Chapter IV
Lie-Algebraic Approach to the Classification and Analysis of Integrable Models

In this chapter we shall summarize and generalize our experience in describing integrable models gained from the study of particular examples. The principal entities of the inverse scattering method and its Hamiltonian interpretation were the auxiliary linear problem operator $L = \frac{d}{dx} - U(x, \lambda)$ and the fundamental Poisson brackets for $U(x, \lambda)$ involving the r-matrix. Similar objects were introduced for lattice models. We will show that these notions have a simple geometric interpretation.

We shall define a natural partition of the integrable models into three families: rational, trigonometric and elliptic, according to the dependence of $U(x, \lambda)$ and $r(\lambda)$ on the spectral parameter λ. We shall interprete the fundamental Poisson brackets for the rational family in terms of an infinite-dimensional Lie algebra associated with the current algebra. The trigonometric and elliptic families will be obtained by averaging over a one- or two-dimensional lattice in the complex plane of the spectral variable λ. There are also similar families for lattice models; we shall discuss the associated fundamental Poisson brackets. Concentrating our attention on the rational case, we shall also discuss the dynamics of integrable models from a general point of view. This will lead to a natural geometric interpretation of the Riemann problem. We shall also give a Lie-algebraic interpretation of the hierarchy of Poisson structures and the associated Λ-operator.

§ 1. Fundamental Poisson Brackets Generated by the Current Algebra

We recall the definition of the standard Poisson structure associated with an arbitrary connected Lie group G, $\dim G = n$. Let $\mathfrak{g}$ be its Lie algebra and let X_a, $a = 1, \ldots, n$, be a set of generators of $\mathfrak{g}$ with structure constants C^c_{ab},

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$$[X_a, X_b] = C^c_{ab} X_c. \tag{1.1}$$

From here on we adopt the convention on summation over repeated indices. The linear space $\mathfrak{g}^*$ dual to $\mathfrak{g}$ has natural coordinates u_a: if $\xi = \xi^a X_a$ lies in $\mathfrak{g}$, then $u(\xi) = (u, \xi) = u_a \xi^a$. In the algebra $\mathscr{A}$ of smooth functions $f(u)$ on $\mathfrak{g}^*$ we define a bracket $\{,\}: \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ by

$$\{f_1, f_2\}(u) = -C^c_{ab} \frac{\partial f_1}{\partial u_a} \frac{\partial f_2}{\partial u_b} u_c, \tag{1.2}$$

which is obviously skew-symmetric and satisfies the Jacobi identity by virtue of the Jacobi identity for the commutator (1.1). Thus (1.2) defines a Poisson structure on the phase space $\mathfrak{g}^*$. The Poisson bracket (1.2) for the coordinates u_a takes the form

$$\{u_a, u_b\} = -C^c_{ab} u_c \tag{1.3}$$

and is called the *Lie-Poisson bracket.*

In general, the Poisson structure (1.2) is degenerate. Its annihilator coincides with the *algebra of Casimir functions* $I(\mathfrak{g})$ which consists of functions $f(u)$ invariant under the coadjoint action $u \to \mathrm{Ad}^* g \cdot u$ of G on $\mathfrak{g}^*$ given by

$$\mathrm{Ad}^* g \cdot u(\xi) = u(g^{-1} \xi g). \tag{1.4}$$

The restriction of the Poisson bracket (1.2) to any orbit of this action is nondegenerate, so that a Poisson submanifold in $\mathfrak{g}^*$ is a union of orbits.

These definitions actually involve only the Lie algebra $\mathfrak{g}$ of G and its coadjoint action $u \to \mathrm{ad}^* \eta \cdot u$ given by

$$\mathrm{ad}^* \eta \cdot u(\xi) = u([\xi, \eta]). \tag{1.5}$$

The orbits of the Ad^* action of G on $\mathfrak{g}^*$ are integral manifolds for the distribution spanned by the vector fields $\mathrm{ad}^* \eta$ for all η in $\mathfrak{g}$. Hence we may speak of the Lie-Poisson structure induced by a Lie bracket.

In the study of integrable systems we shall deal with infinite-dimensional Lie algebras. All the above definitions carry over to this case in a natural manner.

Let us consider the *current algebra* $C(\mathfrak{g})$ *associated with the Lie algebra* $\mathfrak{g}$. It consists of formal Laurent series $\xi(\lambda)$ in the variable λ,

$$\xi(\lambda) = \sum_{k > -\infty}^{\infty} \xi_k \lambda^k, \tag{1.6}$$

with ξ_k in $\mathfrak{g}$ and the symbol $k \gg -\infty$ indicating that the series in powers of λ^{-1} truncates. The commutator in $C(\mathfrak{g})$ is defined in the obvious way

$$[\xi(\lambda), \eta(\lambda)] = \sum_{k \gg -\infty}^{\infty} \sum_{i+j=k} [\xi_i, \eta_j] \lambda^k . \tag{1.7}$$

A set of generators of $C(\mathfrak{g})$ is formed by the elements

$$X_{a,k} = X_a \lambda^k , \qquad a = 1, \ldots, n; \qquad k = -\infty, \ldots, \infty , \tag{1.8}$$

where the X_a are the generators of $\mathfrak{g}$ with structure constants C^c_{ab}. Their commutator clearly is

$$[X_{a,k}, X_{b,l}] = C^c_{ab} X_{c,k+l} . \tag{1.9}$$

Let $u_{a,k}$ denote the coordinates of an element u of the dual space $C^*(\mathfrak{g})$; in agreement with (1.6) we assume that $u_{a,k} = 0$ for large positive k. The corresponding pairing is given by

$$u(\xi) = (u, \xi) = \sum_k u_{a,k} \xi^a_k , \tag{1.10}$$

where the sum over k is always finite. The Lie-Poisson bracket of the coordinates $u_{a,k}$ has the form

$$\{u_{a,k}, u_{b,l}\} = -C^c_{ab} u_{c,k+l} . \tag{1.11}$$

It will be convenient to introduce a generating function $u_a(\lambda)$ for the coordinates $u_{a,k}$ of a point u in $C^*(\mathfrak{g})$ as a formal Laurent series

$$u_a(\lambda) = \sum_{k=-\infty}^{k \ll \infty} u_{a,k} \lambda^{-k-1} . \tag{1.12}$$

The pairing (1.10) is given by a neat formula

$$u(\xi) = \operatorname{Res} u_a(\lambda) \xi^a(\lambda) , \tag{1.13}$$

where the symbol Res indicates the coefficient of λ^{-1} in a Laurent series.

The variable λ determines a *gradation* of $C(\mathfrak{g})$,

$$C(\mathfrak{g}) = \sum_{k \gg -\infty}^{\infty} \mathfrak{g} \lambda^k = \sum_{k \gg -\infty}^{\infty} C_k , \tag{1.14}$$

where

$$[C_k, C_l] \subset C_{k+l}, \tag{1.15}$$

which, in particular, allows us to decompose $C(\mathfrak{g})$ into a linear sum of two subalgebras,

$$C(\mathfrak{g}) = C_+(\mathfrak{g}) + C_-(\mathfrak{g}), \tag{1.16}$$

where

$$C_+(\mathfrak{g}) = \sum_{k=0}^{\infty} C_k, \quad C_-(\mathfrak{g}) = \sum_{k > -\infty}^{k=-1} C_k. \tag{1.17}$$

There is a similar decomposition for $C^*(\mathfrak{g})$,

$$C^*(\mathfrak{g}) = C^*_+(\mathfrak{g}) + C^*_-(\mathfrak{g}), \tag{1.18}$$

which in terms of the generating function $u_a(\lambda)$ becomes

$$u_a(\lambda) = u_a^+(\lambda) + u_a^-(\lambda), \tag{1.19}$$

with

$$u_a^+(\lambda) = \sum_{k=-\infty}^{k=-1} u_{a,k}\lambda^{-k-1}, \quad u_a^-(\lambda) = \sum_{k=0}^{k<\infty} u_{a,k}\lambda^{-k-1}. \tag{1.20}$$

The subspaces $C^*_\pm(\mathfrak{g})$ are orthogonal to $C_\pm(\mathfrak{g})$ relative to the pairing (1.10), and $C^*_\pm(\mathfrak{g}) = (C_\mp(\mathfrak{g}))^*$. The Lie-Poisson bracket (1.11) has a natural restriction to these subspaces.

The resulting Poisson structure on $C^*_\pm(\mathfrak{g})$ has an elegant expression in terms of the generating functions $u_a^\pm(\lambda)$. Namely, multiply both sides of (1.11) by $\lambda^{-k-1}\mu^{-l-1}$ and sum over $k, l < 0$ and $k, l \geq 0$. Then

$$\{u_a^\pm(\lambda), u_b^\pm(\mu)\} = \mp C_{ab}^c \frac{u_c^\pm(\lambda) - u_c^\pm(\mu)}{\lambda - \mu}. \tag{1.21}$$

The corresponding expression for the Poisson bracket $\{u_a^+(\lambda), u_b^-(\mu)\}$ is not so elegant. Fortunately, we shall not need it because we are going to define a new Poisson structure on the phase space $C^*(\mathfrak{g})$. To do so, we use the decomposition (1.16) to define on the vector space $C(\mathfrak{g})$ a new structure of Lie algebra with commutator $[\,,]_0$ by setting

$$[\xi_+, \eta_+]_0 = [\xi_+, \eta_+], \quad [\xi_-, \eta_-]_0 = -[\xi_-, \eta_-] \tag{1.22}$$

and

$$[\xi_+, \eta_-]_0 = 0, \tag{1.23}$$

where $\xi=\xi_+ +\xi_-$, $\eta=\eta_+ +\eta_-$ are elements of $C(\mathfrak{g})$. By introducing the operator

$$R=\tfrac{1}{2}(P_+ - P_-), \tag{1.24}$$

where $P_\pm$ are the projection operators onto $C_\pm(\mathfrak{g})$, $P_+P_- = P_-P_+ = 0$, (1.22)-(1.23) may be combined into a single formula

$$[\xi, \eta]_0 = [R\xi, \eta] + [\xi, R\eta]. \tag{1.25}$$

The infinite-dimensional Lie algebra with the commutator $[\,,]_0$ defined above will be denoted by $C_0(\mathfrak{g})$. This algebra will in fact play a key role in the classification of integrable models.

The corresponding Lie-Poisson brackets $\{,\}_0$ on the phase space $C^*(\mathfrak{g})$ are given by (1.21) without the $\pm$ sign on the right hand side, and

$$\{u_a^+(\lambda), u_b^-(\mu)\} = 0. \tag{1.26}$$

We shall now unite (1.21) and (1.26) into a single elegant formula under the assumption that $\mathfrak{g}$ has a nondegenerate symmetric bilinear form $\langle,\rangle$ invariant under the adjoint action. For example, if $\mathfrak{g}$ is semi-simple, the Killing form may be taken as $\langle,\rangle$.

Consider a nondegenerate matrix K with entries

$$K_{ab} = \langle X_a, X_b\rangle. \tag{1.27}$$

(If $\mathfrak{g}$ is semi-simple and represented as a matrix algebra, we may assume $K_{ab} = \operatorname{tr} X_a X_b$). Let K^{ab} denote the entries of the inverse matrix K^{-1}. Let an element Π of $\mathfrak{g}\otimes\mathfrak{g}$ and elements A^a of $\mathfrak{g}$ be defined by

$$\Pi = K^{ab} X_a \otimes X_b, \tag{1.28}$$

$$A^a = K^{ab} X_b. \tag{1.29}$$

Then we have the relations

$$[\Pi, A^c \otimes I] = -[\Pi, I\otimes A^c] = C^c_{ab} A^a \otimes A^b, \tag{1.30}$$

where the symbols $A\otimes I$ and $I\otimes A$ denote the natural imbeddings of A into $\mathfrak{g}\otimes\mathfrak{g}$. To derive these relations one must use, besides (1.1), the skew-symmetry of the structure constants: the tensor $C^{abc} = K^{aa'}K^{bb'}C^c_{a'b'}$ is totally skew-symmetric, which follows from the invariance of $\langle,\rangle$.

Using Π and A^a we can define an element $r(\lambda)$ of $\mathfrak{g}\otimes\mathfrak{g}$,

$$r(\lambda) = \frac{\Pi}{\lambda}, \tag{1.31}$$

and a formal Laurent series $U(\lambda)$ with coefficients in $\mathfrak{g}^* \otimes \mathfrak{g}$,

$$U(\lambda) = u_a(\lambda) A^a. \tag{1.32}$$

The Lie-Poisson brackets (1.21)–(1.26) associated with $C_0(\mathfrak{g})$ can be written in these terms as a single formula

$$\{U(\lambda) \underset{,}{\otimes} U(\mu)\}_0 = [r(\lambda - \mu), U(\lambda) \otimes I + I \otimes U(\mu)], \tag{1.33}$$

where on the right hand side we have used the natural notation $\{\underset{,}{\otimes}\}_0$ (see § III.1, Part I). Formula (1.33) results from (1.21) upon multiplying by $A^a \otimes A^b$ and using (1.30). From here up to § 4 we shall only deal with the Poisson brackets $\{,\}_0$, where for notational simplicity the index 0 will be dropped.

It is instructive to compare the Lie-Poisson brackets for the Lie algebra $C_0(\mathfrak{g})$ given by (1.33) with the fundamental Poisson brackets for the continuous models of § III.1, Part I, and of §§ II.3, II.6, II.8. These expressions are practically identical; a formal difference is that (1.33) does not contain the spatial variable x. The dependence on x can easily be assimilated in our treatment by considering a direct product of the algebras $C(\mathfrak{g})$ over all x. In a more formal way, one should use the current algebra $\mathscr{C}((\mathfrak{g}))$ that consists of Laurent series $\xi(\lambda, x)$ with coefficients depending on x and satisfying certain boundary conditions (e.g., periodic or rapidly decreasing). The algebra $\mathscr{C}((\mathfrak{g}))$ has generators $X_{a,k}(x)$ with the commutator

$$[X_{a,k}(x), X_{b,l}(y)] = C_{ab}^c X_{c,k+l}(x) \delta(x-y). \tag{1.34}$$

Reproducing the above arguments in the context of $\mathscr{C}((\mathfrak{g}))$, we come down to the *Lie-Poisson brackets on the phase space* $\mathscr{C}^*((\mathfrak{g}))$

$$\{U(x,\lambda) \underset{,}{\otimes} U(y,\mu)\} = [r(\lambda-\mu), U(x,\lambda) \otimes I + I \otimes U(x,\mu)] \delta(x-y), \tag{1.35}$$

which have exactly the same structure as the fundamental Poisson brackets for continuous models.

Nevertheless, (1.35) has a different content than, say, (II.3.8). Thus, the latter deals with a particular matrix $U(x, \lambda)$ in the auxiliary space, which is a rational function of the spectral parameter λ, whereas (1.35) involves a formal Laurent series with coefficients in a given Lie algebra $\mathfrak{g}$. The agreement is reached by noticing that the fundamental Poisson brackets for a particular model are a realization of the Poisson brackets (1.35) in a particular matrix representation of $\mathfrak{g}$ (the representation space playing the role of auxiliary space), with a further restriction to an orbit of the associated algebra

$\mathscr{C}_0((\mathfrak{g}))$ in the phase space $\mathscr{C}^*((\mathfrak{g}))$. Here, for shortness, we speak of the orbits of a Lie algebra having in mind the abovementioned integral submanifolds for the distribution induced by the coadjoint action. The orbits in question are products over x of the orbits of the coadjoint action of the Lie algebra $C_0(\mathfrak{g})$ in $C^*(\mathfrak{g})$. Spatial homogeneity requires these orbits to be the same for all x. We shall therefore suppress the dependence on x in what follows.

For the applications to integrable models, we are mostly interested in finite-dimensional orbits of $C_0(\mathfrak{g})$. (Notice that generic orbits have infinite dimension.) To determine them, it is convenient to specify finite-dimensional Poisson submanifolds of $C^*(\mathfrak{g})$ by imposing constraints on the coordinates $u_{a,k}$ (or on their generating functions $u_a(\lambda)$, $U(\lambda)$) invariant under the coadjoint action of $C_0(\mathfrak{g})$.

