Computational Complexity of the Distance Constrained Labeling Problem for Trees (Extended Abstract)

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Abstract. An $L(p,q)$-labeling of a graph is a labeling of its vertices by nonnegative integers such that the labels of adjacent vertices differ by at least p and the labels of vertices at distance 2 differ by at least q. The span of such a labeling is the maximum label used. Distance constrained labelings are an important graph theoretical approach to the Frequency Assignment Problem applied in mobile and wireless networks.

In this paper we show that determining the minimum span of an $L(p,q)$-labeling of a tree is NP-hard whenever q is not a divisor of p. This demonstrates significant difference in computational complexity of this problem for $q = 1$ and $q > 1$. In addition, we give a sufficient and necessary condition for the existence of an $H(p,q)$-labeling of a tree in the case when the metric on the label space is determined by a strongly vertex transitive graph H. This generalizes the problem of distance constrained labeling in cyclic metric, that was known to be solvable in polynomial time for trees.

1 Introduction

Distance constrained graph labelings stem from the highly practical problem of assigning frequencies to transmitters in order to avoid, or minimize, undesired interference. Suppose that the metric in the frequency space is expressible by a graph H. An $H(p,q)$-labeling of a graph G is defined as a mapping $f : V(G) \to V(H)$ such that $\mathrm{dist}_H(f(u), f(v)) \geq p$ for any two adjacent vertices $u, v \in V(G)$, and $\mathrm{dist}_H(f(u), f(v)) \geq q$ for any two nonadjacent vertices $u, v \in V(G)$ which have a common neighbor (i.e., are at distance 2 in G). Here the vertices of the graph G correspond to the transmitters in the network, and the edges of G express possible interference. This general approach was first studied in the

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connection to locally injective graph homomorphisms [7]. Two special cases have been introduced and intensively studied before — the case of linear metric, where $H = P_{\lambda+1}$ is the path of length λ, and the cyclic metric corresponding to the case $H = C_\lambda$.

These mappings are referred to as $L(p, q)$- and $C(p, q)$-labelings, respectively. In both cases λ is the span of the labeling. We define the linear span $L_{p,q}(G)$ as the minimum span of an $L(p, q)$-labeling of G. Analogously, the cyclic span $C_{p,q}(G)$ is the minimum span of a labeling with the cyclic metric.

The concept of $L(2, 1)$-labeling was introduced by Roberts [18,11]. The exact values for special graphs and graph classes have been determined several works, cf. surveys [14,19,2] Griggs and Yeh [11] conjectured that $L_{2,1}(G) \leq \Delta^2(G)$, where $\Delta(G)$ denotes the maximum degree in G. This upper bound has been recently proven true for every sufficiently large $\Delta(G)$ by Havet et al. [13].

Distance constrained graph labelings provide a rather interesting graph invariant from the computational complexity point of view. Griggs and Yeh [11] proved that it is NP-hard to determine $L_{2,1}(G)$, while Fiala et al.[8] proved that deciding $L_{2,1}(G) \leq k$ is NP-complete for every fixed $k \geq 4$. Rather interesting is the complexity for restricted graph classes. Chang and Kuo [4] described a polynomial time algorithm for determining the $L_{2,1}(G)$ if G is a tree, but already for series-parallel graphs this problem becomes NP-complete [5]. The computational complexity of determining the $L_{p,q}(G)$ if G is a tree for $q > 1$ has been open since then. It was explicitly asked by D. Welsh [private communication during a graph coloring workshop at DIMACS in 1999] with a hope for a generalization of the method of Chang and Kuo. While this works easily for an arbitrary $p > 2$ and $q = 1$, the case $q > 1$ kept resisting all attempts. Intuitively, the difference between $q = 1$ and $q > 1$ relates to the difference between systems of *distinct* and *distant* representatives [10]. Resolving this question is the main result of this paper.

Formally, we consider the following decision problem:

$L(p, q)$-LABELING
Instance: A graph G and an integer λ.
Question: Does G allow an $L(p, q)$-labeling of span λ?

Note that p, q are fixed parameters while λ (and, of course, G) are part of the input. We prove the following result.

