracted at a single depth.

Although hypernormalisation does not hold, not all is lost: Needed-fair reductions, i.e. reductions in which each needed redex that is a residual of another needed redex is contracted within a finite number of steps, are normalising [7].

7 Uniform Normalisation

We next consider uniform normalisation of iCRSs, i.e. the coincidence of weak and strong normalisation. Both the global and local variant are considered, i.e. we consider both iCRSs as a whole and individual terms. As before, we assume *fully-extended*, *orthogonal* iCRSs.

Global Uniform Normalisation. Like orthogonal iTRSs [9], fully-extended, orthogonal iCRSs are uniformly normalising. To show this, we need the following lemma, which is the iCRS analogue of Proposition 21 in [9] and whose proof is identical to the proof of that proposition.

Lemma 7.1. *If there exists an open strongly continuous reduction with an infinite number of steps at a certain depth, then there exists a perpetual reduction.*

We can now prove the iCRS analogue of Theorem 22 in [9]:

Theorem 7.2. *A fully-extended, orthogonal iCRS is weakly normalising iff it is strongly normalising.*

Proof (Sketch). Identical to the proof of Theorem 22 in [9]: That strong normalisation implies weak normalisation is explained on p. 177. For the reverse, reason by contradiction, employing in turn Lemmas 7.1 and 5.5. □

Local Uniform Normalisation. Uniform normalisation does not hold for terms, even under assumption of non-erasure [14,9], i.e. assuming that all variables occurring on the left-hand sides of rules also occur on their right-hand sides. This is contrary to TRSs [13]. That weak normalisation does not imply strong normalisation is the result of iCRSs being both infinite and higher-order.

From the perspective of infinitary rewriting, failure is due to subterms being 'pushed out' of terms (see also Sect. 6). Given the non-erasing rules $a \to a$ and $f(Z) \to g(f(Z))$, it follows that $f(a)$ reduces to the normal form g^ω, but repeatedly contracting the $a \to a$-redex in $f(a)$ yields an open strongly continuous reduction of length ω with an infinite number of reductions at a single depth.

From the perspective of higher-order rewriting, failure is due to erasure by certain variables not occurring bound. Consider $a \to a$ and $f([x]Z(x), Z') \to Z(Z')$. The term $f([x]y, a)$ is weakly normalising, for we have $f([x]y, a) \to y$, where a is erased as x does not occur bound in $[x]y$. The term is not strongly normalising; to see this, repeatedly contract the $a \to a$-redex in $f([x]y, a)$.

As observed by Kennaway et al. [14], albeit without a proof, uniform normalisation holds for terms in iTRSs if all possible reductions are non-erasing. The same holds for iCRSs; to see this we first define non-erasing reductions:

Definition 7.3. *A reduction $s \twoheadrightarrow t$ is* non-erasing *if for every every subterm $s'|_p$ of term s' in $s \twoheadrightarrow t$ either (1) a residual of $s'|_p$ occurs in t or (2) a descendant of p occurs in, or is a variable bound by, the redex pattern of a redex contracted in the suffix $s' \twoheadrightarrow t$ of $s \twoheadrightarrow t$.*

Remark that the second condition applies to a specific residual of a subterm. Any other residual must still satisfy either the first or second condition.

Strengthening the observation by Kennaway et al. [14] slightly, we obtain:

Theorem 7.4. *In fully-extended, orthogonal iCRSs, weak and strong normalisation coincide for terms with only non-erasing reductions starting from them.*

Proof (Sketch). That strong normalisation implies weak normalisation is explained on p. 177. For the reverse, reason by contradiction, employing in turn Lemma 5.5 and Theorem 2.9. □

It is in general undecidable if a term has only non-erasing reductions starting from it. Hence, sufficient, decidable criteria are called for. In the case of iTRSs an obvious criterion is the non-erasure of rules in combination with non-depth increasingness, i.e. each variable occurring on the left-hand side of a rule also occurs on its right-hand side and does so at depth lesser or equal depth.

The criterion no longer suffices for iCRSs. From above, consider the rules $a \to a$ and $f([x]Z(x), Z') \to Z(Z')$ and the term $f([x]y, a)$. Both rules are non-erasing and non-depth increasing, while $f([x]y, a)$ is not uniformly normalising.

8 Conclusion

Using Van Oostrom's technique of essential redexes [8], we showed that terms with perpetual reductions starting from them do not have (head) normal forms. As such, we avoided the use of the Strip Lemma, which is traditionally employed [1,9], but which no longer holds in the higher-order case.

With the help of the above, we showed that needed reductions are normalising for fully-extended, orthogonal iCRSs, extending the classical result by Huet and Lévy [10] and similar ones for iTRSs by Kennaway et al. [1] and for higher-order systems by Glauert and Khasidashvili [11]. We also proved that uniform normalisation holds for these iCRSs and, in case of non-erasing reductions, also for terms, extending results by Klop and De Vrijer [9] and Kennaway et al. [14].

A number of questions remain. For example, what is the relation between strong normalisation in infinite systems — both iTRSs and iCRSs — and root-stabilisation in finite systems [17]? What about weak orthogonality in the case of needed reductions? And, in the case of uniform normalisation can fully-extendedness be dropped or orthogonality be replaced by weak orthogonality?

The dissimilar definitions of finite and infinite reductions pose a problem in the case of root-stabilisation. Fully-extendedness cannot be dropped in case of needed reductions, as Van Raamsdonk [18] already shows for finite systems. This also implies that making the current theory more abstract might be difficult.

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