Clearly, the simplest example of such a Poisson submanifold is the subspace $C^*_{N,M}$ of $C^*(\mathfrak{g})$ defined by

$$u_{a,k}=0 \tag{1.36}$$

if $k\geqslant N$ or $k\leqslant -M-1$, with $N, M\geqslant 0$. The action of $C_0(\mathfrak{g})$ on $C^*_{N,M}$ reduces to the action of a finite-dimensional Lie algebra $C_{N,M}(\mathfrak{g})$ with generators $X_{a,k}$, $-M\leqslant k<N$, and the commutator

$$[X_{a,k}, X_{b,l}]=\begin{cases} C^c_{ab}X_{c,k+l} & \text{if } k,l\geqslant 0, k+l<N, \\ -C^c_{ab}X_{c,k+l} & \text{if } k,l<0, -M-1<k+l, \\ 0 & \text{otherwise}. \end{cases} \tag{1.37}$$

The orbits of this algebra in $C^*_{N,M}$ are the required phase spaces associated with integrable models. In more detail, coordinates in $C^*_{N,M}$ are given by the set $\{u_{a,k}, k=-M,\dots,N-1\}$; their Lie-Poisson brackets are

$$\{u_{a,k}, u_{b,l}\}=\begin{cases} -C^c_{ab}u_{c,k+l} & \text{if } k,l\geqslant 0, k+l<N, \\ C^c_{ab}u_{c,k+l} & \text{if } k,l<0, -M-1<k+l, \\ 0 & \text{otherwise}, \end{cases} \tag{1.38}$$

and Poisson submanifolds are unions of orbits of the Lie algebra $C_{N,M}(\mathfrak{g})$. The problem of specifying such orbits is a finite-dimensional one and can be solved by traditional methods, for instance, by fixing the values of Casimir functions.

The generating function for the $u_{a,k}$ (or $u_{a,k}(x)$ upon restoring the x-dependence) is now a rational function of λ,

$$U(\lambda)=\sum_{k=-M}^{N-1} u_{a,k}A^a\lambda^{-k-1}, \tag{1.39}$$

and if a representation of $\mathfrak{g}$ is chosen, it is a matrix in the representation space. This is what gives the matrix $U(x,\lambda)$ of the auxiliary linear problem for the integrable models to be discussed.

Let us consider several examples.

1. $N=1$, $M=0$.

The corresponding $U(\lambda)$ has the form

$$U(\lambda)=\frac{U_0}{\lambda}=\frac{S_a A^a}{\lambda}, \tag{1.40}$$

where the S_a are dynamical variables on $C^*_{1,0}=\mathfrak{g}^*$ with the Poisson brackets

$$\{S_a, S_b\}=-C^c_{ab}S_c. \tag{1.41}$$

In the simplest case $\mathfrak{g}=su(2)$, there are three dynamical variables S_a, $a=1,2,3$. The fundamental representation of $su(2)$ with generators $X_a=\frac{1}{2i}\sigma_a$, structure constants $C_{abc}=\varepsilon_{abc}$, and matrices $A^a=i\sigma_a$ leads to

$$U(\lambda)=\frac{iS_a\sigma_a}{\lambda}, \tag{1.42}$$

$$r(\lambda)=\frac{1}{2\lambda}\sigma_a\otimes\sigma_a. \tag{1.43}$$

The corresponding orbits are determined by

$$S_1^2+S_2^2+S_3^2=\text{const}. \tag{1.44}$$

The dynamical variables S_a satisfying (1.44) with the Poisson brackets (1.41) were used in the description of the phase space for the HM model in § I.1. The matrix $U(x,\lambda)$ (1.42) and the r-matrix (1.43) turn into their counterparts from § II.3 upon a change $\lambda\to-\frac{2}{\lambda}$ if one recalls that the permutation matrix P in $\mathbb{C}^2\otimes\mathbb{C}^2$ is given by

$$P=\tfrac{1}{2}(I\otimes I+\sigma_a\otimes\sigma_a) \tag{1.45}$$

and drops the irrelevant summand proportional to $I\otimes I$.

For an arbitrary semi-simple Lie algebra $\mathfrak{g}$ the above example leads to an integrable generalization of the HM model, the $\mathfrak{g}$-invariant magnet.

2. $N=0$, $M=2$.

The corresponding $U(\lambda)$ has the form

$$U(\lambda)=U_{-1}+\lambda\, U_{-2}=Q_a A^a+\lambda\, J_a A^a\,, \tag{1.46}$$

where the Q_a, J_a are dynamical variables on the phase space $C^*_{0,2}$ with the Poisson brackets

$$\{Q_a, Q_b\}=C^c_{ab}\, J_c\,, \qquad \{Q_a, J_b\}=\{J_a, J_b\}=0\,. \tag{1.47}$$

We consider the simplest cases $\mathfrak{g}=su(2)$ or $\mathfrak{g}=su(1,1)$ in the fundamental representation with generators $X_a=\frac{1}{2i}\sigma_a$, $a=1,2,3$, and $X_1=\frac{1}{2}\sigma_1$, $X_2=\frac{1}{2}\sigma_2$, $X_3=\frac{1}{2i}\sigma_3$ respectively. The orbit in question is specified by

$$J_1=J_2=0\,, \qquad J_3=\frac{\varepsilon}{2}\,, \tag{1.48}$$

$$Q_3=0\,, \qquad Q_1+iQ_2=\psi\,, \tag{1.49}$$

with $\varepsilon=-1$ for $\mathfrak{g}=su(2)$ and $\varepsilon=1$ for $\mathfrak{g}=su(1,1)$; the only nonvanishing Poisson bracket is

$$\{\psi, \bar{\psi}\}=i\,. \tag{1.50}$$

As a result, $U(\lambda)$ can be written as

$$U(\lambda)=-\frac{\varepsilon\lambda}{2i}\sigma_3+\sqrt{\varepsilon}\begin{pmatrix}0 & \bar{\psi}\\ \psi & 0\end{pmatrix}, \tag{1.51}$$

and after the replacement $\lambda\to-\varepsilon\lambda$ turns into the matrix $U(x,\lambda)$ for the NS model (see § I.2, Part I) with $\varkappa=\varepsilon$. The matrix

$$r(\lambda)=\frac{1}{\lambda}\left(P-\frac{I\otimes I}{2}\right), \tag{1.52}$$

is carried by this replacement into the r-matrix for the NS model of § III.1, Part I (up to an irrelevant unit summand).

Other Lie algebras $\mathfrak{g}$ will give vector and matrix generalization of the NS model.

Thus the general scheme outlined above not only includes the two principal models of the book, but also provides their nontrivial generalizations. We have seen that the NS and HM models are indeed the simplest ones in an infinite sequence of examples: the Lie algebra $\mathfrak{g}$, the integers N and $M\geqslant 0$ and the orbit of the Lie algebra $C_{N,M}(\mathfrak{g})$ may be taken arbitrarily. The auxiliary linear problem with matrix $U(x,\lambda)$ of the form (1.39) and the r-matrix

(1.31) give rise to a zero curvature representation for the corresponding Hamilton equations of motion. A Lie-algebraic interpretation of the associated Hamiltonians and a scheme for solving the equations of motion will be given in § 4.

Here we point out that the above examples do not exhaust all relevant finite-dimensional phase spaces (at fixed x). Let us exhibit another family of interesting examples. To start with, we notice that the Poisson brackets (1.41) may be derived from (1.33) by inserting $U(\lambda)$ as given by (1.40). It does not matter that the pole of $U(\lambda)$ is at $\lambda=0$; the replacement

$$U(\lambda)=\frac{S_a A^a}{\lambda-c} \tag{1.53}$$

leads to the same result. Such a $U(\lambda)$ belongs to $C^*_+(\mathfrak{g})$, and the corresponding coefficients $u_{a,k}$ are nonzero for all $k\leqslant 0$ and are related by

$$u_{a,k-1}=\frac{1}{c}\,u_{a,k}\,, \tag{1.54}$$

which follows from expanding $(\lambda-c)^{-1}$ into a geometric progression. The latter relations are invariant under the coadjoint action of $C_0(\mathfrak{g})$. The same is true for

$$U(\lambda)=\sum_{i=1}^{N}\sum_{k=0}^{n_i}\frac{S^{(i)}_{a,k}A^a}{(\lambda-c_i)^{k_i+1}}\,. \tag{1.55}$$

So, generating functions of the type (1.55) form a Poisson submanifold in $C^*(\mathfrak{g})$ parametrized by the coordinates $S^{(i)}_{a,k}$. In these coordinates the Poisson brackets (1.33) become

$$\{S^{(i)}_{a,k}, S^{(j)}_{b,l}\}=\begin{cases}-C^c_{ab}\,\delta^{ij}S^{(i)}_{c,k+l} & \text{if } k+l\leqslant n_i\,,\\ 0 \quad \text{otherwise}\end{cases} \tag{1.56}$$

and are just the Lie-Poisson brackets of a finite-dimensional Lie algebra, the direct sum of the algebras $C_{n_i+1,0}(\mathfrak{g})$ over all poles.

We have already come across the matrices $U(x,\lambda)$ of the form (1.55) in §§ I.6–I.7 when studying the general solution of the zero curvature equation. Here they have appeared as a result of general Lie-algebraic considerations, and the related integrable models have been endowed with a natural Poisson structure.

Thus, in this section we outlined a general construction of the matrices $U(x,\lambda)$ satisfying the fundamental Poisson brackets with the r-matrix (1.31) and explained their geometric origin. By repeating the reasoning of Part I, with every $U(x, \lambda)$ one can associate a family of integrable models. Specifically, consider the auxiliary linear problem

$$\frac{dF}{dx} = U(x, \lambda) F \tag{1.57}$$

and its monodromy matrix

$$T(\lambda) = \overset{\curvearrowleft}{\exp} \int_{-L}^{L} U(x, \lambda) dx \tag{1.58}$$

(where for definiteness the periodic boundary conditions are assumed). The functionals $\operatorname{tr} T(\lambda)$ and other algebraic invariants of $T(\lambda)$ form an involutive family, and Hamilton's equations of motion induced by any functional of this family admit a zero curvature representation. The geometric meaning of these constructions will be clarified in § 4.

§ 2. Trigonometric and Elliptic r-Matrices and the Related Fundamental Poisson Brackets

In the preceding section, for any semi-simple Lie algebra we introduced an r-matrix,

$$r(\lambda) = \frac{\Pi}{\lambda} = \frac{K^{ab} X_a \otimes X_b}{\lambda}. \tag{2.1}$$

(Here we take the liberty of using the term r-matrix also for an element $r(\lambda)$ of $\mathfrak{g} \otimes \mathfrak{g}$). This is an extension of the r-matrix for the NS and HM models (see § III.1 of Part I and § II.3) that corresponds to the Lie algebra $\mathfrak{g} = su(2)$ and has the form $\frac{P}{\lambda}$ where P is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Still, in other cases such as the SG or LL model (see § II.6 and § II.8) we encounter more complicated r-matrices which depend on the spectral parameter λ through trigonometric or elliptic functions, respectively. It is natural to call (2.1) a *rational* r-matrix and consider the r-matrices of the SG or LL model as examples of *trigonometric* or *elliptic* r-matrices. In § 1 we saw that rational r-matrices determine the structure of the Lie algebra $C_0(\mathfrak{g})$. Now the problem is to describe trigonometric and elliptic r-matrices and find their geometric interpretation, which will be our concern in this section.

We begin by constructing a vast family of such r-matrices. The basic functional equations are

$$r_{12}(-\lambda) = -r_{21}(\lambda) \tag{2.2}$$

and

$$[r_{12}(\lambda-\mu), r_{13}(\lambda)+r_{23}(\mu)]+[r_{13}(\lambda), r_{23}(\mu)] = 0, \tag{2.3}$$

which ensure the skew-symmetry and the Jacobi identity for the fundamental Poisson brackets (see § III.1, Part I). The subscripts 12, 21, 13, 23 indicate a specific imbedding of r from $\mathfrak{g}\otimes\mathfrak{g}$ into $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ (cf. the analogous matrix notation in § III.1, Part I). Obviously, these relations hold for the r-matrix (2.1), with (2.3) being equivalent to the Jacobi identity for the structure constants C^c_{ab}.

The remarkable property of (2.3) is that *it allows averaging over a lattice in the complex* λ*-plane.* More precisely, let θ be an automorphism of a semisimple Lie algebra $\mathfrak{g}$ of finite order q, $\theta^q = I$, and let $\Lambda_1 = \{n\omega, -\infty < n < \infty\}$ be a one-dimensional lattice in $\mathbb{C}$ with generator ω. Let the action of the additive group of translations Λ_1 on the r-matrix (2.1) be defined by

$$r(\lambda) \rightarrow r^{(n)}(\lambda) = (\theta^n \otimes I)r(\lambda - n\omega) = (I \otimes \theta^{-n})r(\lambda - n\omega). \tag{2.4}$$

The last equation in (2.4) reflects the invariance of the r-matrix (2.1) under the diagonal action of θ,

$$(\theta \otimes \theta)r(\lambda) = r(\lambda), \tag{2.5}$$

which is an obvious consequence of

$$(\theta \otimes \theta)\Pi = \Pi. \tag{2.6}$$

Let

$$r^{\Lambda_1}(\lambda) = \sum_{n=-\infty}^{\infty} r^{(n)}(\lambda) \tag{2.7}$$

be the result of averaging the r-matrix (2.1) over Λ_1. The averaged r-matrix, $r^{\Lambda_1}(\lambda)$, is quasi-periodic

$$r^{\Lambda_1}(\lambda+\omega) = (\theta \otimes I)r^{\Lambda_1}(\lambda) = (I \otimes \theta^{-1})r^{\Lambda_1}(\lambda), \tag{2.8}$$

satisfies (2.2) and at first sight also seems to satisfy (2.3). In fact, replace λ in (2.3) by $\lambda - n\omega$, μ by $\mu - m\omega$, and act on the right hand side by the automorphism $\theta^n \otimes \theta^m \otimes I$. Using (2.5), we find

$$[r_{12}^{(n-m)}(\lambda-\mu), r_{13}^{(n)}(\lambda)+r_{23}^{(m)}(\mu)]+[r_{13}^{(n)}(\lambda), r_{23}^{(m)}(\mu)]=0, \tag{2.9}$$

so the relation (2.3) for $r^{\Lambda_1}(\lambda)$ results from summing over n and m.

However, the argument is too naive and in general incorrect. The point is that the series (2.7) converges only as a principal value series,

$$\text{p.v.} \sum_{n=-\infty}^{\infty} = \lim_{N\to\infty} \sum_{n=-N}^{N}, \tag{2.10}$$

so that one cannot replace summation over n and m by summation over $n-m$ and m or over $n-m$ and n.

Let us see what the requirements on θ are in order that (2.7) *do satisfy* (2.3). Using

$$\text{p.v.} \sum_{n=-\infty}^{\infty} \frac{1}{\lambda-n\omega} = \frac{\pi}{\omega} \operatorname{ctg} \frac{\pi\lambda}{\omega}, \tag{2.11}$$

we find for (2.7) the expression

$$r^{\Lambda_1}(\lambda) = \frac{\pi}{q\omega} \sum_{k=0}^{q-1} \operatorname{ctg} \frac{\pi(\lambda-k\omega)}{q\omega} (\theta^k \otimes I)\Pi, \tag{2.12}$$

so that $r^{\Lambda_1}(\lambda)$ is indeed quasi-periodic in the sense of (2.8).