Theorem 1. *For positive integers p and q, the $L(p, q)$-*LABELING *problem restricted to trees is solvable in polynomial time only if q divides p, otherwise it is* NP*-complete.*

The polynomial part is now a folklore. It has been proved already in [11] that an optimum labeling using only labels of the form $ap + bq$ always exists, and hence $L(rp, rq)$-LABELING is equivalent to $L(p, q)$-LABELING. In particular, when q divides p we get the $L(\frac{p}{q}, 1)$-LABELING problem which is solvable in polynomial time by a slight modification of an algorithm by Chang and Kuo [4,3]. We note here that it is sufficient to prove the NP-hardness part of the theorem for the case of mutually prime p and q.

The NP-hardness part of the theorem is proved by a reduction from the problem of deciding the existence of a system of distant representatives in systems of symmetric sets. The construction extends the approach initiated in [9] where the NP-hardness of extending a prelabeling to a complete $L(p,q)$-labeling was proved. The main idea is a construction of trees that allow only specific labels on their roots. The main difficulty, that we successfully managed to overcome, was to keep the size of such trees polynomial. These constructions are presented in Section 3 and the proof of Theorem 1 is concluded in Section 5.

It is interesting to note that $L_{p,q}(T)$ can be approximated well for trees by $q\Delta(T)$. Griggs and Yeh proved that $\Delta(T)q+p-q \le L_{p,q}(T) \le \Delta(T)q+2p-q-1$ holds for positive integers $p \ge q$ and a tree T [11]. But perhaps a bit surprising is the fact that in the cyclic metric the span of a tree is uniquely determined by its maximum degree. Liu and Zhu [16] proved that $C_{p,q}(T) = q\Delta(T)+2p-q$ for every tree T and $p \ge q$. In particular, the $C_{p,q}$-span of a tree can be computed in linear time. We further explore this phenomenon and show that it is due to the fact that cycles are transitive graphs. We prove a necessary and sufficient condition for the existence of an $H(p,q)$-labeling of a tree T when H is a strongly transitive graph in Theorem 2. This result has several applications. For instance, the minimum n such that an input tree T has a $Q_n(p,q)$-labeling (where Q_n is the n-dimensional cube) can be determined in polynomial time for $q=1$ and $q=2$. These results are presented in Section 6.

2 Preliminaries and Notation

For integers i and j, we denote by $[i,j]$ the interval $\{i, i+1, i+2, \dots, j\}$. By convention, $[i,j] = \emptyset$ if $j < i$. Analogously, if $i \equiv j \pmod q$, we denote by $[i,j]_{\equiv q}$ the q-stepped interval $\{i, i+q, i+2q, \dots, j\}$. For a positive integer k, we write $[k] := [1,k]$. The binary operators div and mod stand for the integral division and the remainder of the division.

We consider undirected graphs without loops or multiple edges. In a graph G, the symbol $N_G(u)$ denotes the set of vertices adjacent to u, i.e., the (open) *neighborhood* of u. We also define the *closed neighborhood* as $N_G[u] := N_G(u) \cup \{u\}$. The subscripts G will be omitted if there is no danger of confusion which graph G is being considered. The symbol $\Delta(G)$ stands for the maximum degree of a vertex in the graph G.

A graph is connected if every pair of vertices can be connected by a path. For vertices $u, v \in V_G$, the *distance* $\mathrm{dist}_G(u,v)$ is the length of a shortest path between u and v.

We adopt standard notions from graph theory: the path P_n on n vertices; the cycle C_n; a star — a connected graph with at most one vertex of degree greater than one; a tree — a connected graph with no cycle; and a hypercube Q_n — a graph on binary words of length n where two such words are adjacent if they differ only at one position. For more details we refer to the classical monograph by Harary [12] or to a more recent textbook by Matoušek and Nešetřil [17].

When exploring $L(p,q)$-labelings in the first part of the paper, we assume that the label set is a set $[0,\lambda]$. Thus an $L(p,q)$-*labeling of* G *of span* λ is a mapping $l : V_G \to [0,\lambda]$ such that for any pair of adjacent vertices u and v, it holds that $|l(u)-l(v)| \geq p$, and for any pair of nonadjacent vertices u and v that share a common neighbor, it holds that $|l(u)-l(v)| \geq q$.

For a fixed λ we define the *reversed* mapping on $[0,\lambda]$ by $a \to \bar{a} := \lambda - a$. Observe that for any $L(p,q)$-labeling l of span λ, the reversed labeling $\bar{l}$ defined by $\bar{l}(u) := \overline{l(u)}$ is also an $L(p,q)$-labeling. We extend the reversing to sets by $\overline{S} = \{\bar{a} \mid a \in S\}$.