We now proceed with (2.3) and denote its left hand side by $\Phi(\lambda,\mu)$:

$$\Phi(\lambda,\mu)=[r_{12}^{\Lambda_1}(\lambda-\mu), r_{13}^{\Lambda_1}(\lambda)+r_{23}^{\Lambda_1}(\mu)]+[r_{13}^{\Lambda_1}(\lambda), r_{23}^{\Lambda_1}(\mu)]. \tag{2.13}$$

Consider $\Phi(\lambda,\mu)$ as a function of λ for μ fixed, $\mu\not\equiv 0 (\operatorname{mod}\Lambda_1)$. It satisfies the quasi-periodicity condition

$$\Phi(\lambda+\omega,\mu)=(\theta\otimes I\otimes I)\Phi(\lambda,\mu) \tag{2.14}$$

and may have only simple poles at $\lambda\equiv\mu(\operatorname{mod}\Lambda_1)$ and $\lambda\equiv 0(\operatorname{mod}\Lambda_1)$. We will show that $\Phi(\lambda,\mu)$ is an entire function of λ. In fact, the residue at $\lambda=\mu$ is $[\Pi_{12}, r_{13}^{\Lambda_1}(\mu)+r_{23}^{\Lambda_1}(\mu)]$ and vanishes in view of

$$[\Pi, A\otimes I+I\otimes A]=0 \tag{2.15}$$

(see (1.30)). The case $\lambda=0$ is treated in a similar way; here one should also make use of (2.2). Next, $\Phi(\lambda,\mu)$ is bounded, so that the Liouville theorem yields

$$\Phi(\lambda,\mu)=\Phi(\pm i\infty,\mu) = -\frac{\pi^2}{\omega^2}[\mathscr{P}_{12},\mathscr{P}_{23}] \pm \frac{\pi i}{\omega}[\mathscr{P}_{12}+\mathscr{P}_{13}, r_{23}^{\Lambda_1}(\mu)], \tag{2.16}$$

with

$$\mathscr{P}=\frac{1}{q}\sum_{k=0}^{q-1}(\theta^k\otimes I)\Pi. \tag{2.17}$$

This implies

$$[\mathscr{P}_{12}+\mathscr{P}_{13}, r_{23}^{\Lambda_1}(\mu)]=0 \tag{2.18}$$

and

$$\Phi(\lambda,\mu) = -\frac{\pi^2}{\omega^2}[\mathscr{P}_{12},\mathscr{P}_{23}]. \tag{2.19}$$

Thus we have shown that (2.12) satisfies (2.3) if

$$[\mathscr{P}_{12},\mathscr{P}_{23}]=0. \tag{2.20}$$

This is the desired necessary condition on θ. As is easily seen, this is equivalent to

$$[\tilde{X}_a,\tilde{X}_b]=0 \tag{2.21}$$

for any generators X_a, X_b, where $\tilde{X}$ denotes the average $\tilde{X}=\frac{1}{q}(I+\theta+\ldots+\theta^{q-1})X$ invariant under θ. *Condition* (2.21) *amounts to saying that the subalgebra* $\mathfrak{h}$ *of fixed points of* θ *is abelian.*

We thus obtain a new family of r-matrices of the type (2.12) parametrized by a one-dimensional lattice Λ_1 and an automorphism θ of finite order whose fixed subalgebra is abelian. Formula (2.12) shows that it is natural to refer to such r-matrices as *trigonometric.*

Another family of r-matrices is obtained by averaging over a two-dimensional lattice $\Lambda_2=\left\{n_1\omega_1+n_2\omega_2;\ \operatorname{Im}\frac{\omega_2}{\omega_1}>0,\ n_1,n_2=-\infty,\ldots,\infty\right\}$. Let θ_1 and θ_2 be automorphisms of $\mathfrak{g}$ of order q_1 and q_2, respectively, and let $r^{\Lambda_2}(\lambda)$ be the averaged r-matrix

$$r^{\Lambda_2}(\lambda)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}(\theta_1^{n_1}\theta_2^{n_2}\otimes I)r(\lambda-n_1\omega_1-n_2\omega_2). \tag{2.22}$$

In order for the "naive proof" of (2.3) based on (2.9) to be valid, these automorphisms must commute. However, the series (2.22) should be regarded in the sense of (2.10). By repeating the above derivation of (2.3) which uses the Liouville theorem it can be shown that θ_1 and θ_2 must have no fixed points in common. Such pairs of automorphisms exist only for Lie algebras of type A_{n-1}, i.e. for $\mathfrak{g}=su(n)$, and then $q_1=q_2=n$. In the fundamental (vector) representation of $su(n)$ we have, up to an inner automorphism,

$$\theta_i \xi = T_i \xi T_i^{-1}, \qquad i=1,2, \tag{2.23}$$

where T_1 and T_2 are $n\times n$ matrices with entries

$$(T_1)_{kl}=\zeta^k \delta_{kl}, \qquad (T_2)_{kl}=\delta_{k+1,l} \tag{2.24}$$

and ξ is a primitive n-th root of unity, $k,l=1,\dots,n$, $\delta_{k+n,l}=\delta_{k,l+n}=\delta_{k,l}$.

So the corresponding r-matrix may be fully characterized as a meromorphic matrix function with values in the complexification of $su(n)\times su(n)$, i.e. in $sl(n,\mathbb{C})\times sl(n,\mathbb{C})$, satisfying the quasi-periodicity conditions

$$r^{\Lambda_2}(\lambda+\omega_i)=(T_i\otimes I)\,r^{\Lambda_2}(\lambda)(T_i^{-1}\otimes I), \qquad i=1,2 \tag{2.25}$$

and the requirement

$$r^{\Lambda_2}(\lambda)=\frac{\Pi}{\lambda}+O(1) \tag{2.26}$$

as $\lambda\to 0$. Its matrix entries are elliptic functions with period lattice $n\Lambda_2$ and simple poles at the points of Λ_2. The r-matrices of this kind are naturally called *elliptic*.

We shall consider the simplest examples that correspond to the Lie algebra $su(2)$ in the fundamental representation. Let the generator of the one-dimensional lattice Λ_1 be $\omega=\pi$ and let an automorphism θ be defined by

$$\theta\xi=\sigma_3\xi\sigma_3. \tag{2.27}$$

As follows from (2.12), the associated trigonometric r-matrix is

$$r(\lambda)=\frac{1}{2\sin\lambda}(\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_2+\cos\lambda\,\sigma_3\otimes\sigma_3). \tag{2.28}$$

This r-matrix has already occurred in the study of the partially anisotropic HM model and the SG model (see § II.8 and § II.6; in the latter case λ must be replaced by $i\alpha$).

Let the generators of the two-dimensional lattice Λ_2 be $\omega_1=2K$, $\omega_2=2iK'$ where K and K' are the complete elliptic integrals of moduli k and $k'=\sqrt{1-k^2}$, respectively; we set

$$T_1=\sigma_3, \qquad T_2=\sigma_1. \tag{2.29}$$

The associated elliptic r-matrix is

$$r(\lambda)=\frac{1}{2\,\mathrm{sn}\,(\lambda,k)}(\sigma_1\otimes\sigma_1+\mathrm{dn}\,(\lambda,k)\sigma_2\otimes\sigma_2+\mathrm{cn}\,(\lambda,k)\sigma_3\otimes\sigma_3), \tag{2.30}$$

which can be verified by summing the series (2.22). Of course, relations (2.25)–(2.26) follow immediately from the expression for $r(\lambda)$. This r-matrix occurred in the description of the LL model (see § II.8).

We thus have shown that all the r-matrices that appear in specific models are contained in one of the three families described above: the rational r-matrices having a Lie-algebraic interpretation and trigonometric and elliptic ones resulting from rational r-matrices by averaging.

Each of these r-matrices gives rise to the fundamental Poisson brackets

$$\{U(\lambda)\underset{,}{\otimes}U(\mu)\}=[r(\lambda-\mu),\,U(\lambda)\otimes I+I\otimes U(\mu)], \tag{2.31}$$

where we have again suppressed the x-dependence. The element $U(\lambda)$ of $\mathfrak{g}$ must satisfy the quasi-periodicity conditions

$$U(\lambda+\omega)=\theta\,U(\lambda) \tag{2.32}$$

in the trigonometric case and

$$U(\lambda+\omega_i)=\theta_i\,U(\lambda), \qquad i=1,2, \tag{2.33}$$

in the elliptic case. *A natural method for constructing such a $U(\lambda)$ is to apply the averaging procedure to elements $U(\lambda)$ which belong to a finite dimensional phase space in the rational case.* A large variety of examples of this kind was exhibited in § 1. However, the requirement of convergence of the corresponding series imposes additional constraints on the choice of a rational $U(\lambda)$.

The most representative example is provided by

$$U(\lambda)=\sum_{i=1}^{N}\sum_{k=0}^{n_i}\frac{S^{(i)}_{a,k}A^a}{(\lambda-c_i)^{k+1}}. \tag{2.34}$$

Setting for such $U(\lambda)$

$$U^{A_1}(\lambda)=\sum_{n=-\infty}^{\infty}\theta^n\,U(\lambda-n\omega) \tag{2.35}$$

in the trigonometric case and

$$U^{\Lambda_2}(\lambda)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\theta_1^{n_1}\theta_2^{n_2}\,U(\lambda-n_1\omega_1-n_2\omega_2) \tag{2.36}$$

in the elliptic case (with the summation convention (2.10)) we find that both $U^{\Lambda_1}(\lambda)$ and $U^{\Lambda_2}(\lambda)$ satisfy the fundamental Poisson brackets with the r-matrices $r^{\Lambda_1}(\lambda)$ and $r^{\Lambda_2}(\lambda)$ respectively. In fact, setting

$$U^{(n)}(\lambda)=\theta^n\,U(\lambda-n\omega) \tag{2.37}$$

we get from (2.31)

$$\{U^{(n)}(\lambda)\underset{,}{\otimes} U^{(m)}(\mu)\}=[r^{(n-m)}(\lambda-\mu),\,U^{(n)}(\lambda)\otimes I+I\otimes U^{(m)}(\mu)], \tag{2.38}$$

and summing over n and m we conclude that $U^{\Lambda_1}(\lambda)$ satisfies the fundamental Poisson brackets with the r-matrix $r^{\Lambda_1}(\lambda)$. For a rigorous proof one should compare the poles of the left and right hand sides of the fundamental Poisson brackets and make use of the Liouville theorem. The elliptic case is analyzed in a similar manner.

The simplest illustration of this construction is the matrix $U^{\mathrm{LL}}(\lambda)$ for the LL model obtained by averaging the corresponding matrix $U^{\mathrm{HM}}(\lambda)$ for the HM model,

$$U^{\mathrm{HM}}(\lambda)=\frac{iS_a\sigma_a}{\lambda} \tag{2.39}$$

(see (1.42)). We have

$$\begin{aligned}
U^{\mathrm{LL}}(\lambda)&=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sigma_3^{n_1}\sigma_1^{n_2}\,U^{\mathrm{HM}}(\lambda-2n_1K-2in_2K')\sigma_1^{n_2}\sigma_3^{n_1}\\
&=i\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{(-1)^{n_1}}{\lambda-2n_1K-2in_2K'}S_1\sigma_1\\
&\quad+i\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{(-1)^{n_1+n_2}}{\lambda-2n_1K-2in_2K'}S_2\sigma_2\\
&\quad+i\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{(-1)^{n_2}}{\lambda-2n_1K-2in_2K'}S_3\sigma_3\\
&=\frac{i}{\operatorname{sn}(\lambda,k)}\,(S_1\sigma_1+\operatorname{dn}(\lambda,k)S_2\sigma_2+\operatorname{cn}(\lambda,k)S_3\sigma_3).
\end{aligned} \tag{2.40}$$

In the trigonometric case we obviously obtain the matrix $U(\lambda)$ of the partially anisotropic HM model.

The matrix $U(\alpha)$ of the SG model

$$U(\alpha)=\operatorname{ch}\alpha\, S_1\sigma_1+\operatorname{sh}\alpha\, S_2\sigma_2+S_3\sigma_3, \tag{2.41}$$

where

$$S_1=\frac{m}{2i}\sin\frac{\beta\varphi}{2}, \qquad S_2=\frac{m}{2i}\cos\frac{\beta\varphi}{2}, \qquad S_3=\frac{\beta\pi}{4i}, \tag{2.42}$$

is an example of a quasi-periodic matrix $U(\alpha)$ (with $\omega=i\pi$ and θ given by (2.27)) satisfying the fundamental Poisson brackets with the r-matrix (2.28) (with $\lambda=i\alpha$), which cannot be obtained by a straightforward averaging procedure. It can be derived, however, by contracting the matrix $U^{\Lambda_1}(\lambda)$, which in turn is obtained by averaging a double-pole matrix $U(\lambda)$,

$$U(\lambda)=\frac{S_a^{(1)}\sigma_a}{\lambda-c}+\frac{S_a^{(2)}\sigma_a}{\lambda+c}. \tag{2.43}$$

We may therefore claim that the averaging procedure enables us to build up a classification of continuous models with finite number of degrees of freedom for fixed x, described by trigonometric or elliptic r-matrices, if contraction of phase space is also allowed.

We thus have a scheme for constructing the matrix $U(x,\lambda)$ of the auxiliary linear problem satisfying the fundamental Poisson brackets with an r-matrix from the rational, trigonometric, or elliptic families. In the rational case $U(x,\lambda)$ ranges over an orbit (of finite dimension for fixed x) of the coadjoint action of the Lie algebra $\mathscr{C}((\mathfrak{g}))$. Hence the above scheme may be regarded as a method for classifying the associated integrable models. In the trigonometric and elliptic cases we have outlined an averaging procedure which enables us to produce a large family of quasi-periodic matrices $U(x,\lambda)$.

In the next section we shall extend the discussion to lattice models.

§ 3. Fundamental Poisson Brackets on the Lattice

Our analysis of lattice models in Chapter III was based on the auxiliary linear problem

$$F_{n+1}=L_n(\lambda)F_n \tag{3.1}$$

with the matrix $L_n(\lambda)$ satisfying the fundamental Poisson brackets

$$\{L_n(\lambda) \underset{,}{\otimes} L_m(\mu)\} = [r(\lambda-\mu), L_n(\lambda) \otimes L_n(\mu)]\, \delta_{nm}. \tag{3.2}$$

For (3.2) to be compatible, the r-matrix should satisfy (2.2)–(2.3). These fundamental lattice Poisson brackets have no simple Lie-algebraic interpretation. Still, the corresponding r-matrices have been already described in connection with continuous models. We shall therefore discuss here only the choice of finite-dimensional phase spaces for fixed n, and the associated $L_n(\lambda)$. The index n will be suppressed in what follows.

Let us first consider rational r-matrices and concentrate for simplicity on the Lie algebra $\mathfrak{g}=su(N)$ in the fundamental representation. The corresponding r-matrix (2.1) (up to an irrelevant unit summand) is given by

$$r(\lambda) = \frac{P}{\lambda}, \tag{3.3}$$

where P is the permutation matrix in $\mathbb{C}^N \otimes \mathbb{C}^N$. The simplest matrix satisfying (3.2) is

$$L(\lambda) = I + \frac{S_a A^a}{\lambda}, \tag{3.4}$$

where the dynamical variables S_a have the Lie-Poisson brackets of the Lie algebra $su(N)$,

$$\{S_a, S_b\} = -C^c_{ab} S_c, \tag{3.5}$$

and may be restricted to an orbit. The verification of (3.2) is straightforward: in addition to (1.30) one should use

$$[P, A \otimes A] = 0. \tag{3.6}$$

Notice that (3.4) is the naive lattice version of the operator L of the continuous auxiliary linear problem,

$$L = \frac{d}{dx} - \frac{S_a \sigma_a}{\lambda}, \tag{3.7}$$

which appeared in the HM model for $\mathfrak{g}=su(2)$ (see § 1). Thus (3.4) provides a lattice version of the continuous $\mathfrak{g}$-invariant magnet for the case $\mathfrak{g}=su(N)$.

Since (3.2) is multiplicative, a product $L(\lambda)$ of the simplest matrices,

$$L(\lambda) = L^{(1)}(\lambda + c_1) \ldots L^{(m)}(\lambda + c_m), \tag{3.8}$$

also satisfies the fundamental Poisson brackets (3.2). Of course, the dynamical variables $S_a^{(i)}$ entering into $L^{(i)}(\lambda)$ are in involution for distinct i. Expression (3.8) is the lattice analogue of the multi-pole matrix $U(\lambda)$ in (1.55) with simple poles. Conversely, by taking various continuum limits of (3.8) one can obtain a rational matrix $U(\lambda)$ of the general type, in particular with multiple poles.

We thus have constructed a large set of matrices $L(\lambda)$ that describe integrable lattice models and are associated with rational r-matrices for the Lie algebra $\mathfrak{g}=su(N)$. The construction of $L(\lambda)$ satisfying the fundamental Poisson brackets (3.2) with rational r-matrices for other classical series of Lie algebras is more difficult and will not concern us here.

We shall now turn to the fundamental Poisson brackets (3.2) with trigonometric and elliptic r-matrices. Here we can only suggest to look for suitable combinations generalizing (3.4) and satisfying the quasi-periodicity conditions

$$L(\lambda+\omega_i)=\theta_i L(\lambda), \qquad i=1,2. \tag{3.9}$$

In the case of $\mathfrak{g}=su(2)$ and the fundamental representation, we already know the simplest matrix $L(\lambda)$: this is the one for the LLL model (see § III.5). Its trigonometric degenerate case gives the matrix $L(\lambda)$ for the partially anisotropic LHM model, or for the LSG model if one takes another real form. Similar expressions exist for $\mathfrak{g}=su(N)$ for any N, but we will not write them explicitly. Knowing the simplest $L(\lambda)$'s we may construct more complicated ones using (3.8).

Notice that the averaging procedure of § 2 introduced for continuous models can also be extented to the lattice case. Let $L(\lambda)$ satisfy the fundamental Poisson brackets (3.2) with a rational r-matrix. Setting

$$L^{(n)}(\lambda)=\theta^n L(\lambda-n\omega), \tag{3.10}$$

we can write (3.2) as

$$\{L^{(n)}(\lambda) \underset{,}{\otimes} L^{(m)}(\mu)\}=[r^{(n-m)}(\lambda-\mu), L^{(n)}(\lambda)\otimes L^{(m)}(\mu)], \tag{3.11}$$

where $r^{(n-m)}(\lambda)$ was defined in (2.4). Taking a formal product of (3.11) over all n and m we come down to relation (3.2) that involves the matrix

$$L^{\Lambda_1}(\lambda)=\prod_{n=-\infty}^{\infty} L^{(n)}(\lambda) \tag{3.12}$$

and the r-matrix $r^{\Lambda_1}(\lambda)$ given by (2.7). This method, however, needs substantial justification which is beyond the purposes of the book.

These few remarks about lattice models is all we wanted to say here. The classes of $L_n(\lambda)$ displayed above include the models discussed in the book and allow for interesting generalizations.

Thus, in this and the two previous sections, the auxiliary linear problems for integrable models and the corresponding fundamental Poisson brackets were discussed from a general standpoint. The most complete geometric interpretation was obtained for continuous models with a rational r-matrix. The associated fundamental Poisson brackets were shown to be generated by a special infinite-dimensional Lie algebra $\mathscr{C}((\mathfrak{g}))$; the corresponding continuous models have the highest degree of symmetry. Continuous models with trigonometric and elliptic r-matrices have a partially broken symmetry and were described in less detail. In particular, the analogue of the algebra $\mathscr{C}_0((\mathfrak{g}))$ was not defined. Finally, when dealing with lattice models, we in fact restricted our analysis to suitable substitutions. Nevertheless, at least for the rational r-matrix associated with $su(N)$, these substitutions in the continuum limit give the whole class of the matrices $U(x,\lambda)$ described in § 1. The role of the Lie algebra $\mathscr{C}_0((\mathfrak{g}))$ is transferred in lattice models to an object which is rather a Lie group, but a detailed discussion of this topic cannot be persued here. This brings us to the end of the exposition of our classification scheme for integrable models.

§ 4. Geometric Interpretation of the Zero Curvature Representation and the Riemann Problem Method

The Riemann problem of factorization of matrix-valued functions played a significant role in our book. Firstly, it was used to solve the inverse problem – the problem of inverting the mapping $\mathscr{F}$ (see §§ II.1–3, II.6 of Part I and §§ II.2, II.5) and so was a constituent part of the method for solving the initial value problem for integrable nonlinear equations. In particular, the zero curvature representation for these equations was a consequence of the Riemann problem formalism. Secondly, in § I.6 the Riemann problem was used to outline a method for constructing a large class of special solutions of the general zero curvature equation, the dressing procedure. In this section the method for solving the initial value problem for integrable nonlinear equations and the dressing procedure will be discussed from a general standpoint. More specifically, we shall outline a geometric scheme which gives rise to Hamilton's equations possessing a rich set of integrals of the motion in involution and admitting a zero curvature representation. The dressing procedure is incorporated in the scheme in a natural way. The key role in the scheme is played by the infinite-dimensional Lie algebra $\mathscr{C}((\mathfrak{g}))$ with two commutators $[\,,]$ and $[\,,]_0$, and its central extension.

The construction will be divided into several steps. First, in Subsection 1 we shall consider a model situation, starting with a finite-dimensional Lie algebra $\mathfrak{g}$. This will serve to introduce and illustrate the geometric techniques for constructing integrable equations and their solution via a factorization problem in the Lie group G. Then in Subsection 2 $\mathfrak{g}$ will be replaced by the infinite-dimensional Lie algebra $\mathscr{C}(\mathfrak{g})$ consisting of functions of x with values in $\mathfrak{g}$, and its central extension. This will give rise to the zero curvature representation and the monodromy matrix of the auxiliary linear problem in a natural way. The final scheme of Subsection 3 will result from replacing $\mathfrak{g}$ in Subsection 2 by the current algebra $C(\mathfrak{g})$ thus leading to the Lie algebra $\mathscr{C}((\mathfrak{g}))$. So the variables x and λ are brought into action in reverse order compared to § 1. The abstract factorization problem of Subsection 1 will become in Subsection 3 the traditional Riemann problem of analytic factorization of matrix-functions.

1. The factorization problem as a method for constructing integrable Hamiltonian equations and their solutions

Let G be a finite-dimensional Lie group such that its Lie algebra $\mathfrak{g}$ can be split into a linear sum of two subalgebras

$$\mathfrak{g}=\mathfrak{g}_+ +\mathfrak{g}_- . \tag{4.1}$$

Let $P_\pm$ be the corresponding projection operators,

$$P_\pm\,\mathfrak{g}_\pm=\mathfrak{g}_\pm\,, \qquad P_\pm\,\mathfrak{g}_\mp=0. \tag{4.2}$$

Let

$$R=\tfrac{1}{2}(P_+ -P_-) \tag{4.3}$$

and define a second Lie structure on $\mathfrak{g}$ with the commutator given by

$$[\xi,\eta]_0=[R\xi,\eta]+[\xi,R\eta], \tag{4.4}$$

where $[\,,]$ is the original commutator in $\mathfrak{g}$ (cf. § 1). On the phase space $\mathfrak{g}^*$ we consider the Lie-Poisson brackets $\{,\}$ and $\{,\}_0$ associated with the Lie brackets $[\,,]$ and $[\,,]_0$, respectively. Let $I(\mathfrak{g})$ denote the annihilator of the Poisson structure $\{,\}$, the algebra of Casimir functions consisting of invariants of the coadjoint action Ad^* of G on $\mathfrak{g}^*$: for $f(u)$ in $I(\mathfrak{g})$ we have

$$f(\mathrm{Ad}^*g\cdot u)=f(u) \tag{4.5}$$

for all g in G.

The remarkable property of the Poisson structure $\{,\}_0$ is that $I(\mathfrak{g})$ *is an involutive algebra with respect to* $\{,\}_0$.

To prove this we shall use an invariant definition of the Lie-Poisson bracket given by (1.2). For any function $f(u)$ on $\mathfrak{g}^*$ let $\nabla f(u)$ denote its gradient which is an element of $\mathfrak{g}$ defined by

$$\nabla f(u) = \frac{\partial f(u)}{\partial u_a} X_a . \tag{4.6}$$

With this notation, (1.2) can be written as

$$\{f_1, f_2\}(u) = -(u, [\nabla f_1(u), \nabla f_2(u)]) \tag{4.7}$$

where u is a point of $\mathfrak{g}^*$ at which the Poisson bracket is evaluated, and (,) is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ (see § 1). In particular, setting $f_1(u) = u_a \xi^a$ and $f_2(u) = f(u)$ with $f(u)$ in $I(\mathfrak{g})$, we find

$$(u, [\nabla f(u), \xi]) = 0 \tag{4.8}$$

for any element $\xi = \xi_a X^a$ of $\mathfrak{g}$. Now the involutive property of the algebra $I(\mathfrak{g})$ is obvious: for any functions $f_1(u)$, $f_2(u)$ in $I(\mathfrak{g})$ we have

$$\{f_1, f_2\}_0(u) = -(u, [R\,\nabla f_1(u), \nabla f_2(u)]) - (u, [\nabla f_1(u), R\,\nabla f_2(u)]) = 0, \tag{4.9}$$

since every term on the right hand side vanishes in view of (4.8).

We thus have obtained a large family of functions which are in involution with respect to the Poisson bracket $\{,\}_0$. It is natural to look at Hamilton's equations of motion defined by these functions. They have the form

$$\frac{du_a}{dt} = \{f, u_a\}_0(u), \tag{4.10}$$

where a function f from $I(\mathfrak{g})$ is the Hamiltonian. In view of (4.8) these equations may be written as

$$\frac{du_a}{dt} = -(u, [R\,\nabla f(u), \nabla u_a]) = (\mathrm{ad}^*(R\,\nabla f(u)) \cdot u)_a, \tag{4.11}$$

or

$$\left(\frac{du}{dt}, \xi\right) = -(u, [R\,\nabla f(u), \xi]) \tag{4.12}$$

for all ξ in $\mathfrak{g}$. In the semisimple case this equation can be given a more elegant form: for a point U in $\mathfrak{g}$,

$$U = u_a A^a, \tag{4.13}$$

(cf. (1.32)) it becomes

$$\frac{dU}{dt} = [R \nabla f(u), U]. \tag{4.14}$$

The representation of (4.10) in the form (4.12) or (4.14) is of crucial importance. We shall presently see that it leads to a method for solving the initial value problem $u_a(t)|_{t=0} = u_a^0$ for the nonlinear equation (4.10) in terms of a factorization problem in the Lie group G.

Let $G_\pm$ be the subgroups of G associated with the Lie subalgebras $\mathfrak{g}_\pm$. For any g in G sufficiently close to I there is a factorization

$$g = g_+ g_-, \tag{4.15}$$

with $g_\pm$ in $G_\pm$; the decomposition is unique if $g_\pm$ are also supposed to be close to the unit element of G. *The factorization problem (4.15) is the abstract analogue of the Riemann problem.*

We will show how (4.15) allows us to solve the nonlinear equation (4.10). *Consider a one-parameter subgroup of G given by $g(t) = \exp\{-t \nabla f(u^0)\}$ and the associated family of factorization problems (for t small enough)*

$$g(t) = g_+(t)\, g_-(t), \tag{4.16}$$

with $g_\pm(t)|_{t=0} = I$. The solution of the equations of motion (4.10) with initial condition u^0 is then given by

$$u(t) = \mathrm{Ad}^* g_+^{-1}(t) \cdot u^0 = \mathrm{Ad}^* g_-(t) u^0, \tag{4.17}$$

or

$$(u(t), \xi) = (u^0, g_+(t)\, \xi\, g_+^{-1}(t)) = (u^0, g_-^{-1}(t)\, \xi\, g_-(t)) \tag{4.18}$$

for all ξ in $\mathfrak{g}$.

The two representations for $u(t)$ in (4.17) or (4.18) coincide due to the relation

$$\mathrm{Ad}^* g(t) \cdot u^0 = u^0, \tag{4.19}$$

whose infinitesimal version is given by (4.8).

For the proof let us differentiate (4.16) with respect to t and write the result as

$$g_+^{-1}(t)\frac{dg}{dt}(t)g^{-1}(t)g_+(t)=g_+^{-1}(t)\frac{dg_+}{dt}(t)+\frac{dg_-}{dt}(t)g_-^{-1}(t). \tag{4.20}$$

Recalling that $\frac{dg}{dt}(t)g^{-1}(t)=-\nabla f(u^0)$ and setting

$$\xi_+(t)=g_+^{-1}\frac{dg_+}{dt}(t), \quad \xi_-(t)=\frac{dg_-}{dt}(t)g_-^{-1}(t), \tag{4.21}$$

we find

$$-g_+^{-1}(t)\nabla f(u^0)g_+(t)=\xi_+(t)+\xi_-(t). \tag{4.22}$$

It is not hard to see that the left hand side of this equation equals $-\nabla f(u(t))$. To show this one must use that $f(u)$ is Ad*-invariant, hence

$$f(u)=f(\mathrm{Ad}^* g_+^{-1}(t)u), \tag{4.23}$$

then differentiate this equation with respect to u and use (4.17). In other words, the gradient of an Ad*-invariant function "transforms by conjugation". As a result, we get the decomposition

$$-\nabla f(u(t))=\xi_+(t)+\xi_-(t), \tag{4.24}$$

so that

$$\xi_\pm(t)=-P_\pm(\nabla f(u(t))). \tag{4.25}$$

Now, differentiating (4.18) with respect to t we find

$$\begin{aligned}\left(\frac{du(t)}{dt},\xi\right)&=\left(u^0,\frac{dg_+}{dt}(t)\xi g_+^{-1}(t)\right)\\&\quad-\left(u^0,g_+(t)\xi g_+^{-1}(t)\frac{dg_+}{dt}(t)g_+^{-1}(t)\right)\\&=(u(t),[\xi_+,\xi])\end{aligned} \tag{4.26}$$

and

$$\left(\frac{du(t)}{dt},\xi\right)=-(u(t),[\xi_-,\xi]). \tag{4.27}$$

Since

$$\tfrac{1}{2}(\xi_+(t)-\xi_-(t))=-R\,\nabla f(u(t)), \tag{4.28}$$

by virtue of (4.26)–(4.27) we conclude that $u(t)$ satisfies (4.12) or (4.10).

Thus we have shown that the solution of the nonlinear equation (4.10) reduces to constructing a one-parameter subgroup $g(t)$ and solving the factorization problem (4.16). Both these problems are linear. In other words, in a model finite-dimensional situation we have explained the linearization procedure for Hamilton's equations of special form (4.10). As we have seen, it already contains the principal aspects of the inverse scattering method, i.e. the Hamiltonian structure of the equations of motion, the existence of a set of integrals of the motion in involution, and a method for solving the initial value problem by means of the factorization problem. The scheme is based, in essence, on a single formula that describes the splitting of the original Lie algebra into the sum of two its subalgebras.

2. A central extension of the Lie algebra $\mathscr{C}(\mathfrak{g})$ and the zero curvature equation

Consider the Lie algebra $\mathscr{C}(\mathfrak{g})$ of functions $\xi(x)$ with values in a finite dimensional Lie algebra $\mathfrak{g}$. We shall assume, for definiteness, that $\xi(x)$ satisfies the periodic boundary conditions

$$\xi(x+2L)=\xi(x). \tag{4.29}$$

The algebra $\mathscr{C}(\mathfrak{g})$ is spanned by the generators $X_a(x)$, $-L\leqslant x<L$, with the commutator

$$[X_a(x), X_b(y)]=C^c_{ab}X_c(x)\delta(x-y) \tag{4.30}$$

(cf. (1.34)).

We shall suppose that $\mathfrak{g}$ has an invariant bilinear form $\langle , \rangle$ and introduce a *central extension* $\tilde{\mathscr{C}}(\mathfrak{g})$ *of the Lie algebra* $\mathscr{C}(\mathfrak{g})$ *defined by the Maurer-Cartan 2-cocycle*

$$\omega(\xi,\eta)=\int_{-L}^{L}\left\langle \xi(x), \frac{d\eta}{dx}(x)\right\rangle dx. \tag{4.31}$$

The elements of $\tilde{\mathscr{C}}(\mathfrak{g})$ are representend by pairs $\tilde{\xi}=(\xi(x),\sigma)$ where $\xi(x)$ lies in $\mathscr{C}(\mathfrak{g})$ and σ is a complex number; the commutator is

$$[\tilde{\xi},\tilde{\eta}]=([\xi(x),\eta(x)],\omega(\xi,\eta)). \tag{4.32}$$

The Jacobi identity for (4.32) holds in view of the cocycle property

$$\omega([\xi,\eta],\zeta)+\omega([\zeta,\xi],\eta)+\omega([\eta,\zeta],\xi)=0, \tag{4.33}$$

which follows from (4.31) by integrating by parts and using the invariance of $\langle , \rangle$. Elements of the form $(0,\sigma)$ constitute the center of $\tilde{\mathscr{C}}(\mathfrak{g})$.

The generators of $\tilde{\mathscr{C}}(\mathfrak{g})$ are the $\tilde{X}_a(x)$ and I; their commutation relations are

$$[\tilde{X}_a(x), \tilde{X}_b(y)] = C^c_{ab}\tilde{X}_c(x)\delta(x-y) + K_{ab}\delta'(x-y)I, \tag{4.34}$$

$$[\tilde{X}_a(x), I] = 0 \tag{4.35}$$

(cf. (4.30)) where $\delta'(x-y)$ indicates the derivative of the delta function $\delta(x-y)$ with respect to the argument. Here

$$K_{ab} = \langle X_a, X_b \rangle, \tag{4.36}$$

where the X_a are the generators of $\mathfrak{g}$.