We extend any mapping f defined on vertices of a graph G into a mapping on sets of vertices by letting $f(W) := \bigcup_{u\in W}\{f(u)\}$ for each $W \subset V_G$.

In the context of fixed p,q and λ we say that a label $a \in [0,\lambda]$ is *feasible* for a vertex $u \in V_G$ if there exists an $L(p,q)$-labeling l of G of span λ such that $l(u) = a$. The set of feasible labels for u is called the *feasible set of* u *in* G. Observe that every feasible set S is reversable: $S = \overline{S}$. If a symmetric set S is expressed as the union of a set and its reverse $S = S' \cup \overline{S'}$, we abbreviate this expression by the notation $S = S' \cup \overline{\cdots}$. Finally, we say that a label a is *forced on a vertex* $u \in V_G$ if $\{a, \bar{a}\}$ is the feasible set of u.

3 Auxiliary Constructions for the Case $p > q$

Throughout the coming three sections we assume that $p > q > 1$ and that p and q are relatively prime. In our construction we also use a third parameter, an integer k, whose value will be specified later in the polynomial reduction. We will use $\lambda := 2p + kq$ and $d := k + (p \text{ div } q) + 1$.

Construction 1. *For given* $p > q > 1$ *and* $k \geq 2$*, let* T^1 *be the only tree with a vertex* u *of degree two which is a common neighbor of vertices* w *and* w' *of degree* d*. The other neighbors of* w *and* w' *are of degree* $k+2$*. All remaining vertices are leaves.*

Construction 2. *For given* $p > q > 1$ *and* $k \geq 2$*, let* T^2 *be the tree obtained from* T^1 *by adding* k *leaves to* u*. Denote by* v *some leaf adjacent to* u*. (See Fig. 1.)*

Lemma 1. *If* p, q *and* k *satisfy the assumptions of Constructions 1 and 2, then* $[p,\bar{p}]_{\equiv q}$ *is the feasible set for* u *in* T_1 *and* $[q,\overline{2p}]_{\equiv q} \cup \overline{\cdots}$ *is the feasible set for* v *in* T_2*.*

Note that for each choice of $a \in [p,\bar{p}]_{\equiv q}$ and $b \in [q, a-p]_{\equiv q} \cup [a+p, \bar{q}]_{\equiv q}$, a labeling l of T^2 exists where $l(u) = a$ and $l(v) = b$.

To simplify the next construction we define $d' := d - 2 = k - 1 + (p \text{ div } q)$.

Construction 3. *For given* $p > q > 1$*,* $k \geq \frac{2p}{q} + 1$ *and* $i \in [k-1]$*, let* T_i^3 *be the tree constructed as follows: Take the disjoint union of* i *copies of the tree* T^1 *and*

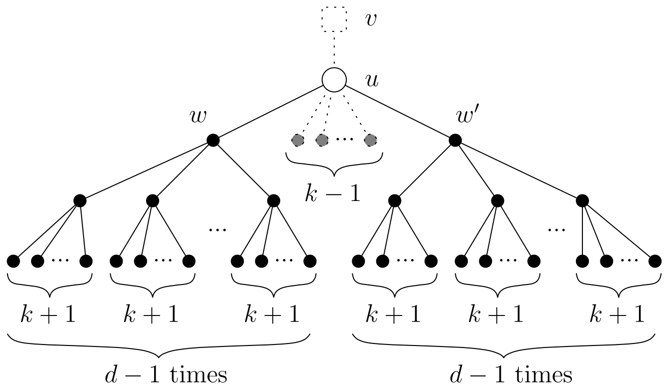

Fig. 1. The tree T^1 (solid lines) and tree T^2 (solid and dotted lines)

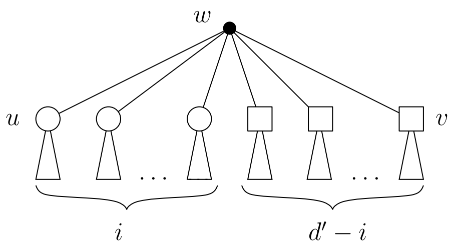

Fig. 2. The tree T_i^3

$d' - i$ copies of T^2. Then insert a new vertex w and connect it to the i vertices u of trees T^1 as well as to the $d' - i$ vertices v of trees T^2.