The dual space $\tilde{\mathscr{C}}^*(\mathfrak{g})$ of the Lie algebra $\tilde{\mathscr{C}}(\mathfrak{g})$ is formed by elements $\tilde{u}$ with coordinates $(u_a(x), c)$; the corresponding pairing is given by

$$\tilde{u}(\tilde{\xi}) = (\tilde{u}, \tilde{\xi}) = \int_{-L}^{L} u_a(x)\xi^a(x)\,dx + c\sigma. \tag{4.37}$$

The coadjoint action of the center of $\tilde{\mathscr{C}}(\mathfrak{g})$ is trivial, so that the $\tilde{\mathrm{a}}\mathrm{d}^*$-action of $\tilde{\mathscr{C}}(\mathfrak{g})$ reduces to the action of $\mathscr{C}(\mathfrak{g})$ which will be denoted by the same symbol. By definition, we have

$$\tilde{\mathrm{a}}\mathrm{d}^*\xi\cdot\tilde{u}(x) = \left(C^c_{ab}u_c(x)\xi^b(x) + cK_{ab}\frac{d\xi^b}{dx}, 0\right), \tag{4.38}$$

where $\xi(x) = \xi^a(x)X_a$. If K_{ab} is nondegenerate, which will be assumed from here on, (4.38) can be written in an elegant manner

$$(\tilde{\mathrm{a}}\mathrm{d}^*\xi\cdot U)(x) = c\frac{d\xi(x)}{dx} + [\xi(x), U(x)], \tag{4.39}$$

where $U(x)$ is a $\mathfrak{g}$-valued function associated with $\tilde{u} = (u_a(x), c)$ according to

$$U(x) = u_a(x)A^a, \qquad A^a = K^{ab}X_b \tag{4.40}$$

(cf. (4.14)). The last formula defines an identification of the dual space $\mathscr{C}^*(\mathfrak{g})$ with $\mathscr{C}(\mathfrak{g})$.

The action $\tilde{\mathrm{a}}\mathrm{d}^*$ of the Lie algebra $\mathscr{C}(\mathfrak{g})$ lifts to the action of the Lie group $\mathscr{C}(G)$ consisting of periodic functions $g(x)$ with values in the Lie group G:

$$(\tilde{\mathrm{A}}\mathrm{d}^*g\cdot U)(x) = c\frac{dg(x)}{dx}g^{-1}(x) + g(x)U(x)g^{-1}(x), \tag{4.41}$$

which is an extension of the usual action Ad* (by conjugation).

It is appropriate to compare the last formula with the gauge transformation introduced in § I.2 of Part I. The comparison shows that the element $\tilde{u}=(u_a(x), c)$ is conveniently associated with the differential operator

$$L=c\frac{d}{dx}-U(x). \tag{4.42}$$

Then the $\tilde{\mathrm{A}}\mathrm{d}^*$*-action of* $\mathscr{C}(G)$ *can be expressed as*

$$\tilde{\mathrm{A}}\mathrm{d}^* g\cdot L=g(x)Lg^{-1}(x), \tag{4.43}$$

where the right hand side is understood to be the composition of multiplication operators by the functions $g(x)$ and $g^{-1}(x)$ with the differential operator L. The monodromy "matrix"

$$T(\tilde{u})=\overset{\curvearrowleft}{\exp}\frac{1}{c}\int_{-L}^{L}U(x)\,dx \tag{4.44}$$

is a G-valued functional on $\tilde{\mathscr{C}}^*(\mathfrak{g})$; its transformation under $\tilde{\mathrm{A}}\mathrm{d}^*$ is given by $T\to g(L)Tg^{-1}(L)$ (where we have used that $g(x)$ is periodic). *Hence the invariants of the finite-dimensional action* Ad *of* G

$$\mathrm{Ad}\, g\cdot T=gTg^{-1} \tag{4.45}$$

are invariant under the action $\tilde{\mathrm{A}}\mathrm{d}^*$ *of* $\mathscr{C}(G)$ *and generate the algebra* $I(\tilde{\mathscr{C}}(\mathfrak{g}))$ *of Casimir functions of* $\tilde{\mathscr{C}}(\mathfrak{g})$. In fact, if the monodromy matrices $T(\tilde{u}_1)$ and $T(\tilde{u}_2)$ are conjugate to one another in G,

$$T(\tilde{u}_2)=g\,T(\tilde{u}_1)g^{-1}, \tag{4.46}$$

then

$$g(x)=F_2(x)gF_1^{-1}(x), \tag{4.47}$$

where

$$L_1F_1(x)=0, \qquad L_2F_2(x)=0 \tag{4.48}$$

and

$$F_1(x)|_{x=-L}=F_2(x)|_{x=-L}=I, \tag{4.49}$$

is a periodic function and so belongs to $\mathscr{C}(G)$, and

$$\tilde{\mathrm{A}}\mathrm{d}^* g\cdot U_1(x)=U_2(x). \tag{4.50}$$

Suppose now that the Lie algebra $\mathfrak{g}$ admits the splitting (4.1). Then in the reasoning of Subsection 1 $\mathfrak{g}$ may be replaced by $\mathscr{C}(\mathfrak{g})$. More precisely, in accord with (4.1) we split $\mathscr{C}(\mathfrak{g})$ into a linear sum of two subalgebras,

$$\mathscr{C}(\mathfrak{g})=\mathscr{C}_+(\mathfrak{g})+\mathscr{C}_-(\mathfrak{g}), \tag{4.51}$$

where $\mathscr{C}_\pm(\mathfrak{g})=\mathscr{C}(\mathfrak{g}_\pm)$, and define a second Lie bracket on $\tilde{\mathscr{C}}(\mathfrak{g})$ by

$$\begin{aligned}&[(\xi(x),\sigma),(\eta(x),\tau)]_0\\&=([\xi_+(x),\eta_+(x)]-[\xi_-(x),\eta_-(x)],\omega(\xi_+,\eta_+)-\omega(\xi_-,\eta_-)),\end{aligned} \tag{4.52}$$

where $\xi(x)=\xi_+(x)+\xi_-(x)$, $\eta(x)=\eta_+(x)+\eta_-(x)$. The corresponding Lie algebra will be denoted by $\tilde{\mathscr{C}}_0(\mathfrak{g})$. It is obtained from the Lie algebra $\mathscr{C}_0(\mathfrak{g})$ with the commutator

$$[\xi(x),\eta(x)]_0=[R\,\xi(x),\eta(x)]+[\xi(x),R\eta(x)] \tag{4.53}$$

(cf. (4.4)), where the operator R is determined by the splitting (4.51) according to (4.3), by means of a central extension with the 2-cocycle

$$\omega_0(\xi,\eta)=\omega(R\xi,\eta)+\omega(\xi,R\eta). \tag{4.54}$$

Now we define the Lie-Poisson brackets $\{,\}_0$ on the phase space $\tilde{\mathscr{C}}^*(\mathfrak{g})$ and consider Hamilton's equations of motion

$$\frac{\partial\tilde{u}}{\partial t}=\{f,\tilde{u}\}_0, \tag{4.55}$$

with $\tilde{u}=(u_a(x),c)$ and $f(\tilde{u})$ lying in $I(\tilde{\mathscr{C}}(\mathfrak{g}))$. Comparing the general formula (4.11) of Subsection 1 with (4.39) we find that in terms of $U(x)=u_a(x)A^a$ equation (4.55) becomes

$$\frac{\partial U(x)}{\partial t}=c\frac{\partial V(x)}{\partial x}+[V(x),U(x)], \tag{4.56}$$

$$\frac{dc}{dt}=0, \tag{4.57}$$

with

$$V(x)=R\,\nabla f(u) \tag{4.58}$$

and

$$\nabla f(u) = \frac{\delta f}{\delta u_a(x)} X_a . \tag{4.59}$$

We see that the phase space $\tilde{\mathscr{C}}^*(\mathfrak{g})$ *stratifies into the Poisson submanifolds* $c = \text{const}$. *On the reduced phase space* $\mathscr{C}^*(\mathfrak{g})$ *with* $c=1$ *Hamilton's equation (4.55) turns into the zero curvature equation. Its integrals of the motion are given by elements of the Casimir algebra* $I(\tilde{\mathscr{C}}(\mathfrak{g}))$, *i.e. by invariants of the monodromy matrix (4.44) (for* $c=1$*). The functional dimension of* $I(\tilde{\mathscr{C}}(\mathfrak{g}))$ *is equal to the dimension of the Cartan subalgebra of* $\mathfrak{g}$.

This closes our description of the general scheme which gives rise to Hamilton's equations possessing integrals of the motion in involution and admitting a zero curvature representation. However, the previously discussed zero curvature representations connected with integrable models involve the matrices U and V that depend not only on x and t but also on the spectral parameter λ. This is a specialization of the general scheme for the case when the current algebra $C(\mathfrak{g})$ (see § 1) is taken in place of the Lie algebra $\mathfrak{g}$.

3. Realization of the general scheme for the Lie algebra $\mathscr{C}((\mathfrak{g}))$; the Riemann problem and a family of Poisson structures

Let us replace the Lie algebra $\mathfrak{g}$ in the discussion of Subsection 2 by the current algebra $C(\mathfrak{g})$ with the splitting (see § 1)

$$C(\mathfrak{g}) = C_+(\mathfrak{g}) + C_-(\mathfrak{g}) . \tag{4.60}$$

To be able to define the corresponding current group, we have to modify the definition given in § 1. Specifically, we shall suppose $C(\mathfrak{g})$ to consist of functions $\xi(\lambda)$ with values in $\mathfrak{g}$ which are analytic in $\mathbb{C}\backslash\{0\}$. The subalgebras $C_+(\mathfrak{g})$ and $C_-(\mathfrak{g})$ will then consist of functions $\xi_+(\lambda)$ and $\xi_-(\lambda)$ analytic in $\mathbb{C}$ or $(\mathbb{C}\backslash\{0\})\cup\{\infty\}$, respectively, with $\xi_-(\infty)=0$. (Other definitions of $C(\mathfrak{g})$ and $C_\pm(\mathfrak{g})$ are also possible, one of them will be encountered in § 5.) The Lie groups $C(G)$ and $C_\pm(G)$ consist of G-valued functions $g(\lambda)$ and $g_\pm(\lambda)$ analytic in their respective domains with the normalization condition $g_-(\infty) = I$. The factorization problem

$$g(\lambda) = g_+(\lambda) g_-(\lambda) \tag{4.61}$$

in the Lie group $C(G)$ may be interpreted as a Riemann problem for a contour which separates the points $\lambda = 0$ and $\lambda = \infty$.

Every invariant bilinear form $\langle , \rangle$ *on* $\mathfrak{g}$ *gives rise to an infinite family of invariant forms* $\langle , \rangle_p$ *on the current algebra* $C(\mathfrak{g})$ *given by*

$$\langle \xi, \eta \rangle_p = \operatorname{Res} \lambda^p \langle \xi(\lambda), \eta(\lambda) \rangle, \qquad p = -\infty, \ldots, \infty . \tag{4.62}$$

Each such form gives rise to a 2-cocycle on the Lie algebra $\mathscr{C}((\mathfrak{g})) = \mathscr{C}(C(\mathfrak{g}))$,

$$\omega_p(\xi,\eta) = \int_{-L}^{L} \operatorname{Res} \lambda^p \left\langle \xi(\lambda,x), \frac{d}{dx}\eta(\lambda,x)\right\rangle dx, \tag{4.63}$$

which defines its central extension $\tilde{\mathscr{C}}_p((\mathfrak{g}))$. The Lie algebra $\tilde{\mathscr{C}}_p((\mathfrak{g}))$ is spanned by the generators $X_{a,k}(x)$ and I with the commutator

$$[X_{a,k}(x), X_{b,l}(y)]_p = C^c_{ab} X_{c,k+l}\delta(x-y) + \delta_{k+l,-p-1} K_{ab}\delta'(x-y) I, \tag{4.64}$$

$$[X_{a,k}(x), I]_p = 0. \tag{4.65}$$

The splitting (4.60) allows us to introduce a second structure of a Lie algebra on $\tilde{\mathscr{C}}((\mathfrak{g}))$ and a related *family of Lie-Poisson brackets* $\{,\}_p$ on the reduced dual space $\mathscr{C}^*((\mathfrak{g}))$ with coordinates $u_{a,k}(x)$:

$$\{u_{a,k}(x), u_{b,l}(y)\}_p = \begin{cases} -C^c_{ab} u_{c,k+l}(x)\delta(x-y) - K_{ab}\delta_{k+l,-p-1}\delta'(x-y) \\ \text{for} \quad k,l \geqslant 0, \\ C^c_{ab} u_{c,k+l}(x)\delta(x-y) + K_{ab}\delta_{k+l,-p-1}\delta'(x-y) \\ \text{for} \quad k,l<0, \quad 0 \quad \text{otherwise}. \end{cases} \tag{4.66}$$

Of course, the non-ultralocal term $\pm K_{ab}\delta'(x-y)$ appears in only one line on the right hand side of (4.66), namely in the upper one for $p<0$ and in the lower one for $p>0$. For $p=0$ it does not appear at all, and so we recover the ultralocal Poisson brackets of § 1.

Thus, the discussion in § 1 is completely covered by the general scheme. It also becomes clear why it is natural to consider the differential operator L of the auxiliary linear problem and the invariants of the associated monodromy matrix.

In the general case, the coadjoint action $\tilde{\mathrm{ad}}^*_p$ of the Lie algebra $\mathscr{C}((\mathfrak{g}))$ on the reduced phase space $\mathscr{C}^*((\mathfrak{g}))$ is given by

$$\tilde{\mathrm{ad}}^*_p \xi \cdot U(x,\lambda) = \lambda^p \frac{d\xi}{dx}(x,\lambda) + [\xi(x,\lambda), U(x,\lambda)] \tag{4.67}$$

(cf. (4.39)). *Hence Hamilton's equations of motion with a Hamiltonian f from the Casimir algebra* $I(\tilde{\mathscr{C}}((\mathfrak{g})))$

$$\frac{\partial U(x,\lambda)}{\partial t} = \{f, U(x,\lambda)\}_p = \tilde{\mathrm{ad}}^* R\,\nabla f \cdot U(x,\lambda) \tag{4.68}$$

take the form of a zero curvature equation upon replacing U by

$$U_p(x,\lambda)=\lambda^{-p}\,U(x,\lambda). \tag{4.69}$$

Equations (4.68) for different p are essentially equivalent. More precisely, Hamilton's equation (4.68) for $p=p_1$ with the Hamiltonian f_1 can be written as an equation of the same form for $p=p_2$ with the Hamiltonian f_2 which is simply related to f_1. For instance, if f_1 is given by

$$f_1(U(x,\lambda))=\operatorname{Res}\lambda^N P(T(U(\cdot,\lambda))), \tag{4.70}$$

where P is some invariant of the monodromy matrix $T(U(\cdot,\lambda))$, then f_2 is given by

$$f_2(U(x,\lambda))=\operatorname{Res}\lambda^{N+p_2-p_1}P(T(U(\cdot,\lambda)\lambda^{p_1-p_2})). \tag{4.71}$$

For the proof we notice that since λ is a parameter in the definition of the monodromy matrix (4.44), there is a general expression for ∇f_1,

$$\nabla f_1(U(x,\lambda))=M(x,\lambda)\lambda^N, \tag{4.72}$$

where $M(x,\lambda)$ is a function of x and λ with values in $\mathfrak{g}$ which depends on λ only through $U(x,\lambda)$, $M(x,\lambda)=M(U(\cdot,\lambda),x)$. It follows that

$$\nabla f_2(U(x,\lambda)\lambda^{p_2-p_1})=\nabla f_1(U(x,\lambda)) \tag{4.73}$$

so that the zero curvature equations engendered by the corresponding Hamilton equations (4.68) coincide upon replacing $U(x,\lambda)\to U(x,\lambda)\lambda^{p_2-p_1}$.

The Poisson submanifolds $C^*_{N,M}$ for the Lie-Poisson bracket $\{,\}_0$ defined in § 1 are also Poisson submanifolds for the Lie-Poisson brackets $\{,\}_p$ for $p=-N,\ldots,M$. The expressions for the corresponding Lie-Poisson brackets may be obtained from (4.66) by setting $u_{a,k}=0$ for $k\geqslant N$, $k<-M$. Formula (4.69) shows that Hamilton's equation (4.68) expressed as a zero curvature equation is more conveniently studied on the phase space $C^*_{N-p,M+p}$. The corresponding elements $U_p(x,\lambda)$ in this case all have the same structure

$$U_p(x,\lambda)=\sum_{k=-M}^{N-1}u^{(p)}_{a,k}(x)A^a\lambda^{-k-1} \tag{4.74}$$

and the phase spaces can be identified with one another in a natural way. In the next section we shall see for the NS model, where $N=0$, $M=2$, that the brackets $\{,\}_p$ give rise to the Λ-operator and the hierarchy of Poisson structures introduced in § III.5 of Part I.