Rename the vertices such that only u of the first copy of T^1 and v of the last copy of T^2 keep their names. (See Fig. 2.)

Lemma 2. *Assume that p, q, k, and i satisfy the assumptions of Construction 3. Then*

- *the set $[p + q, p + iq]_{\equiv q} \cup \overline{\cdots}$ is the feasible set for u in T_i^3,*
- *$[\overline{(d' - i)q}, \bar{q}]_{\equiv q} \cup \overline{\cdots}$ is the feasible set for v in T_i^3, and*
- *q is forced on w.*

Proof. Let l be a hypothetical labeling. We show that this labeling is unique upto an isomorphism of T_i^3 and upto reversion of the labeling.

Observe that for the closed neighborhood of w it holds that $l(N[w]) \subseteq [q, \bar{q}]$, since for every vertex in this set we identify vertices labeled by 0 and λ at distance at most two. (This follows from the labeling described in Lemma 1.)

As the degree of w is at least k we exclude the case $l(w) \in [p + q, \overline{p + q}]$ by the same argument as in Lemma 1. Assume without loss of generality that $l(w) \in [q, p + q]$. Consequently, for the open neighborhood of w we get that $l(N(w)) \subseteq [p + q, \bar{q}]$.

We also assume without loss of generality that $l(u) < l(v)$: if the maximal label on $N(w)$ was on some vertex u of the first copies, then $l(N(w)) \subset [p, \bar{p}]$. This interval is not long enough to accommodate $k+1$-many q-distant labels.

If we choose v such that it receives the maximal label on $N(w)$, we see that $l(v) \equiv 2p \pmod{q}$. Hence, at least once the distance between consecutive labels on $N(w)$ is at least $q + (p \bmod q)$.

The only way how labels of $N(w)$ can be arranged into the interval $[p+q, \bar{q}]$ is to use the arithmetic progression $[p+q, p+iq]_{\equiv q}$ on the first i copies of u, and then after the gap $q + (p \bmod q)$ to use the set $[\overline{(d'-i)q}, \bar{q}]_{\equiv q}$ on the copies of v. In all other ways a gap greater than q would be used at least twice, and the above arrangement is already tight: the smallest possible label on v is $p + iq + q + (p \bmod q) = 2p + kq - (k - 1 + (p \operatorname{div} q) - i)q$.

As the smallest label used on $N(w)$ is $p+q$, the label q is forced on w.

The above described labeling of $N(w)$ can be extended to the rest of the tree T_i^3. In the copies of T_1, the labels of u and w comply with the labelings mentioned after Lemma 1. In the remaining $d'-i$ copies of T^2, we choose $l(u) = p+q$ if $l(v) > 2p+q$ and $l(u) = \bar{p}$ otherwise. The choice of k in the Definition 3 assures that this partial labeling can be extended on each copy of T^2.

In the following two lemmas we show constructions of trees that force exact labels on some vertices.

Construction 4. *For $p > q > 1$ and an even $k \geq \frac{2p}{q} + 1$, take the disjoint union of $k-2$ trees: one copy of T_1^3, one copy of $T_{\frac{k}{2}}^3$, and two copies of each tree T_i^3 for $i \in [2, \frac{k}{2} - 1]$. Insert an extra new vertex w and make it adjacent to each of the $k-2$ vertices u. The resulting graph is the tree T^4. Rename the vertices such that u_i is the vertex u of T_i^3, i.e., of one of the two isomorphic copies when $i \in [2, \frac{k}{2} - 1]$. (See Fig. 3.)*

Lemma 3. *If p, q and k satisfy the assumptions of Construction 4, then in T^4, $p + iq$ is forced on u_i for each $i \in [\frac{k}{2}]$, and $2q$ is forced on w.*

Construction 5. *For $p > q > 1$ and an even $k \geq \frac{2p}{q} + 1$, construct the tree T^5 from the disjoint union of pairs of trees T_i^3 for $i \in [\frac{k}{2}]$ by adding an extra new vertex w and making it adjacent to all vertices v. Rename the vertices such that for each i, v_i is the vertex v of one of the two copies of T_i^3.*