4. Geometric interpretation of the dressing procedure

The dressing procedure for the general zero curvature equation

$$\frac{\partial U(\lambda)}{\partial t} - \frac{\partial V(\lambda)}{\partial x} + [U(\lambda), V(\lambda)] = 0 \tag{4.75}$$

was explained in § I.6, with the following result. Given the input data $U(x, t, \lambda)$ and $V(x, t, \lambda)$ which are rational functions of λ with values in the Lie algebra $\mathfrak{g}$ of the Lie group G and satisfy (4.75), we were able to construct a new solution, $U^g(\lambda)$ and $V^g(\lambda)$, of (4.75) parametrized by an element g of $C(G)$, the Lie group of functions $g(\lambda)$ on the contour Γ in $\mathbb{C}$ with values in G. To do so we took a solution of the compatible system of equations

$$\frac{\partial F}{\partial x} = U(x, t, \lambda) F, \tag{4.76}$$

$$\frac{\partial F}{\partial t} = V(x, t, \lambda) F \tag{4.77}$$

and solved for each x and t the factorization problem

$$F g F^{-1}(\lambda) = (F g F^{-1})_+(\lambda)(F g F^{-1})_-(\lambda), \tag{4.78}$$

where the functions $h_+(\lambda) = (F g F^{-1})_+(\lambda)$ and $h_-(\lambda) = (F g F^{-1})_-(\lambda)$ have an analytic continuation inside or outside the contour Γ, respectively. After that we set

$$F^g = h_+^{-1} F = h_- F g^{-1}, \tag{4.79}$$

and then $U^g(\lambda)$ and $V^g(\lambda)$ were determined by

$$U^g = \frac{\partial F^g}{\partial x}(F^g)^{-1}, \qquad V^g = \frac{\partial F^g}{\partial t}(F^g)^{-1}. \tag{4.80}$$

This operation preserves the pole divisors of $U(\lambda)$ and $V(\lambda)$ or, in our new terminology, the Poisson submanifolds that have finite dimension for fixed x (see § I.6 and § 1).

Here we shall elucidate the meaning of dressing transformations regarded as transformations in the phase space $\mathscr{C}^*((\mathfrak{g}))$ of functions $U(x, \lambda)$.

It is not hard to see that if two successive transformations are defined by the functions $g_1(\lambda)$ and $g_2(\lambda)$ then

$$F^{g_1 g_2}(x, \lambda) = (F^{g_2})^{g_1}(x, \lambda). \tag{4.81}$$

In fact, using both expressions (4.79) and the elementary relations

$$(g_+f)_-=f_-, \quad (gf_-)_-=g_-f_- \tag{4.82}$$

we find

$$\begin{aligned}(F^{g_2})^{g_1}&=(F^{g_2}g_1F^{g_2^{-1}})_-F^{g_2}g_1^{-1}\\&=((Fg_2F)_+^{-1}Fg_1g_2F^{-1}(Fg_2F^{-1})_-^{-1})_-\cdot(Fg_2F^{-1})_-Fg_2^{-1}g_1^{-1}\\&=(Fg_1g_2F^{-1})_-\cdot F(g_1g_2)^{-1}=F^{g_1g_2}.\end{aligned} \tag{4.83}$$

However, in general there is no such group property for elements U^g of phase space. The point is that the expression for U in terms of F given by (4.76) is invariant with respect to right multiplication $F\to Fg$ which does not commute with the above action of $C(G)$, $F\to F^g$. This may be remedied by fixing the value of $F(x,\lambda)$ at one point x, say, at $x=-L$,

$$F(x,\lambda)|_{x=-L}=I. \tag{4.84}$$

The modified dressing

$$F^g=h_+^{-1}Fg_+=h_-Fg_-^{-1} \tag{4.85}$$

preserves the boundary condition (4.84) but is no longer an action of the group $C(G)$.

Remarkably, *there exists a group such that (4.85) defines its group action. It coincides with $C(G)$ as a point set, but has a different multiplication law*

$$g\circ f=f_+g_+g_-f_-, \tag{4.86}$$

where $g=g_+g_-$ and $f=f_+f_-$ (under the assumption that the factorization problem in $C(G)$ has a unique solution). We shall denote this group by $C_0(G)$, and let $C_+(G)$ and $C_-(G)$ be the subgroups whose elements have the same analyticity properties as g_+ and g_-, respectively. These subgroups commute with one another in $C_0(G)$ and the multiplication law in each of them is given by

$$g_+\circ f_+=f_+g_+, \quad g_-\circ f_-=g_-f_-. \tag{4.87}$$

A comparison with (1.22)–(1.23) shows that the Lie algebras of $C_0(G)$ and $C_\pm(G)$ coincide (up to the sign of the commutator) respectively with the Lie algebras $C_0(\mathfrak{g})$ and $C_\pm(\mathfrak{g})$ introduced in § 1.

Let us now verify that (4.85) *defines an action of* $C_0(G)$. The verification is quite similar to (4.83):

$$\begin{aligned}(F^f)^g &= (F^f g F^{f^{-1}})_- F^f g_-^{-1} \\ &= ((FfF^{-1})_+^{-1} Ff_+ gf_- F^{-1}(FfF^{-1})_-^{-1})_- \cdot Ff_-^{-1} g_-^{-1} \\ &= (Ff_+ g_+ g_- f_- F^{-1})_- \cdot F(g_- f_-)^{-1} = F^{g \circ f}.\end{aligned} \tag{4.88}$$

The formula

$$U^g(x, \lambda) = \frac{d}{dx} F^g (F^g)^{-1} = \frac{d}{dx} h_- \cdot h_-^{-1} + h_- U h_-^{-1} = \widetilde{\mathrm{Ad}}^* h_- \cdot U(x, \lambda) \tag{4.89}$$

(cf. (4.41)) carries this action over to the phase space $\mathscr{C}^*((\mathfrak{g}))$. Moreover, the transformation $U \to U^g$ is only defined in a certain extension of $C^*((\mathfrak{g}))$ since in general it violates the periodicity condition. We will not go into details here.

It is appropriate to compare the dressing formulae (4.78), (4.89) with the formulae (4.16)–(4.17) for the solutions of the equations of motion within the general scheme of Subsection 1. They have practically the same structure but the matrices to be factorized in (4.78) and (4.16) differ at first sight: the dressing procedure involves an arbitrary matrix $g(\lambda)$ conjugated by the solution $F(x, \lambda)$ of the auxiliary linear problem, whereas in order to solve the equations of motion we must factorize the function $\exp\{-t\,\nabla f(U(x, \lambda))\}$, where $f(U)$ belongs to the Casimir algebra $I(\widetilde{\mathscr{C}}((\mathfrak{g})))$. Nevertheless, it can be shown that

$$\exp\{-t\,\nabla f(U)\} = \hat{F} g_0(\lambda, t) \hat{F}^{-1}, \tag{4.90}$$

where $\hat{F}(x, \lambda)$ is a solution of the auxiliary linear equation for the initial value $U(x, \lambda)$ and $g_0(\lambda, t)$ takes values in a Cartan subgroup K of G that does not depend on t. In fact, consider a function $h(\lambda)$ which carries the monodromy matrix into a function with values in a fixed Cartan subgroup K,

$$T(U(\cdot, \lambda)) = h(\lambda) \hat{T}(U(\cdot, \lambda)) h^{-1}(\lambda) = h(\lambda) \exp C(\lambda) h^{-1}(\lambda), \tag{4.91}$$

where $C(\lambda)$ takes values in the corresponding Cartan subalgebra $\mathfrak{k}$. It follows from (4.46)–(4.50) that

$$U(x, \lambda) = \widetilde{\mathrm{Ad}}^* h(x, \lambda) \cdot \hat{U}_0(\lambda), \tag{4.92}$$

where

$$\hat{U}_0(\lambda) = \frac{1}{2L} C(\lambda), \quad h(x, \lambda) = F(x, \lambda) h(\lambda) \exp\left(-\frac{x+L}{2L} C(\lambda)\right) \tag{4.93}$$

and $F(x,\lambda)$ satisfies the auxiliary linear equation with the boundary condition (4.84). Recalling that the gradient of an invariant function transforms by conjugations, we have

$$\nabla f(U(x,\lambda)) = F(x,\lambda)\, h(\lambda)\, \nabla f(\hat{U}_0)\, h^{-1}(\lambda)\, F^{-1}(x,\lambda)\,. \tag{4.94}$$

Now, $\nabla f(\hat{U}_0)$ lies in the same Cartan subalgebra $\mathfrak{k}$ as $\hat{U}_0$. Hence (4.90) is an immediate consequence of (4.94). This observation shows that the dynamics flow is naturally incorporated in the general dressing group. In particular, this implies once again that the dressing procedure carries the set of solutions of the equations of motion into itself. However, unlike the dynamical transformations, the general dressing transformation is not a Hamiltonian one.

With this we close the description of a general geometric set-up for the inverse scattering method. Of course, our presentation was incomplete. The analytical justification of the formal infinite-dimensional constructions developed above is beyond the scope of the book. Nevertheless, we decided to outline the general scheme because it seems to be a faily elegant one and illuminates the main structures of the inverse scattering method used throughout the book in the study of particular models.

§ 5. The General Scheme as Illustrated with the NS Model

To close the book, we shall return to our basic example – the NS model – in order to show how the method for solving it developed in Part I agrees with the general geometric discussion of this chapter. More precisely, we will show in what sense the Riemann problem that was used to solve the initial value problem in Chapter III of Part I is interpreted as a factorization problem of § 4. Besides, we will relate the hierarchy of Poisson structures and the generating Λ-operator defined in § III.5, Part I, to the family of Poisson structures in Subsection 3 of § 4. This will provide a proof for the Jacobi identity as was promised in § III.5 of Part I.

When dealing with the Riemann problem we shall restrict our attention to the rapidly decreasing case and assume that there is no discrete spectrum. The solution of the initial problem for the NS equation via the Riemann problem given in § III.3, Part I, amounts to the following: given the initial data $\psi(x)$, $\bar{\psi}(x)$ one determines the transition coefficient $b(\lambda)$ and solves a family of regular Riemann problems

$$G(x,t,\lambda) = G_+(x,t,\lambda)\, G_-(x,t,\lambda)\,, \tag{5.1}$$

with

$$G(x,t,\lambda)=\begin{pmatrix} 1 & \varepsilon\bar{b}(\lambda)e^{-i\lambda x+i\lambda^2 t} \\ -b(\lambda)e^{i\lambda x-i\lambda^2 t} & 1 \end{pmatrix}$$
$$=e^{\frac{i\lambda^2 t}{2}\sigma_3}G(x,\lambda)e^{-\frac{i\lambda^2 t}{2}\sigma_3}, \quad \varepsilon=\operatorname{sign}\varkappa \tag{5.2}$$

and

$$G(x,\lambda)=G(x,0,\lambda). \tag{5.3}$$

The contour Γ is taken to be the real axis; the solution $G_{\pm}(x,t,\lambda)$ is required to have an analytic continuation into the half-plane $\pm\operatorname{Im}\lambda>0$ which is nondegenerate and normalized to I as $|\lambda|\to\infty$:

$$G_{\pm}(x,t,\infty)=I. \tag{5.4}$$

The matrix $U(x,t,\lambda)$ of the auxiliary linear problem is expressed in terms of $G_{\pm}(x,t,\lambda)$ as

$$U(x,t,\lambda)=-G_+^{-1}(x,t,\lambda)\frac{\partial G_+}{\partial x}(x,t,\lambda)+\frac{\lambda}{2i}G_+^{-1}(x,t,\lambda)\sigma_3 G_+(x,t,\lambda)$$
$$=\frac{\partial G_-}{\partial x}(x,t,\lambda)G_-^{-1}(x,t,\lambda)+\frac{\lambda}{2i}G_-(x,t,\lambda)\sigma_3 G_-^{-1}(x,t,\lambda). \tag{5.5}$$

With the notation of § 4 these formulae are compactly written as

$$U(t)=\widetilde{\mathrm{Ad}}^* G_+^{-1}(t)\left(\frac{\lambda\sigma_3}{2i}\right)=\widetilde{\mathrm{Ad}}^* G_-(t)\left(\frac{\lambda\sigma_3}{2i}\right), \tag{5.6}$$

where the dependence on x and λ is suppressed, as will often be done is what follows.

Using (5.6) and the group property of $\widetilde{\mathrm{Ad}}^*$, we find an expression for the solution $U(t)$ in terms of the initial data $U_0=U(t)|_{t=0}$,

$$U(t)=\widetilde{\mathrm{Ad}}^* h_+^{-1}(t)\,U_0=\widetilde{\mathrm{Ad}}^* h_-(t)\,U_0, \tag{5.7}$$

where the matrices $h_{\pm}(t)$ are given by

$$h_+(x,t,\lambda)=G_+^{-1}(x,0,\lambda)G_+(x,t,\lambda), \tag{5.8}$$

$$h_-(x,t,\lambda)=G_-(x,t,\lambda)G_-^{-1}(x,0,\lambda). \tag{5.9}$$

These matrices solve the factorization problem

$$h(t)=h_+(t)h_-(t), \tag{5.10}$$

where $h(t)$ is given by

$$h(x,t,\lambda)=G_+^{-1}(x,0,\lambda)e^{\frac{i\lambda^2 t}{2}\sigma_3}G_+(x,0,\lambda)$$
$$\times G_-(x,0,\lambda)e^{-\frac{i\lambda^2 t}{2}\sigma_3}G_-^{-1}(x,0,\lambda) \qquad (5.11)$$

and so is expressed through the solution of the Riemann problem (5.1) for $t=0$ which is uniquely determined by the initial data U_0.

Expressions (5.7) coincide with the general formulae (4.17) from Subsection 1 of § 4 for the solution of the initial value problem for the abstract Hamilton equation (4.10), which is a zero curvature equation as shown in Subsection 2 of § 4. However, the general factorization problem (4.16) differs from the Riemann problem (5.10): the former deals with the factorization of a one-parameter matrix subgroup $g(t)=\exp\{-t\nabla H(U_0)\}$ whereas the matrices $h(t)$ involved in the latter problem do not make up a one-parameter subgroup.

To coordinate the two ways of solving the initial value problem we notice that (5.7) determines $h_+^{-1}(t)$ and $h_-(t)$ up to right factors from the centralizer of U_0 with respect to the action $\widetilde{\mathrm{Ad}}^*$. Obviously, the matrices $F(x,\lambda)C(\lambda)F^{-1}(x,\lambda)$, where $F(x,\lambda)$ is a solution of the auxiliary linear problem with matrix $U_0(x,\lambda)$ and $C(\lambda)$ is an arbitrary matrix, lie in the centralizer. With this in mind we introduce the matrices

$$g_+(x,t,\lambda)=h_+(x,t,\lambda), \qquad (5.12)$$

$$g_-(x,t,\lambda)=h_-(x,t,\lambda)G_-(x,0,\lambda)e^{\frac{i\lambda^2 t}{2}\sigma_3}G_-^{-1}(x,0,\lambda), \qquad (5.13)$$

which satisfy

$$g_+(t)g_-(t)=g(t), \qquad (5.14)$$

with

$$g(x,t,\lambda)=G_+^{-1}(x,0,\lambda)e^{\frac{i\lambda^2 t}{2}\sigma_3}G_+(x,0,\lambda). \qquad (5.15)$$

The matrices $g(t)$ now form a one-parameter subgroup, and comparison with (4.90) shows that

$$g(t)=e^{-t\nabla H(U_0)}. \qquad (5.16)$$

Thus (5.14) *gives a realization of the abstract factorization problem for the case of the NS model.* Expressions (5.12)–(5.13) indicate the functional classes that contain the required matrices $g_\pm(x,t,\lambda)$: these have an analytic continuation into the half-planes $\pm\mathrm{Im}\,\lambda>0$ with the following asymptotic behaviour, as $|\lambda|\to\infty$,

$$g_+(x,t,\lambda)=I+O\left(\frac{1}{|\lambda|}\right), \tag{5.17}$$

$$g_-(x,t,\lambda)=\left(I+O\left(\frac{1}{|\lambda|}\right)\right)e^{\frac{i\lambda^2 t}{2}\sigma_3}\left(I+O\left(\frac{1}{|\lambda|}\right)\right). \tag{5.18}$$

We thus have a formal agreement between the concrete Riemann problem for the NS model and the abstract factorization problem of Subsection 1, § 4. It should be noted, however, that we have not stated the abstract factorization problem in the infinite-dimensional group $\mathscr{C}((G))$ (or even in the corresponding Lie algebra) for the case of rapidly decreasing boundary conditions. Therefore the above reasoning should be regarded as defining such a problem on a particular orbit associated with the NS model. This example shows that the application of the general scheme of § 4 to a given nonlinear equation associated with a particular orbit requires additional analytical investigation of the corresponding auxiliary linear problem, which would give rise to a suitable Riemann problem. This concludes our discussion of the role of the factorization problem in solving the initial value problem for integrable nonlinear equations.