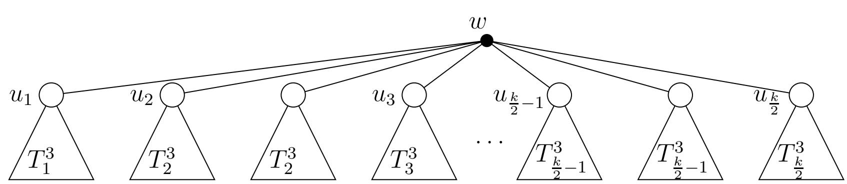

Fig. 3. The Tree T^4

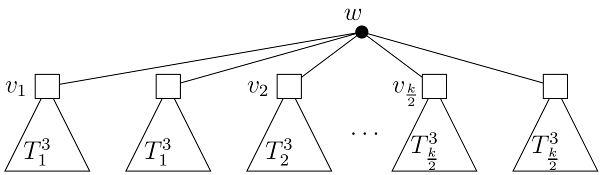

Fig. 4. The Tree T^5

Lemma 4. *Suppose* p, q, *and* k *satisfy the assumptions of Construction 5. Then in* T^5, $p + \frac{k}{2}q$ *is forced on* w, *and for every* $i \in [\frac{k}{2}]$, iq *is forced on* v_i.

Let $\Lambda(k,p,q)$ (called the *set of applicable labels*) be the set of labels that are forced on some vertex u_i of T^4 or on some v_i of T^5. By Lemmas 4 and 5,

$$\Lambda(k,p,q) := \left(\left[p+q, p+\frac{kq}{2}\right]_{\equiv q} \cup \left[q, \frac{kq}{2}\right]_{\equiv q} \right) \cup \overline{\cdots}$$

4 Symmetric Systems of q-Distant Representatives

As a technical tool for proving NP-hardness results we use the following problem of finding distant representatives:

System of q-distant representatives — Sq-DR
Parameter: A positive integer q.
Instance: A collection of sets $R_i, i \in [m]$ of integers.
Question: Is there a collection of elements $r_i \in R_i$, $i \in [m]$ that pairwise differ by at least q

It is known that the S1-DR problem allows a polynomial time algorithm (by finding a maximum matching in a bipartite graph), while for all $q \geq 2$ the Sq-DR problem is NP-complete, even if each set R_i has at most three elements [10].

We adjust the Sq-DR problem so that it can be used to prove NP-hardness for the $L(p,q)$-Labeling problem for trees. Since every $L(p,q)$-labeling l comes together with the reversed labeling $\overline{l}$, we need a stronger concept of systems of q-distant representatives where the sets are λ-symmetric, i.e., $R_i = \overline{R_i}$ for every $i \in [m]$. Moreover, we say that a set R is $2q$-sparse if the distance between any two elements in R is at least $2q$.

Lemma 5. *For any* $p > q > 1$, *the* Sq-DR *problem remains* NP-*complete even when restricted to instances where each* R_i

- *is of size at most 6,*
- *is* $2q$-*sparse*
- *is* λ-*symmetric for* $\lambda = 2p + (6n + 2(p \text{ div } q) + 4)q$, *and*
- *is a subset of* $\Lambda(6n + 2(p \text{ div } q) + 4, p, q) \cap \left([p+2q, p+3nq] \cup \overline{\cdots}\right)$

where n *is a suitable integer.*

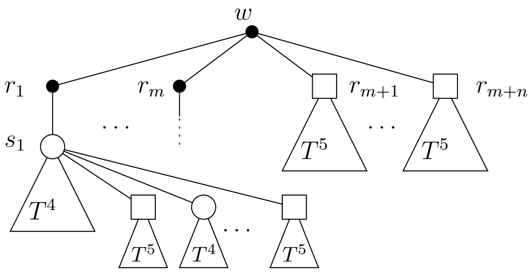

Fig. 5. The final tree T^6

5 Proof of Theorem 1 for $p > q$

Proof. Assume an instance $(R_i)_{i\in[m+n]}$ of the Sq-DR problem with properties described in Lemma 5, and let $|R_i| = 4$ or 6 for $i \in [m]$ and $|R_{m+i}| = 2$ for $i \in [n]$. We construct a tree T^6 with special vertices r_i such that these vertices share a common neighbor and force for every $i \in [m+n]$ that r_i can not be labeled by any label outside the set R_i under any $L(p,q)$-labeling of span λ of T^6.

Assume that the set $R_i, i \in [m]$ consists of elements $\{a, b, \bar{b}, \bar{a}\}$ with $a < b$.

We choose an auxiliary set $S_i \subset [q, p+(3n+1)q]$ of applicable labels such that $\{q, a-q, a+q, b-q, b+q, p+(3n+1)q\} \subset S_i$. The set S_i contains also sufficiently many other labels such that the distance between consecutive elements of S_i is at least q but strictly less than $2q$ with only two exceptions: $a-q, a+q$ and $b-q, b+q$. (A construction of such set can be given explicitly, but we would like to avoid excessive formalism.) As the set R_i is $2q$-sparse, each S_i is nonempty.

Analogously, we construct the sets S_i for all R_i with six elements.

We construct tree T^6 as follows:

1. For each R_i with four or six elements
   (a) Insert into T^6 a new copy of T^4 and rename its vertex $u_{\frac{k}{2}}$ by s_i
   (b) For each element $p + jq$ of S_i add two copies of the tree T^4 and make s_i adjacent to both vertices u_j.
   (c) Analogously, for each $jq \in S_i$ add two copies of the tree T^5 and make s_i adjacent to both vertices v_j.
   (d) Add an extra new leaf r_i adjacent to s_i
2. For each $R_i = \{jq, \overline{jq}\}$ add a copy of T^5 and rename its v_j by r_i.
3. Finally, connect these $m + n$ partial trees by a new common neighbor w of all vertices $r_i, i \in [m+n]$. (See Fig. 5.)

We argue that for every $i \in [n]$ only the set R_i is feasible for the vertex r_i in each partial tree constructed in the first step of the construction. As $\frac{\lambda}{2}$ is forced on s_i and both 0 and λ appear on $l(N(s_i))$ inside the copy of T^4 we get that $l(r_i) \in [q, \frac{kq}{2}] \cup \overline{\cdots}$.

The elements of S_i are forced on $N(s_i)$, so only $l(r_i) \in (0 \cup R_i \cup [p + (3n + 2)q, p + \frac{kq}{2}]) \cup \overline{\cdots}$. By the choice of $k = 6n + 2(p \text{ div } q) + 4$ we have $\frac{kq}{2} = (3n + p \text{ div } q + 2)q < p + (3n + 2)q$ and hence $l(r_i) \in R_i$.

Observe that the labelings giving pairs of u_j of step 1b) λ-symmetric labels can be simply combined altogether with any label of r_i from the set R_i to get an $L(p, q)$-labeling of the partial tree.

We conclude the proof by showing that the entire T^6 allows an $L(p, q)$-labeling of span λ if and only if the set system $(R_i)_{i \in [m+n]}$ allows a system of q-distant representatives.

Given a labeling l, the labels of vertices r_i provide valid representatives for R_i. This is since vertices r_i are mutually at distance two and we have shown that their labels can only belong to sets R_i (for $i > m$ this follows directly from Lemma 4).

In the opposite direction, we label each r_i by the representative of R_i and use the corresponding labelings of the partial trees described above. Then w can be labeled by 0 as $l(r_i) > q$ for every $i \in [m + n]$ as well as it holds that $0 \notin N(l(v_j))$ for every feasible labeling of T^5 which was added in the second step (consult Lemma 4).

Though the practical motivation for $L(p, q)$-labelings implies that $p \geq q$, the notion itself makes sense also for $p < q$. The NP-hardness result prevails as well.

6 $H(p, q)$-Labelings for Transitive Target Graphs

Consider the following graph labeling problem with a condition at distance two:

(p, q)-Distance Labeling	(p, q)-DL
Instance: Graphs G and H.	
Question: Does G allow an $H(p, q)$-labeling?	

We emphasize that the target graph H is a part of the input of the problem. The problem of determining $L_{p,q}(G)$ and $C_{p,q}(G)$ is equivalent to the (p, q)-DL problem restricted to graphs H being paths and cycles. In contrary to our former result on NP-completeness of the (p, q)-DL problem for $q > 1$, G being a star and an arbitrary target graph H [6], the (p, q)-DL problem becomes easy when G is a tree and the target graph H is a cycle:

Proposition 1 (Leese and Noble [15], Liu and Zhu [16]). *Let T be a tree, and $p \geq q$ be nonnegative integers. Then a $C_\lambda(p, q)$-labeling of T exists if and only if $\lambda \geq q\Delta(T) + 2p - q$.*

We show now that the change of the complexity of the labeling problem for linear and circular metrics follows from the fact that cycles are vertex-transitive. Recall that a graph H is (strongly) *vertex transitive* if for every two vertices $x, y \in V_H$, the graph H allows an automorphism f swapping vertices x and y, i.e., $f(x) = y$ and $f(y) = x$.