We shall now proceed to describe the geometric meaning of the Λ-operator defined in § III.5, Part I, and the associated hierarchy of Poisson structures. We recall the relevant definitions (for simplicity, we consider only the rapidly decreasing case and assume $\varkappa=-1$). The phase space $\mathscr{M}_0$ with coordinates $\psi(x)$, $\bar{\psi}(x)$ is equipped with the basic Poisson structure

$$\{f,g\}=\langle \operatorname{grad} f, \operatorname{grad} g\rangle = i\int_{-\infty}^{\infty} \operatorname{tr}(\operatorname{grad} f(x)\sigma_3 \operatorname{grad} g(x))\,dx, \tag{5.19}$$

where for any observable f

$$\operatorname{grad} f(x)=\frac{1}{i}\left(\frac{\delta f}{\delta\psi(x)}\sigma_+ + \frac{\delta f}{\delta\bar{\psi}(x)}\sigma_-\right). \tag{5.20}$$

In addition to (5.19) we have introduced a hierarchy of Poisson structures

$$\begin{aligned}\{f,g\}_{(k)}&=\langle \operatorname{grad} f, \Lambda^k \operatorname{grad} g\rangle\\ &= i\int_{-\infty}^{\infty} \operatorname{tr}(\operatorname{grad} f(x)\sigma_3\Lambda^k \operatorname{grad} g(x))\,dx,\end{aligned} \tag{5.21}$$

$k=-\infty,\dots,\infty$. Here Λ is an integro-differential operator acting on off-diagonal matrices $F(x)$ according to

$$\Lambda F(x)=i\sigma_3\left(\frac{dF}{dx}(x)-[U_0(x),d^{-1}([U_0(\cdot),F(\cdot)])(x)]\right), \qquad (5.22)$$

where

$$U_0(x)=i(\psi(x)\sigma_-+\bar{\psi}(x)\sigma_+) \qquad (5.23)$$

and $\langle,\rangle$ stands for the bilinear form given by the integral in (5.19). Obviously, $\{,\}=\{,\}_{(0)}$.

In § III.5, Part I, the Jacobi identity for the Poisson brackets $\{,\}_{(k)}$ was left unverified. *Here we shall indicate the geometric meaning of the Poisson structure* $\{,\}_{(1)}$ *and prove the Jacobi identity.*

In Subsection 3 of the preceding section we defined a family of Poisson structures $\{,\}_p, p=-N,\dots,M$, on the phase space $C^*_{\mathfrak{N},M}$. In particular, it was shown that the zero curvature equation for a matrix $U(x,\lambda)$ of the form

$$U(x,\lambda)=\lambda J+Q(x), \qquad (5.24)$$

where for $\mathfrak{g}=su(2)$ we have

$$J=iJ_a\sigma_a, \qquad Q(x)=iQ_a(x)\sigma_a, \qquad (5.25)$$

may be written in Hamiltonian form in three ways; in what follows we shall be interested in only two of them. The first one involves the phase space $C^*_{\mathfrak{d},2}$ consisting of matrices $U(x,\lambda)$ of the form (5.24) with the Poisson bracket $\{,\}_0$:

$$\{J_a,J_b\}_0=0, \qquad \{J_a,Q_b(x)\}_0=0, \qquad (5.26)$$

$$\{Q_a(x),Q_b(y)\}_0=\varepsilon_{abc}J_c\delta(x-y), \qquad (5.27)$$

the second one makes use of the phase space $C^*_{\mathfrak{f},1}$ consisting of matrices $\tilde{U}(x,\lambda)=J+\dfrac{Q(x)}{\lambda}$ with the Poisson bracket $\{,\}_{-1}$:

$$\{J_a,J_b\}_{-1}=0, \qquad \{J_a,Q_b(x)\}_{-1}=0, \qquad (5.28)$$

$$\{Q_a(x),Q_b(y)\}_{-1}=-\varepsilon_{abc}Q_c(x)\delta(x-y)+\tfrac{1}{2}\delta_{ab}\delta'(x-y). \qquad (5.29)$$

The zero curvature equation in the latter case is written for the matrix $U(x,\lambda)=\lambda\,\tilde{U}(x,\lambda)$ which has the same structure as (5.24).

The NS model is associated with a particular orbit in $C^*_{\mathfrak{d},2}$ given by

$$J_1=J_2=0, \qquad J_3=-\tfrac{1}{2}, \qquad Q_3(x)=0, \qquad (5.30)$$

which is identified with the phase space $\mathscr{M}_0$ by setting

$$\psi(x)=Q_1(x)+iQ_2(x) \tag{5.31}$$

(see example 2 in § 1). However, $\mathcal{M}_0$ is not a Poisson submanifold with respect to the Poisson bracket $\{,\}_{-1}$.

We may, nevertheless, reduce the Poisson structure $\{,\}_{-1}$ to the manifold $\mathcal{M}_0$ if we regard the equations $Q_3(x)=0$ as constraints. To do so we shall calculate explicitly the corresponding Poisson-Dirac bracket given by

$$\{f,g\}^*_{-1}=\{f,g\}_{-1}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{f,Q_3(x)\}_{-1}K^{-1}(x,y)\{Q_3(y),g\}_{-1}\,dx\,dy, \tag{5.32}$$

where $K^{-1}(x,y)$ is the kernel of the integral operator K^{-1} inverse to the operator K with the kernel $K(x,y)=\{Q_3(x),Q_3(y)\}_{-1}$. The right hand side of (5.32) must be restricted to the level surface $Q_3(x)=0$ of the constraints. From (5.29) we find

$$K(x,y)=\tfrac{1}{2}\delta'(x-y), \tag{5.33}$$

so that

$$K^{-1}(x,y)=\varepsilon(x-y), \tag{5.34}$$

where

$$\varepsilon(x)=\begin{cases}1 & \text{for } x>0,\\ -1 & \text{for } x<0.\end{cases} \tag{5.35}$$

In particular, letting formally $f=\psi(x)$ and $g=\psi(y)$ or $\bar{\psi}(y)$ we get the Poisson-Dirac brackets of the coordinates $\psi(x)$, $\bar{\psi}(x)$

$$\{\psi(x),\psi(y)\}^*_{-1}=\psi(x)\psi(y)\varepsilon(x-y), \tag{5.36}$$

$$\{\psi(x),\bar{\psi}(y)\}^*_{-1}=\delta'(x-y)-\psi(x)\bar{\psi}(y)\varepsilon(x-y). \tag{5.37}$$

The same result is obtained by calculating the Poisson brackets of $\psi(x)$, $\bar{\psi}(x)$ according to (5.21) for $k=1$. *Thus we have established that the Poisson structures* $\{,\}_{(1)}$ *and* $\{,\}^*_{-1}$ *coincide. This implies, in particular, the Jacobi identity for the Poisson bracket* $\{,\}_{(1)}$.

To prove the Jacobi identity for all the Poisson structures $\{,\}_{(k)}$ we shall apply the following method. Observe that the Poisson brackets $\{,\}_0$ and $\{,\}_{-1}$ on $C^*_{0,2}$ *are compatible in the following sense: for any* α *the bracket* $\{,\}^{(\alpha)}=\{,\}_{-1}+\alpha\{,\}_0$ *satisfies the Jacobi identity.* This is best verified in the coordinates $Q_a(x)$, $a=1,2,3$. From (5.26)-(5.27) and (5.28)-(5.29) we derive

$$\{Q_a(x), Q_b(y)\}^{(\alpha)} = \{Q_a(x), Q_b(y)\}_{-1} + \alpha\{Q_a(x), Q_b(y)\}_0$$
$$= -\varepsilon_{abc}\tilde{Q}_c(x)\delta(x-y) + \tfrac{1}{2}\delta_{ab}\delta'(x-y) \tag{5.38}$$

where $\tilde{Q}_a(x) = Q_a(x) = \alpha J_a$, $a = 1, 2, 3$. Hence the bracket $\{,\}^{(\alpha)}$ results from the Poisson bracket $\{,\}_{-1}$ by a change of variables $Q_a(x) \to \tilde{Q}_a(x)$ in phase space and therefore satisfies the Jacobi identity. The reduced Poisson brackets $\{,\}_{(0)}$ and $\{,\}_{(1)}$ on the phase space $\mathscr{M}_0$ are compatible as well. Let us now consider the symplectic form Ω_α associated with the Poisson bracket $\{,\}^{(\alpha)}$. It is a bilinear form in off-diagonal matrices $\xi(x)$, $\eta(x)$ given by

$$\Omega_\alpha(\xi, \eta) = \Omega_{(0)}(\xi, (\Lambda + \alpha)^{-1}\eta), \tag{5.39}$$

where Ω_0 is the symplectic form of the Poisson bracket $\{,\}_{(0)}$. Since Ω_α is closed, it follows that all the forms

$$\Omega_{(k)}(\xi, \eta) = \Omega_{(0)}(\xi, \Lambda^{-k}\eta), \tag{5.40}$$

are closed, which can easily be seen by expanding $(\Lambda + \alpha)^{-1}$ in powers of α near $\alpha = 0$ and $\alpha = \infty$. The form $\Omega_{(k)}$ is associated with the Poisson bracket $\{,\}_{(k)}$ so that the fact that the former is closed implies the Jacobi identity for the latter.

Of course, these arguments also show *the validity of the Jacobi identity for the more general Poisson bracket*

$$\{f, g\}_\varphi = \langle \operatorname{grad} f, \varphi(\Lambda)\operatorname{grad} g\rangle, \tag{5.41}$$

where φ is an arbitrary smooth function. Therefore the Poisson brackets $\{,\}_\varphi$ and $\{,\}_\chi$ are compatible for any functions φ and χ. The formal proof given above can be made precise by assuming that Λ has an inverse; if Λ has a nontrivial null-space (this is the case for the Λ-operator (5.22)), then the phase space $\mathscr{M}_0$ must be reduced by fixing the values of the functionals from the annihilator.

In § III.5 of Part I, we also indicated another role of Λ, that of a generating operator for a family of integrals of the motion I_n in involution related by

$$\operatorname{grad} I_n(x) = \Lambda \operatorname{grad} I_{n-1}(x). \tag{5.42}$$

Here we will show how this formula, which was obtained by a straightforward calculation in § III.5, Part I, results from simple geometric considerations and may serve to define the family I_n.

Suppose we are given two functionals I_1 and I_2 such that the associated Hamilton equations of motion on the phase space $\mathscr{M}_0$ relative to the Poisson *brackets* $\{,\}_{(1)}$ *and* $\{,\}_{(0)}$, *respectively, coincide with one another, i.e. for any observable f we have*

$$\{I_1, f\}_{(1)} = \{I_2, f\}_{(0)}. \tag{5.43}$$

Using the compatibility of the Poisson brackets $\{,\}_{(0)}$ *and* $\{,\}_{(1)}$ *we will show that this equation implies the existence of a family of functionals* I_n *that are in involution with respect to both Poisson brackets* $\{,\}_{(0)}$ *and* $\{,\}_{(1)}$ *and satisfy*

$$\{I_n, f\}_{(1)} = \{I_{n+1}, f\}_{(0)}. \tag{5.44}$$

For the proof it suffices to show the existence of a functional I_3 such that

$$\{I_2, f\}_{(1)} = \{I_3, f\}_{(0)}. \tag{5.45}$$

For that purpose we will show that the vector field

$$Xf = \{I_2, f\}_{(1)} \tag{5.46}$$

is (locally) Hamiltonian with respect to the Poisson bracket $\{,\}_{(0)}$ i.e.

$$X\{f, g\}_{(0)} = \{Xf, g\}_{(0)} + \{f, Xg\}_{(0)}. \tag{5.47}$$

The last formula can be written as

$$\{I_2, \{f, g\}_{(0)}\}_{(1)} = \{\{I_2, f\}_{(1)}, g\}_{(0)} + \{f, \{I_2, g\}_{(1)}\}_{(0)} \tag{5.48}$$

and follows from the Jacobi identity for the Poisson bracket $\{,\}_{(0)} + \{,\}_{(1)}$ and the equation

$$\{I_2, \{f, g\}_{(1)}\}_{(0)} = \{\{I_2, f\}_{(0)}, g\}_{(1)} + \{f, \{I_2, g\}_{(0)}\}_{(1)}, \tag{5.49}$$

which is derived from (5.43) in a similar way to (5.48).

The explicit expressions (5.19) and (5.21) for $\{,\}_{(0)}$ and $\{,\}_{(1)}$ yield that the functional I_3 can be determined from

$$\operatorname{grad} I_3(x) = \Lambda \operatorname{grad} I_2(x). \tag{5.50}$$

The above argument may be regarded as an existence proof for this equation (in a simply connected phase space).

The final expression for I_n is

$$I_n = \operatorname{grad}^{-1} \Lambda^{n-1} \operatorname{grad} I_1 = \operatorname{grad}^{-1} \Lambda^{n-m} \operatorname{grad} I_m. \tag{5.51}$$

Together with the definition (5.21) of the Poisson brackets $\{,\}_{(k)}$ this implies that the functionals I_n are in involution with respect to all these Poisson structures; we also have a relation generalizing (5.44),

$$\{I_k, f\}_{(l)} = \{I_m, f\}_{(n)}, \tag{5.52}$$

for $k+l=m+n$.

For the NS model, I_1 and I_2 may be chosen to be the number of particles N

$$N = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \tag{5.53}$$

and the momentum P

$$P = \frac{1}{2i} \int_{-\infty}^{\infty} \left(\bar{\psi} \frac{d\psi}{dx} - \psi \frac{d\bar{\psi}}{dx}\right) dx. \tag{5.54}$$

Relation (5.43) for the Poisson brackets $\{,\}_{(0)}$ and $\{,\}_{(1)}$ given by (5.19) and (5.36)-(5.37) is easily verified. *Thus the above discussion yields the existence of a Λ-operator, a hierarchy of Poisson structures $\{,\}_{(k)}$, and a family of functionals I_n which are in involution with respect to all these Poisson brackets.*

So, coming back to the NS model we have cast a new light on the related structures. The chain of ideas developed in the text is now closed, and this brings the book to its natural end.

§ 6. Notes and References

1. The Lie-Poisson bracket of the form (1.3) on the phase space $\mathfrak{g}^*$ was introduced and studied by S. Lie [L 1888]. This Poisson structure and the corresponding symplectic structure on the orbits of the coadjoint action of the Lie algebra $\mathfrak{g}$ was subsequently rediscovered by different people [B 1967], [K 1970], [S 1970], [Ki 1972]. For an up-dated treatment of the properties of the Lie-Poisson bracket and the theory of Poisson manifolds see [W 1983].

2. The term current algebra for the infinite-dimensional Lie algebra $C(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]$ is borrowed from quantum field theory. In the mathematical literature, the Lie algebra $C(\mathfrak{g})$ is called a loop algebra.

3. The construction of integrable systems using the decomposition of a Lie algebra into a linear sum of two subalgebras was proposed by B. Kostant for the case of the Toda model with free ends [K 1979a]. In [A 1979],

[RSF 1979], [RS 1979], [MM 1979], [S 1980], [RS 1981] the scheme was developed further and applied to a wide class of Lie algebras including infinite dimensional ones. The second Lie algebra structure with commutator $[\,,]_0$ was introduced in [MM 1979], [R 1980].

4. The connection between the r-matrix formulation and the Lie algebra $C_0(\mathfrak{g})$ was found in [RF 1983].

5. Formula (1.25) suggests an abstract definition of the r-matrix as given in [S 1983]: an r-matrix is an operator R in a Lie algebra $\mathfrak{g}$ such that the commutator defined by (1.25) satisfies the Jacobi identity. An important class of r-matrices consists of those operators R that satisfy the equation

$$[R\xi, R\eta]-R([R\xi,\eta]+[\xi,R\eta])=-[\xi,\eta] \tag{6.1}$$

for any ξ,η in $\mathfrak{g}$, which is the so-called modified Yang-Baxter equation (see [S 1983]). This is precisely the equation that holds for the integral operator R in $C(\mathfrak{g})$ with the kernel $r(\lambda-\mu)$ of the form (1.31), the integral being taken in the sense of principal value; for $\lambda\neq\mu$ it coincides with the usual Yang-Baxter equation (2.3).