Theorem 2. *Let H be a vertex transitive graph, and p, q be positive integers. An $H(p,q)$-labeling of a tree T exists if and only if the graph H contains vertices $x, y_1, y_2, \ldots, y_{\Delta(T)}$ such that $\text{dist}_H(x, y_i) \geq p$ and $\text{dist}_H(y_i, y_j) \geq q$ hold for every distinct $i, j \in [\Delta(T)]$.*

Proof. Observe that the existence of an $H(p,q)$-labeling of T immediately gives the existence of vertices $x, y_1, \ldots, y_{\Delta(T)}$.

For the opposite implication we construct the labeling by induction. The $H(p,q)$-labeling will satisfy the property that for every vertex u of T and its neighbors $v_1, \ldots v_k$, the graph H allows an automorphism g such that the labels satisfy $l(u) = g(x)$ and for every $i \in [k]$ it holds that $l(v_i) = g(y_j)$ for some $j \in \Delta(T)$.

Firstly, select an arbitrary vertex r of T, and make the tree rooted in r. Also define $l(r) = x$ and extend the labeling l on $N(r)$ such that distinct neighbors of r are mapped onto distinct y_i's. Clearly, the required automorphism g is the identity.

Assume that the labeling is already defined on a vertex u and its parent v, but not yet at the children of u. Let g be the automorphism of H used to distribute labels on $N[v]$ and f be the automorphism swapping vertices $l(u)$ and $l(v)$. We now compose both automorphisms $h := f \circ g$ and extend the labeling on the whole neighborhood of u by using distinct vertices of $h(y_1), \ldots, h(y_{\Delta(T)})$ as labels.

Note that Proposition 1 is a corollary of Theorem 2 since every cycle is vertex transitive. The (p,q)-DL problem for trees is solvable in polynomial time for those classes of target graphs for which the condition of the existence of vertices x and $y_1, \ldots, y_{\Delta(T)}$ can be answered in polynomial time. In particular, this applies for vertex transitive graphs of restricted treewidth as follows from well known results by Arnborg et al. [1].

Corollary 1. *Let $\mathcal{G}$ be a class of vertex transitive graphs with bounded treewidth. Then the (p,q)-DL problem, restricted to input graphs G that are trees and graph H from the class $\mathcal{G}$, can be solved in polynomial time.*

We now consider the case when H is an n-dimensional hypercube Q_n and $q = 1, 2$.

Corollary 2. *Let T be a tree, and H be an n-dimensional hypercube Q_n. Then a tree T allows an $H(p,1)$-labeling if and only if*

$$\Delta(T) \leq \binom{n}{p} + \binom{n}{p+1} + \cdots + \binom{n}{n}.$$

Also, T has an $H(p,2)$-labeling if and only if

$$\Delta(T) \leq \binom{n}{p} + \binom{n}{p+2} + \binom{n}{p+4} + \ldots.$$

Proof. Every hypercube is vertex transitive. Choose $x \in V_{Q_n}$ arbitrarily.

The first claim follows directly from the fact that the number of vertices in Q_n that are at distance at least p from x is exactly $\binom{n}{p} + \binom{n}{p+1} + \cdots + \binom{n}{n}$.

Let U_i be the set of vertices at distance i from x. It is well known that $|U_i| = \binom{n}{i}$. Define $U := U_p \cup U_{p+2} \cup U_{p+4} \cup \ldots$. This U is an independent set and every vertex of U is at distance at least p from x. Let $W := U_{p-2} \cup U_{p-4} \cup U_{p-6} \cup \ldots$. As the union $U \cup W$ is a maximum independent set in Q_n, the set U is a maximum independent set among vertices that are at distance at least p from x. By Theorem 2 the tree T allows an $H(p, 2)$-labeling if and only if $\Delta(T) \leq |U|$ and the other claim follows.

Note that for $q \geq 3$ the problem of finding $x, y_1, \ldots, y_{\Delta(T)}$ in Q_n becomes harder, since it requires to compute the set of vertices that are pairwise at distance at least three — in particular none of the layers $U_i, i < n$ can be taken completely.

Finally, observe that for $p = q = 2$ the search for $x, y_1, \ldots, y_{\Delta(T)}$ in a general graph H is equivalent to the test whether H allows an independent set of size at least $\Delta(T) + 1$.

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