6. In order to attach precise meaning to the symbols $A\otimes I$ and $I\otimes A$ in § 1, the Lie algebra $\mathfrak{g}$ must be assumed embedded into an associative algebra with unity (for instance, into the universal enveloping algebra $U(\mathfrak{g})$); clearly, this notation is "functorial".

7. A description of the algebra of Casimir functions of the Lie algebra $C_{N,M}(\mathfrak{g})$ is given in [T 1982], [KR 1983].

8. Multi-pole Poisson submanifolds discussed in § 1 have a simple interpretation in terms of the Lie algebra $\mathfrak{g}_{\mathbb{A}}$ over the adele ring $\mathbb{A}$ of the field $\mathbb{C}(\lambda)$ of rational functions on $\mathbb{C}$ (see [C 1983]). The analogue of the decomposition (1.16) is given by splitting $\mathfrak{g}_{\mathbb{A}}$ into the sum of the subalgebra of principal adeles $\mathfrak{g}(\lambda)$ consisting of rational functions in λ with values in $\mathfrak{g}$, and the subalgebra of integral adeles $\mathfrak{g}_{\mathbb{A}}^0$ (a reformulation of the resolution of a rational function into partial fractions). The elements $U(\lambda)$ of the form (1.55) make up a Poisson submanifold of $\mathfrak{g}(\lambda)$ with respect to the action of the subalgebra $\mathfrak{g}_{\mathbb{A}}^0$.

9. The classification of solutions of (2.2)–(2.3) with values in a simple algebra $\mathfrak{g}$ is given in [BD 1982]. This paper describes all trigonometric and elliptic r-matrices of the form $r(u)=X^{ab}(u)\,t_a\otimes t_b$ where the matrix $X^{ab}(u)$ is not identically degenerate and satisfies $X^{ab}(u)=\dfrac{\delta^{ab}}{u}+O(1)$ as $u\to 0$, and $\{t_a\}$ is an orthonormal basis in $\mathfrak{g}$ with respect to the Killing form; also, a large family of rational r-matrices is constructed. There turned out to be a close connection between the problem of classification of trigonometric r-matrices and the structure theory of affine Lie algebras.

10. A method for constructing trigonometric and elliptic r-matrices and the associated fundamental Poisson brackets by means of the averaging pro-

cedure was proposed in [RF 1983]. In fact, one does not get all the trigonometric r-matrices in this way; nevertheless, it may be shown that any such matrix is obtained by combining the averaging procedure with extension theory of linear operators due to von Neumann [BD 1982], [S 1983].

11. We point out that if two automorphisms of finite order, θ and θ', of the Lie algebra $\mathfrak{g}$ with abelian fixed point subalgebras differ by an inner automorphism, then the associated trigonometric r-matrices are equivalent [BD 1982].

12. The elliptic r-matrix for $n=2$ was introduced by E. K. Sklyanin in [S 1979] and extended to higher dimensions in [Be 1980]. In [T 1984] it was interpreted as a matrix analogue of the Weierstrass zeta function.

13. There is an analogue of the Lie algebra $C_0(\mathfrak{g})$ for the trigonometric and elliptic cases. The algebra $C(\mathfrak{g})$ and its subalgebra $C_+(\mathfrak{g})$ are the same as in the rational case, while the analogue of $C_-(\mathfrak{g})$ is defined by using the lattice Λ_1 or Λ_2. For instance, in the elliptic case it is defined as follows. Let $\mathscr{E}(\mathfrak{g})$ be the algebra of meromorphic functions $\xi(\lambda)$ on $\mathbb{C}$ with values in $\mathfrak{g}$ that satisfy the quasi-periodicity conditions

$$\xi(\lambda+\omega_i)=\theta_i\,\xi(\lambda), \qquad i=1,2, \tag{6.2}$$

and whose poles are at the points of the lattice Λ_2. We identify $\mathscr{E}(\mathfrak{g})$ with a subalgebra of $C(\mathfrak{g})$ by assigning to a function in $\mathscr{E}(\mathfrak{g})$ its Laurent series at $\lambda=0$. A function in $\mathscr{E}(\mathfrak{g})$ is uniquely determined by its principal part at $\lambda=0$, so that we have the decomposition

$$C(\mathfrak{g})=C_+(\mathfrak{g})+\mathscr{E}(\mathfrak{g}) \tag{6.3}$$

which defines the Lie algebra $C_0(\mathfrak{g})$. The operator $R=\frac{1}{2}(P_+-P_-)$ (see (1.24)) gives the elliptic r-matrix $r^{\Lambda_2}(\lambda)$. The linear space $\mathscr{E}(\mathfrak{g})$ is dual to the Lie algebra $C_+(\mathfrak{g})$ relative to the pairing (1.13), so the orbits of the coadjoint action of the subalgebra $C_+(\mathfrak{g})$ described in § 1 acquire a new functional realization in the space $\mathscr{E}(\mathfrak{g})$. In particular, the simplest orbit associated with the HM model turns into the one associated with the LL model. This interpretation is due to A. G. Reyman and M. A. Semenov-Tian-Shansky [RS 1986].

14. The simplest matrices $L(\lambda)$ that satisfy the fundamental Poisson brackets with a rational r-matrix for classical Lie algebras are given in [RF 1983].

15. An analytical justification of the multiplicative averaging (3.12) for the case of $\mathfrak{g}=sl(2)$ and the matrix $L(\lambda)$ of the LHM model was carried out in [RF 1983]. The infinite product (3.12) can be computed explicitly and gives the matrix $L(\lambda)$ for the LSG model discussed in § III.5.

16. The fundamental Poisson brackets in one site define a Poisson structure on the Lie group $C(G)$: the elements $L(\lambda)$ for all λ may be thought of as generators of the ring of functions on $C(G)$, and the Poisson bracket may be

extended by the "Leibnitz rule" to the whole ring of functions. The resulting Poisson bracket is one example in the class of Poisson brackets introduced in [D 1983]. The key property of such Poisson structures is that the operation of group multiplication is a Poisson mapping. This property formalizes the fact that the monodromy matrix $T_N(\lambda)$ for lattice models has the same Poisson brackets as the matrices $L_n(\lambda)$ (see § III.1). A Lie group with such a Poisson structure is called a Poisson-Lie group (or Hamilton-Lie group in [D 1983]). The quadratic Poisson bracket defined in [GD 1978] also provides an example of a Poisson-Lie group [S 1983].

17. A geometric theory of integrable lattice models is developed in [S 1983], [S 1985b]; the corresponding Poisson submanifolds and orbits have been described by V. G. Drinfeld (see [S 1985b]).

18. The fact that Casimir functions lying in $I(\mathfrak{g})$ are in involution with respect to the Poisson bracket $\{,\}_0$ – the "involutivity theorem" – is essentially contained in [K 1979a]. Its r-matrix formulation is given in [S 1983].

19. A method for solving the Hamilton equations of motion (4.10) induced by a Casimir function relative to the Poisson bracked $\{,\}_0$ by means of the factorization problem (4.16) in the Lie group G (the "factorization theorem") was proposed in [RS 1979], [RS 1981]. The idea of the method goes back to the work of V. E. Zakharov and A. B. Shabat [ZS 1979].

20. Finite-dimensional simple Lie algebras lead to integrable systems that generalize the Toda model with free ends to the case of an arbitrary root system; these models were introduced in [B 1976], where a Lax representation was found. The solution of the corresponding equations of motion and the study of the asymptotic dynamics was carried out in [K 1979b].

21. The central extension $\tilde{\mathscr{C}}(\mathfrak{g})$ of the current algebra $C(\mathfrak{g})$ (when $\mathfrak{g}$ is a simple Lie algebra) is an example of a Kac-Moody algebra – an affine Lie algebra of height 1 [K 1983]. The reason for introducing it is that the coadjoint action of the Lie group $\mathscr{C}(G)$ is given by gauge transformations and leads to the equations of motion in the form of a zero curvature condition. This was first observed in [RS 1980a] where these algebras were used for constructing integrable equations.

22. The introduction of the spectral parameter λ leads to infinitely many Casimir functions and hence enables one to construct meaningful examples of nonlinear equations which are completely integrable Hamiltonian systems. If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then associated with the Lie algebra $C(\mathfrak{g})$ (no x-dependence) there is a series of interesting finite-dimensional integrable systems: the generalized periodic Toda models, multi-dimensional tops in potential fields, and systems of interacting tops [RS 1979], [R 1980], [AM 1980], [RS 1981], [B 1984], [RS 1986]. In this case the factorization theorem immediately yields the linearization of the equations of motion on the Jacobian of the spectral curve, an algebraic curve defined by the equation $\det(U(\lambda)-\mu)=0$ [RS 1981]. This links the Lie-algebraic approach to integrable systems with the theory of finite-gap integration and Novikov's equations [N 1974], [DMN 1976].

23. "Switching on" the x-dependence may be regarded as the "two-dimensional extension" of finite-dimensional integrable systems. Thus, for instance, there is a natural two-dimensional extension of the periodic Toda models [Mi 1979]; the two-dimensional extension of tops leads to systems such as the matrix sine-Gordon model (see § I.8).

24. If in the zero curvature equation

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0 \tag{6.4}$$

the dependence on t disappears (the stationary case), then (6.4) takes the Lax form with respect to the matrix $V(x, \lambda)$,

$$\frac{dV}{dx} = [U, V]. \tag{6.5}$$

The Lie-algebraic scheme provides a simple interpretation for the Poisson structure of these stationary equations determined in [GD 1975], [BN 1976]: it coincides with the Lie-Poisson bracket for the finite-dimensional orbit passing through $V(x, \lambda)$ [R 1983], [FNR 1983b]. In addition, there is a compatible system of higher equations (6.4),

$$\frac{\partial U}{\partial t_n} - \frac{\partial V_n}{\partial x} + [U, V_n] = 0, \tag{6.6}$$

where each equation is equipped with its own time variable t_n, so that $t = t_N$, $V = V_N$; the NS model corresponds to $N = 3$. Since (6.6) is a compatible system, the matrices $V_n(x, t_1, \ldots; \lambda)$ also satisfy the equations

$$\frac{\partial V_n}{\partial t_k} - \frac{\partial V_k}{\partial t_n} + [V_n, V_k] = 0. \tag{6.7}$$

On the manifold of stationary solutions $\left(\frac{\partial}{\partial t_k} = 0\right)$ this gives rise to a compatible system of Novikov's equations

$$\frac{\partial V}{\partial t_n} = [V_n, V] \tag{6.8}$$

which also have Hamiltonian form with respect to the same Lie-Poisson bracket on the orbit [R 1983], [FNR 1983b].

25. The observation that different cocycles ω_p give different Poisson structures $\{,\}_p$ which produce the same family of integrable equations is due to [KR 1983].

26. Equations (6.6) may be put in Hamiltonian form, so that the densities $h_n(x, t_1, \ldots)$ of their Hamiltonians restricted to solutions of the equations of motion satisfy

$$\frac{\partial h_n}{\partial t_k} = \frac{\partial p_n^k}{\partial x}, \tag{6.9}$$

where the $p_n^k(x, t_1, \ldots)$ are the densities of the Hamiltonians for (6.7). The densities h_n can be chosen in such a way (by adding perfect derivatives, if necessary) that they are given, as well as the p_n^k, by a generating function $\tau(x, t_1, \ldots)$,

$$h_n = \frac{\partial^2}{\partial t_n \partial x} \log \tau, \quad p_n^k = \frac{\partial^2}{\partial t_n \partial t_k} \log \tau \tag{6.10}$$

[R 1983], [FNR 1983 b, c]. (In the case of the NS model, h_n should be taken as in (III.5.42), Part I). The function τ is the well-known τ-function of the Japanese authors [DJKM 1981] who introduced it on the basis of representation theory for affine Lie algebras. Originally the τ-function was defined by R. Hirota [H 1976] in a special setting, as a solution of a system of bilinear equations (Hirota's equations). These equations have subsequently got a natural interpretation in terms of representations of affine Lie algebras [DJKM 1981], [K 1983]. A different approach to the τ-function based on the analyticity properties of solutions of the auxiliary linear problem has been developed in [SW 1985].

27. The general Lie-algebraic scheme of § 4 applies not only to the current algebra but also the algebra of formal pseudo-differential operators [A 1979], [LM 1979]. This provides a natural interpretation for the Poisson structures introduced in [GD 1975]. The two-dimensional extension that uses the Maurer-Cartan 2-cocycle leads in this case to equations such as the Kadomtsev-Petviashvili equation [RS 1984].

28. The behaviour of Poisson brackets under dressing transformations has a natural explanation based on the theory of Poisson-Lie groups. Specifically, dressing transformations define a Poisson action with respect to a certain Poisson structure on the Lie group $C_0(G)$ [S 1985 b]. Infinitesimal dressing transformations may also be defined by direct methods. The vector fields that realize them are often associated with "hidden symmetries"; for more details see the survey [D 1984]. We notice that in general these vector fields are not Hamiltonian [S 1985 b].

29. An important class of dressing transformations which can be given explicitly are the so-called Bäcklund transformations defined by A. Bäck-

lund for the sine-Gordon equation [B 1882]; they are frequently met in modern literature on the inverse scattering method and soliton theory [M 1976].

30. Expression (5.16) can be made quite rigorous if, under the rapidly decreasing boundary conditions, one considers the current group $\mathscr{C}((G))$ with triangular asymptotics as $x \to \pm\infty$ (cf. the corresponding asymptotic formulae for the matrices $G_{\pm}(x, t, \lambda)$ in § II.1 of Part I). The invariants of the coadjoint action in this case are the minors of the reduced monodromy matrix $T(\lambda)$ (i.e. the coefficient $a(\lambda)$ for the NS model).

31. The fact that the Poisson bracket $\{,\}_{(1)}$ is a reduction in the sense of Dirac [D 1964] of the Lie-Poisson bracket $\{,\}_{-1}$ was observed in [RS 1980b].

32. The notion of compatible Poisson brackets was introduced in [Ma 1978] and elaborated in [GD 1979], [GD 1981] (we point out that the Jacobi identity for the Poisson bracket $\{,\}^{(\alpha)}$ for every α is a consequence of the Jacobi identity for a single value of $\alpha \neq 0$). The derivation of the Jacobi identity for the Poisson brackets $\{,\}_{(k)}$ given in § 5 follows [GD 1979].

33. The construction of the family of functionals I_n and the hierarchy of compatible Poisson brackets $\{,\}_{(k)}$ given in § 5 goes back to [Ma 1978] (see also [KR 1978]).

34. The HM model also possesses a Λ-operator and a hierarchy of Poisson structures. The corresponding second Poisson bracket $\{,\}_{(1)}$ is obtained from the Lie-Poisson bracket

$$\{S_a(x), S_b(y)\}_{(1)} = \tfrac{1}{2}\,\delta_{ab}\,\delta'(x-y) \tag{6.11}$$

by reducing to the orbit $\vec{S}^2 = 1$; the Λ-operator that arises coincides with the one given in [GY 1984].

35. The general Lie-algebraic scheme of § 4 also leads to interesting results if the Lie algebra $\mathfrak{g}$ is the algebra of vector fields on the circle, and the central extension of $C(\mathfrak{g})$ is determined by the Gelfand-Fuks 2-cocycle [GF 1968], which gives the Virasoro algebra. In particular, the second Poisson structure for the KdV equation associated with the Λ-operator coincides with the corresponding Lie-Poisson bracket [S 1985a].

36. The connection between the basic and the second Poisson structures for the KdV equation is established by the well-known Miura transformation [M 1968]. This was generalized in [DS 1981] to equations in the algebra of formal pseudo-differential operators, associated with higher order differential operators. More precisely, any affine Lie algebra gives rise to a sequence of the KdV-type equations and to several sequences of the modified KdV-type equations, their solutions being related by a generalized Miura transformation. The structure of this Miura transformation is determined by the Dynkin diagram of the affine Lie algebra [DS 1984].

37. A different approach to the classification of integrable equations having an a priori assigned functional form was proposed in [MS 1985] relying on classical methods of the theory of Lie-Bäcklund transformations.

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